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Parabolic Obstacle Problem with Measurable Data in Generalized Morrey Spaces

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Abstract. We study the global regularity in generalized Morrey spaces of the solutions to variational inequality and obstacle problem related to divergence form parabolic operator in bounded non-smooth domain. We impose minimal regularity conditions as to the coefficients of the operator so also to the boundary of the domain.

Keywords. Parabolic obstacle problem, generalized Morrey spaces, measurable coefficients, small BMO, Reifenberg flat domain

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1. Introduction

The obstacle problem for partial differential equations arises naturally in the classical elasticity theory as one of the simplest unilateral problems in the study of mechanics of elastic membranes. Roughly speaking, it aims to find the equilibrium position of an elastic membrane, the boundary of which is keeping fixed and which is constrained to stay above a prescribed obstacle. More generally, the obstacle problems provide a basic analytic tool in the study of variational inequalities and free boundary problems for PDEs and are involved in various geometric and potential theory problems such as capacities of sets or minimal surfaces. Their applications cover a broad spectrum of problems of modern technology, among them the study of fluid filtration in porous media, constrained heating, elasto-plasticity, optimal control problems in the theory of Brownian motion, phase transitions, groundwater hydrology, financial mathematics, and

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so on. We refer the reader to the classical texts by Murthy and Stampacchia [18], Kinderlehrer and Stampacchia [15], Friedman [12], Rodrigues [23] and Caffarelli [9] for further discussions and more details.

The present paper deals with regularity in generalized Morrey spaces of the weak solutions to variational inequalities for divergence form parabolic systems with measurable coefficients in non-smooth domains. In that sense, it provides a natural extension of the results in [3–6,8,11,13] which studied equations/systems without obstacle in the framework of different functional spaces.

Our work is motivated by the recent papers [1, 2, 24] where the authors developed a sort of Calderón–Zygmund theory for nonlinear elliptic and parabolic problems with irregular obstacles. To the difference of [1,2,24], we deal with differential operators having coefficients *only* measurable in *one* variable, say x^1, allowing this way quite arbitrary discontinuities in that direction, while in the other variables (x', t) they have *small mean oscillation* (small BMO). This situation is closely related to the equilibrium equations of linearly elastic laminates and composite materials which have been widely applied to various fields, see [10,17]. Even if there have been recently a lot of works in this direction, most of the obtained results consider single equations without obstacles. Another point of difference with [1, 2, 24] consists of the fact that we derive here a generalized version of the gradient estimate in the settings of the generalized Morrey spaces. Regarding the non-smooth domain considered here, we suppose that its boundary is flat in the sense of Reifenberg [22]. Loosely speaking, this means that the boundary is well approximated by hyperplanes at each point and at each scale, and is a sort of "minimal regularity" of the boundary guaranteeing the main results of the geometric analysis continue to hold true. For instance, C^1-smooth or Lipschitz continuous boundaries with small Lipschitz constants belong to that category. The class of Reifenberg flat domains extends beyond these common examples and contains domains with rough fractal boundaries such as the von Koch snowflake. In addition a domain which is flat in the sense of Reifenberg is also Jones-flat and possesses the extension properties. Moreover the Reifenberg condition implies the two-sided (A) condition (11) that ensures the existence of extension operator and hence also the trace operator on the boundary of Ω.

Turning back to our problem, let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$ and $Q = \Omega \times (0, T]$ be a cylinder in $\mathbb{R}^n \times \mathbb{R}_+$. Denote by ∂Q the usual parabolic boundary $\big\{\Omega \times \{t = 0\}\big\} \cup \big\{\partial\Omega \times [0, T]\big\}$. For a given vector function $\psi = (\psi^1, \dots, \psi^m) : Q \to \mathbb{R}^m$ satisfying

$$\begin{aligned} &\psi \in L^2(0, T; H^1(\Omega, \mathbb{R}^m)), \quad \psi_t \in L^2(Q; \mathbb{R}^m), \\ &\psi^i \leq 0 \text{ a.e. on } \partial Q, \quad i = 1, \dots, m, \end{aligned}$$

we define the admissible set $\mathcal{A}$ consisting of vector functions

$$\phi = (\phi^1, \dots, \phi^m) \in C^0(0, T; L^2(\Omega; \mathbb{R}^m)) \cap L^2(0, T; H_0^1(\Omega, \mathbb{R}^m))$$

such that

$$\phi^i(\cdot,0)=0 \text{ a.e. in } \Omega \quad \text{and} \quad \phi^i\geq\psi^i \text{ a.e. in } Q, \quad i=1,\dots,m.$$

Hereafter we adopt the standard summation convention on the repeated indexes, with $1\leq\alpha,\beta\leq n$ and $1\leq i,j\leq m$ where $m\geq 1$.

We are interested in vector-functions $\mathbf{u}=(u^1,\dots,u^m)\colon Q\to\mathbb{R}^m$ lying in $\mathcal{A}$ and such that

$$\begin{aligned}&\int_0^T\langle\phi^i_t,\phi^i-u^i\rangle\,dt+\int_Q A^{\alpha\beta}_{ij}(x,t)D_\beta u^j\cdot D_\alpha\left(\phi^i-u^i\right)\,dxdt\\&\geq\int_Q f^\alpha_i(x,t)\cdot D_\alpha\left(\phi^i-u^i\right)\,dxdt\end{aligned}\tag{1}$$

for all $\phi\in\mathcal{A}$ with $\phi_t\in L^2(0,T;H^{-1}(\Omega,\mathbb{R}^m))$, where $\mathbf{F}=\{f^\alpha_i\}\in L^2(Q,\mathbb{R}^{mn})$ is a given non-homogeneous term and $\langle\cdot,\cdot\rangle$ denotes the pairing between H^{-1} and H^1_0. Such a function $\mathbf{u}$ is called a *weak solution to the variational inequality* (1).

Throughout the paper, the tensor coefficients $A^{\alpha\beta}_{ij}\colon Q\to\mathbb{R}^{mn\times mn}$ are assumed to be uniformly elliptic and uniformly bounded, namely, we suppose that there exist positive constants λ and Λ such that

$$\lambda|\xi|^2\leq A^{\alpha\beta}_{ij}(x,t)\xi^i_\alpha\xi^j_\beta \quad \text{and} \quad \|A^{\alpha\beta}_{ij}\|_{L^\infty(Q,\mathbb{R}^{mn\times mn})}\leq\Lambda\tag{2}$$

for all matrices $\xi\in\mathbb{M}^{m\times n}$ and for almost every point $(x,t)\in Q$.

According to the classical theory of the variational inequalities ([1,9,15,24]), if $\mathbf{F}\in L^2(Q,\mathbb{R}^{mn})$, there exists a unique weak solution $\mathbf{u}\in\mathcal{A}$ of (1) satisfying the estimate

$$\left\||D\mathbf{u}|^2\right\|_{L^1(Q)}\leq c\left(\left\||\mathbf{F}|^2\right\|_{L^1(Q)}+\left\||\psi_t|^2\right\|_{L^1(Q)}+\left\||D\psi|^2\right\|_{L^1(Q)}\right)\tag{3}$$

with a positive constant c depending only on λ,Λ,m and $|Q|$.

This paper addresses the question of how the estimate (3) in L^1 can be replaced with the one in the generalized Morrey space under minimal regularity requirements on $A^{\alpha\beta}_{ij}$ and a lower level geometric assumptions on $\partial\Omega$, which will be specified in the next section.

2. Generalized parabolic Morrey spaces

Let us start with the description of the spaces that we are going to use and the definitions of the families of domains that we need:

- *parabolic cylinder* centered in $(y,\tau)\in\mathbb{R}^{n+1}$ and of radius $r>0$:

  $$\mathcal{I}_r(y,\tau)=\mathcal{B}_r(y)\times(\tau-r^2,\tau+r^2)=\left\{(x,t)\in\mathbb{R}^{n+1}\colon\ |x-y|<r,\ |t-\tau|<r^2\right\}$$

  with Lebesgue measure $|\mathcal{I}_r|=c(n)r^{n+2}$. For each fixed $(y,\tau)\in Q$ we write $Q_r=Q\cap\mathcal{I}_r(y,\tau)$.

- *parabolic cube* centered in $(y,\tau)=(y^1,y',\tau)$, $y'=(y_2,\dots,y^n)$:

  $$\mathcal{C}_r(y,\tau)=\big\{(x^1,x',t)\in\mathbb{R}^{n+1}:\ |x^1-y^1|<r,\ |x'-y'|<r,\ |t-\tau|<r^2\big\}$$

  with $|\mathcal{C}_r|=c(n)r^{n+2}$.
- x^1-*slice of* $\mathcal{C}_r(y,\tau)$ for some fixed $x^1\in(y^1-r,y^1+r)$:

  $$\mathcal{C}_r^{x^1}(y,\tau)=\big\{(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}:\ (x^1,x',t)\in\mathcal{C}_r(y,\tau)\big\}.$$

- *elliptic cubes* in $\mathbb{R}^n$ centered in $y=(y^1,y')$:

  $$\mathcal{C}'_r(y)=\big\{(x^1,x')\in\mathbb{R}^n:\ |x^1-y^1|<r,\ |x'-y'|<r\big\}$$

  with Lebesgue measure $|\mathcal{C}'_r|=c(n)r^n$.

In what follows, we use the letter c to denote a constant that can be explicitly computed in terms of known quantities such as λ,Λ,m,n,p and $|Q|$.

We call *weight* a positive measurable function $\varphi:\mathbb{R}^{n+1}\times\mathbb{R}_+\to\mathbb{R}_+$.

Definition 2.1. Let Q be a cylinder in $\mathbb{R}^{n+1}$. A function $f\in L^q(Q)$, $1<q<\infty$, belongs to the *generalized Morrey space* $L^{q,\varphi}(Q)$ if the following norm is finite

$$\|f\|_{L^{q,\varphi}(Q)}=\sup_{\substack{(y,\tau)\in Q\\ r>0}}\left(\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r}|f(x,t)|^q\,dxdt\right)^{\frac1q}.$$

If $\varphi\equiv r^\lambda$, $\lambda\in(0,n+2)$, then $L^{q,\varphi}$ coincides with the classical Morrey space $L^{q,\lambda}$. However, there exist examples of weights of more general form as $\varphi(r)=r\ln(r+2)$ or $\varphi(\mathcal{I}_r(y,\tau))=\Big(\int_{\mathcal{I}_r(y,\tau)}w(x,t)\,dxdt\Big)^\alpha$, $0<\alpha<1$, where $w\in A_q$ is Muckenhoupt weight with $q\in\big(1,\frac1\alpha\big)$ (see [19]). One more example is the following: the function $f(x)=\chi_{[-1,1]}|x|^{-\frac12}\in L^{1,\varphi}(\mathbb{R})$ with $\varphi(\mathcal{I})=\int_{\mathcal{I}}|x|^\alpha\,dx$, for $\alpha\in(-1,-\frac12)$, where $\mathcal{I}$ is any interval in $\mathbb{R}$.

Let $\mathcal{M}$ denote the *Hardy-Littlewood maximal operator* in $\mathbb{R}^{n+1}$. For any $f\in L^1_{\rm loc}(\mathbb{R}^{n+1})$ we have

$$\mathcal{M}f(y,\tau)=\sup_{r>0}\frac{1}{|\mathcal{I}_r(y,\tau)|}\int_{\mathcal{I}_r(y,\tau)}|f(x,t)|\,dxdt.$$

If $D\subset\mathbb{R}^{n+1}$ is a bounded domain and $f\in L^1(D)$, then $\mathcal{M}f=\mathcal{M}\tilde f$, where $\tilde f$ is the zero extension of f in $\mathbb{R}^{n+1}$. It is well known that $\mathcal{M}$ is a bounded sublinear operator from L^q into itself. Precisely, if $f\in L^q(\mathbb{R}^{n+1}),q\in(1,\infty)$, then

$$\int_{\mathbb{R}^{n+1}}|f(x,t)|^q\,dxdt\le\int_{\mathbb{R}^{n+1}}|\mathcal{M}f(x,t)|^q\,dxdt\le c(q,n)\int_{\mathbb{R}^{n+1}}|f(x,t)|^q\,dxdt.$$

Moreover, the following weak type estimate holds

$$\left|\{(x,t)\in\mathbb{R}^{n+1}:\mathcal{M}f(x,t)>\mu\}\right|\le\frac{c_q}{\mu^q}\int_{\mathbb{R}^{n+1}}|f(x,t)|^q\,dxdt,\tag{4}$$

for any $1\le q<\infty$ and any $\mu>0$.

Lemma 2.2 (Maximal inequality, [19]). *Assume that there are constants κ_1, κ_2, $\kappa_3>0$ such that for any fixed $(y,\tau)\in\mathbb{R}^{n+1}$ and any $r>0$ we have*

$$\kappa_1\le\frac{\varphi(\mathcal{I}_s(y,\tau))}{\varphi(\mathcal{I}_r(y,\tau))}\le\kappa_2\quad\text{for all } r\le s\le 2r,\tag{5}$$

$$\int_r^\infty\frac{\varphi(\mathcal{I}_s(y,\tau))}{s^{n+3}}\,ds\le\kappa_3\,\frac{\varphi(\mathcal{I}_r(y,\tau))}{r^{n+2}}.\tag{6}$$

Then, for any $1<q<\infty$, there is a constant $c_q>0$ such that

$$\|f\|_{L^{q,\varphi}(\mathbb{R}^{n+1})}\le\|Mf\|_{L^{q,\varphi}(\mathbb{R}^{n+1})}\le c_q\|f\|_{L^{q,\varphi}(\mathbb{R}^{n+1})}\quad\text{for all } f\in L^{q,\varphi}(\mathbb{R}^{n+1}).$$

Impose in addition a kind of monotonicity condition on φ, precisely

$$\varphi(\mathcal{I}_r(y,\tau))\le\varphi(\mathcal{I}_s(z,\xi))\quad\text{for all}\quad\mathcal{I}_r(y,\tau)\subset\mathcal{I}_s(z,\xi).\tag{7}$$

This implies the boundedness of the quantity

$$\sup_{\substack{(y,\tau)\in Q\\ r>0}}\frac{|\mathcal{I}_r(y,\tau)\cap Q|}{\varphi\left(\mathcal{I}_r(y,\tau)\right)}\le\kappa_4,\tag{8}$$

with a positive constant κ_4 depending on n, φ and Q. In fact, since Q is a bounded domain, there exists $d>0$ such that $Q\subset\mathcal{I}_d(0,0)$. Then, if $r\ge 2d$ for any $(y,\tau)\in Q$ we have

$$\frac{|\mathcal{I}_r(y,\tau)\cap Q|}{\varphi(\mathcal{I}_r(y,\tau))}\le\frac{|Q|}{\varphi(\mathcal{I}_d(0,0))}.$$

On the other hand, if $0<r<2d$, then we see from (6) that $\kappa_3\frac{\varphi(\mathcal{I}_r(y,\tau))}{r^{n+2}}\ge\int_{2d}^\infty\frac{\varphi(\mathcal{I}_s(y,\tau))}{s^{n+3}}\,ds\ge\varphi(\mathcal{I}_{2d}(y,\tau))\int_{2d}^\infty\frac{1}{s^{n+3}}\,ds\ge\varphi(\mathcal{I}_d(0,0))\frac{1}{(n+2)(2d)^{n+2}}$. It implies that for some positive constant $c=c(n)$ it holds

$$\frac{|\mathcal{I}_r(y,\tau)\cap Q|}{\varphi(\mathcal{I}_r(y,\tau))}\le\frac{cr^{n+2}}{\varphi\left(\mathcal{I}_r(y,\tau)\right)}\le\frac{c\kappa_3(n+2)(2d)^{n+2}}{\varphi(\mathcal{I}_d(0,0))}.$$

Suppose now that $f\in L^{p,\varphi}(Q)$ with $p\in(2,\infty)$ and φ satisfying (7), then $f\in L^p(Q)$. Precisely, for a fixed $(y,\tau)\in Q$ we have

$$\sup_{(z,\xi)\in Q}\max\left\{|y-z|,\sqrt{|\tau-\xi|}\right\}<\operatorname{diam}Q.$$

Hence there exists $r^* < \operatorname{diam} Q$ such that $Q \subset \mathcal{I}_{r^*}(y,\tau)$ and this gives that

$$\|f\|_{L^p(Q)} \le \varphi(\mathcal{I}_{r^*}(y,\tau))^{\frac{1}{p}}\|f\|_{L^{p,\varphi}(Q)} \le \varphi(\mathcal{I}_{2d}(0,0))^{\frac{1}{p}}\|f\|_{L^{p,\varphi}(Q)}.$$

Then the Hölder inequality implies

$$\|f\|^2_{L^2(Q)} \le |Q|^{1-\frac{2}{p}}\||f|^2\|_{L^{\frac{p}{2}}(Q)} \le |Q|^{1-\frac{2}{p}}\varphi(\mathcal{I}_{2d}(0,0))^{\frac{2}{p}}\||f|^2\|_{L^{\frac{p}{2},\varphi}(Q)}. \tag{9}$$

3. Statement of the problem and main result

Our goal is to derive regularity estimate for the weak solution to the variational inequality (1) in the framework of the generalized Morrey spaces. More precisely, under additional regularity assumptions on the coefficients in (1) and a suitable geometric condition on the boundary of Ω, we will show that for every $p \in (2,\infty)$ and for every φ satisfying (5)–(7), it holds that

$$|D\mathbf{u}|^2 \in L^{\frac{p}{2},\varphi}(Q)$$

provided

$$|\mathbf{F}|^2 \in L^{\frac{p}{2},\varphi}(Q) \quad \text{and} \quad |\psi_t|^2,\ |D\psi|^2 \in L^{\frac{p}{2},\varphi}(Q).$$

To do this, we first define integral average of $A^{\alpha\beta}_{ij}$ over x^1-slice of $\mathcal{C}_r(y,\tau)$, $x^1 \in (y^1 - r, y^1 + r)$,

$$\overline{A}^{\alpha\beta}_{ij\ \mathcal{C}^{x^1}_r(y,\tau)}(x^1) = \frac{1}{|\mathcal{C}^{x^1}_r(y,\tau)|}\int_{\mathcal{C}^{x^1}_r(y,\tau)} A^{\alpha\beta}_{ij}(x^1,x',t)\,dx'dt.$$

Definition 3.1. We say that $(A^{\alpha\beta}_{ij},\Omega)$ are (δ,R)-vanishing of codimension 1, if the following properties are satisfied:

- For every point $(y,\tau) \in Q$ and for every number $r \in (0,\frac{1}{3}R]$ with

  $$\operatorname{dist}(y,\partial\Omega) = \min_{x\in\partial\Omega}\operatorname{dist}(y,x) > \sqrt{2}r,$$

  there exists a coordinate system depending on (y,τ) and r, whose variables we still denote by (x,t) so that in this new coordinate system $(y,\tau) \equiv (0,0)$ is the origin and

  $$\frac{1}{|\mathcal{C}_r(0,0)|}\int_{\mathcal{C}_r(0,0)}\left|A^{\alpha\beta}_{ij}(x,t) - \overline{A}^{\alpha\beta}_{ij\ \mathcal{C}^{x^1}_r(0,0)}(x^1)\right|^2 dxdt \le \delta^2.$$

- For any point $(y,\tau) \in Q$ and for every number $r \in (0, \frac{1}{3}R]$ such that

$$\operatorname{dist}(y,\partial\Omega) = \min_{x\in\partial\Omega} \operatorname{dist}(y,x) = \operatorname{dist}(y,x_0) \le \sqrt{2}r,$$

there exists a coordinate system depending on (y,τ) and r, whose variables we still denote by (x,t) such that in this new coordinate system $(x_0,\tau) \equiv (0,0)$ is the origin and Ω verifies the *Reifenberg condition*

$$\Omega\cap\{x\in\mathcal{C}'_{3r}(0)\colon x^1 > 3r\delta\} \subset \Omega\cap\mathcal{C}'_{3r}(0) \subset \Omega\cap\{x\in\mathcal{C}'_{3r}(0)\colon x^1 > -3r\delta\} \quad (10)$$

while the coefficients have small BMO with respect to (x',t)

$$\frac{1}{|\mathcal{C}_{3r}(0,0)|}\int_{\mathcal{C}_{3r}(0,0)} \left| A_{ij}^{\alpha\beta}(x,t) - \overline{A}_{ij}^{\alpha\beta}{}_{\mathcal{C}_{3r}^{x^1}(0,0)}(x^1)\right|^2 dxdt \le \delta^2.$$

Some remarks are in order to clarify the notion just introduced. If $(A_{ij}^{\alpha\beta},\Omega)$ are (δ,R)-*vanishing of codimension* 1, then, for each point and for each small scale, there is a coordinate system such that the coefficients have *small bounded mean oscillation* (briefly BMO) in the (x',t)-variables with *no regularity* required with respect to x^1, that is, the coefficients can be only *measurable* in the direction x^1.

The domain Ω is (δ,R)-*Reifenberg flat* (see [22,26]). Moreover, (10) implies (cf. [20,21]) existence of a constant $\gamma = \gamma(\delta,n,\partial\Omega) \in (0,\frac{1}{2})$ such that

$$\gamma\,|\mathcal{C}'_{3r}(0)| \le |\mathcal{C}'_{3r}(0)\cap\Omega| \le (1-\gamma)|\mathcal{C}'_{3r}(0)| \quad (11)$$

for each cube $\mathcal{C}'_{3r}(0)$ centered in some point $x_0 \in \partial\Omega$ that we call 0 and $r \in (0,\frac{R}{3}]$.

The constant δ will be determined later, it belongs to $(0,\frac{1}{8})$ and it is invariant under a scaling (see Lemma 4.3). Moreover, by means of the scaling invariant property of the problem (1), (2), the constant R can be assumed to be any value greater than or equal to 1.

Finally, the numbers $\sqrt{2}r$ and $3r$ are choosen on purpose since we need enough space to make rotate the cylinder $\mathcal{C}_r(y,\tau)$ in any spatial direction.

Theorem 3.2. *For any given* $p \in (2,\infty)$ *and weight* φ *satisfying* (5)–(7)*, suppose that* $|\mathbf{F}|^2 \in L^{\frac{p}{2},\varphi}(Q)$ *and* $|\psi_t|^2$, $|D\psi|^2 \in L^{\frac{p}{2},\varphi}(Q)$*. Then there exists a small constant* $\delta = \delta(\lambda,\Lambda,m,n,p,\varphi)$ *such that if the couple* $\left(A_{ij}^{\alpha\beta},\Omega\right)$ *is* (δ,R)*-vanishing of codimension* 1*, then* $|D\mathbf{u}|^2 \in L^{\frac{p}{2},\varphi}(Q)$ *and we have the estimate*

$$\left\||D\mathbf{u}|^2\right\|_{L^{\frac{p}{2},\varphi}(Q)} \le c\left(\left\||\mathbf{F}|^2\right\|_{L^{\frac{p}{2},\varphi}(Q)} + \left\||\psi_t|^2\right\|_{L^{\frac{p}{2},\varphi}(Q)} + \left\||D\psi|^2\right\|_{L^{\frac{p}{2},\varphi}(Q)}\right), \quad (12)$$

where c *is a positive constant depending on* $\lambda,\Lambda,m,n,p,\varphi$ *and* Q.

4. Auxiliary Results

In this section, we prove several preliminary results that we are going to use in the rest of the paper. The main tools in our approach are the Hardy-Littlewood maximal inequality and a Vitali type covering lemma.

Because of the scaling invariance property of the Reifenberg domain (cf. [6, Lemma 5.2]), we can take $R = 1$ hereafter. Fix $(y_0, \tau_0) \in Q$, take a parabolic cylinder $\mathcal{I}_r(y_0, \tau_0)$ and denote $Q_r = \mathcal{I}_r(y_0, \tau_0) \cap Q$. For the weak solution $\mathbf{u}$ of (1), we define the super-level sets

$$\mathfrak{C} = \{(x,t) \in Q_r \colon \mathcal{M}(|D\mathbf{u}|^2) > N^2\} \tag{13}$$

and

$$\begin{aligned}\mathfrak{D} = \{(x,t) \in Q_r \colon \mathcal{M}(|D\mathbf{u}|^2) > 1\} \cup \{(x,t) \in Q_r \colon \mathcal{M}(|\mathbf{F}|^2) > \delta^2\} \\ \cup \{(x,t) \in Q_r \colon \mathcal{M}(|\psi_t|^2 + |D\psi|^2) > \delta^2\}.\end{aligned} \tag{14}$$

Let us note that the sets are defined locally and for $N > 1$ the following inclusion holds

$$\mathfrak{C} \subset \mathfrak{D} \subset Q_r.$$

For a.a. $(y, \tau) \in \mathfrak{C}$ and for each $\rho > 0$ we define the function

$$\Theta(\rho) = \frac{|\mathfrak{C} \cap \mathcal{C}_\rho(y,\tau)|}{|\mathcal{C}_\rho(y,\tau)|}.$$

Then $\Theta \in C^0(0,\infty)$ and by the Lebesgue Differentiation Theorem

$$\Theta(0) = \lim_{\rho \to 0_+} \Theta(\rho) = 1, \qquad \lim_{\rho \to +\infty} \Theta(\rho) = 0.$$

Lemma 4.1. *Let Ω be a bounded $(\delta, 1)$-Reifenberg flat domain. Suppose that*

(i) *there exists $\varepsilon \in (0,1)$ such that $\Theta(1) < \varepsilon$ for a.a. $(y,\tau) \in \mathfrak{C}$;*

(ii) *for each $\rho > 0$ such that $\Theta(\rho) \geq \varepsilon$ it holds $Q_r \cap \mathcal{C}_\rho(y,\tau) \subset \mathfrak{D}$.*

Then

$$|\mathfrak{C}| \leq \varepsilon \left(\frac{10\sqrt{2}}{1-\delta} \right)^{n+2} |\mathfrak{D}|.$$

Proof. Since $\Theta(1) < \varepsilon$, there exists $\rho_{(y,\tau)} \in (0,1)$ such that $\Theta(\rho_{(y,\tau)}) = \varepsilon$ and $\Theta(\rho) < \varepsilon$ for all $\rho > \rho_{(y,\tau)}$.

Consider a family of parabolic cubes $\{\mathcal{C}_{\rho_{(y,\tau)}}(y,\tau)\}_{(y,\tau)\in\mathfrak{C}}$ which forms an open covering of $\mathfrak{C}$. By the Vitali covering lemma, there exists a disjoint subcollection $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\geq 1}$ with $\rho_i = \rho_{(y_i,\tau_i)} \in (0,1)$, $(y_i,\tau_i) \in \mathfrak{C}$ such that $\Theta(\rho_i) = \varepsilon$,

$$\sum_{i\geq 1} |\mathcal{C}_{\rho_i}(y_i,\tau_i)| \geq c(n)|\mathfrak{C}| \quad \text{and} \quad \mathfrak{C} \subset \bigcup_{i\geq 1} \mathcal{C}_{5\rho_i}(y_i,\tau_i).$$

Since $\Theta(5\rho_i) < \varepsilon$, we have

$$|\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)| < \varepsilon|\mathcal{C}_{5\rho_i}(y_i,\tau_i)| = \varepsilon 5^{n+2}|\mathcal{C}_{\rho_i}(y_i,\tau_i)|.$$

Furthermore, making use of the measure density condition obtained from the $(\delta,1)$-flatness condition, see [6], we get

$$|\mathcal{C}_{\rho_i}(y_i,\tau_i)| \leq \left(\frac{2\sqrt{2}}{1-\delta}\right)^{n+2} |Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)|.$$

Now we have $|\mathfrak{C}| = \left|\bigcup_{i\geq 1}\left(\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)\right)\right| \leq \sum_{i\geq 1}|\mathfrak{C} \cap \mathcal{C}_{5\rho_i}(y_i,\tau_i)| < \varepsilon\sum_{i\geq 1}|\mathcal{C}_{5\rho_i}(y_i,\tau_i)| \leq \varepsilon 5^{n+2}\sum_{i\geq 1}|\mathcal{C}_{\rho_i}(y_i,\tau_i)| \leq \varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}\sum_{i\geq 1}|Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)|$. Having in mind that $\Theta(\rho_i) = \varepsilon$, $\{\mathcal{C}_{\rho_i}(y_i,\tau_i)\}_{i\geq 1}$ are mutually disjoint, and condition (ii), we get

$$|\mathfrak{C}| \leq \epsilon_1\left|\bigcup_{i\geq 1} Q_r \cap \mathcal{C}_{\rho_i}(y_i,\tau_i)\right| \leq \epsilon_1|\mathfrak{D}|$$

with $\epsilon_1 = \varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}$. □

The next result follows from the standard measure theory.

Lemma 4.2. *Let $h \in L^1(Q)$ be a nonnegative function, φ be a weight satisfying (5)–(7), $q \in (1,\infty)$ and $\zeta > 0, \theta > 1$ be constants. Then $h \in L^{q,\varphi}(Q)$ if and only if*

$$\mathcal{S} := \sup_{\substack{(y,\tau)\in Q\\ r>0}} \sum_{k\geq 1} \frac{\theta^{kq}|\{(x,t)\in Q_r\colon h(x,t) > \zeta\theta^k\}|}{\varphi(\mathcal{I}_r(y,\tau))} < \infty.$$

Moreover,

$$\frac{1}{c}\mathcal{S} \leq \|h\|^q_{L^{q,\varphi}(Q)} \leq c(1+\mathcal{S}),$$

where $c = c(\theta,\zeta,q,\varphi,Q)$.

Proof. For a.a. $(y,\tau) \in Q$ we have

$$\begin{aligned}\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r} h^q(x,t)\,dxdt &= \frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{\{(x,t)\in Q_r: h\leq\zeta\theta\}} h^q(x,t)\,dxdt\\ &\quad+\sum_{k\geq 1}\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{\{(x,t)\in Q_r:\zeta\theta^k<h\leq\zeta\theta^{k+1}\}} h^q(x,t)\,dxdt\end{aligned}$$

and hence

$$
\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r} h^q(x,t)\,dxdt \\
&\le (\zeta\theta)^q \frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))} + \sum_{k\ge 1} \frac{(\zeta\theta^{k+1})^q}{\varphi(\mathcal{I}_r(y,\tau))} \left|\{(x,t)\in Q_r : h(x,t) > \zeta\theta^k\}\right| \\
&= (\zeta\theta)^q \left(\frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))} + \sum_{k\ge 1} \frac{\theta^{kq}\left|\{(x,t)\in Q_r : h(x,t) > \zeta\theta^k\}\right|}{\varphi(\mathcal{I}_r(y,\tau))} \right).
\end{aligned}
$$

Taking the supremum over $(y,\tau)\in Q, r>0$ and making use of (8), we get

$$
\|h\|^q_{L^{q,\varphi}(Q)} \le c\,(1+\mathcal{S})
$$

with a constant depending on $q, n, \varphi, \zeta, \theta$ and Q. On the other hand

$$
\begin{aligned}
&\frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r} h^q(x,t)\,dxdt \\
&= \frac{q}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r} \left(\int_0^{h(x,t)} \xi^{q-1} d\xi \right) dxdt \\
&= \frac{q}{\varphi(\mathcal{I}_r(y,\tau))}\int_0^{\infty} \left|\{(x,t)\in Q_r : h(x,t) > \xi\}\right| \xi^{q-1}\,d\xi \\
&\ge \frac{q}{\varphi(\mathcal{I}_r(y,\tau))} \sum_{k\ge 1} \left|\{(x,t)\in Q_r : h(x,t) > \zeta\theta^k\}\right| \int_{\zeta\theta^{k-1}}^{\zeta\theta^k} \xi^{q-1}\,d\xi \\
&= \zeta^q(1-\theta^{-q}) \frac{1}{\varphi(\mathcal{I}_r(y,\tau))} \sum_{k\ge 1} \theta^{kq}\left|\{(x,t)\in Q_r : h(x,t) > \zeta\theta^k\}\right|.
\end{aligned}
$$

Taking again the supremum over $(y,\tau)\in Q, r>0$, we get $\|h\|^q_{L^{q,\varphi}(Q)} \ge \frac{1}{c}\mathcal{S}$. □

In the proof of our main theorem, we employ the fact that the obstacle problem here considered is invariant under scaling and normalization. This property follows by straightforward calculations.

Lemma 4.3. *Let* $\mathbf{u}\in\mathcal{A}$ *be the weak solution to the problem* (1), (2). *Assume that* $(A_{ij}^{\alpha\beta},\Omega)$ *are* (δ,R)*-vanishing of codimension* 1. *Fix* $M>1$, $0<\rho<1$, *and define the rescaled maps*

$$
\begin{aligned}
\widetilde{A}_{ij}^{\alpha\beta}(x,t) &= A_{ij}^{\alpha\beta}(\rho x,\rho^2 t), \quad \tilde{\mathbf{u}}(x,t) = \frac{\mathbf{u}(\rho x,\rho^2 t)}{M\rho}, \\
\tilde{\mathbf{F}}(x,t) &= \frac{\mathbf{F}(\rho x,\rho^2 t)}{M}, \qquad \tilde{\psi}(x,t) = \frac{\psi(\rho x,\rho^2 t)}{M\rho},
\end{aligned}
$$

and the sets $\widetilde{\Omega} = \left\{ \frac{x}{\rho} : x \in \Omega \right\}$, $\widetilde{Q} = \widetilde{\Omega} \times (0, \tilde{T}] = \left\{ \left(\frac{x}{\rho}, \frac{t}{\rho^2} \right) : (x,t) \in Q \right\}$ *and*

$$\begin{aligned} \tilde{\mathcal{A}} = \{ \tilde{\phi} \in C^0(0, \tilde{T}; L^2(\widetilde{\Omega}, \mathbb{R}^m)) \cap L^2(0, \tilde{T}; H_0^1(\widetilde{\Omega}, \mathbb{R}^m)) : \\ \tilde{\phi}^i(\cdot, 0) = 0 \text{ a.e. in } \tilde{\Omega} \text{ and } \tilde{\phi}^i \geq \tilde{\psi}^i \text{ a.e. in } \tilde{Q},\ i = 1, \ldots, m \}. \end{aligned}$$

Then

1. $\widetilde{A}_{ij}^{\alpha\beta} : \widetilde{Q} \to \mathbb{R}^{mn \times mn}$ *satisfy the basic condition* (2) *with the same constants* λ *and* Λ.
2. $\left(\widetilde{A}_{ij}^{\alpha\beta}, \widetilde{\Omega} \right)$ *are* $(\delta, \frac{R}{\rho})$*-vanishing of codimension* 1.
3. $\tilde{\mathbf{u}} \in \tilde{\mathcal{A}}$ *is the weak solution to the resulting variational inequality:*

$$\int_0^{\tilde{T}} \langle \tilde{\phi}_t^i, \tilde{\phi}^i - \tilde{u}^i \rangle \, dt + \int_{\widetilde{Q}} \widetilde{A}_{ij}^{\alpha\beta}(x,t) D_\beta \tilde{u}^j \cdot D_\alpha \left(\tilde{\phi}^i - \tilde{u}^i \right) dxdt \geq \int_{\widetilde{Q}} \tilde{f}_i^\alpha \cdot D_\alpha \left(\tilde{\phi}^i - \tilde{u}^i \right) dxdt,$$

for all $\tilde{\phi} \in \tilde{\mathcal{A}}$ *with* $\tilde{\phi}_t \in L^2(0, \tilde{T}; H^{-1}(\tilde{\Omega}, \mathbb{R}^m))$.

5. Global gradient estimate

Let $\mathbf{u} \in \mathcal{A}$ be weak solution to (1), (2). Fix $p \in (2, \infty)$ and take φ satisfying (5)–(7). Suppose that

$$|\mathbf{F}|^2 \in L^{\frac{p}{2}, \varphi}(Q) \quad \text{and} \quad |\psi_t|^2, \, |D\psi|^2 \in L^{\frac{p}{2}, \varphi}(Q).$$

We will show that $D\mathbf{u} \in L^{\frac{p}{2}, \varphi}(Q)$ with the estimate (12) under the regularity requirements staying in Definition 3.1. Recall that $Q_r = Q \cap \mathcal{I}_r(y_0, \tau_0)$ for a fixed point $(y_0, \tau_0) \in Q$. Denote in addition $\Omega_r = \Omega \cap \mathcal{B}_r(y_0)$, $\partial\Omega_r = \partial\Omega \cap \mathcal{B}_r(y_0)$ and $\partial Q_r = \partial Q \cap \mathcal{I}_r(y_0, \tau_0)$.

Now, in order to apply Lemma 4.1 we need the following result.

Lemma 5.1. *There exists a large constant* $N = N(\lambda, \Lambda, m, n) > 1$ *such that for each* $0 < \varepsilon < 1$ *there exists a small* $\delta > 0$*, depending on known quantities, such that if* $\left(A_{ij}^{\alpha\beta}, \Omega \right)$ *are* (δ, R)*-vanishing of codimension* 1 *and if* $\mathcal{C}_\rho(y, \tau)$ *with* $(y, \tau) \in Q_r$ *and* $\rho \in (0,1)$ *satisfies*

$$\left| \left\{ (x,t) \in Q_r \colon \mathcal{M}(|D\mathbf{u}|^2) > N^2 \right\} \cap \mathcal{C}_\rho(y,\tau) \right| \geq \varepsilon \, |\mathcal{C}_\rho(y,\tau)| \tag{15}$$

then

$$\begin{aligned} Q_r \cap \mathcal{C}_\rho(y,\tau) \subset \left\{ (x,t) \in Q_r \colon \mathcal{M}(|D\mathbf{u}|^2) > 1 \right\} \cup \left\{ (x,t) \in Q_r \colon \mathcal{M}(|\mathbf{F}|^2) > \delta^2 \right\} \\ \cup \left\{ (x,t) \in Q_r \colon \mathcal{M}(|\psi_t|^2 + |D\psi|^2) > \delta^2 \right\}. \end{aligned} \tag{16}$$

Proof. Because of the scaling invariance property of Ω (cf. [6, Lemma 5.2]) we can take arbitrary $R \geq 1$. For technical convenience we choose $R = 49$ and take also ρ and r such that $\mathcal{C}'_{6\rho} \subset \mathcal{B}_r(y_0)$ with $y \in \Omega_r$.

We argue by contradiction supposing that in Q_r the maximal functions do not satisfy (16), hence $\mathcal{C}_\rho(y,\tau)$ satisfies (15) but the claim (16) is false. Then there exists a point $(y_1, \tau_1) \in Q_r \cap \mathcal{C}_\rho(y,\tau)$ such that for every $\sigma > 0$ we have

$$\begin{gathered}\frac{1}{|\mathcal{C}_\sigma(y_1,\tau_1)|}\int_{Q_r\cap\mathcal{C}_\sigma(y_1,\tau_1)} |D\mathbf{u}(x,t)|^2\,dxdt \leq 1,\\ \frac{1}{|\mathcal{C}_\sigma(y_1,\tau_1)|}\int_{Q_r\cap\mathcal{C}_\sigma(y_1,\tau_1)} |\mathbf{F}(x,t)|^2\,dxdt \leq \delta^2 \\ \frac{1}{|\mathcal{C}_\sigma(y_1,\tau_1)|}\int_{Q_r\cap\mathcal{C}_\sigma(y_1,\tau_1)} \left(|\psi_t(x,t)|^2 + |D\psi(x,t)|^2\right)\,dxdt \leq \delta^2.\end{gathered} \tag{17}$$

We consider only the lateral boundary case $\mathcal{C}'_{6\rho}(y) \cap \partial\Omega_r \neq \emptyset$. The interior case $\mathcal{C}'_{6\rho}(y) \cap \partial\Omega_r = \emptyset$ can be handled in a simpler way, since there is no issue related to the boundary. The estimates on the corner and on the bottom can be treated in the same way as in the estimates near the lateral boundary with a proper extension of $\mathbf{F}$ and ψ defined on $(0,T)$ to $\mathbb{R}$. Take a lateral boundary point $y_2 \in \mathcal{C}'_{6\rho}(y)\cap\partial\Omega_r$. According to Definition 3.1, there exist a new coordinate system, modulo reorientation of the axes and translation whose variables we denote by (z,ξ) such that in this new coordinate system, the origin is $(y_2 + 42\rho\delta\vec{n}_0, \tau) \equiv (0,0)$. Here $\vec{n}_0$ is unit inward vector at y_2 denoting the normal direction z^1. Then in the new coordinate system the considered points become $(y,\tau) \equiv (w,\zeta)$, $(y_1,\tau_1) \equiv (w_1,\zeta_1)$ and $(w_1,\zeta_1) \subset \mathcal{C}_\rho(w,\zeta) \subset \mathcal{C}_{7\rho}(0,0)$ for $\delta < \frac{1}{8}$. According to Definition 3.1 we can write

$$\Omega_r \cap \{\mathcal{C}'_{42\rho}(0) : z^1 > 0\} \subset \Omega_r \cap \mathcal{C}'_{42\rho}(0) \subset \Omega_r \cap \left\{\mathcal{C}'_{42\rho}(0) : z^1 > -84\rho\delta\right\}, \tag{18}$$

and

$$\fint_{\mathcal{C}_{42\rho}(0,0)} \left| A^{\alpha\beta}_{ij}(z,\xi) - \overline{A^{\alpha\beta}_{ij}}_{\mathcal{C}^{z^1}_{42\rho}(0,0)}(z^1)\right|^2 dzd\xi \leq \delta^2. \tag{19}$$

Since $\mathcal{C}_{42\rho}(0,0) \subset \mathcal{C}_{49\rho}(w_1,\zeta_1)$, by (17) we get

$$\begin{aligned}&\frac{1}{|\mathcal{C}_{42\rho}(0,0)|}\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r} |D\mathbf{u}(z,\xi)|^2\,dzd\xi \\ &\leq 2\left(\frac{49}{42}\right)^{n+2} \frac{1}{|\mathcal{C}_{49\rho}(w_1,\zeta_1)|}\int_{\mathcal{C}_{49\rho}(w_1,\zeta_1)\cap Q_r} |D\mathbf{u}(z,\xi)|^2\,dzd\xi \\ &< 2^{n+3}.\end{aligned} \tag{20}$$

In a similar manner, we get

$$\frac{1}{|\mathcal{C}_{42\rho}(0,0)|}\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}|\mathbf{F}(z,\xi)|^2\,dzd\xi<2^{n+3}\delta^2,$$
$$\frac{1}{|\mathcal{C}_{42\rho}(0,0)|}\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}\left(|\psi_\xi(z,\xi)|^2+|D\psi(z,\xi)|^2\right)\,dzd\xi<2^{n+3}\delta^2. \tag{21}$$

Using now the concept of *localizable solutions*, introduced in [24, 25], we find that

$$\begin{aligned}&\int_{\zeta_2}^{\zeta_3}\langle\phi^i_\xi,\phi^i-u^i\rangle\,d\xi+\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}A^{\alpha\beta}_{ij}(z,\xi)D_\beta u^j\cdot D_\alpha\left(\phi^i-u^i\right)\,dzd\xi\\ &\geq\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}f^\alpha_i(z,\xi)\cdot D_\alpha\left(\phi^i-u^i\right)\,dzd\xi\end{aligned} \tag{22}$$

with $\zeta_2=-(42\rho)^2$, $\zeta_3=(42\rho)^2$ and $i=1,\dots,m$. The inequality (22) holds for all $\phi\in C^0\left(\zeta_2,\zeta_3;L^2(\mathcal{C}'_{42\rho}(0)\cap\Omega_r,\mathbb{R}^m)\right)\cap L^2\left(\zeta_2,\zeta_3;H^1_0(\mathcal{C}'_{42\rho}(0)\cap\Omega_r,\mathbb{R}^m)\right)$ with $\phi_t\in L^2(\zeta_2,\zeta_3;H^{-1}(\mathcal{C}'_{42\rho}(0)\cap\Omega_r,\mathbb{R}^m))$ such that $\phi^i(\cdot,\zeta_2)=0$ a.e. in $\mathcal{C}'_{42\rho}(0)\cap\Omega_r$ and $\phi^i\geq\psi^i$ a.e. in $\mathcal{C}_{42\rho}(0,0)\cap Q_r$. Let us note that the weak solution in (1) coincides with the localizable solution in (22) under a H^1-extension property of the domain, as it was shown in [24, 25]. Needless to say, δ-Reifenberg domain enjoys such a property, see [14, 16, 21].

Let $\mathbf{k}\in C^0(\zeta_2,\zeta_3;L^2(\mathcal{C}'_{42\rho}(0)\cap\Omega_r,\mathbb{R}^m))\cap L^2(\zeta_2,\zeta_3;H^1_0(\mathcal{C}'_{42\rho}(0)\cap\Omega_r,\mathbb{R}^m))$ be the weak solution of the system

$$\begin{cases}k^i_\xi-D_\alpha\Big(A^{\alpha\beta}_{ij}(z,\xi)D_\beta k^j\Big)=\psi^i_\xi-D_\alpha\Big(A^{\alpha\beta}_{ij}(z,\xi)D_\beta\psi^j\Big) & \text{in } \mathcal{C}_{42\rho}(0,0)\cap Q_r\\ k^i=u^i & \text{on } \mathcal{C}_{42\rho}(0,0)\cap\partial Q_r\end{cases} \tag{23}$$

with $i=1,\dots,m$. Remembering $k^i=u^i\geq\psi^i$ on $\mathcal{C}_{42\rho}(0,0)\cap\partial Q_r$ and employing the *comparison principle* (see [1, Lemma 2.8]), we deduce $k^i\geq\psi^i$ a.e. in $\mathcal{C}_{42\rho}(0,0)\cap Q_r$. We then substitute $\phi=\mathbf{k}$ into (22) to deduce

$$\begin{aligned}&\int_{\zeta_2}^{\zeta_3}\langle k^i_\xi,k^i-u^i\rangle\,d\xi+\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}A^{\alpha\beta}_{ij}(z,\xi)D_\beta u^j\cdot D_\alpha\big(k^i-u^i\big)\,dzd\xi\\ &\geq\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}f^\alpha_i(z,\xi)\cdot D_\alpha\big(k^i-u^i\big)\,dzd\xi.\end{aligned} \tag{24}$$

Taking as a test function $\mathbf{k}-\mathbf{u}$ for (23), we have

$$\begin{aligned}&\int_{\zeta_2}^{\zeta_3}\langle k^i_\xi,k^i-u^i\rangle\,d\xi+\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}A^{\alpha\beta}_{ij}(z,\xi)D_\beta k^j\cdot D_\alpha\big(k^i-u^i\big)\,dzd\xi\\ &=\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}\psi^i_\xi\big(k^i-u^i\big)\,dzd\xi+\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r}A^{\alpha\beta}_{ij}(z,\xi)D_\beta\psi^jD_\alpha\big(k^i-u^i\big)\,dzd\xi\end{aligned} \tag{25}$$

Combining (24) and (25), we obtain

$$\begin{aligned}
&\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r} A_{ij}^{\alpha\beta}(z,\xi) D_\beta\big(k^j - u^j\big)\cdot D_\alpha\big(k^i - u^i\big)\, dzd\xi \\
&\le \int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r} \psi_\xi^i\big(k^i - u^i\big)\, dzd\xi + \int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r} A_{ij}^{\alpha\beta}(z,\xi) D_\beta \psi^j D_\alpha\big(k^i - u^i\big)\, dzd\xi \\
&\quad - \int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r} f_i^\alpha(z,\xi)\cdot D_\alpha\big(k^i - u^i\big)\, dzd\xi.
\end{aligned}$$

We then use the uniform ellipticity condition (2) and the smallness assumptions (20), (21), to conclude

$$\frac{1}{|\mathcal{C}_{42\rho}(0,0)|}\int_{\mathcal{C}_{42\rho}(0,0)\cap Q_r} |D(\mathbf{k}-\mathbf{u})|^2\, dzd\xi \le c\delta^2.$$

Since $\mathbf{k}$ is the weak solution to the parabolic system, it follows from an argument, very similar to that of [6, Lemma 4.8], that we can find a small positive constant $\delta = \delta(\varepsilon, \lambda, \Lambda, m, n) > 0$ and a function $\mathbf{v}$ such that for such δ satisfying (18), (19) and (21), one has

$$\frac{1}{|\mathcal{C}_{14\rho}(0,0)|}\int_{\mathcal{C}_{14\rho}(0,0)\cap Q_r} |D(\mathbf{k}-\mathbf{v})|^2\, dzd\xi \le c\delta^\sigma$$

for some $\sigma = \sigma(\lambda, \Lambda, m, n) > 0$, and

$$\|D\mathbf{v}\|^2_{L^\infty(\mathcal{C}_{21\rho}(0,0)\cap Q_r)} \le N_1^2, \tag{26}$$

where N_1 is a universal constant depending on $\lambda, \Lambda, m,$ and n. Therefore, for any small $\eta > 0$ and $\delta = \delta(\eta)$ we get

$$\frac{1}{|\mathcal{C}_{14\rho}(0,0)|}\int_{\mathcal{C}_{14\rho}(0,0)\cap Q_r} |D(\mathbf{u}-\mathbf{v})|^2\, dzd\xi \le c(\delta^2 + \delta^\sigma) = \eta^2. \tag{27}$$

Now, taking $N = \max\left\{2N_1, 2^{\frac{n+2}{2}}\right\}$, we get as follows.

$$\begin{aligned}
&\frac{|\{(z,\xi)\in Q_r\colon \mathcal{M}(|D\mathbf{u}|^2) > N^2\}\cap \mathcal{C}_{7\rho}(0,0)|}{|\mathcal{C}_{7\rho}(0,0)|} \\
&\le \frac{|\{(z,\xi)\in Q_r\colon \mathcal{M}(|D(\mathbf{u}-\mathbf{v})|^2) > N_1^2\}\cap \mathcal{C}_{7\rho}(0,0)|}{|\mathcal{C}_{7\rho}(0,0)|} \\
&\quad + \frac{|\{(z,\xi)\in Q_r\colon \mathcal{M}(|D\mathbf{v}|^2) > N_1^2\}\cap \mathcal{C}_{7\rho}(0,0)|}{|\mathcal{C}_{7\rho}(0,0)|} \\
&\overset{(26)}{\le} \frac{c}{|\mathcal{C}_{14\rho}(0,0)|}\int_{\mathcal{C}_{14\rho}(0,0)\cap Q_r} |D(\mathbf{u}-\mathbf{v})|^2\, dzd\xi \\
&\overset{(27)}{\le} c\eta^2,
\end{aligned}$$

which implies

$$\left|\left\{(z,\xi)\in Q_r\colon \mathcal{M}(|D\mathbf{u}|^2)>N^2\right\}\cap \mathcal{C}_\rho(w,\zeta)\right|\le c\eta^2|\mathcal{C}_\rho(w,\zeta)|,$$

since $\mathcal{C}_\rho(w,\zeta)\subset \mathcal{C}_{7\rho}(0,0)$. However, this contradicts (15), since this estimate is invariant under the change of variables and $\eta>0$ is arbitrary, which completes the proof. □

Fix now $\varepsilon>0$ and take δ and N as given in Lemma 5.1. Making use of Lemma 4.1, we will obtain power decay estimate of the super-level sets of $\mathcal{M}(|D\mathbf{u}|^2)$.

Lemma 5.2. *Under the assumptions of Lemma* 5.1*, suppose in addition that* $\Theta(1)<\varepsilon$ *for each* $(y,\tau)\in Q_r$*. Then for all positive integers* k*, we have*

$$\begin{aligned}&\left|\left\{(x,t)\in Q_r\colon \mathcal{M}(|D\mathbf{u}|^2)>N^{2k}\right\}\right|\\ &\le \epsilon_1^k\left|\left\{(x,t)\in Q_r\colon \mathcal{M}(|D\mathbf{u}|^2)>1\right\}\right|\\ &\quad+\sum_{i=1}^k \epsilon_1^i\left|\left\{(x,t)\in Q_r\colon \mathcal{M}(|\mathbf{F}|^2)>\delta^2N^{2(k-i)}\right\}\right)\\ &\quad+\sum_{i=1}^k \epsilon_1^i\left|\left\{(x,t)\in Q_r\colon \mathcal{M}(|\psi_t|^2+|D\psi|^2)>\delta^2N^{2(k-i)}\right\}\right|\end{aligned}\tag{28}$$

with $\epsilon_1=\varepsilon\left(\frac{10\sqrt{2}}{1-\delta}\right)^{n+2}$.

Proof. The Lemma 5.1 and assumption $\Theta(1)<\varepsilon$ ensure the validity of the hypothesis of Lemma 4.1 for the sets $\mathfrak{C}$ and $\mathfrak{D}$, which gives immediately (28) for $k=1$. Further, we proceed with the proof by induction, in a similar manner as in [7]. Suppose that (28) holds true for some $k>1$. Define the vector-functions $\mathbf{u}_1=\frac{\mathbf{u}}{N}$, and $\mathbf{F}_1=\frac{\mathbf{F}}{N}$ and $\psi_1=\frac{\psi}{N}$. It is easy to see that $\mathbf{u}_1$ is a weak solution to the problem (1) with a right-hand side $\mathbf{F}_1$. Hence, Lemma 5.1 and the assumption $\Theta(1)<\varepsilon$ hold with sets $\mathfrak{C}$ and $\mathfrak{D}$ corresponding to $\mathbf{u}_1$ as defined in (13) and (14). Then (28) holds true for $\mathbf{u}_1$ with the same $k>1$. The definitions of $\mathbf{u}_1,\mathbf{F}_1$ and ψ_1 ensure the inductive passage from k to $k+1$ for $\mathbf{u}$. □

We are in a position now to prove Theorem 3.2.

Proof of Theorem 3.2*.* Assume that the norms of $\mathbf{F}$ and ψ are small enough. More precisely, we can obtain this by taking

$$K:=\||\mathbf{F}|^2\|_{L^{\frac{p}{2},\varphi}(Q)}+\||\psi_t|^2\|_{L^{\frac{p}{2},\varphi}(Q)}+\||D\psi|^2\|_{L^{\frac{p}{2},\varphi}(Q)},$$

$$\widetilde{\mathbf{u}}=\frac{\delta\mathbf{u}(x,t)}{\sqrt{K}},\quad \widetilde{\mathbf{F}}=\frac{\delta\mathbf{F}(x,t)}{\sqrt{K}},\quad \widetilde{\psi}=\frac{\delta\psi(x,t)}{\sqrt{K}}\tag{29}$$

instead of $\mathbf{u}$, $\mathbf{F}$ and ψ in (1).

Consider the super-level set $\mathfrak{C}$ defined for $\widetilde{\mathbf{u}}$. For each $(y,\tau)\in\mathfrak{C}$ holds $\frac{|\mathfrak{C}\cap\mathcal{C}_1(y,\tau)|}{|\mathcal{C}_1(y,\tau)|} \le c|\mathcal{C}| \le c\int_{Q_r}\mathcal{M}(|D\widetilde{\mathbf{u}}|^2)(x,t)\,dxdt \le c\int_{Q_r}|D\widetilde{\mathbf{u}}(x,t)|^2\,dxdt \le c\int_Q|D\widetilde{\mathbf{u}}(x,t)|^2\,dxdt \le c\int_Q\Big(|\widetilde{\mathbf{F}}(x,t)|^2+|\widetilde{\psi}_t|^2+|D\widetilde{\psi}|^2\Big)\,dxdt \le c\Big(\||\widetilde{\mathbf{F}}|^2\|_{L^{\frac{p}{2},\varphi}(Q)}+\||\widetilde{\psi}_t|^2\|_{L^{\frac{p}{2},\varphi}(Q)}+\||D\widetilde{\psi}|^2\|_{L^{\frac{p}{2},\varphi}(Q)}\Big)\le c\delta^2$ where we have used (9). Taking δ small enough we get

$$\Theta(1)=\frac{|\mathfrak{C}\cap\mathcal{C}_1(y,\tau)|}{|\mathcal{C}_1(y,\tau)|}\le c\delta^2<\varepsilon.$$

Now we can apply Lemma 4.2 with $h=\mathcal{M}(|D\widetilde{\mathbf{u}}|^2)$, $\theta=N^2$, $\lambda=1$ and $q=\frac{p}{2}$. Thus, using Lemma 5.2, we get

$$\begin{aligned}
\Sigma &\equiv \sum_{k=1}^{\infty}\frac{N^{2k\frac{p}{2}}\left|\left\{(x,t)\in Q_r\colon \mathcal{M}(|D\widetilde{\mathbf{u}}|^2)>N^{2k}\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))}\\
&\le \sum_{k=1}^{\infty}\frac{N^{kp}\epsilon_1^k\,|\{(x,t)\in Q_r:\mathcal{M}(|D\widetilde{\mathbf{u}}|^2)>1\}|}{\varphi(\mathcal{I}_r(y,\tau))}\\
&\quad+\sum_{k=1}^{\infty}N^{kp}\sum_{i=1}^{k}\frac{\epsilon_1^i\left|\left\{(x,t)\in Q_r:\mathcal{M}(|\widetilde{\mathbf{F}}|^2)>\delta^2N^{2(k-i)}\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))}\\
&\quad+\sum_{k=1}^{\infty}N^{kp}\sum_{i=1}^{k}\frac{\epsilon_1^i\left|\left\{(x,t)\in Q_r:\mathcal{M}(|\widetilde{\psi}_t|^2+|D\widetilde{\psi}|^2)>\delta^2N^{2(k-i)}\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))}\\
&\le \sum_{k=1}^{\infty}(N^p\epsilon_1)^k\,\frac{|Q_r|}{\varphi(\mathcal{I}_r(y,\tau))}\\
&\quad+\sum_{i=1}^{\infty}(N^p\epsilon_1)^i\underbrace{\sum_{k=i}^{\infty}N^{(k-i)p}\frac{\left|\left\{(x,t)\in Q_r:\mathcal{M}(|\widetilde{\mathbf{F}}|^2)>\delta^2N^{2(k-i)}\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))}}_{\Sigma'}\\
&\quad+\sum_{i=1}^{\infty}(N^p\epsilon_1)^i\underbrace{\sum_{k=i}^{\infty}N^{(k-i)p}\frac{\left|\left\{(x,t)\in Q_r:\mathcal{M}(|\widetilde{\psi}_t|^2+|D\widetilde{\psi}|^2)>\delta^2N^{2(k-i)}\right\}\right|}{\varphi(\mathcal{I}_r(y,\tau))}}_{\Sigma''}\\
&\overset{(8)}{\le}\sum_{k=1}^{\infty}(N^p\epsilon_1)^k\,(\kappa_4+\Sigma'+\Sigma'')\,.
\end{aligned}$$

To estimate Σ' and Σ'' we apply the maximal inequality, Lemma 2.2, and the measure estimate Lemma 4.2. Precisely

$$\begin{aligned}\Sigma' &= \frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\sum_{k=i}^{\infty} N^{2(k-i)\frac{p}{2}}\left|\left\{(x,t)\in Q_r : \mathcal{M}\Big(\frac{|\widetilde{\mathbf{F}}|^2}{\delta^2}\Big) > N^{2(k-i)}\right\}\right| \\ &\le \frac{c}{\varphi(\mathcal{I}_r(y,\tau))}\left(|Q_r| + \int_{Q_r}\mathcal{M}\Big(\frac{|\widetilde{\mathbf{F}}|^2}{\delta^2}\Big)^{\frac{p}{2}}(x,t)\,dxdt\right) \\ &\le \frac{c}{\varphi(\mathcal{I}_r(y,\tau))}\left(|Q_r| + \int_{Q_r}\frac{|\widetilde{\mathbf{F}}(x,t)|^p}{\delta^p}\,dxdt\right).\end{aligned}$$

In a similar way we estimate also Σ''.

$$\Sigma'' \le \frac{c}{\varphi(\mathcal{I}_r(y,\tau))}\left(|Q_r| + \int_{Q_r}\frac{|\widetilde{\psi}_t(x,t)|^p + |D\widetilde{\psi}(x,t)|^p}{\delta^p}\,dxdt\right).$$

Unifying the above estimates and applying again (8), we get

$$\begin{aligned}\Sigma \le c\sum_{k=1}^{\infty}(N^p\epsilon_1)^k &\left[\kappa_4 + \frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r}\frac{|\widetilde{\mathbf{F}}(x,t)|^p}{\delta^p}\,dxdt\right. \\ &\left. + \frac{1}{\varphi(\mathcal{I}_r(y,\tau))}\int_{Q_r}\frac{|\widetilde{\psi}_t(x,t)|^p + |D\widetilde{\psi}(x,t)|^p}{\delta^p}\,dxdt\right].\end{aligned}$$

Since $\mathcal{S}$ is the supremum of Σ over $(y,\tau)\in Q$ and $r>0$, we obtain

$$\begin{aligned}\mathcal{S} &\le c\sum_{k=1}^{\infty}(N^p\epsilon_1)^k\left[1 + \frac{1}{\delta^p}\|\widetilde{\mathbf{F}}\|^p_{L^{p,\varphi}(Q)} + \frac{1}{\delta^p}\Big(\|\psi_t\|^p_{L^{p,\varphi}(Q)} + \|D\widetilde{\psi}\|^p_{L^{p,\varphi}(Q)}\Big)\right] \\ &\le c\sum_{k=1}^{\infty}(N^p\epsilon_1)^k,\end{aligned}$$

where we have used (29) in the last inequality above.

We now recall $\epsilon_1 = \varepsilon\Big(\frac{10\sqrt{2}}{1-\delta}\Big)^{n+2}$ and $0<\delta<\frac{1}{8}$, to see that $\epsilon_1 \le c_1\varepsilon$, the constant c_1 depending only on n. Taking ε small enough such that $N^p\epsilon_1 \le N^p c_1\varepsilon < 1$, we get $\mathcal{S}<\infty$. In view of Lemma 5.1 and the substitution (29), we get

$$\|\mathcal{M}(|D\mathbf{u}|^2)\|^{\frac{p}{2}}_{L^{\frac{p}{2},\varphi}(Q)} \le c\left(\||\mathbf{F}|^2\|^{\frac{p}{2}}_{L^{\frac{p}{2},\varphi}(Q)} + \||\psi_t|^2\|^{\frac{p}{2}}_{L^{\frac{p}{2},\varphi}(Q)} + \||D\psi|^2\|^{\frac{p}{2}}_{L^{\frac{p}{2},\varphi}(Q)}\right),$$

which gives (12) through the maximal inequality. □

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