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The Profile of Blow-Up for a Neumann Problem of Nonlocal Nonlinear Diffusion Equation with Reaction

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Abstract. In this note, we consider a Neumann problem for nonlocal nonlinear diffusion equation with reaction, which may be seen as a significant generalization of the usual Neumann problem for the heat equation. For the blow-up solutions, the blow-up rate estimates and spacial localization of blow-up set are studied.

Keywords. Nonlocal nonlinear diffusion, blow-up rate, blow-up set
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1. Introduction

In recent years, various nonlocal problems governed by differential equations have been studied deeply due to their important values in sciences and technologies, and many interesting results have been established, please see, for example, [8, 10, 13–15, 17]) and some references mentioned below. In particular, in the study of the population biology, a class of nonlocal diffusion equations was proposed to model the spatial diffusion of certain types of populations (see, e.g., [9]). Such class of equations shares many properties with the classical heat equation and has, in some cases, better effects in applications than the traditional models such as reaction-diffusion equations, a prototype of which involves the distribution u of the density of a single population satisfying the following integral equation

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)\mathrm{d}y - u(x,t), \tag{1}$$

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where $J(x-y)$ is interpreted as the probability distribution of jumping from location y to location x, $(J*u)(x,t)$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^n} J(y-x)u(x,t)\mathrm{d}y$ is the rate at which they are leaving location x to travel to all other sites. Neumann problem (1) is called a nonlocal diffusion equation, since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J*u$.

Equations (1) and its variations have been extensively studied through the last decades. In view of the biologically meaningful question whether or not populations spontaneously form aggregates, some studies focused on the issue whether solutions exist globally or blow up. Some significant contributions along this line have been made, among them for instance are [6, 9, 16] and the references therein.

A significant generalization for Neumann problem (1) is following the nonlinear nonlocal diffusion equation with reaction

$$u_t(x,t)=\int_\Omega\Big(J\Big(\frac{x-y}{u^\alpha(y,t)}\Big)u^{1-n\alpha}(y,t)-J\Big(\frac{x-y}{u^\alpha(x,t)}\Big)u^{1-n\alpha}(x,t)\Big)\mathrm{d}y+\chi u^k(x,t), \quad (2)$$

$x\in\Omega$, $t>0$, where Ω is a connected bounded smooth domain in $\mathbb{R}^n$ $(n\geq 1)$, $\chi>0$, $0<\alpha\leq\frac{1}{n}$ are constants, and the kernel $J:\mathbb{R}^n\to\mathbb{R}$, with compact support in the unit ball and $\int_{\mathbb{R}^n} J(x)\mathrm{d}x=1$, is a non-negative, smooth, symmetric radially and strictly decreasing function.

Let us note that in (2), the integral is only in Ω, which means the diffusion takes place only in Ω, no individual may enter or leave the domain, this is called Neumann boundary conditions (see [1] for details).

Here, some recent work closely related to Neumann problem (2) is mentioned. Pazoto and Rossi [11] studied the global well-posedness and asymptotic behavior of the solutions for (2) when $\alpha=0$, $\chi=1$ and $\Omega=\mathbb{R}^n$. Pézez-Llanos and Rossi [12] considered the blow-up conditions, blow-up rates, and blow-up set for (2) when $\alpha=0$, $\chi=1$ and Ω is bounded. As to $\alpha=1$, $\chi=0$ and $\Omega=[-L,L]$, Bogoya et al. [4] established the existence and uniqueness of solutions and a comparison principle for (2), and proved that the solutions approach the mean value of the initial conditions asymptotically as $t\to\infty$. Moreover, from Bogoya et al. [5] and Cortazar et al. [7] one can find results on the asymptotic behavior of the solutions for a nonlinear nonlocal diffusion operator under blowing-up boundary conditions of Dirichlet or Neumann type. Let us notice in particular that in [3] the blow-up phenomenon for Neumann problem (2) with a more general source function is analyzed and the blow-up rate estimates are given for some particular sources. However, the author did not consider the spacial location of the set where the solution blows up.

Following these work, in the present paper we are interested in studying the

Neumann problem (2) equipped with an initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

The line, which we will go along is that we establish the blow-up rate estimates for the blow-up solutions, which allows us to localize the blow-up set near any point in Ω just by taking an initial datum being very large near that point and not so large in the rest of the domain. The theorems formulated are extensions of many previous results on the nonlocal diffusion equations.

2. Results and proofs

Denote $Q_T = \overline{\Omega} \times (0,T)$. We begin by introducing the Banach space $Y_T = C([0,T]; C(\overline{\Omega}))$ equipped the norm

$$\|z\|_{Y_T} = \max_{0\le t\le T} \|z(\cdot,t)\|_{L^\infty(\overline{\Omega})} = \max_{0\le t\le T} \max_{\overline{\Omega}} |z(x,t)|.$$

We use the following nature definition of solution, sub- and supersolution.

Definition 2.1. A pair of nonnegative functions $\tilde{u}, \hat{u} \in C([0,T]; C(\overline{\Omega}))$ are, respectively, called supersolution and subsolution of problem (2), (3) in Q_T if $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t(x,t) \ge \displaystyle\int_\Omega \Big(J\Big(\frac{x-y}{\tilde{u}^\alpha(y,t)}\Big) \tilde{u}^{1-n\alpha}(y,t) - J\Big(\frac{x-y}{\tilde{u}^\alpha(x,t)}\Big)\tilde{u}^{1-n\alpha}(x,t)\Big) \mathrm{d}y \\ \qquad\qquad + \chi \tilde{u}^k(x,t), \\ \tilde{u}(x,0) \ge u_0(x), \end{cases} \tag{4}$$

and $\hat{u}$ satisfies (4) in the reverse order. Further, we say that u is a solution of problem (2), (3) if it is both a subsolution and a supersolution of (2)- (3) in Q_T.

On Y_T we define the mapping Γ as

$$\begin{aligned} \Gamma(u)(x,t) = &\int_0^t \int_\Omega \Big(J\Big(\frac{x-y}{u^\alpha(y,\tau)}\Big) u^{1-n\alpha}(y,\tau) - J\Big(\frac{x-y}{u^\alpha(x,\tau)}\Big) u^{1-n\alpha}(x,\tau)\Big) \mathrm{d}y\mathrm{d}\tau \\ &+ \chi \int_0^t |u|^{k-1} u(x,\tau)\mathrm{d}\tau + u_0(x) \end{aligned}$$

for each $u \in Y_T$.

Similar to [12, Theorem 1.1], we can prove that Γ is well defined. Moreover, Γ is a strict contraction on an appropriate ball of Y_T. This enables us to obtain the following result.

Theorem 2.2. *Given a positive initial datum u_0. Then there exists $t_{max} = t_{max}(u_0) > 0$ and a unique solution $u \in C([0, t_{max}); C(\overline{\Omega}))$ of problem* (2), (3). *Also, if $t_{max} < +\infty$, then*

$$\limsup_{t \to t_{max}} \|u(\cdot, t)\|_{L^\infty(\overline{\Omega})} = +\infty.$$

Moreover, we have

$$\int_\Omega u(x,t)\mathrm{d}x = \int_\Omega u_0(x)\mathrm{d}x + \chi \int_\Omega \int_0^t u^k(x,\tau)\mathrm{d}\tau\mathrm{d}x$$

for every $t \in [0, t_{max})$.

We shall also need the following comparison principle for Neumann problem (2), (3).

Theorem 2.3. *Let $\tilde{u}, \hat{u}$ be supersolution and subsolution of Neumann problem* (2), (3) *in Q_T, respectively. Then $\tilde{u}(x,t) \geq \hat{u}(x,t)$ for all $(x,t) \in \overline{Q}_T$.*

Proof. The proof is a trivial modification of that of [2, Theorem 3.6], hence we omit it here. □

Corollary 2.4. *Let $u \in C([0, t_{max}); C(\overline{\Omega}))$ be a solution of problem* (2), (3) *with a positive initial datum u_0. Then $u(x,t) > 0$ for all $(x,t) \in Q_{t_{max}}$.*

In the following theorem we will analyze the blow-up condition and blow-up rate.

Theorem 2.5. *Given a positive initial datum u_0. We have the following assertions:*

(1) *if $0 < k \leq 1$, then the solution of problem* (2), (3) *is global.*
(2) *if $k > 1$, then the solution u of problem* (2), (3) *blows up in a finite time t_{max}. Moreover, we have*

$$\begin{aligned} \lim_{t \to t_{max}} (t_{max} - t)^{\frac{1}{k-1}} \max_{x \in \overline{\Omega}} u(x,t) &= \left(\frac{1}{\chi(k-1)}\right)^{\frac{1}{k-1}}, \\ t_{max} &\leq \frac{1}{\chi(k-1)} \left(\frac{|\Omega|}{\int_\Omega u_0(x)\mathrm{d}x}\right)^{k-1}. \end{aligned} \tag{5}$$

Proof. Let us consider the following ODE

$$w'(t) = \chi w^k(t). \tag{6}$$

It is easy to verify that if $0 < k \leq 1$, then the continuous solution of (6) with

$$w(0) = \max_{x \in \overline{\Omega}} u_0(x)$$

is a global supersolution of problem (2), (3), which together with Theorem 2.3 proves that the assertion (1) holds.

For the case when $k > 1$, we see that the continuous solution of (6) with

$$w(0) = \min_{x\in\overline{\Omega}} u_0(x) > 0$$

is a subsolution of problem (2), (3), which, thanks to Theorem 2.3, implies that the corresponding solution blows up in a finite time.

In the sequel, let $t_{max} < \infty$ be the maximal existence time of the solution u corresponding to a positive initial datum u_0. To establish the blow-up rate, we can estimate

$$u_t(x,t) \geq -\int_{\mathbb{R}^n} J\Big(\frac{x-y}{u^\alpha(x,t)}\Big)u^{1-N\alpha}(x,t)\mathrm{d}y + \chi u^k(x,t) = -u(x,t) + \chi u^k(x,t).$$

So,

$$\max_{x\in\overline{\Omega}} u_t(x,t) \geq \max_{x\in\overline{\Omega}} u^k(x,t)\Big(\chi - \Big(\max_{x\in\overline{\Omega}} u(x,t)\Big)^{1-k}\Big). \tag{7}$$

On the other hand, letting $x^* \in \overline{\Omega}$ such that $\max_{x\in\overline{\Omega}} u(x,t) = u(x^*,t)$, we have

$$J\Big(\frac{x^*-y}{u^\alpha(y,t)}\Big)u^{1-n\alpha}(y,t) \leq J\Big(\frac{x^*-y}{u^\alpha(x^*,t)}\Big)u^{1-n\alpha}(x^*,t), \quad y\in\overline{\Omega}, t\in(0,t_{max}).$$

Accordingly, one finds

$$\begin{aligned} u_t(x^*,t) &= \int_\Omega \Big(J\Big(\frac{x^*-y}{u^\alpha(y,t)}\Big)u^{1-n\alpha}(y,t) - J\Big(\frac{x^*-y}{u^\alpha(x^*,t)}\Big)u^{1-n\alpha}(x^*,t)\Big)\mathrm{d}y + \chi u^k(x^*,t) \\ &\leq \chi u^k(x^*,t), \end{aligned}$$

which enables us to conclude that $u(x^*,t) \geq \Big(\frac{1}{\chi(k-1)(t_{max}-t)}\Big)^{\frac{1}{k-1}}$. Therefore, invoking (7) it follows that

$$\max_{x\in\overline{\Omega}} u_t(x,t) \geq \max_{x\in\overline{\Omega}} u^k(x,t)\chi(1-(k-1)(t_{max}-t)).$$

Integrating the inequality above in (t, t_{max}), we thus have

$$\max_{x\in\overline{\Omega}} u(x,t) \leq (\chi(t_{max}-t))^{\frac{1}{1-k}}\Big((k-1) - \frac{(k-1)^2(t_{max}-t)}{2}\Big)^{\frac{1}{1-k}}. \tag{8}$$

Taking the limit as $t \to t_{max}$, we obtain the first assertion in (5).

Finally, integrating problem (2), (3) in Ω and applying Fubini's theorem one has

$$\frac{\partial}{\partial t}\int_\Omega u(x,t)\mathrm{d}x = \chi\int_\Omega u^k(x,t)\mathrm{d}x \geq \chi|\Omega|^{1-k}\Big(\int_\Omega u(x,t)\mathrm{d}x\Big)^k,$$

which implies that the second assertion in (5) remains true. □

Next we deal with the spacial location of the set where the solution blows up. The blow-up set of solution u will be denoted by

$$S(u) = \{x \in \overline{\Omega} : \text{there exists } x_n \to x,\ t_n \to t_{max},\ u(x_n, t_n) \to \infty\},$$

where t_{max} is the maximal existence time of u.

We denote $D_r(x)$ the ball in $\overline{\Omega}$ centered at x with radius r. Below the letters C_i will denote various positive constants.

Theorem 2.6. *Let $k \geq 2 - n\alpha$. Then for every $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$, there exists an initial datum u_0 such that $S(u)$ is contained in $D_\varepsilon(x_0)$.*

Proof. Given $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$. Let

$$u_0(x) = Mg(x) + \mu, \quad x \in \overline{\Omega},$$

where g is a nonnegative smooth function with $\text{supp}(g) \subset D_{\frac{\varepsilon}{2}}(x_0)$ and $M, \mu > 0$ are constants to be specified later. Note that $u_0(x) = \mu$ for every $x \in \overline{\Omega} \backslash D_\varepsilon(x_0)$. Also, since $t_{max} \leq \frac{C_1}{M^{k-1}}$ due to (5), one can choose an appropriate M such that t_{max} is as small as we need below.

From (8) it follows that

$$\max_{x\in\overline{\Omega}} u(x,t) \leq C_2(\chi(t_{max} - t))^{-\frac{1}{k-1}}$$

for small t_{max} as appropriate, which enables us to estimate

$$u_t(x,t) \leq \int_\Omega J\Big(\frac{x-y}{u^\alpha(y,t)}\Big)u^{1-n\alpha}(y,t)\mathrm{d}y + \chi u^k(x,t) \leq C_3(t_{max}-t)^{-\frac{1-n\alpha}{k-1}} + \chi u^k(x,t).$$

Let v be the continuous solution of the following ODE:

$$v'(t) = C_3(t_{max} - t)^{-\frac{1-n\alpha}{k-1}} + \chi v^k(t)$$

with initial datum $v(0) = \mu$. Then from Theorem 2.3 it follows that $u(x,t) \leq v(t)$ for $x \in \overline{\Omega} \setminus D_\varepsilon(x_0)$.

In the sequel, we write

$$\lambda(s) = e^{-\frac{1-n\alpha}{k-1}s}v(t),$$

where $s = -\ln(t_{max} - t)$. It is clear that $\lambda(s) > 0$. Moreover, we have

$$\lambda'(s) = -\frac{1-n\alpha}{k-1}\lambda(s) + C_3e^{-s} + \chi e^{-n\alpha s}\lambda^k(s), \tag{9}$$

which implies that $\lambda'(-lnt_{max}) < 0$ for μ, t_{max} small enough.

We claim that

$$\lambda'(s) < 0 \quad \text{for all } s > -lnt_{max}. \tag{10}$$

Indeed, if this is not the case, there exists a first time s^* such that $\lambda'(s^*) = 0$. At the time s^* we obtain

$$\begin{aligned}\lambda''(s^*) &= -\frac{1-n\alpha}{k-1}\lambda'(s^*) - C_3 e^{-s^*} + \chi e^{-n\alpha s^*}\lambda^{k-1}(s^*)(k\lambda'(s^*) - n\alpha\lambda(s^*)) \\ &= -C_3 e^{-s^*} - \chi n\alpha e^{-n\alpha s^*}\lambda^k(s^*).\end{aligned}$$

From which we see that $\lambda''(s^*) < 0$, a contradiction. Hence, from (9) and (10) one finds that

$$\lambda(s) \to 0 \quad \text{as s} \to +\infty. \tag{11}$$

Assume that γ is a constant satisfying

$$\frac{1-kn\alpha}{k} \leq \gamma(1-n\alpha) \leq \frac{(k-1)(1-n\alpha)}{k}.$$

From (9) we have

$$\lambda'(s) + \frac{\gamma(1-n\alpha)}{k-1}\lambda(s) = \lambda(s)\Big(-\frac{(1-\gamma)(1-n\alpha)}{k-1} + \chi e^{-n\alpha s}\lambda^{k-1}(s)\Big) + C_3 e^{-s}.$$

Letting s be large enough, we have, thanks to (11), that

$$\lambda'(s) + \frac{\gamma(1-n\alpha)}{k-1}\lambda(s) \leq C_3 e^{-s}.$$

Accordingly, $\lambda(s) \leq C_4 e^{-\frac{\gamma(1-n\alpha)}{k-1}s}$, which together with (9) enables us to obtain

$$\lambda'(s) + \frac{1-n\alpha}{k-1}\lambda(s) \leq C_3 e^{-s} + \chi C_4^k e^{-\left(\frac{k\gamma(1-n\alpha)}{k-1}+n\alpha\right)s}.$$

So, $\lambda'(s) + \frac{1-n\alpha}{k-1}\lambda(s) \leq C_5 e^{-\left(\frac{k\gamma(1-n\alpha)}{k-1}+n\alpha\right)s}$. Therefore, we get

$$\lambda(s) \leq C_6 e^{-\frac{1-n\alpha}{k-1}s},$$

which implies that $v(t) \leq C_6$ and hence $u(x,t)$ is bounded for every $x \in \overline{\Omega}\backslash B_\mu(x_0)$, as desired. This completes the proof. □

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