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An Example of a Non-Trivial, Non-Self-Adjoint Huygens Differential Equation in Four Independent Variables

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We dedicate this paper to the memory of Volkmar Wünsch

Abstract. A family of non-trivial, essentially non-self-adjoint wave equations which satisfy Huygens' principle is given. It is constructed on a 4-dimensional Lorentzian space which is a product of two 2-dimensional spaces of constant curvature. Prior to this example, the only known non-trivial Huygens equation was the scalar wave equation on the exact plane wave spacetime as presented by Günther.

Keywords. Initial value problem, Huygens' principle, Hadamard's problem, non-self-adjoint wave equation

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1. Introduction

This paper concerns Hadamard's problem of diffusion of waves for second order, linear, homogeneous, partial differential equations of normal hyperbolic type for an unknown function u in n independent variables. Such an equation may be written in coordinate invariant form as

$$F(u) := g^{ij}\nabla_i\nabla_j u + A^i\nabla_i u + Cu = 0\,, \tag{1}$$

where g^{ij} are the contravariant components of the metric tensor $\mathbf{g}$ of a Lorentzian space $(M, \mathbf{g})$ of signature $2-n$ and ∇_i denotes the covariant derivative with respect to the Lorentzian connection. The coefficients g^{ij}, A^i, and C as well as M are assumed to be of class C^∞.

This problem arises in the study of *Cauchy's problem* for the equation (1) which is the problem of determining a solution which assumes given values for u

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and its normal derivative on a given space-like $(n-1)$-dimensional submanifold S. These given values are called the *Cauchy data*. The first general solution to Cauchy's problem for (1) was given by Hadamard [18]. Hadamard's theory is local in the sense that it is restricted to geodesic simply convex neighbourhoods of M. A modern treatment is given by Friedlander [15]. The considerations of this paper are entirely local.

The question of how the value of the solution u at a point $x_0 \in M$ depends on the Cauchy data is of considerable interest. Hadamard shows that in general $u(x_0)$ depends on the data on and in the interior of the intersection of the retrograde characteristic conoid $C^-(x_0)$ with the initial surface S. If the solution depends only on the data in an arbitrarily small neighbourhood of $S \cap C^-(x_0)$ for every Cauchy problem and for every x_0, one says that the equation satisfies *Huygens' principle* or is a *Huygens equation*. Examples of such equations are the ordinary wave equations

$$\frac{\partial^2 u}{\partial {x^1}^2} - \sum_{i=2}^{2m} \frac{\partial^2 u}{\partial {x^i}^2} = 0\,, \tag{2}$$

in an even number of variables $n = 2m \geqslant 4$, which we denote by $\mathrm{E_M}$. Hadamard asked the fundamental question: for which equations is Huygens' principle true? This is called *Hadamard's problem* in the literature. He showed that in order for Huygens' principle to be valid it is necessary that n be even and $\geqslant 4$. He further showed that a necessary and sufficient condition for its validity is that the elementary solution contain no logarithmic term. Since none other than (2) were known, he suggested that as a first step one should attempt to prove that every Huygens equation is equivalent to some equation of the form (2). This suggestion has been called *Hadamard's conjecture* in the literature (see Courant and Hilbert [11, p. 765]).

Recall that two equations of the form (1) are said to be *equivalent* if they are related by one of the following transformations called *trivial transformations* that preserve the Huygens' property of the equation:

(a) a transformation of coordinates,
(b) multiplication of the equation by a non-vanishing factor $e^{-2\phi}$, where ϕ is a function on M (this transformation induces a conformal transformation of the metric),
(c) replacement of the unknown function u by λu, where λ is a non-vanishing function on M.

An equation (1) which is equivalent to an equation (2) is said to be *trivial.* Any equation (1) which is equivalent to a self-adjoint equation, defined by $A^i = 0$, is said to be *essentially self-adjoint.*

Hadamard's conjecture has been proven in the physically interesting case $n = 4$, g^{ij} *constant* by Mathisson [20, 21], Hadamard [19], and Asgeirson [3].

However, it is known not to be true in general. The first counter-examples were given by Stellmacher [26, 27] for $n \geqslant 6$. Further important results have been obtained by Berest [4, 5] and Berest and Winternitz [6] in this case. Counter examples for $n = 4$ have been given by Günther [16]. These examples arise from the wave equation

$$\Box u = 0\,, \tag{3}$$

where $\Box = g^{ij}\nabla_i\nabla_j$ denotes the wave operator, on the Lorentzian space with metric

$$\mathrm{d}s^2 = 2\mathrm{d}x^1\mathrm{d}x^2 - a_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta, \quad (\alpha, \beta = 3, 4) \tag{4}$$

where the symmetric matrix $(a_{\alpha\beta})$ is positive definite with elements that are functions only of x^1. The above metric may interpreted in the framework of general relativity as an exact plane wave solution of the vacuum or Einstein-Maxwell field equations. It has been studied in this context by Ehlers and Kundt [14] in a different coordinate system where it has the form

$$\mathrm{d}s^2 = 2\mathrm{d}v[\mathrm{d}u + (Dz^2 + \overline{D}\overline{z}^2 + ez\overline{z})\mathrm{d}v] - 2\mathrm{d}z\mathrm{d}\overline{z}\,, \tag{5}$$

where D and $e = \overline{e}$ are functions only of v. It should be noted that the Ricci scalar R vanishes identically for the above metric and that the corresponding Weyl curvature tensor C_{ijkl} is Petrov type N [28] in general. We shall denote equation (3), where g^{ij} is given by (4) or (5), by E_{PW}.

2. The example

Until the present, E_{PW} (and those equations equivalent to it) was the only known non-trivial Huygens equation. Indeed, a number of results [1, 2, 8–10, 13, 22–24, 29] suggest that it might be the unique non-trivial such equation for $n = 4$. However, the following example, announced recently in [12][1] without proof, shows that this supposition is not true. Consider the Lorentzian space $(M, \mathbf{g}) = (M_1, \underset{1}{\mathbf{g}}) \times (M_2, \underset{2}{\mathbf{g}})$, where $(M_1, \underset{1}{\mathbf{g}})$ is a 2-dimensional Lorentzian space of constant curvature and $(M_2, \underset{2}{\mathbf{g}})$ is a 2-dimensional Riemannian space of constant curvature. There exist systems of coordinates (u, v) on M_1 and $(z, \bar{z})$ on M_2 with respect to which the metrics $\underset{1}{\mathbf{g}}$ and $\underset{2}{\mathbf{g}}$ have the respective forms

$$\mathrm{d}{s_1}^2 = \frac{2\mathrm{d}u\mathrm{d}v}{\left[1 - \frac{1}{8}(R + \beta)uv\right]^2}\,,$$

$$\mathrm{d}{s_2}^2 = -\frac{2\mathrm{d}z\mathrm{d}\overline{z}}{\left[1 + \frac{1}{8}(R - \beta)z\overline{z}\right]^2}\,,$$

[1] The example given therein is incomplete, since the vector field A^i is undefined.

where R (the Ricci scalar) and β are any real constants. Since $(M,\mathbf{g})$ is a product, the metric $(M,\mathbf{g})$ is expressible as

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}{s_1}^2 + \mathrm{d}{s_2}^2\,, \tag{6}$$

where g_{ij} denotes the covariant components of $\mathbf{g}$. The properties of $(M,\mathbf{g})$ that are key for the construction of our example are the following (see Stephani et al. [28, Sections 12.1, 35.2], and Cahen and McLenaghan [7]):

(i) it possesses an isometry group G_6 which acts transitively on M,

(ii) it has a non-vanishing Weyl conformal curvature tensor of Petrov type D iff $R \neq 0$,

(iii) it is a Riemannian symmetric space, that is the covariant derivative of the Riemann curvature tensor vanishes identically.

Consider also the one-form

$$\begin{aligned}\mathbf{A} =& A_i\mathrm{d}x^i\\ =& \frac{1}{2}(H_3+\overline{H}_3)(1-\alpha uv)^{-1}(v\mathrm{d}u-u\mathrm{d}v)+\frac{1}{2}(\overline{H}_3-H_3)(1+\delta z\overline{z})^{-1}(\overline{z}\mathrm{d}z-z\mathrm{d}\overline{z}),\end{aligned} \tag{7}$$

where A_i denotes the covariant components of $\mathbf{A}$ and[2]

$$|H_3| = \left(\frac{\beta R}{60}\right)^{\frac{1}{2}}, \quad H_3 - \overline{H}_3 = \left(\frac{\beta^2}{630}\left(5\frac{R}{\beta} - 3\right)\left(2\frac{R}{\beta} - 3\right)\right)^{\frac{1}{2}},$$
$$\alpha = \frac{1}{8}(R+\beta)\,, \quad \delta = \frac{1}{8}(R-\beta)\,.$$

The main purpose of the present paper is to prove the following theorem:

Theorem 2.1. *The equation* (1)*, where* g^{ij} *is given by* (6)*, where* $A^i = g^{ij}A_j$ *with* A_i *given by* (7)*, and where*

$$C = \frac{1}{2}A^i{}_{;i} + \frac{1}{4}A^iA_i + \frac{1}{6}R\,, \tag{8}$$

satisfies Huygens' principle if

$$\frac{R}{\beta} = \frac{3}{5}\,. \tag{9}$$

Proof. It follows from (9) that the metric and one-form $\mathbf{A}$ may be simplified to

$$\mathrm{d}s^2 = \frac{2\mathrm{d}u\mathrm{d}v}{\left[1-\frac{1}{3}Ruv\right]^2} - \frac{2\mathrm{d}z\mathrm{d}\overline{z}}{\left[1-\frac{1}{12}Rz\overline{z}\right]^2} \tag{10}$$

and

$$\mathbf{A} = H_3(1-\alpha uv)^{-1}(v\mathrm{d}u - u\mathrm{d}v)\,, \tag{11}$$

[2]In [12] the value of the quantity $H_3 - \overline{H}_3$ is not given.

respectively, where H_3 is real with

$$H_3 = \pm\frac{1}{6}R \tag{12}$$

and

$$\alpha = \frac{1}{3}R\,, \quad \delta = -\frac{1}{12}R\,. \tag{13}$$

We shall denote the equation (1) where g^{ij} is defined by (10), A^i by (11), C by (8), and (12) and (13) are satisfied, by E_{RB}.

To prove that Huygens' principle is satisfied by E_{RB}, we utilize Hadamard's necessary and sufficient condition [19] (see also Friedlander [15, Theorem 5.7.1] or Günther [17, Theorem 1.4, p. 233])

$$[G(V)(x_0, x)] = 0, \quad \forall x_0 \in M, \tag{14}$$

where the brackets $[\cdots]$ signify the restriction of the enclosed function to the null conoid

$$C(x_0) = C^+(x_0) \cup C^-(x_0)\,,$$

and where G denotes the adjoint operator for (1) defined by

$$G(v) = \Box v - (A^i v)_{;i} + Cv\,. \tag{15}$$

The function V is defined by

$$V(x_0, x) = \frac{1}{2\pi}\exp\left\{-\frac{1}{4}\int_0^{s(x)} (g^{ij}\Gamma_{;ij} - 8 - A^i\Gamma_{,i})\frac{dt}{t}\right\}, \tag{16}$$

where the integration is along the geodesic joining x_0 and x, $\Gamma(x_0, x)$ is, up to a sign, the square of the geodesic distance between x_0 and x, and s is an affine parameter. We note that $\Gamma(x_0, x) = 0$, iff x_0 and x are connected by a null geodesic; hence, the null conoid $C(x_0)$ is defined by $\Gamma(x_0, x) = 0$, for all $x \in M$.

The convenience of the condition (14) results in part from the fact that the function V defined by (16) may be expressed as

$$V(x_0, x) = \frac{1}{2\pi}(\rho(x_0, x))^{-\frac{1}{2}}\exp\left\{\frac{1}{4}\int_0^{s(x)} A^i\Gamma_{,i}\frac{dt}{t}\right\}, \tag{17}$$

where

$$\rho(x_0, x) = 8(g(x)g(x_0))^{\frac{1}{2}}\left|\det\left(\frac{\partial^2\Gamma}{\partial x^i \partial x_0^j}\right)\right|^{-1}$$

is the discriminant function, and $g(x) = \det(g_{ij}(x))$. In what follows, we choose x_0 to be the point with coordinates $u = v = z = \overline{z} = 0$; this can be done without loss of generality, since the isometry group G_6 acts transitively on M, that is given any two points $x, y \in M$ there exists $g \in G_6$ such that $x = g(y)$.

Furthermore, since M is decomposable into two 2-dimensional spaces of constant curvature, we have by a result of Ruse et al. ([25, p. 215]),

$$\Gamma = \Gamma_1 + \Gamma_2$$

where Γ_1 is the square of the geodesic distance on the Lorentzian 2-space with metric

$$\mathrm{d}{s_1}^2 = \frac{2\mathrm{d}u\mathrm{d}v}{\left[1 - \frac{1}{3}Ruv\right]^2} \tag{18}$$

and Γ_2 is the square of the distance on the Riemannian 2-space with metric

$$\mathrm{d}{s_2}^2 = -\frac{2\mathrm{d}z\mathrm{d}\overline{z}}{\left[1 - \frac{1}{12}Rz\overline{z}\right]^2}\,. \tag{19}$$

We denote the covariant components of the metric tensor for (18) by $\underset{1}{g}_{ab}$ (where the indices a, b range over $x^1 = u, x^2 = v$), and for (19) by $\underset{2}{g}_{cd}$ (where the indices c, d range over $x^3 = z, x^4 = \overline{z}$). We may then explicitly determine Γ_1 following Ruse et al. ([25, p. 14]). We let y^a be a system of normal coordinates for (18) about the origin, and

$$\underset{1}{\overset{*}{g}}_{ab} = \frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b}\underset{1}{g}_{\alpha\beta}$$

be the components of $\underset{1}{\mathbf{g}}$ relative to these coordinates. Then

$$\Gamma_1 = (\underset{1}{\overset{*}{g}}_{ab})_0 y^a y^b$$

where we have $(\underset{1}{\overset{*}{g}}_{ab})_0 = (\underset{1}{g}_{ab})_0$. Such a system is given by

$$y^a = \frac{1}{\sqrt{\alpha uv}}\tanh^{-1}(\sqrt{\alpha uv})x^a\,, \tag{20}$$

which follows since

$$\underset{1}{\overset{*}{g}}_{ab}\,y^b = (\underset{1}{\overset{*}{g}}_{ab})_0\,y^b$$

(see ([25, p. 12]). We remark that the complex function $\tanh^{-1}$ is analytic in a neighborhood of the origin, with McLaurin expansion

$$\tanh^{-1}(w) = \sum_{k=0}^{\infty}\frac{w^{2k+1}}{2k+1}\,, \quad |w| < 1\,.$$

It follows that the coordinate transformation (20) is real, since $\tanh^{-1}(\sqrt{\alpha uv})$ is pure imaginary when $\sqrt{\alpha uv}$ is pure imaginary. Moreover, it also follows that y^1, y^2 defined by (20) are analytic, and that the Jacobian of the transformation becomes the identity matrix at the origin. Thus, the coordinate system defined by (20) is regular. We then have

$$\Gamma_1 = \frac{2}{\alpha}\left[\tanh^{-1}(\sqrt{\alpha uv}\,)\right]^2 , \tag{21}$$

which is real regardless of whether $\sqrt{\alpha uv}$ is real or pure imaginary. In similar fashion we may then also obtain

$$\Gamma_2 = \frac{2}{\delta}\left[\tanh^{-1}(\sqrt{-\delta z\bar{z}})\right]^2 \tag{22}$$

for the space with metric (19). We observe that the functions Γ_1, Γ_2 then satisfy

$$\underset{1}{g}^{ab}\Gamma_{1,a}\Gamma_{1,b} = 4\Gamma_1 , \quad \underset{2}{g}^{cd}\Gamma_{2,c}\Gamma_{2,d} = 4\Gamma_2 ; \tag{23}$$

hence, we also have $g^{ij}\Gamma_{,i}\Gamma_{,j} = 4\Gamma$. Before proceeding with the evaluation of (14), we further note that

$$\begin{aligned} A^i\Gamma_{,i} &= g^{ij}A_j\Gamma_{,i} \\ &= (1-\alpha uv)^2\left[-H_3(1-\alpha uv)^{-1}u\,\Gamma_{,u} + H_3(1-\alpha uv)^{-1}v\,\Gamma_{,v}\right] \\ &= 0 . \end{aligned} \tag{24}$$

In view of (24), the function V given in (17) takes the form

$$V = \frac{1}{2\pi}\rho^{-\frac{1}{2}} . \tag{25}$$

In order to calculate the final term of (14), as determined by (15), we first compute

$$\begin{aligned} A^i{}_{;i} &= g^{-\frac{1}{2}}(g^{\frac{1}{2}}g^{ij}A_j)_{,i} \\ &= g^{-\frac{1}{2}}[(g^{\frac{1}{2}}g^{uv}A_u)_{,v} + (g^{\frac{1}{2}}g^{vu}A_v)_{,u}] \\ &= (1-\alpha uv)^2 A_{u,v} + (1-\alpha uv)^2 A_{v,u} \\ &= 0 \end{aligned}$$

and $A^iA_i = 2g^{uv}A_uA_v = -2{H_3}^2uv$. Thus, in view of (12) we have

$$C = -\frac{1}{72}R^2uv + \frac{1}{6}R . \tag{26}$$

Next, for the middle term of (14) we use (25) to compute (noting that $A^i{}_{;i} = 0$)

$$(A^iV)_{;i} = A^i{}_{;i}V + A^iV_{;i} = g^{ij}A_jV_{,i} = \frac{1}{2\pi}\left[g^{uv}A_u\frac{\partial\rho^{-\frac{1}{2}}}{\partial v} + g^{vu}A_v\frac{\partial\rho^{-\frac{1}{2}}}{\partial u}\right]. \quad (27)$$

Since our space is decomposable into spaces of constant curvature, we have (see [25, p. 215 and p. 30]) that

$$\rho = \rho_1\rho_2\,, \quad (28)$$

where

$$\rho_1 = \left(\frac{\sin^2\sqrt{\kappa_1\Gamma_1}}{\kappa_1\Gamma_1}\right)^{\frac{1}{2}}, \quad (29)$$

$$\rho_2 = \left(\frac{\sin^2\sqrt{\kappa_2\Gamma_2}}{\kappa_2\Gamma_2}\right)^{\frac{1}{2}}, \quad (30)$$

and

$$\kappa_1 = -2\alpha = -\frac{2}{3}R\,, \quad \kappa_2 = -2\delta = \frac{1}{6}R\,. \quad (31)$$

Note that since $\kappa_1\Gamma_1$ is real, the expression $\sqrt{\kappa_1\Gamma_1}$ is either real or pure imaginary according to the sign of $\kappa_1\Gamma_1$. The complex sine function satisfies $\sin(iw) = i\sinh(w)$, so in the latter case the quantity $\sin^2\sqrt{\kappa_1\Gamma_1}$ in (29) will be real and negative. It follows that ρ_1 (and by a similar argument, ρ_2) will be real. Using the equations (28)–(31) along with (21), (22), we obtain from (27) (after some lengthy but straightforward computation)

$$\begin{aligned}(A^iV)_{;i} &= -\frac{1}{4\pi}\rho^{-\frac{3}{2}}\left[g^{uv}A_u\rho_2\frac{\partial\rho_1}{\partial v} + g^{vu}A_v\rho_2\frac{\partial\rho_1}{\partial u}\right]\\ &= -\frac{1}{4\pi}\rho^{-\frac{3}{2}}\left[g^{uv}\rho_2\frac{\partial\rho_1}{\partial\Gamma_1}\left(A_u\frac{\partial\Gamma_1}{\partial v} + A_v\frac{\partial\Gamma_1}{\partial u}\right)\right]\\ &= 0\,. \end{aligned} \quad (32)$$

It remains to calculate the first term of (14). Since our space is decomposable, we have

$$\Box V = \Box_1 V + \Box_2 V\,,$$

where

$$\Box_1 V = \underset{1}{g}^{-\frac{1}{2}}\partial_a\left(\underset{1}{g}^{\frac{1}{2}}\underset{1}{g}^{ab}\partial_b V\right)$$

and

$$\Box_2 V = \underset{2}{g}^{-\frac{1}{2}}\partial_c\left(\underset{2}{g}^{\frac{1}{2}}\underset{2}{g}^{cd}\partial_d V\right),$$

and $\underset{1}{g}$, $\underset{2}{g}$ denote the determinants of the metrics in (18), (19) respectively. Then from (25) and (28) we have

$$
\begin{aligned}
2\pi\Box V &= \left(\rho_2^{-\frac{1}{2}}\Box_1\rho_1^{-\frac{1}{2}} + \rho_1^{-\frac{1}{2}}\Box_2\rho_2^{-\frac{1}{2}}\right) \\
&= \rho_2^{-\frac{1}{2}}\left[-\frac{1}{2}\rho_1^{-\frac{3}{2}}\left(-\frac{3}{2}\rho_1^{-1}\underset{1}{g}^{ab}\partial_a\rho_1\partial_b\rho_1 + \Box_1\rho_1\right)\right] \\
&\quad + \rho_1^{-\frac{1}{2}}\left[-\frac{1}{2}\rho_2^{-\frac{3}{2}}\left(-\frac{3}{2}\rho_2^{-1}\underset{2}{g}^{cd}\partial_c\rho_2\partial_d\rho_2 + \Box_2\rho_2\right)\right] \\
&= -\frac{1}{2}\rho^{-\frac{1}{2}}\Big(\rho_1^{-1}\left[2\Gamma_1(2\rho_1'' - 3\rho_1^{-1}(\rho_1')^2) + \rho_1'\Box_1\Gamma_1\right] \\
&\quad + \rho_2^{-1}\left[2\Gamma_2(2\rho_2'' - 3\rho_2^{-1}(\rho_2')^2) + \rho_2'\Box_2\Gamma_2\right]\Big),
\end{aligned}
$$

where the indices a, b and c, d range as appropriate for the metrics (18), (19) respectively, and the equations (29), (30), and (23) have also been used. In order to proceed further, we require the expressions

$$\Box_1\Gamma_1 = 2 + 2\left(\kappa_1\Gamma_1\right)^{\frac{1}{2}}\cot\left(\kappa_1\Gamma_1\right)^{\frac{1}{2}}\ , \tag{33}$$

and

$$\Box_2\Gamma_2 = 2 + 2\left(\kappa_2\Gamma_2\right)^{\frac{1}{2}}\cot\left(\kappa_2\Gamma_2\right)^{\frac{1}{2}} \tag{34}$$

(see [25, p. 30]). We note that the right hand sides of both (33) and (34) are real because of the identity $\cot(iw) = -i\coth(w)$ satisfied by the complex cotangent function. Using (33) and (34) along with (29) and (30), we then obtain

$$2\pi\Box V = -\frac{1}{2}\rho^{-\frac{1}{2}}\left[-\frac{\kappa_1}{2}(2+\cot^2(\kappa_1\Gamma_1)^{\frac{1}{2}}) - \frac{\kappa_2}{2}(2+\cot^2(\kappa_2\Gamma_2)^{\frac{1}{2}}) + \frac{1}{2}(\Gamma_1^{-1}+\Gamma_2^{-1})\right]. \tag{35}$$

In order to simplify the first term in the above equation, we use (21) to write $\kappa_1\Gamma_1 = -4\left(\tanh^{-1}\sqrt{\alpha uv}\right)^2$, so that

$$\tan\left(\kappa_1\Gamma_1\right)^{\frac{1}{2}} = \tan\left(2i\tanh^{-1}\sqrt{\alpha uv}\right) = i\tanh\left(2\tanh^{-1}\sqrt{\alpha uv}\right) = \frac{2i\sqrt{\alpha uv}}{1+\alpha uv}\,.$$

Thus,

$$\cot^2\left(\kappa_1\Gamma_1\right)^{\frac{1}{2}} \;=\; \frac{(1+\alpha uv)^2}{-4\alpha uv}\,.$$

We likewise obtain

$$\cot^2\left(\kappa_2\Gamma_2\right)^{\frac{1}{2}} \;=\; \frac{(1-\delta z\bar{z})^2}{4\delta z\bar{z}}\,.$$

Hence, the equation (35) simplifies to

$$\Box V = \left[-\frac{3}{16}R + \frac{1}{8}\left(\frac{1}{uv} - \frac{1}{z\bar{z}}\right) + \frac{1}{8}(\alpha^2 uv - \delta^2 z\bar{z}) - \frac{1}{4}(\Gamma_1^{-1} + \Gamma_2^{-1})\right]V\,. \tag{36}$$

Finally, in order to verify the necessary and sufficient condition (14), we must evaluate the quantity $G(V)$ on the conoid $C(x_0)$. We therefore set

$$\Gamma = \Gamma_1 + \Gamma_2 = 0\,. \tag{37}$$

Using (21) and (22), the equation (37) may be written as

$$\tanh^{-1}\left(\frac{1}{3}\sqrt{3Ruv}\right) = \pm 2\tanh^{-1}\left(\frac{1}{6}\sqrt{3Rz\overline{z}}\right).$$

Applying the function tanh to both sides and squaring, we obtain

$$uv = \frac{144z\overline{z}}{(12+Rz\overline{z})^2}\,. \tag{38}$$

Thus, using (36)–(38), (32), and (26), the equation (14) becomes

$$[G(V)] = \left(-\frac{1}{6}R + \frac{2R^2z\overline{z}}{(12+Rz\overline{z})^2} - \frac{1}{4}\frac{\Gamma_1+\Gamma_2}{\Gamma_1\Gamma_2} - \frac{2R^2z\overline{z}}{(12+Rz\overline{z})^2} + \frac{1}{6}R\right)[V] = 0\,,$$

which completes the proof. □

We observe that E_{RB} is an essentially non-self-adjoint equation; that is, it is not equivalent to any self-adjoint equation. To see this it is enough to note that, since $R \neq 0$ by (9), it follows that $d\mathbf{A} \neq 0$ by (11) and (12). Thus $\mathbf{A}$ is not a closed one-form; however, by [12, Equation (3.3)], $\mathbf{A}$ closed is a necessary condition for the existence of a trivial transformation to set $\bar{\mathbf{A}} = 0$. *We thus have proved that* E_{RB} *is a Huygens equation which is equivalent to neither* E_M *nor* E_{PW}*, both of which are self-adjoint equations.*

3. Conclusion

It is known [9, 24, 29] that there exist no Petrov type D spaces on which the essentially self-adjoint equation (1) satisfies Huygens' principle. Thus Hadamard's problem is completely solved in this case. However, the existence of E_{RB} shows that the problem remains open for the essentially non-self-adjoint equation (1) on type D background spaces. In a subsequent paper, it will be shown that E_{RB} is (up to a trivial transformation) the *unique* non-trivial, essentially non-self-adjoint Huygens equation on a general conformally symmetric space of Petrov type D.

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