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A Rigorous Interpretation of Approximate Computations of Embedded Eigenfrequencies of Water Waves

Sergei A. Nazarov and Keijo M. Ruotsalainen

Abstract. In this paper, we will investigate embedded eigenvalues in the framework of the linearized theory of water waves. We assume that an approximation of an embedded eigenvalue is provided. To the question, whether there is a trapped mode near the computed solution, we provide an affirmative answer.
We will prove that, under certain assumptions on the data for a water wave problem in an infinite channel Ω^0, the result λ^0 of the numerical computation can be justified as follows: if the computational error ε is sufficiently small, there exists a water domain Ω^ε which is a local regular perturbation of Ω^0 and has an eigenvalue $\lambda^\varepsilon \in [\lambda^0 - c\varepsilon, \lambda^0 + c\varepsilon]$ embedded in the continuous spectrum. This conclusion is made by means of an asymptotic analysis of the augmented scattering matrix, whose properties guarantee a sufficient condition for the existence of trapped mode.

Keywords. Water waves, approximate computation of embedded eigenvalues, augmented scattering matrix, asymptotic analysis, perturbation techniques
Mathematics Subject Classification (2010). Primary 74J05, secondary 74J20, 76B15, 76M45

1. Introduction

1.1. The problem. Localized oscillations in unbounded media, so called trapped modes, are generated by eigenvalues of the corresponding boundary value problem which are naturally divided into two classes with completely different

S. A. Nazarov: St. Petersburg State University, Universitetskaya nab., 7–9, St. Petersburg, 1990343, Russia; Peter the Great Saint-Petersburg State Polytechnical University, Polytechnicheskaya ul., 29, St. Petersburg, 195251, Russia; and Institute of Problems of Mechanical Engineering RAS, V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia; srgnazarov@yahoo.com.uk
K. M. Ruotsalainen: University of Oulu, Applied and Computational Mathematics Research Group, Faculty of Science, P.O. Box 4500, 90014 Oulu, Finland; keijo.ruotsalainen@oulu.fi

properties. The first class contains eigenvalues in the discrete spectrum. They are stable with respect to the small perturbations of the problem data. On the contrary, eigenvalues embedded in continuous spectrum, which form the second class, are intrinsically unstable. For those arbitrarily small perturbations may remove them out of the spectrum and turn them into the points of complex resonances (cf. examples in [1, 3, 21, 22] and the review paper [13]).

In the framework of the linearized theory of water waves, there are several examples of trapped modes and eigenvalues, see e.g. [29,30] and [15]. Moreover, various approaches to find out stable eigenvalues and several criteria for the existence of embedded eigenvalues have been developed. However, the most used tools for detecting eigenvalues in the water wave problems are based on numerical approximation schemes. For eigenvalues in the discrete spectrum, the error estimates are supported by the above-mentioned stability property, whereas for the embedded eigenvalues such estimates cannot subsist because of the instability. In this paper, based on the concept of *enforced stability of embedded eigenvalues* (cf. [22, 23]), we will provide a solid scheme to interpret the approximate computations of trapped modes. Similar results were obtained in [24] for an elementary problem in a two-dimensional acoustic waveguide.

Mathematically, we assume that a plane surface wave

$$e^{ikx}\varphi(y,z), \quad k \geqslant 0,\ i = \sqrt{-1}$$

propagates over a water layer of constant depth $d > 0$ with an infinite cylinder directed along the x-axis. It is either submerged (Figure 1a) or immovable surface-piercing (Figure 1b). According to the linear theory of water waves, the velocity potential satisfies, after separation of variables, the Helmholtz equation

$$-\Delta\varphi(y,z) + k^2\varphi(y,z) = 0, \quad (y,z) \in \Omega, \tag{1}$$

where Ω is the cross-section of the water domain in the (y,z)-plane. On the free surface $\Gamma = \{(y,z) \in \partial\Omega :\ z = 0\} \subset \mathbb{R} \times \{0\}$, the velocity potential fulfils the kinematic (Steklov) boundary condition

$$\partial_z\varphi(y,0) = \lambda\varphi(y,0), \quad (y,0) \in \Gamma, \tag{2}$$

and the Neumann boundary condition (no normal flow)

$$\partial_\nu\varphi(y,z) = 0, \quad (y,z) \in \Sigma, \tag{3}$$

at the union of the bottom and the wetted surface of the obstacle denoted by Σ (Figure 1). The domain Ω, open and connected set, is supposed to have a Lipschitz boundary $\partial\Omega$ and to coincide with the strip $\Pi = \mathbb{R} \times (-d, 0)$ outside a rectangle of width $R > 0$:

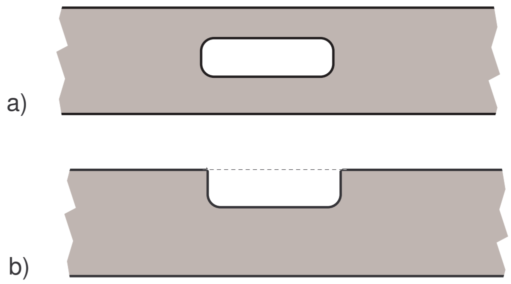

Figure 1: a) Submerged cylinder, b) surface piercing cylinder

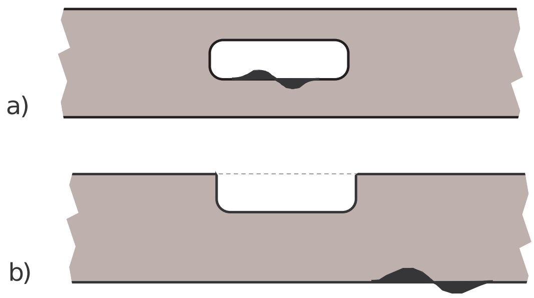

Figure 2: Perturbated domain

$$\begin{aligned}\Omega(R) = \{(y,z) \in \Omega : \, |y| < R\}, \quad \Omega \setminus \overline{\Omega(R)} = \Pi_+(R) \cup \Pi_-(R) \\ \Pi_\pm(R) = \{(y,z) \in \Pi : \, \pm y > R\}.\end{aligned} \tag{4}$$

In the Steklov condition (2), $\lambda = \frac{\omega^2}{g}$ is the spectral parameter, where $\omega > 0$ is the angular frequency of the time-harmonic oscillations and $g > 0$ the acceleration due to gravity. In (3), ∂_ν is the directional derivative along the outward normal ν which is defined almost everywhere on $\partial\Omega$ and equals $\partial_z = \frac{\partial}{\partial z}$ on the free surface Γ.

If the obstacle is absent, the plane wave in the straight layer $\mathbb{R} \times \Pi$ takes the form

$$\mathrm{e}^{\mathrm{i}kx} w(y,z) = \mathrm{e}^{\mathrm{i}kx} \mathrm{e}^{\mathrm{i}ly} W(z),$$

where $l \in \mathbb{R}$ and

$$W(z) = \mathrm{e}^{mz} + \mathrm{e}^{-m(z+2d)} \tag{5}$$

with $m = \sqrt{k^2 + l^2} > 0$. The spectral parameter $\lambda = \lambda(m)$ in this case is given by

$$\lambda(m) = m \frac{1 - \mathrm{e}^{-2md}}{1 + \mathrm{e}^{-2md}} = m \tanh(2md).$$

Notice that the mapping $\mathbb{R}_+ \ni m \mapsto \lambda(m) \in \mathbb{R}_+$ is one-to-one.

It is known, [4] and [11], that the spectrum σ of the problem (1)–(3) consists of the continuous spectrum $\sigma_c = [\lambda_\dagger, +\infty)$ with the cut-off point $\lambda_\dagger = \lambda(k)$ and of the discrete spectrum $\sigma_d \subset (0, \lambda_\dagger)$, which is composed of finite number of eigenvalues. There also may exist so called *embedded eigenvalues* which together with σ_d form the *point* spectrum σ_p of the problem (1)–(3). Each point in σ_p gives rise to a *trapped mode*, that is a solution of the problem (1)–(3) with the exponential decay at infinity.

1.2. Assumptions and the main goal of the paper. Let $\lambda^0 \in (\lambda_\dagger, \infty)$ and let a function φ^0, which belongs to the Sobolev space $H^1(\Omega)$ and is normalized in the Lebesgue space $L^2(\Omega)$, satisfy the variational formulation of the inhomogeneous problem (1)–(3)

$$(\nabla\varphi^0, \nabla\psi)_\Omega + k^2(\varphi^0, \psi)_\Omega - \lambda^0(\varphi^0, \psi)_\Gamma = f^0(\psi), \quad \forall\psi \in H^1(\Omega), \tag{6}$$

where ∇ = grad, $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ are the standard scalar products in the Lebesgue spaces $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. We assume that the anti-linear continuous functional $f^0 \in H^1(\Omega)^*$ has a compact support in $\overline{\Omega(R)}$ and a relatively small norm, i.e.

$$f^0(\psi) = 0 \quad \forall\psi \in H^1(\Omega) : \psi(y, z) = 0,\ |y| < R,$$

and

$$\varepsilon := \|f^0; H^1(\Omega)^*\| \in (0, 1], \tag{7}$$
$$\|\varphi^0; H^1(\Omega)\| = 1. \tag{8}$$

In this paper, under certain assumptions, we will detect positive constants ε_0 and c_0 such that, in the case where ε in (7) belongs to the range $(0, \varepsilon_0)$, then one can construct a small local perturbation Ω^ε of $\Omega^0 = \Omega$ for which the problem

$$-\Delta\varphi^\varepsilon(y, z) + k^2\varphi^\varepsilon(y, z) = 0, \qquad (y, z) \in \Omega^\varepsilon, \tag{9}$$
$$\partial_z\varphi^\varepsilon(y, 0) = \lambda^\varepsilon\varphi^\varepsilon(y, 0), \quad (y, 0) \in \Gamma, \tag{10}$$
$$\partial_{\nu^\varepsilon}\varphi^\varepsilon(y, z) = 0, \qquad (y, z) \in \Sigma^\varepsilon, \tag{11}$$

has an eigenvalue $\lambda^\varepsilon \in (\lambda_\dagger, +\infty)$ subject to the estimate

$$|\lambda^\varepsilon - \lambda^0| \leqslant c_0\varepsilon. \tag{12}$$

The couple $\{\lambda^0, \varphi^0\}$ can be regarded as a computational result. Since the computations are usually performed in a bounded domain $\Omega(R_c)$, the function φ^0 is obtained by an extension over $\Pi_\pm(R_c)$. In this way the small discrepancy f^0 involves both the computational and extension errors.

The perturbed domain Ω^ε will be determined as follows, cf. Figures 2a and 2b. Let Υ be a smooth open arc in $\partial\Omega\setminus\overline{\Gamma}$ and let its tubular neighbourhood $U_\delta = \{(y,z) : s \in \Upsilon, |n| < \delta\}$ be such that it does not intersect the free surface Γ for some $\delta > 0$. Here (n, s) are the intrinsic curvi-linear coordinates, n is the oriented distance to Υ ($n < 0$ inside Ω) and s is the arc length on Υ. The boundary $\partial\Omega^\varepsilon$ coincides with $\partial\Omega$ outside the neighbourhood U_δ, but inside U_δ is defined via the equation

$$n = \varepsilon h(s), \quad s \in \Upsilon, \tag{13}$$

where $h \in C_0^\infty(\Upsilon)$ is a smooth profile function vanishing near the endpoints of Υ. The perturbed curve $\Upsilon^\varepsilon = \partial\Omega^\varepsilon \setminus (\partial\Omega \setminus \Upsilon)$ is situated at a positive distance from the free surface Γ. From this it follows that the Steklov conditions (2) and (10) are imposed on the same set.

To prove the existence of the eigenvalue λ^ε, some assumptions must be put on the profile function h. This will be done in Section 4.2. In other words, the choice of an appropriate perturbation profile requires "very fine" tuning of parameters.

Since f^0 in (6) has a support in $\overline{\Omega(R)}$, the Fourier method gives us the representation

$$\begin{aligned} &\varphi^0(y,z) = \sum_{\tau=\pm} \chi_\tau(y) K_\tau \mathrm{e}^{-\tau\theta_1|y|} \cos(t_1(\tfrac{z}{d}+1)) + \widetilde{\varphi}^0(y,z) \\ &|\widetilde{\varphi}^0(y,z)| \leqslant C^0 \mathrm{e}^{-\theta_2|y|}, \quad (y,z) \in \Pi_\pm(R+d), \end{aligned} \tag{14}$$

where $\chi_\pm$ are smooth cut-off functions:

$$\begin{gathered} \chi_\pm(y) = 1 \text{ for } \pm y > R+d, \quad \chi_\pm(y) = 0 \text{ for } \pm y < R, \\ 0 \leqslant \chi_\pm(y) \leqslant 1, \quad \chi_+(y) = \chi_-(-y). \end{gathered} \tag{15}$$

$K_\pm$ are some coefficients and $\theta_j = \sqrt{k^2 + d^{-2}t_j^2}$, where $t_j \in (\pi(j-\frac{1}{2}), \pi j)$ are positive roots of the transcendental equation

$$-t\tan(t) = \lambda^0 d. \tag{16}$$

Selecting either the real or imaginary part, we may always consider φ^0 and $K_\pm$ as real valued. After a preparatory work we will modify in Section 2.1 the normalization condition (8) in terms of the coefficients $K_\pm$ from (14) which, by our assumption, cannot vanish simultaneously.

Moreover, we suppose that the problem (1)–(3) in Ω^0 has no trapped mode with the decay rate $O(\mathrm{e}^{-\theta_2|y|})$ as $|y| \to \infty$, cf. formula (33) in Section 2.3. In other words, neither our approximate solution φ^0 nor any trapped mode at the computed frequency $\omega^0 = \sqrt{g\lambda^0}$ gets too fast exponential decay at infinity.

We will discuss in Section 5.2, how to avoid these restrictions. We emphasize that the existence of a trapped mode $\varphi(y,z) = o(\mathrm{e}^{-\theta_1|y|})$ just assures that λ^0 is an embedded eigenvalue. However, this trapped mode has no relation to the computations performed and this is just the reason to introduce the condition above.

1.3. Structure of the paper. In Section 2, we proceed with the different operator formulations of the problem (9)–(11) with appropriate radiation conditions, see Theorems 2.1 and 2.2, bearing in mind two purposes. First, to give a criterion for the existence of a trapped mode in Theorem 2.5, which is based on the notion of the augmented scattering matrix S^ε (cf. [9]) and involves the linear combinations of exponential waves (wave "packets"). Second, to apply the perturbation theory of linear operators we will use the technique of weighted spaces with detached asymptotics, see e.g. [17, 19] which has never before applied in the theory of water waves. With the help of the perturbation theory, we conclude the smooth (actually analytic) dependence of S^ε on several parameters, which is needed for the fine tuning of the profile function and detection of the eigenvalue. The latter allows us to prove in Section 4.3 the existence of the desired perturbed water domain Ω^ε.

In Section 3 we perform a simple asymptotic analysis to construct the asymptotics of special solutions to the problem (9)–(11) as well as the augmented scattering matrix. We emphasize that the utilization of the weighted spaces helps us to justify the asymptotics of S^ε.

To make use of the derived asymptotic formulae, we start Section 4 with reformulating the criterion for a trapped mode in Lemma 4.1. This reformulated criterion is written in more convenient vector form which directly gives us a system of non-linear equations to find the desired profile function h in (13). The proof that the system of non-linear equations is solvable is given in Theorem 4.2 and it uses the contraction principle. The main result on the embedded eigenvalue λ^ε is formulated in Theorem 4.3. In the final section we provide an argument that an eigenvalue λ^ε satisfying the estimate (12) can be found under the assumptions (6)–(8) with a sufficiently small ε "almost always", that is when the numerical computations have been performed accurately enough. We finish the paper with some conclusive remarks collected in Section 5.

As a preliminary remark to the quite technical presentation and rigorous proofs of assertions, we can make the following conclusion. If the numerical result is sufficiently accurate, i.e. the aggregate numerical error ε in (7) will not exceed a certain bound ε_0, which depends on λ^0 and Ω^0, then the mildly sloped perturbation Ω^ε of Ω^0 supports, indeed, a trapped mode with an eigenvalue $\lambda^\varepsilon \approx \lambda^0$, see (12) and (13). We emphasize that in case $\lambda^0 \in (0, \lambda_\dagger)$ perturbation of the water domain is not needed whereas in the case $\lambda^0 > \lambda_\dagger$ the perturbation compensates the intrinsic instability of embedded eigenvalues.

2. Waves and scattering matrices

2.1. Oscillating waves and exponential wave packets. In the semi-strips $\Pi_\pm(R)$ we introduce the functions

$$w^{\varepsilon,out}_{0,\pm}(y,z) = \chi_\pm(y) N_{0,\varepsilon}^{-\frac{1}{2}} e^{\pm i l^\varepsilon y} W(z), \quad w^{\varepsilon,in}_{0,\pm}(y,z) = \chi_\pm(y) N_{0,\varepsilon}^{-\frac{1}{2}} e^{\mp i l^\varepsilon y} W(z), \tag{17}$$

which are called propagating waves. According to the sign convention $w^{\varepsilon,out}_{0,\pm}$ are the *outgoing waves* and $w^{\varepsilon,in}_{0,\pm}$ the *incoming waves* in the water domain Ω^ϵ. The cut-off functions $\chi_\pm$ are used in (17) to localise the waves to the semi-infinite parts $\Pi_\pm(R)$ of the channel. The function W^ε is determined through the formula (5) with the parameters

$$l^\varepsilon = \sqrt{(m^\varepsilon)^2 - k^2} > 0, \; m^\varepsilon > k, \quad \lambda(m^\varepsilon) = \lambda^\varepsilon > \lambda_\dagger.$$

The normalisation factor $N_{0,\varepsilon}$ is chosen as

$$N_{0,\varepsilon} = 2l^\varepsilon \|W^\varepsilon; L^2(-d,0)\|^2 = 2l^\varepsilon \left(2d e^{-2m^\varepsilon d} + \frac{1 - e^{-4m^\varepsilon d}}{2m^\varepsilon} \right) > 0.$$

Moreover, we consider the perturbed spectral parameter

$$\lambda^\varepsilon = \lambda^0 + \varepsilon \Lambda, \tag{18}$$

where the correction term Λ will be found out in Section 4.2. The previous formulae are valid in the case $\varepsilon = 0$ as well. Then W^0 is just given by (5).

The exponential waves, which are depicted in formula (14), must also be normalized:

$$v^\varepsilon_\pm(y,z) = e^{\pm \theta^\varepsilon_1 y} V^\varepsilon(z), \quad V^\varepsilon(z) = N_{1,\varepsilon}^{-\frac{1}{2}} \cos(t^\varepsilon_1(d^{-1}z + 1)), \tag{19}$$

$$N_{1,\varepsilon} = 2\theta^\varepsilon_1 \int_{-d}^0 |\cos(t^\varepsilon_1(d^{-1}z+1))|^2 dz = \theta^\varepsilon_1 d \left(1 + \frac{\sin(2t^\varepsilon_1)}{2t^\varepsilon_1} \right) > 0.$$

Here and in the sequel $\theta^\varepsilon_j = \sqrt{k^2 + d^{-2}(t^\varepsilon_j)^2}$ and $t^\varepsilon_j \in \big(\pi(j - \frac{1}{2}), \pi j\big)$ are the roots of the transcendental equation (16) with the obvious replacement $\lambda^0 \mapsto \lambda^\varepsilon$.

We multiply the approximate solution φ^0 with an appropriate constant, but still denote the product as φ^0. Then the decomposition (14) can be rewritten in the form

$$\varphi^0(y,z) = \sum_{\alpha=\pm} \chi_\alpha(y) K_\alpha v^0_{-\alpha}(y,z) + \widetilde{\varphi}^0(y,z) \tag{20}$$

involving now the normalised exponential waves for $\varepsilon = 0$. After multiplying the computed solution φ^0 with a constant, it will convenient to change the normalization condition (8) for

$$|K| = \sqrt{K_+^2 + K_-^2} = 1. \tag{21}$$

In this way we also change the quantity ε in (7), but still assume it to be small in the sense of Theorem 4.3.

Following the approach of [9], see also [22, 23] and Remark 2.1 below, we introduce the following exponential wave packets

$$
\begin{aligned}
&w^{\varepsilon,out}_{1,-}(y,z)\\
&=\frac{1}{2}\Big(K_+\chi_+(y)-\mathrm{i}K_-\chi_-(y)\Big)v^\varepsilon_+(y,z)+\frac{1}{2}\Big(K_-\chi_-(y)-\mathrm{i}K_+\chi_+(y)\Big)v^\varepsilon_-(y,z)\\
&w^{\varepsilon,in}_{1,-}(y,z)\\
&=\frac{1}{2}\Big(K_+\chi_+(y)+\mathrm{i}K_-\chi_-(y)\Big)v^\varepsilon_+(y,z)+\frac{1}{2}\Big(K_-\chi_-(y)+\mathrm{i}K_+\chi_+(y)\Big)v^\varepsilon_-(y,z)\\
&w^{\varepsilon,out}_{1,+}(y,z)\\
&=-\frac{1}{2}\Big(K_-\chi_+(y)+\mathrm{i}K_+\chi_-(y)\Big)v^\varepsilon_+(y,z)+\frac{1}{2}\Big(K_+\chi_-(y)+\mathrm{i}K_-\chi_+(y)\Big)v^\varepsilon_-(y,z)\\
&w^{\varepsilon,in}_{1,+}(y,z)\\
&=-\frac{1}{2}\Big(K_-\chi_+(y)-\mathrm{i}K_+\chi_-(y)\Big)v^\varepsilon_+(y,z)+\frac{1}{2}\Big(K_+\chi_-(y)-\mathrm{i}K_-\chi_+(y)\Big)v^\varepsilon_-(y,z).
\end{aligned}
\tag{22}
$$

We call these exponential wave packets outgoing and incoming similar to standard oscillating waves. They form a new basis in the linear hull $\operatorname{span}\{\chi_\pm v^\varepsilon_\pm, \chi_\mp v^\varepsilon_\pm\}$ of exponential waves in $\Pi_-(R)\cup\Pi_+(R)$. The reason for this modification of the basis becomes obvious when we consider the symplectic form (i.e. anti-hermitian sesqui-linear form)

$$
Q(u,v)=\sum_{\tau=\pm}\tau\int_{-d}^{0}\Big(\overline{v(\tau\rho,z)}\partial_y u(\tau\rho,z)-u(\tau\rho,z)\overline{\partial_y v((\tau\rho,z)}\Big)\mathrm{d}z
$$

which appears as a line integral in the Green formula on the truncated domain $\Omega(\rho)$ defined by (4). The symplectic form is independent on $\rho>\rho_0$, if the functions u and v satisfy the Helmholtz equation (9) in the semi-strips $\Pi_\pm(\rho_0)$ and the boundary conditions (10), (11) on their lateral sides. Due to the normalization factors $N_{j,\varepsilon}$, $j=0,1$, both the oscillating waves and exponential wave packets enjoy the relations

$$
\begin{aligned}
Q(w^{\varepsilon,out}_{j,\alpha},w^{\varepsilon,out}_{k,\tau})&=\mathrm{i}\delta_{j,k}\delta_{\tau,\alpha}, &\quad Q(w^{\varepsilon,in}_{j,\alpha},w^{\varepsilon,in}_{k,\tau})&=-\mathrm{i}\delta_{j,k}\delta_{\tau,\alpha},\\
Q(w^{\varepsilon,out}_{j,\alpha},w^{\varepsilon,in}_{k,\tau})&=0, &\quad Q(w^{\varepsilon,in}_{j,\alpha},w^{\varepsilon,out}_{k,\tau})&=0,
\end{aligned}
\tag{23}
$$

where $\delta_{p,q}$ is the Kronecker symbol, $\alpha,\tau=\pm$, and $j,k\in\{0,1\}$. The calculations of the previous relations are straightforward and we omit them here, cf. [22,23].

The direction of propagation of the water waves can be determined by the Sommerfeld principle. According to it, the wave propagates along the channel from $\mp\infty$ to $\pm\infty$ depending on the sign $\pm$ of the wave number l^ε so that $w^{\varepsilon,out}_{0,\pm}$

are outgoing waves in $\Pi_\pm(R)$ and $w_{0,\pm}^{\varepsilon,in}$ incoming. After a simple calculation, one readily observes that this sign is also given by the sign of $\operatorname{Im} Q(w,w)$. At the same time a much more cumbersome, but still a straightforward, calculation shows that we can still distinguish the wave packets in (22) as outgoing and incoming due to the fact that $K_\pm \in \mathbb{R}$ and $|K| = 1$.

Remark 2.1. Clearly, there exist many possible bases which meet the normalisation and orthogonality conditions (23) and therefore provide more useful conclusions in Theorems 2.2 and 2.3. Our choice of the bases is adapted to the decomposition (20) of the approximate solution φ^0. The advantages of the choice will become clear in the next subsections. In the papers [22–24] much simpler structures were applied for other purposes.

2.2. The standard and augmented scattering matrices. It is known (see [32], [25, Chapter 5] and [28]) that each of the incoming waves in (17) generates the solutions of the problem (9)–(11) which have the asymptotic form

$$\zeta_\pm^\varepsilon(y,z) = w_{0,\pm}^{\varepsilon,in}(y,z) + \sum_{\alpha=\pm} w_{0,\pm}^{\varepsilon,out}(y,z)s_{\alpha\pm} + \widetilde{\zeta}_\pm^\varepsilon(y,z),$$

where $\widetilde{\zeta}_\pm^\varepsilon \in H^1(\Omega^\varepsilon)$ is a remainder with the decay rate $O(\mathrm{e}^{-\theta_1^\varepsilon|y|})$. The transmission $s_{\mp\pm}^\varepsilon$ and reflection $s_{\pm\pm}^\varepsilon$ coefficients compose the scattering matrix

$$s^\varepsilon = \begin{bmatrix} s_{++}^\varepsilon & s_{+-}^\varepsilon \\ s_{-+}^\varepsilon & s_{--}^\varepsilon \end{bmatrix}$$

which is, owing to the normalisation factors in (17), unitary and symmetric:

$$(s^\varepsilon)^{-1} = (s^\varepsilon)^* := (\overline{s^\varepsilon})^\top,\ s^\varepsilon = (s^\varepsilon)^\top,$$

where $(\cdot)^\top$ stands for the transposition and $\overline{a}$ for complex conjugation as usual.

The incoming wave packets in (22) give rise to similar solutions. To simplify the notation, we introduce the vectors

$$\begin{aligned} w^{\varepsilon,out/in} &= (w_{0,+}^{\varepsilon,out/in}, w_{0,-}^{\varepsilon,out/in}, w_{1,+}^{\varepsilon,out/in}, w_{1,-}^{\varepsilon,out/in}) \\ &= (w_1^{\varepsilon,out/in}, w_2^{\varepsilon,out/in}, w_3^{\varepsilon,out/in}, w_4^{\varepsilon,out/in}). \end{aligned} \tag{24}$$

Notice that we have enumerated the waves with just one subscript $i \in \{1,2,3,4\}$. The corresponding row vector of solutions is denoted by

$$Z^\varepsilon = (Z_1^\varepsilon, Z_2^\varepsilon, Z_3^\varepsilon, Z_4^\varepsilon) = (Z_{0,+}^\varepsilon, Z_{0,-}^\varepsilon, Z_{1,+}^\varepsilon, Z_{1,-}^\varepsilon). \tag{25}$$

and has the decomposition

$$Z^\varepsilon(y,z) = w^{\varepsilon,in}(y,z) + w^{\varepsilon,out}S^\varepsilon + \widetilde{Z}^\varepsilon(y,z). \tag{26}$$

Here the remainder term $\widetilde{Z}^\varepsilon = (\widetilde{Z}_1^\varepsilon, \widetilde{Z}_2^\varepsilon, \widetilde{Z}_3^\varepsilon, \widetilde{Z}_4^\varepsilon)$ gets the decay rate $O(\mathrm{e}^{-\theta_2^\varepsilon |y|})$. The matrix S^ε is a 4×4-matrix which we call an *augmented scattering matrix.* As we shall later see, it becomes an algebraic indicator for trapped modes, see [9, 22–24]. To derive the general properties of the augmented scattering matrix, we will need an operator formulation of the problem (9)–(11) in *weighted spaces with detached asymptotics*, cf. [16, 19].

2.3. Operator formulation. Let $W^1_\beta(\Omega^\varepsilon)$ be the Kondratiev space with the exponentially weighted norm

$$\|u; W^1_\beta(\Omega^\varepsilon)\| = \Big(\|\mathrm{e}^{\beta|y|}\nabla u; L^2(\Omega^\varepsilon)\|^2 + \|\mathrm{e}^{\beta|y|}u; L^2(\Omega^\varepsilon)\|^2 \Big)^{\frac{1}{2}}, \tag{27}$$

where $\beta \in \mathbb{R}$ is the weight index. The space consists of all functions in $H^1_{loc}(\overline{\Omega^\varepsilon})$ with the finite norm (27). Obviously, $W^1_0(\Omega^\varepsilon) = H^1(\Omega^\varepsilon)$, but for $\beta > 0$ the space is smaller than $H^1(\Omega^\varepsilon)$, because it includes functions with exponential decay at infinity. The weak solution $\phi^\varepsilon \in W^1_\beta(\Omega^\varepsilon)$ of the inhomogeneous problem (9)–(11) satisfies the integral identity, see e.g. [19],

$$(\nabla\phi^\varepsilon, \nabla\psi)_{\Omega^\varepsilon} + k^2(\phi^\varepsilon, \psi)_{\Omega^\varepsilon} - \lambda^\varepsilon(\phi^\varepsilon, \psi)_\Gamma = F^\varepsilon(\psi)\ \forall \psi \in W^1_{-\beta}(\Omega^\varepsilon). \tag{28}$$

If the right hand side of (28) is a continuous functional in $W^1_{-\beta}(\Omega^\varepsilon)^*$ (notice the change of the weight index), all terms in (28) become defined properly due to the evident trace inequality

$$\|\mathrm{e}^{\beta|y|}\phi^\varepsilon; L^2(\Gamma)\| \leqslant c\|\phi^\varepsilon; W^1_\beta(\Omega^\varepsilon)\|$$

and the extension of the scalar product $(\cdot,\cdot)$ by duality between the appropriate weighted Lebesgue spaces $L^2_\beta(\Omega^\varepsilon)$ and $L^2_{-\beta}(\Omega^\varepsilon)$. As a result, the problem (28) is associated with the continuous mapping

$$\mathcal{A}^\varepsilon_\beta : W^1_\beta(\Omega^\varepsilon) \to W^1_{-\beta}(\Omega^\varepsilon)^*, \quad \mathcal{A}^\varepsilon_\beta(\lambda^\varepsilon)\phi^\varepsilon = F^\varepsilon. \tag{29}$$

By the theory of elliptic problems in domains with cylindrical outlets to infinity, it has the Fredholm property if and only if the model water wave problem in the strip Π has no exponential solution of the form $\mathrm{e}^{\theta y}V(z)$ such that $\operatorname{Re}\theta = -\beta$. Recalling the decompositions of the waves in (14) and (19), we fix the weight index

$$\beta \in (\theta^\varepsilon_1, \theta^\varepsilon_2), \tag{30}$$

It is important to note that a fixed $\beta \in (3\frac{\pi}{2}, 2\pi)$ verifies (30) for any $\varepsilon \geqslant 0$.

Our choice of the weight index β is based on the following observations on $\{\lambda^0, \varphi^0\}$ in Section 1.2. First of all, waves (17) and (22) are contained in the space $W^1_{-\beta}(\Omega^\varepsilon)$, but do not belong to the space $W^1_\beta(\Omega^\varepsilon)$. Secondly, the solution $\varphi^0 \in H^1(\Omega^0)$ of the problem (6) does not belong to the space $W^1_\beta(\Omega^0)$ because of the assumption (21), cf. (16), but the remainder term $\widetilde{\varphi}^0$ does. Finally, owing to our assumption on the absence of trapped modes in problem (1)–(3) with the decay rate $o(\mathrm{e}^{-\theta_1^0|y|})$, the operator $\mathcal{A}_\beta^0(\lambda^0)$ is an isomorphism and below we are able to extend this property to $\mathcal{A}_\beta^\varepsilon(\lambda^\varepsilon)$ for any $\varepsilon \in (0, \varepsilon_0]$.

As was mentioned above, the following assertion is proved in [10], see also [25, Chapter 3 and 5].

Theorem 2.2. *Assume that the weight index β satisfies* (30)*. Then the operators $\mathcal{A}_\beta^\varepsilon(\lambda^\varepsilon)$ and $\mathcal{A}_{-\beta}^\varepsilon(\lambda^\varepsilon)$ are Fredholm. Moreover, if $\phi^\varepsilon \in W_{-\beta}^1(\Omega^\varepsilon)$ is a solution of the problem* (28) *with β replaced by $-\beta$ and $F^\varepsilon \in W_{-\beta}^1(\Omega^\varepsilon)^*$, we have*

$$\phi^\varepsilon(y,z) = \sum_{\tau=\pm}\sum_{j=0,1}\Big(a_{j,\tau}^\varepsilon w_{j,\tau}^{\varepsilon,out}(y,z) + b_{j,\tau}^\varepsilon w_{j,\tau}^{\varepsilon,in}(y,z)\Big) + \widetilde{\phi}^\varepsilon(y,z) \tag{31}$$

for some coefficients $a_{j,\tau}^\varepsilon$, $b_{j,\tau}^\varepsilon \in \mathbb{C}$ and $\widetilde{\phi}^\varepsilon \in W_\beta^1(\Omega^\varepsilon)$. Furthermore, there holds the estimate

$$\|\widetilde{\phi}^\varepsilon; W_\beta^1(\Omega^\varepsilon)\| + \sum_{\tau=\pm}\sum_{j=0,1}(|a_{j,\tau}^\varepsilon| + |b_{j,\tau}^\varepsilon|) \leqslant c_\varepsilon\Big(\|F^\varepsilon; W_{-\beta}^1(\Omega^\varepsilon)^*\| + \|\phi^\varepsilon; W_{-\beta}^1(\Omega^\varepsilon)\|\Big). \tag{32}$$

Note that the dual space $W_{-\beta}^1(\Omega^\varepsilon)^*$ is included into $W_\beta^1(\Omega^\varepsilon)^*$ and contains continuous functionals defined on the space of exponentially growing functions. So in some sense the functional F^ε decays exponentially as well and (31) implies an asymptotics for the exponentially growing solution ϕ^ε. A proof of the next assertion, which is based on the assumption introduced above on the absence of trapped modes with fast decay that

$$\ker \mathcal{A}_\beta^0(\lambda^0) = 0, \tag{33}$$

will be divided into two steps. The case $\varepsilon = 0$ is treated first. For small ε we use the perturbation argument.

Theorem 2.3. *There exist positive constants ε_0 and c_0 such that, for $\varepsilon \in [0, \varepsilon_0]$, the problem* (28) *with $F^\varepsilon \in W_{-\beta}^1(\Omega^\varepsilon)^*$ and the replacement $\beta \mapsto -\beta$ has a unique solution $\phi^\varepsilon \in W_{-\beta}^1(\Omega^\epsilon)$ admitting the asymptotic form*

$$\phi^\varepsilon(y,z) = \sum_{\tau=\pm}\sum_{j=0,1} a_{j,\tau}^\varepsilon w_{j,\tau}^{\varepsilon,out}(y,z) + \widetilde{\phi}^\varepsilon(y,z) \tag{34}$$

and there hold the estimate

$$\|\widetilde{\phi}^\varepsilon; W_\beta^1(\Omega^\varepsilon)\| + \sum_{\tau=\pm}\sum_{j=0,1} |a_{j,\tau}^\varepsilon| \leqslant c_0 \|F^\varepsilon; W_{-\beta}^1(\Omega^\varepsilon)^*\|. \tag{35}$$

Proof. Since by definition $\mathcal{A}^{\varepsilon}_{-\beta}(\lambda^{\varepsilon})$ is the adjoint of $\mathcal{A}^{\varepsilon}_{\beta}(\lambda^{\varepsilon})$, the assumption (33) and Theorem 2.2 ensure that $\mathcal{A}^{0}_{-\beta}(\lambda^{0})$ and $\mathcal{A}^{0}_{\beta}(\lambda^{0})$ are Fredholm operators and isomorphisms, respectively. Thus the problem (28) at $\varepsilon = 0$ admits a solution of the form (31). Moreover, the formula on the index increment, see [25, Theorem 4.1.4], shows that

$$\dim\ker(\mathcal{A}^{0}_{-\beta}(\lambda^{0})) = \dim\ker(\mathcal{A}^{0}_{\beta}(\lambda^{0})) + 4.$$

Here the integer 4 is just half of the number of linearly independent oscillating and exponential waves in the outlets $\Pi_{\pm}(R)$, which fall into $W^{1}_{-\beta}(\Omega^{\varepsilon}) \setminus W^{1}_{\beta}(\Omega^{\varepsilon})$. Together with (33) this means that to find a particular solution in the form (34), one has to verify that any solution (31) of the homogeneous ($F^{\varepsilon} = 0$) problem (28), having either $b^{0}_{j\tau} = 0$ or $a^{0}_{j\tau} = 0$, is necessarily trivial.

In the first case ($b^{0}_{j,\pm} = 0$), we insert ϕ^{0} on both positions in the Green formula on the truncated channel $\Omega(\rho)$ and obtain

$$0 = Q(\phi^{0}, \phi^{0}) = Q\Big(\sum_{\alpha=\pm}\sum_{j=0,1} a^{0}_{j,\alpha} w^{0,out}_{j,\alpha}, \sum_{\tau=\pm}\sum_{k=0,1} a^{0}_{k,\tau} w^{0,out}_{k,\tau}\Big) = \mathrm{i}\sum_{\alpha=\pm}\sum_{j=0,1} |a^{0}_{j,\alpha}|^{2}, \quad (36)$$

i.e. $a^{0}_{j,\pm} = 0$. In proving the previous equality we have used the relations (23) and the fact that the input of remainders with fast decay vanishes as $\rho \to +\infty$. Hence $\phi^{\varepsilon} = \widetilde{\phi}^{\varepsilon} \in W^{1}_{\beta}(\Omega^{\varepsilon})$ and thus, by the assumption (33), $\phi^{\varepsilon} = 0$. The estimate (35) at $\varepsilon = 0$ follows from the estimate (32).

To cover the case $\varepsilon > 0$, we introduce the space $\mathfrak{W}^{1,\varepsilon}_{\beta}(\Omega^{\varepsilon})$ consisting of functions in the form (34). It will become a Banach space when endowed with the norm

$$\|\phi; \mathfrak{W}^{1,\varepsilon}_{\beta}(\Omega^{\varepsilon})\| = \|\widetilde{\phi}^{\varepsilon}; W^{1}_{\beta}(\Omega^{\varepsilon})\| + \sum_{\tau=\pm}\sum_{j=0,1} |a^{\varepsilon}_{j,\tau}|.$$

It is called *weighted space with detached asymptotics*, see [25, Chapter 5] and [19]. It depends on the spectral parameter λ^{ε} through formulae (16)–(20) and (19), (22) for waves, but can be identified with $\mathbb{C}^{4} \times W^{1}_{\beta}(\Omega^{\varepsilon})$ both algebraically and topologically.

The decomposition (34) ought to be regarded as a *radiation condition* which permits only outgoing waves $w^{\varepsilon,out}_{j,\pm}$ in the asymptotics of the solution. The operator formulation of the problem (28) with such a radiation condition dwells upon the mapping

$$\mathfrak{A}^{\varepsilon}_{\beta}(\lambda^{\varepsilon}) : \mathfrak{W}^{1,\varepsilon}_{\beta}(\Omega^{\varepsilon}) \to W^{1}_{-\beta}(\Omega^{\varepsilon})^{*}. \quad (37)$$

Due to the first part of the theorem, the operator $\mathcal{A}^{0}_{\beta}(\lambda^{0})$ is an isomorphism. Now we want to interpret $\mathcal{A}^{\varepsilon}_{\beta}(\lambda^{\varepsilon})$ as a small perturbation of $\mathcal{A}^{0}_{\beta}(\lambda^{0})$ reducing both operators onto $\mathbb{C}^{4} \times W^{1}_{\beta}(\Omega^{0})$, which evidently concludes the proof.

In the tubular neighbourhood U_{δ} of Υ, we make the coordinate change $(y, z) \mapsto (\mathfrak{y}^{\varepsilon}, \mathfrak{z}^{\varepsilon})$ from the Cartesian coordinates to the curvilinear coordinates, where the point $(\mathfrak{y}^\varepsilon, \mathfrak{z}^\varepsilon)$ has the coordinates

$$\mathfrak{n}^\varepsilon = n - \varepsilon h(s), \quad \mathfrak{s}^\varepsilon = s. \tag{38}$$

We then set

$$(y^\varepsilon, z^\varepsilon) = (\mathfrak{y}^\varepsilon, \mathfrak{z}^\varepsilon)\chi_\Upsilon(y,z) + (y,z)(1 - \chi_\Upsilon(y,z)), \tag{39}$$

where $\chi_\Upsilon \in C_0^\infty(U_\delta)$ is a cut-off function, $\chi_\Upsilon = 1$ in the neighbourhood of Υ. Comparing (38) and (13), we see that the coordinate change $(y,z) \mapsto (y^\varepsilon, z^\varepsilon)$ transforms Ω^ε to Ω^0. By (38) and (39), it becomes non-degenerate for a small ε. Furthermore, it is "almost identical", i.e. it's Jacobi matrix satisfies

$$J^\varepsilon(y,z) = \frac{d(y^\varepsilon, z^\varepsilon)}{d(y,z)}, \quad |J^\varepsilon(y,z) - \mathbb{I}_2| \leqslant c_0\varepsilon, \quad |\nabla^p J^\varepsilon(y,z)| \leqslant c_p\varepsilon,$$

where $p = 1, 2$ and $\mathbb{I}_N$ is the $N \times N$ unit matrix. Hence the Laplace operator Δ in Ω^ε and the normal derivative $\partial_{\nu^\varepsilon}$ on Σ^ε in the coordinates (35) turn into the second and first order differential operators $\mathcal{L}^\varepsilon(y^\varepsilon, z^\varepsilon, \nabla_{(y^\varepsilon,z^\varepsilon)})$ and $\mathcal{N}^\varepsilon(y^\varepsilon, z^\varepsilon, \nabla_{(y^\varepsilon,z^\varepsilon)})$, which differ from $\Delta_{(y^\varepsilon,z^\varepsilon)}$ and $\partial_{\nu(y^\varepsilon,z^\varepsilon)}$ by differential operators with small coefficients supported in $\overline{\Omega} \cap U_\delta$.

We also observe that the waves $w_{j,\pm}^{\varepsilon,out}$ differ by $O(\varepsilon)$ from the waves $w_{j,\pm}^{0,out}$ on the compact sets $\{(y,z) :\ \pm y \in [R, R+d],\ z \in [-d, 0]\} \supset \operatorname{supp}|\nabla\chi_\pm|$ by (15). This and the coordinates (39) demonstrate that the changes $(y,z) \mapsto (y^\varepsilon, z^\varepsilon)$ and $w_{j,\pm}^{\varepsilon,out} \mapsto w_{j,\pm}^{0,out}$ turn (37) into an operator

$$\widehat{\mathfrak{A}}_\beta^\varepsilon(\lambda^\varepsilon) : \mathbb{C}^4 \times W_\beta^1(\Omega^0) \to W_{-\beta}^1(\Omega^0)^*$$

satisfying the following estimate in the operator norm $\|\widehat{\mathfrak{A}}_\beta^\varepsilon(\lambda^\varepsilon) - \mathfrak{A}_\beta^0(\lambda^0)\| \leqslant c_0\varepsilon$. As a result, $\mathfrak{A}_\beta^\varepsilon(\lambda^\varepsilon)$ inherits the isomorphism property from $\mathfrak{A}_\beta^0(\lambda^0)$ for a small ε. Besides the estimate (35) is valid with a constant c_0 independent on ε. □

2.4. Properties of the augmented scattering matrix and the criterion for trapped modes. Theorems 2.2 and 2.3 demonstrate that $\mathfrak{W}_\beta^{1,\varepsilon}(\Omega^\varepsilon)$ belongs to the pre-image $\mathcal{A}_{-\beta}^\varepsilon(\lambda^\varepsilon)^{-1}W_{-\beta}^1(\Omega^\varepsilon)^*$ of the subset $W_{-\beta}^1(\Omega)^* \subset W_\beta^1(\Omega)^*$ and, moreover,

$$\mathcal{A}_{-\beta}^\varepsilon(\lambda^\varepsilon)^{-1}W_{-\beta}^1(\Omega^\varepsilon)^* = \mathfrak{W}_\beta^{1,\varepsilon}(\Omega^\varepsilon) \oplus \operatorname{span}\{Z_{0,\pm}^\varepsilon, Z_{1,\pm}^\varepsilon\}, \tag{40}$$

where $\{Z_{j,\pm}^\varepsilon\}$ is a basis in the subspace $\ker(\mathcal{A}_{-\beta}^\varepsilon(\lambda^\varepsilon))$, whose dimension is 4 (cf. the proof of Theorem 2.3). To describe a particular basis we observe that by the definitions of the waves and cut-off functions in formulae (15), (17) and (22), $w_{j,\pm}^{j,in}$ satisfies the boundary conditions (10), (11) and

$$f_{j,\pm}^{\varepsilon,in} = -\Delta w_{j,\pm}^{\varepsilon,in} - k^2 w_{j,\pm}^{\varepsilon,in}$$

is a smooth function with a support in $\overline{\Omega^\varepsilon(R+d)}$. Thus the problem (28) with the right hand side

$$F^\varepsilon \in W^1_{-\beta}(\Omega^\varepsilon)^*, \quad F^\varepsilon(\psi) = (f^{\varepsilon,in}_{j,\pm}, \psi)_{\Omega^\varepsilon}$$

has a solution $\mathfrak{Z}^\varepsilon_{j,\pm} \in \mathfrak{W}^{1,\varepsilon}_\beta(\Omega^\varepsilon)$ such that the sum $Z^\varepsilon_{j,\pm} = w^{\varepsilon,in}_{j,\pm} + \mathfrak{Z}^\varepsilon_{j,\pm} \in W^1_{-\beta}(\Omega^\varepsilon)$ becomes a non-trivial solution of the homogeneous problem. Furthermore, these four solutions inherit the linear independence from the incoming waves and their row vector (25) admits the representation (26).

Theorem 2.4. *The 4×4-matrix S^ε in (26) is unitary and symmetric.*

Proof. First we prove that the column vectors of the matrix S^ε are orthonormal. This can be seen by the calculation similar to (36):

$$\begin{aligned} 0 &= Q(Z^\varepsilon_{j,\alpha}, Z^\varepsilon_{k,\tau}) \\ &= Q\Big(w^{\varepsilon,in}_{j,\alpha} + \sum_{\kappa=\pm}\sum_{p=0,1} w^{\varepsilon,out}_{p,\kappa} S^\varepsilon_{p\kappa,j\alpha}, w^{\varepsilon,in}_{k,\tau} + \sum_{\eta=\pm}\sum_{q=0,1} w^{\varepsilon,out}_{q,\eta} S^\varepsilon_{q\eta,k\tau}\Big) \\ &= -\mathrm{i}\delta_{j,k}\delta_{\alpha,\tau} + \mathrm{i}\sum_{\eta=\pm}\sum_{p=0,1} \overline{S^\varepsilon_{p\eta,j\alpha}} S^\varepsilon_{p\eta,k\tau}, \quad j,k=0,1;\ \alpha,\tau=\pm. \end{aligned}$$

To verify the symmetry, we observe that due to formulae (17), (19) and (22) the row vectors of waves in (24) have the relationship $w^{\varepsilon,in}(y,z) = \overline{w^{\varepsilon,out}(y,z)}$. Using this we obtain

$$\overline{Z^\varepsilon}(\overline{S^\varepsilon})^{-1} = (\overline{w^{\varepsilon,in}} + \overline{w^{\varepsilon,out}S^\varepsilon})(\overline{S^\varepsilon})^{-1} + \overline{\widetilde{Z}^\varepsilon}(\overline{S^\varepsilon})^{-1} = w^{\varepsilon,out}(\overline{S^\varepsilon})^{-1} + w^{\varepsilon,in} + \overline{\widetilde{Z}^\varepsilon}(\overline{S^\varepsilon})^{-1}.$$

Therefore, the difference

$$Z^\varepsilon - \overline{Z^\varepsilon}(\overline{S^\varepsilon})^{-1} = w^{\varepsilon,out}(S^\varepsilon - (\overline{S^\varepsilon})^{-1}) + \widetilde{Z}^\varepsilon - \overline{\widetilde{Z}^\varepsilon}(\overline{S^\varepsilon})^{-1} \in \mathfrak{W}^{1,\varepsilon}_\beta(\Omega^\varepsilon)^4$$

satisfies the homogeneous problem (28) and is zero by Theorem 2.3. In other words, we get

$$S^\varepsilon = (\overline{S^\varepsilon})^{-1} = \overline{(S^\varepsilon)^*} = (S^\varepsilon)^\top$$

which proves the symmetry. □

Repeating the calculation (36) again we get the following assertion.

Proposition 2.5. *Let $a^\varepsilon = (a^\varepsilon_{0+}, a^\varepsilon_{0-}, a^\varepsilon_{1,+}, a^\varepsilon_{1,-})^\top \in \mathbb{C}^4$ be a column vector, where its elements are the coefficients in the decomposition (34) of the solution $\phi^\varepsilon \in \mathfrak{W}^{1,\varepsilon}_\beta(\Omega^\varepsilon)$ to the problem (28) with the right hand side $F^\varepsilon \in W^1_{-\beta}(\Omega^\varepsilon)^*$. Then a^ε satisfies the equation*

$$\mathrm{i}(S^\varepsilon)^* a^\varepsilon = F^\varepsilon(Z^\varepsilon). \tag{41}$$

The next theorem will be our main tool for identification of the eigenvalue λ^ε. For the reader's convenience we present here a condensed proof of a sufficient condition for an embedded eigenvalue, which was detected in [9] for general elliptic boundary value problems and applied in [22, 23] for other concrete problems in mathematical physics.

We split the augmented scattering matrix into blocks of size 2×2:

$$S^\varepsilon = \begin{bmatrix} S^\varepsilon_{\circ\circ} & S^\varepsilon_{\circ\bullet} \\ S^\varepsilon_{\bullet\circ} & S^\varepsilon_{\bullet\bullet} \end{bmatrix} \tag{42}$$

and divide the solution and the wave row vector in the same way:

$$Z^\varepsilon = (Z^\varepsilon_\circ, Z^\varepsilon_\bullet),\ w^{\varepsilon,in/out} = (w^{\varepsilon,in/out}_\circ, w^{\varepsilon,in/out}_\bullet), \tag{43}$$

where $Z^\varepsilon_\circ = (Z^\varepsilon_{0+}, Z^\varepsilon_{0-})$, $Z^\varepsilon_\bullet = (Z^\varepsilon_{1,+}, Z^\varepsilon_{1,-})$ and so on.

Theorem 2.6. *A spectral parameter λ^ε is an eigenvalue of the problem* (9)–(11) *if and only if -1 is an eigenvalue of the 2×2-block $S^\varepsilon_{\bullet\bullet}$. If $a^\varepsilon_\bullet = (a^\varepsilon_{1,+}, a^\varepsilon_{1,-})^\top \in \mathbb{C}^2$ is the corresponding eigenvector, then the linear combination*

$$Z^\varepsilon_\bullet a^\varepsilon_\bullet = a^\varepsilon_{1,+} Z^\varepsilon_{1,+} + a^\varepsilon_{1,-} Z^\varepsilon_{1,-}$$

belongs to $H^1(\Omega^\varepsilon)$ and therefore is a trapped mode.

Proof. According to (42) and (43), we can rewrite the decomposition (26) as follows:

$$\begin{aligned} Z^\varepsilon_\circ &= w^{\varepsilon,in}_\circ + w^{\varepsilon,out}_\circ S_{\circ\circ} + w^{\varepsilon,out}_\bullet S_{\bullet\circ} + \widetilde{Z}^\varepsilon_\circ, \\ Z^\varepsilon_\bullet &= w^{\varepsilon,in}_\bullet + w^{\varepsilon,out}_\circ S_{\circ\bullet} + w^{\varepsilon,out}_\bullet S_{\bullet\bullet} + \widetilde{Z}^\varepsilon_\bullet. \end{aligned} \tag{44}$$

Let then $a^\varepsilon_\bullet \in \ker(S^\varepsilon_{\bullet\bullet} + \mathbb{I}_2)$ be a non-zero vector and set $a^\varepsilon_\circ = (0,0)^\top$. Then for a vector $a^\varepsilon = (a^\varepsilon_\circ, a^\varepsilon_\bullet)^\top$ we obtain by the unitary property of the augmented scattering matrix

$$0 = |a^\varepsilon|^2 - |a^\varepsilon_\bullet|^2 = |S^\varepsilon a^\varepsilon|^2 - |a^\varepsilon_\bullet|^2 = |S^\varepsilon_{\circ\bullet} a^\varepsilon_\bullet|^2 + |S^\varepsilon_{\bullet\bullet} a^\varepsilon_\bullet|^2 - |a^\varepsilon_\bullet|^2 = |S^\varepsilon_{\circ\bullet} a^\varepsilon_\bullet|^2.$$

Hence $S^\epsilon_{0\bullet} a^\epsilon_\bullet = 0$ and we derive from (44) that

$$Z^\varepsilon_\bullet a^\varepsilon_\bullet = w^{\varepsilon,in}_\bullet a^\varepsilon_\bullet + w^{\varepsilon,out}_\circ S^\varepsilon_{\circ\bullet} a^\varepsilon_\bullet + w^{\varepsilon,out}_\bullet S^\varepsilon_{\bullet\bullet} a^\varepsilon_\bullet + \widetilde{Z}^\varepsilon_\bullet a^\varepsilon_\bullet = w^{\varepsilon,in}_\bullet - w^{\varepsilon,out}_\bullet + \widetilde{Z}^\varepsilon_\bullet a^\varepsilon_\bullet. \tag{45}$$

Recalling the formulae (22), we see that both exponentially growing waves $\chi_\pm v^\varepsilon_\pm$ in $\Pi_\pm$ disappear from the difference $w^{\varepsilon,in}_{1\pm} - w^{\varepsilon,out}_{1\pm}$. Therefore the function (45) gets exponential decay at infinity, falls into $H^1(\Omega^\varepsilon)$, and becomes a trapped mode.

On the contrary, based on the representation (40), any non-trivial solution $\varphi^\varepsilon \in H^1(\Omega^\varepsilon) \subset W^1_{-\beta}(\Omega^\varepsilon)$ of the homogeneous problem (28) is a linear combination $Z^\varepsilon a^\varepsilon$ for some $a^\varepsilon \in \mathbb{C}^4 \setminus \{0\}$. Thus

$$\begin{aligned}\varphi^\varepsilon &= Z^\varepsilon_\circ a^\varepsilon_\circ + Z^\varepsilon_\bullet a^\varepsilon_\bullet \\ &= w^{\varepsilon,in}_\circ a^\varepsilon_\circ + w^{\varepsilon,out}_\circ (S^\varepsilon_{\circ\circ} a^\varepsilon_\circ + S^\varepsilon_{\circ\bullet} a^\varepsilon_\bullet) + w^{\varepsilon,in}_\bullet a^\varepsilon_\bullet + w^{\varepsilon,out}_\bullet (S^\varepsilon_{\bullet\circ} a^\varepsilon_\circ + S^\varepsilon_{\bullet\bullet} a^\varepsilon_\bullet) + \widetilde{\varphi}^\varepsilon, \end{aligned} \quad (46)$$

where $\widetilde{\varphi}^\varepsilon \in W^1_\beta(\Omega^\varepsilon)$.

The inclusion $\varphi^\varepsilon \in H^1(\Omega^\varepsilon)$, in particular, requires that the coefficients of the oscillatory waves $w^{\varepsilon,in/out}_{0\pm}$ vanish. It means that $a^\varepsilon_\circ = 0$ and $S^\varepsilon_{\circ\bullet} a^\varepsilon_\bullet = 0$. Then the equation (46) converts into

$$\varphi^\varepsilon = w^{\varepsilon,in}_\bullet a^\varepsilon_\bullet + w^{\varepsilon,out}_\bullet (S^\varepsilon_{\bullet\circ} a^\varepsilon_\circ + S^\varepsilon_{\bullet\bullet} a^\varepsilon_\bullet) + \widetilde{\varphi}^\varepsilon. \quad (47)$$

By the formulae (22) we conclude that the growing waves $\chi_\pm v^\varepsilon_\pm$ are absent in (47) provided $a^\varepsilon_\bullet + S^\varepsilon_{\bullet\bullet} a^\varepsilon_\bullet = 0$. Hence the theorem is proved. □

We emphasize that the inequality

$$\dim \ker(S^\varepsilon_{\bullet\bullet} + \mathbb{I}_2) > 0 \quad (48)$$

is a criterion for trapped modes only under the condition (33) which ensures the absence of trapped modes with fast decay. Otherwise, (48) reduces to a *sufficient condition*, cf. [9]. Ignoring this fact may lead to mistakes in numerical schemes to compute the embedded eigenvalue, see a discussion in [18]. If an information on $\ker \mathcal{A}^0_\beta$ is not available, it is worth to employ so called *fictitious scattering operator*, an infinite dimensional analog of scattering matrices, cf. [18] and [26] for water waves.

3. Asymptotic analysis

3.1. Preliminaries. Since $f^0 \in H^1(\Omega^0)^*$ has a compact support and therefore belongs to $W^1_{-\beta}(\Omega^0)^*$, Theorem 2.3 provides a unique solution $\varphi(f) \in \mathfrak{W}^{1,0}_\beta(\Omega^0)$ of the problem (28), where β is replaced by $-\beta$ and F^ε by f^0. By (7) and (35), the column vector $a^f \in \mathbb{C}^4$ and the remainder $\widetilde{\varphi}(f) \in W^1_\beta(\Omega^0)$ satisfy the estimate

$$|a^f_\circ|^2 + |a^f_\bullet|^2 + \|\widetilde{\varphi}(f); W^1_\beta(\Omega^0)\| \leqslant c\|f^0; W^1_{-\beta}(\Omega^0)^*\| \leqslant c\|f^0; H^1(\Omega^0)^*\| \leqslant c\varepsilon. \quad (49)$$

Clearly, $\varphi^0 - \varphi(f) \in W^1_{-\beta}(\Omega^0)$ is a solution of the homogeneous problem (28). Hence it is a linear combination of the solutions $Z_{j,\pm}$. However, according to (22) we have

$$w^{\varepsilon,in}_{1,-} - w^{\varepsilon,out}_{1,-} = \mathrm{i}K_+ \chi_+ v^\varepsilon_- + \mathrm{i}K_- \chi_- v^\varepsilon_+.$$

Moreover, the equality (20) can be written as

$$\varphi^0 = -\mathrm{i}(w_{1,-}^{0,in} - w_{1,-}^{0,out}) + \widetilde{\varphi}^0. \tag{50}$$

Thus

$$\varphi^0 - \varphi(f) = -\mathrm{i}Z_{1,-}^0 = -\mathrm{i}Z_4^0 \tag{51}$$

and comparing the coefficients in the decompositions of φ^0, $\varphi(f)$ and $Z_{1,-}^0$ yields

$$-a_\sharp^f = -\mathrm{i}S_{\sharp 4}^0,\ \mathrm{i} - a_4^f = -\mathrm{i}S_{44}^0. \tag{52}$$

Here we have used the alternative notation (cf. (24), (25)) in which

$$a_\sharp^f = (a_{0+}^f, a_{0-}^f, a_{1,+}^f) = (a_1^f, a_2^f, a_3^f) \quad \text{and} \quad a_4^f = a_{1,-}^f.$$

Hence the augmented scattering matrix has a following block structure:

$$S^\varepsilon = \begin{bmatrix} S_{\sharp\sharp}^\varepsilon & S_{\sharp 4}^\varepsilon \\ S_{4\sharp}^\varepsilon & S_{44}^\varepsilon \end{bmatrix}, \tag{53}$$

where $S_{\sharp\sharp}^\varepsilon$ is 3×3-block, $S_{44}^\varepsilon \in \mathbb{C}$ and according to the symmetry of S^ε

$$S_{4\sharp}^\varepsilon = (S_{\sharp 4}^\varepsilon)^\top = (S_{1-,0+}^\varepsilon, S_{1-,0-}^\varepsilon, S_{1-,1+}^\varepsilon) = (S_{41}^\varepsilon, S_{42}^\varepsilon, S_{43}^\varepsilon).$$

Here we have $\varepsilon = 0$, but the notation will be used also in the case $\varepsilon > 0$. The formulae (49) and (52) mean that

$$|S_{44}^0 + 1| \leqslant c\varepsilon, \quad |S_{\sharp 4}^0| \leqslant c\varepsilon. \tag{54}$$

The immediate objective is to find a perturbed water domain Ω^ε such that

$$S_{44}^\varepsilon = -1, \tag{55}$$

$$S_{\sharp 4}^\varepsilon = 0 \in \mathbb{C}^3. \tag{56}$$

Note that owing to (55) the vector $(0,1)^\top \in \mathbb{C}^2$ belongs to $\ker(S_{\bullet\bullet}^\varepsilon + \mathbb{I}_2)$ and therefore the criterion (48) is satisfied. Equality (56) is a direct consequence of (55). In order to fulfil (55) we will derive the asymptotics

$$S^\varepsilon = S^0 + \varepsilon S' + \varepsilon \widetilde{S}^\varepsilon \tag{57}$$

to detect an explicit form of the last column S_4' in S' and estimate the remainder $\widetilde{S}^\varepsilon$.

3.2. Asymptotic ansätze. The small perturbations (13) of the boundary $\partial\Omega^0$ and (18) provoke small changes in the special solutions $Z^0_{j\pm}$ of the homogeneous problems so that in a finite part of the domain we may employ the standard and simple asymptotic expansion

$$Z^\varepsilon(y,z) = Z^0(y,z) + \varepsilon Z'(y,z) + \cdots \tag{58}$$

for the solution vector (25), where Z' is the correction term to be found. The dots stand for the higher order terms which are inessential in our asymptotic procedure. We emphasize that the coordinate functions of the vectors Z^0 and Z' are originally defined in the domain Ω^0, but are extended across the smooth arc Υ to $\Omega^0 \cup U_\delta \supset \Omega^\varepsilon$ with preservation of differential properties.

Clearly, the sum $Z^0(y,z) + \varepsilon Z'(y,z)$ on the right of (58) cannot get the necessary asymptotic form (26) at infinity due to the Taylor formula

$$w^{\varepsilon,in/out}(y,z) = w^{0,in/out}(y,z) + \varepsilon w'^{,in/out} + \cdots \tag{59}$$

for waves and wave packets (17) and (22) caused by the perturbation (18) of the spectral parameter. At this point, we consider (58) as the *inner expansion* in the framework of the method of *matched asymptotic expansions*, cf. [7, 31] and interpretation of the method in [20, 21]. Besides the ansätze (57) and (59) convert (26) into the *outer expansion* which is acceptable only at a large distance from the obstacle:

$$\begin{aligned} Z^\varepsilon &= w^{0,in} + \varepsilon w'^{,in} + \cdots + (w^{0,out} + \varepsilon w'^{,out} + \cdots)(S^0 + \varepsilon S' + \cdots) \\ &= w^{0,in} + w^{0,out}S^0 + \varepsilon(w'^{,in} + w'^{,out}S^0 + w^{0,out}S') + \cdots . \end{aligned}$$

The matching procedure for the inner and outer expansions provides the representation formulae

$$Z^0 = w^{0,in} + w^{0,out}S^0 + \cdots , \tag{60}$$

$$Z' = w'^{,in} + w'^{,out}S^0 + w^{0,out}S' + \cdots . \tag{61}$$

Let us describe the correction term in (59). The waves (17) and (22) involve the cut-off functions $\chi_\pm$ and we specify these definitions as

$$w^{\varepsilon,in/out}_{j\pm} = \chi(y)\mathbf{w}^{\varepsilon,in/out}_{j\pm}, \tag{62}$$

where $\mathbf{w}^{\varepsilon,in/out}_{j\pm}$ is defined only in $\Pi_+(R) \cup \Pi_-(R)$ and $\chi(y)$ stands for $\chi_+(y)$ in $\Pi_+(R)$ and for $\chi_-(y)$ in $\Pi_-(R)$. Moreover, $\mathbf{w}^{\varepsilon,in/out}_{j\pm}$ satisfies

$$\begin{aligned} -\Delta\mathbf{w}^{\varepsilon,in/out}_{j\pm}(y,z) + k^2\mathbf{w}^{\varepsilon,in/out}_{j\pm}(y,z) &= 0, \quad (y,z) \in \Pi_+(R) \cup \Pi_-(R) \\ \partial_z\mathbf{w}^{\varepsilon,in/out}_{j\pm}(y,0) - \lambda^\varepsilon\mathbf{w}^{\varepsilon,in/out}_{j\pm}(y,0) &= 0, \quad (y,0) \in \Gamma_+(R) \cup \Gamma_-(R) \\ -\partial_z\mathbf{w}^{\varepsilon,in/out}_{j\pm}(y,-d) &= 0, \quad (y,-d) \in \Sigma_+(R) \cup \Sigma_-(R). \end{aligned} \tag{63}$$

The boundary parts $\Gamma_\pm(R)$ and $\Sigma_\pm(R)$ are defined analogously to (4). Applying the Taylor formula in ε to equations (63) and using them at $\varepsilon = 0$ gives us the equations for the correction terms $\mathbf{w}'^{,in/out}$:

$$\begin{aligned}
&-\Delta \mathbf{w}_{j\pm}'^{,in/out}(y,z) + k^2 \mathbf{w}_{j\pm}'^{,in/out}(y,z) = 0, && (y,z) \in \Pi_+ \cup \Pi_-(R) && (64)\\
&(\partial_z - \lambda^0)\mathbf{w}_{j\pm}'^{,in/out}(y,0) = \Lambda \mathbf{w}_{j\pm}^{0,in/out}(y,0), && (y,0) \in \Gamma_+(R) \cup \Gamma_-(R) && (65)\\
&-\partial_z \mathbf{w}_{j\pm}'^{,in/out}(y,-d) = 0, && (y,-d) \in \Sigma_+(R) \cup \Sigma_-(R), && (66)
\end{aligned}$$

where

$$w_{j\pm}'^{,in/out}(y,z) = \chi(y)\mathbf{w}_{j\pm}'^{,in/out}(y,z). \tag{67}$$

3.3. Determination of the correction term. Clearly, the correction term Z' in (58) must satisfy the Helmholtz equation

$$-\Delta Z'(y,z) + k^2 Z'(y,z) = 0, \quad (y,z) \in \Omega^0, \tag{68}$$

and the inhomogeneous Steklov condition

$$\partial_z Z'(y,0) - \lambda^0 Z'(y,0) = \Lambda Z^0(y,0), \quad (y,0) \in \Gamma, \tag{69}$$

at the free surface. Both of these are obtained directly by inserting the ansätze into the problem (9)–(11) and extracting the coefficients of ε. What is left is to determine the right hand side in the Neumann boundary condition

$$\partial_\nu Z'(y,z) = G(y,z), \quad (y,z) \in \Sigma. \tag{70}$$

To do so, we need to transfer the homogeneous condition (11) from Σ^ε onto Σ. Since the perturbation of the boundary occurs in the neighbourhood U_δ of the smooth curve Υ, where both Z^0 and Z' are smooth functions, we may apply the Taylor formula in the n-variable. First of all, we recall the formula for the normal derivative $\partial_{\nu^\varepsilon}$ on the curve Υ^ε given by (13):

$$\frac{\partial}{\partial \nu^\varepsilon} = \Big(1 + \frac{\varepsilon^2 |\partial_s h(s)|^2}{(1+n\kappa(s))^2}\Big)^{-\frac{1}{2}} \Big(\frac{\partial}{\partial n} - \frac{\varepsilon \partial_s h(s)}{(1+n\kappa(s))^2}\frac{\partial}{\partial s}\Big) \tag{71}$$

where $\kappa(s)$ is the curvature of Υ at the point s. By (13) we know that $n = O(\varepsilon)$ on Υ^ε and, hence,

$$\begin{aligned}
&\frac{\partial}{\partial \nu^\varepsilon}\Big(Z^0 + \varepsilon Z' + \cdots\Big)\Big|_{n=\varepsilon h(s)}\\
&= \frac{\partial Z^0}{\partial n}\Big|_{n=\varepsilon h(s)} + \varepsilon\Big(-\partial_s h \frac{\partial Z^0}{\partial s} + \frac{\partial Z'}{\partial n}\Big)\Big|_{n=\varepsilon h(s)} + \cdots\\
&= \frac{\partial Z^0}{\partial n}\Big|_{n=0} + \varepsilon h \frac{\partial^2 Z^0}{\partial n^2}\Big|_{n=0} + \varepsilon\Big(-\partial_s h \frac{\partial Z^0}{\partial s} + \frac{\partial Z'}{\partial n}\Big)\Big|_{n=\varepsilon h(s)} + \cdots .
\end{aligned} \tag{72}$$

The Helmholtz equation in the curvilinear coordinates

$$-\frac{1}{1+n\kappa}\frac{\partial}{\partial n}(1+n\kappa)\frac{\partial Z^0}{\partial n} - \frac{1}{1+n\kappa}\frac{\partial}{\partial s}\frac{1}{1+n\kappa}\frac{\partial Z^0}{\partial s} + k^2 Z^0 = 0$$

shows that $\partial_n^2 Z^0 = -\partial_z^2 Z^0 + k^2 Z^0$ at $n = 0$, because $\partial_n Z^0 = 0$ when $n = 0$, cf. the problem (1)–(3) for Z^0. Collecting the coefficients of ε in (72), we obtain the boundary condition (70) with the right-hand side

$$\begin{aligned} G(y,z) &= \partial_s h(s)\partial_s Z^0(y,z) - h(s)\partial_n^2 Z^0(y,z) \\ &= \partial_s h(s)\partial_s Z^0(y,z) + h(s)\partial_s^2 Z^0(y,z) - k^2 Z^0(y,z) \\ &= \partial_s(h(s)\partial_s Z^0(y,z)) - k^2 Z^0(y,z),\ (y,z) \in \Sigma. \end{aligned} \tag{73}$$

Notice that the profile function h is extended in (73) by zero from Υ over the whole Σ so that the boundary condition (70) is inhomogeneous on Υ only.

The right hand side ΛZ^0 in the boundary condition (69) has a fast growth at infinity and at the moment Theorem 2.3 does not apply. However, searching for a solution for (68)–(70) in the form

$$Z' = \mathbf{Z} + (\chi \mathbf{w}'^{,in} + \chi \mathbf{w}'^{,out} S^0), \tag{74}$$

we take the relations (64)–(67) into account and conclude that the new unknown $\mathbf{Z}$ satisfies the boundary value problem

$$-\Delta \mathbf{Z}(y,z) + k^2 \mathbf{Z}(y,z) = F_1(y,z), \qquad (y,z) \in \Omega \tag{75}$$

$$\partial_z \mathbf{Z}(y,0) - \lambda^0 \mathbf{Z}(y,0) = \Lambda \widetilde{Z}^0(y,0), \quad (y,0) \in \Gamma \tag{76}$$

$$\partial_\nu \mathbf{Z}(y,z) = G(y,z), \qquad (y,z) \in \Sigma, \tag{77}$$

where

$$F_1(y,z) = [\Delta, \chi](\mathbf{w}'^{,in}(y,z) + \mathbf{w}'^{,out}(y,z) S^0) \tag{78}$$

and

$$[\Delta, \chi] = \Delta\chi + 2\nabla\chi \cdot \nabla$$

is the commutator of the Laplace operator and the cut-off function χ. We remind here our convention on the factor χ in (62) and (67).

The boundary conditions in (76) require some explanation. First, the substitution $Z' \mapsto \mathbf{Z}$ changes the right hand side of (69) into the product $\Lambda \widetilde{Z}^0$ due to formulae (26) and (65). Secondly, the commutators $[\partial_z, \chi_\pm]$ in (76) and (77) vanish, because the cut-off functions (15) depend only on y and on $\Pi_\pm(R)$ the normal derivative $\partial_\nu = -\partial_z$. There also the equalities (65) and (66) are valid.

The boundary value problem (75)–(77) can be reformulated as the integral identity

$$(\nabla \mathbf{Z}, \nabla\psi)_\Omega + k^2(\mathbf{Z}, \psi)_\Omega - \lambda^0(\mathbf{Z}, \psi)_\Gamma = \mathbf{F}(\psi) \quad \forall \psi \in W_\beta^1(\Omega). \tag{79}$$

The linear functional

$$\mathbf{F}(\psi) = (F_1, \psi)_\Omega + (G, \psi)_\Sigma + \Lambda(\widetilde{Z}^0, \psi)_\Gamma$$

is continuous in $W^1_{-\beta}(\Omega)^*$ due to the inclusion $\widetilde{Z}^0 \in W^1_\beta(\Omega)^4$ and because F_1 and G have compact supports. Now, for $\varepsilon = 0$, we may apply Theorem 2.3 which delivers a unique solution $\mathbf{Z} \in \mathfrak{W}^{1,0}_\beta(\Omega)^4$. It has an asymptotic behaviour, which according to (61) and (74), must be written as follows:

$$\mathbf{Z}(y,z) = w^{0,out}(y,z)S' + \widetilde{\mathbf{Z}}(y,z), \quad \widetilde{\mathbf{Z}} \in W^1_\beta(\Omega)^4. \tag{80}$$

The correction term (74) in (58) is constructed while the coefficient matrix S' in (80) is calculated by means of formula (41) in Proposition 2.5 with $\varepsilon = 0$:

$$\begin{aligned} &\mathrm{i}(S^0)^* S' = J, \\ &J = (F_1, Z^0)_\Omega + (G, Z^0)_\Sigma + \Lambda(\widetilde{Z}^0, Z^0)_\Gamma. \end{aligned} \tag{81}$$

Since S^0 is unitary, we finally obtain the desired correction term in (57):

$$S' = -\mathrm{i} S^0 J. \tag{82}$$

3.4. Justification of the asymptotics. Applying asymptotic structures developed in [20], see also [22, 23], we introduce the global approximation of the solution vector Z^ε in (25):

$$\begin{aligned} Z^{as}(y,z) = {}& X_\varepsilon(y)(Z^0(y,z) + \varepsilon Z'(y,z)) \\ &+ \chi(y)\mathbf{w}^{\varepsilon,in}(y,z) + \chi(y)\mathbf{w}^{\varepsilon,out}(y,z)(S^0 + \varepsilon S') \\ &- X_\varepsilon(y)\Big(\chi(y)\mathbf{w}^{0,\in}(y,z) + \chi(y)\mathbf{w}^{0,out}(y,z) + \chi(y)\mathbf{w}^{\prime,in}(y,z) \\ &+ \chi(y)\mathbf{w}^{0,out}(y,z)S^0 + \varepsilon \mathbf{w}^{0,out}(y,z)S'\Big) \\ =:{}& X_\varepsilon(y)\mathcal{Z}^0(y,z) + \chi(y)\mathcal{Z}^\infty(y,z) - X_\varepsilon(y)\chi(y)\mathcal{Z}^m(y,z) \end{aligned} \tag{83}$$

where $\chi = \chi_\pm$ are cut-off functions as in (15) and $X_\varepsilon \in C^\infty(\mathbb{R})$ is such that

$$X_\varepsilon(y) = 1, \text{ for } |y| \leqslant \varepsilon^{-1}, \quad X_\varepsilon(y) = 0, \text{ for } |y| \geqslant d + \varepsilon^{-1}, \quad |\partial_y^k X_\varepsilon(y)| \leqslant c_k. \tag{84}$$

Note that these cut-off functions have overlapping supports, cf. Figure 3. Due to this fact the truncated inner $\mathcal{Z}^0$ and outer $\mathcal{Z}^\infty$ expansions are overlapping in the zones $\{(y,z) : R < |y| < d + \varepsilon^{-1},\ z \in (-d, 0)\}$ and we have to compensate by subtracting $\mathcal{Z}^m$ which involves all terms matched in the previous section.

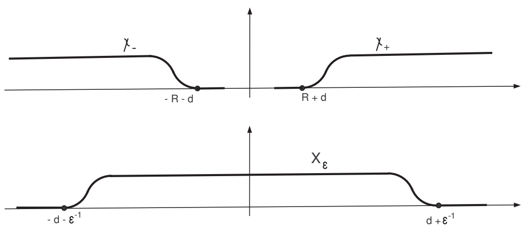

Figure 3: The graphs of the cut-off functions

Next we show that $(\Delta - k^2)Z^{as}$ is small. A direct calculation gives us

$$(\Delta-k^2)Z^{as}=X_\varepsilon(\Delta-k^2)\mathcal{Z}^0+\chi(\Delta-k^2)\mathcal{Z}^\infty-X_\varepsilon\chi(\Delta-k^2)\mathcal{Z}^m \\ +[\Delta, X_\varepsilon](\mathcal{Z}^0(y,z)-\mathcal{Z}^m(y,z))+[\Delta,\chi](\mathcal{Z}^\infty(y,z)-\mathcal{Z}^m(y,z)).$$

Here, the second and third terms vanish, since $\mathcal{Z}^\infty$ and $\mathcal{Z}^m$ are composed of the waves $\mathbf{w}_{j\pm}^{\varepsilon,in/out}$, $\mathbf{w}_{j\pm}^{0,in/out}$ and $\mathbf{w}_{j\pm}^{\prime,in/out}$ which satisfy the Helmholtz equation. The first term is zero in Ω^0, but nonzero in $\Omega^\varepsilon \setminus \Omega^0$. However, Z^0 and Z' were extended smoothly to $\Omega^0 \cup U_\delta \supset \Omega^\varepsilon$ and hence

$$-\Delta\mathcal{Z}^0(y,z) + k^2\mathcal{Z}^0(y,z) = O(|n|) = O(\varepsilon), \quad (y,z) \in \Omega^\varepsilon \setminus \Omega^0,$$

because the curved thin strip $\Omega^\varepsilon \setminus \Omega^0$ has the width of order ε in the n-direction, cf. the over-shadowed region in Figure 2. By the Newton-Leibnitz formula the inequality

$$\|\psi; L^2(\Omega^\varepsilon \setminus \Omega^0)\|^2 \leqslant c\varepsilon\|\psi; H^1(\Omega^\varepsilon \cap \Omega^\varepsilon)\|^2$$

is valid for all $\psi \in H^1(\Omega^\varepsilon)$. With the previous bounds we get

$$\left|(X_\varepsilon(\Delta - k^2)\mathcal{Z}^0, \psi)_{\Omega^\varepsilon}\right| \leqslant c\varepsilon \int_{\Omega^\varepsilon\setminus\Omega^0} |\psi(y,z)|\mathrm{d}z\mathrm{d}y \leqslant c\varepsilon^2\|\psi; H^1(\Omega^\varepsilon(R)\|. \quad (85)$$

Coefficients of the first order differential operator $[\Delta, \chi]$ are located in the set $\{(y,z) : R \leqslant |y| \leqslant R+d,\ z \in [-d,0]\}$. On the other hand, the difference $\mathcal{Z}^\infty(y,z) - \mathcal{Z}^m(y,z)$ become small for $|y| \ll \varepsilon^{-1}$. In fact, $\mathcal{Z}^\infty(y,z) - \mathcal{Z}^m(y,z)$ gets order ε^2 due to the Taylor formula (59) with the remainder $O(\varepsilon^2(1+|y|)^2)$ which turns into $O(\varepsilon^2)$ under the constrain $|y| \leqslant R+d$. Hence we obtain the estimate

$$|([\Delta,\chi](\mathcal{Z}^\infty - \mathcal{Z}^m), \psi)_{\Omega^\varepsilon} \leqslant c\varepsilon^2\|\psi; L^2(\Omega^\varepsilon(R+d))\|. \quad (86)$$

Both the bounds in (85) and (86) can be replaced by $c\varepsilon^2\|\psi; W^1_{-\gamma}(\Omega^\varepsilon)\|$ with any $\gamma \in \mathbb{R}$ due to the boundedness of $\Omega^\varepsilon(R) \subset \Omega^\varepsilon(R+d)$. In particular, we can choose $\gamma = \beta$.

The smallness of $[\Delta, X_\varepsilon](\mathcal{Z}^0 - \mathcal{Z}^m)$ follows similarly. Here the coefficients of $[\Delta, X_\varepsilon]$ are nonzero only in the set $\{(y,z) : \varepsilon^{-1} \leqslant |y| \leqslant d+\varepsilon^{-1},\ z \in [-d,0]\}$.

By choosing ε small enough, we have $R \ll \varepsilon^{-1}$. Now the difference $\mathcal{Z}^0(y,z) - \mathcal{Z}^m(y,z) = \widetilde{Z}^0(y,z) + \varepsilon\widetilde{\mathbf{Z}}(y,z)$ becomes exponentially small for $|y| \geqslant \varepsilon^{-1}$. However, to express the smallness, we need to diminish the weight index and choose

$$\gamma \in (\pi, \beta) \subset (\theta_1^\varepsilon, \theta_2^\varepsilon).$$

Then we can write

$$\begin{aligned}
&\big|([\Delta, X_\varepsilon](\mathcal{Z}^0 - \mathcal{Z}^m), \psi)_{\Omega^\varepsilon}\big| \\
&\leqslant c\|\mathrm{e}^{\beta|y|}(\widetilde{Z}^0 + \varepsilon\widetilde{Z}); L^2(\Omega^\varepsilon)\|\,\|\mathrm{e}^{-\beta|y|}\psi; L^2(\Omega^\varepsilon(d+\varepsilon^{-1}) \setminus \Omega^\varepsilon(\varepsilon^{-1}))\| \\
&\leqslant c\mathrm{e}^{\frac{\gamma-\beta}{\varepsilon}}\|\mathrm{e}^{-\gamma|y|}\psi; L^2(\Omega^\varepsilon)\| \\
&\leqslant c\mathrm{e}^{\frac{\gamma-\beta}{\varepsilon}}\|\mathrm{e}^{-\gamma|y|}\psi; W^1_{-\gamma}(\Omega^\varepsilon)\|.
\end{aligned}$$

Finally, collecting all the previous estimates, we obtain $|((\Delta - k^2)Z^{as}, \psi)_{\Omega^\varepsilon}| \leqslant c\varepsilon^2\|\psi; W^1_{-\gamma}(\Omega^\varepsilon)\|$, which implies

$$\|(\Delta - k^2)Z^{as}; W^1_{-\gamma}(\Omega^\varepsilon)^*\| \leqslant c\varepsilon^2. \tag{87}$$

In a similar manner we process the discrepancy in the Steklov condition:

$$\begin{aligned}
(\partial_z - \lambda^\varepsilon)Z^{as} &= X_\varepsilon(\partial_z - \lambda^\varepsilon)(\mathcal{Z}^0 - \chi\mathcal{Z}^m) + \chi(\partial_z - \lambda^\varepsilon)\mathcal{Z}^\infty \\
&= X_\varepsilon(\partial_z - \lambda^\varepsilon)(\widetilde{Z}^0 + \varepsilon\widetilde{\mathbf{Z}}) \\
&= X_\varepsilon\left(\partial_z\widetilde{Z}^0 - \lambda^0\widetilde{Z}^0 + \varepsilon(\partial_z\widetilde{\mathbf{Z}} - \lambda^0\widetilde{\mathbf{Z}} - \Lambda\widetilde{Z}^0) - \varepsilon^2\Lambda\widetilde{\mathbf{Z}}\right) \\
&= -\varepsilon^2\Lambda X_\varepsilon\widetilde{\mathbf{Z}}.
\end{aligned} \tag{88}$$

In deriving this we have used several issues. First, $\mathcal{Z}^\infty$ is composed of waves (17) and (22), which verify the Steklov condition (10). Similarly, $\widetilde{Z}^0$ satisfies the Steklov condition by its definition (26). The factor in ε-term vanishes because of (76) which hold for $\widetilde{\mathbf{Z}}$ as well. With the help of (88) we finally deduce the estimate

$$\begin{aligned}
|((\partial_z - \lambda^\varepsilon)Z^{as}, \psi)_{\Omega^\varepsilon}| &\leqslant c\varepsilon^2\|X_\varepsilon\widetilde{\mathbf{Z}}; W^1_\beta(\Omega^0)\|\|\psi; W^1_{-\beta}(\Omega^\varepsilon)\| \\
&\leqslant c\varepsilon^2\|\psi; ; W^1_{-\gamma}(\Omega^\varepsilon)\|.
\end{aligned} \tag{89}$$

All the terms in (83) meet the homogeneous Neumann condition on $\Sigma^\varepsilon \setminus \Upsilon$. Furthermore, the cut-off functions $X_\varepsilon = 1$ and $\chi_\pm = 0$ on Υ^ε. Hence, we have

$$\partial_{\nu^\varepsilon}Z^{as}(y,z) = \partial_{\nu^\varepsilon}Z^0(y,z) + \varepsilon\partial_{\nu^\varepsilon}Z'(y,z),\ (y,z) \in \Upsilon^\varepsilon. \tag{90}$$

Since Z^0 and Z' are smooth in $\Omega^0 \cap U_\delta$ our calculations in (71)–(73) above demonstrate that the right hand side of (90) is nothing but $O(\varepsilon^2)$. Therefore we get

$$|(\partial_{\nu^\varepsilon}Z^{as}, \psi)_{\Sigma^\varepsilon}| \leqslant c\varepsilon^2\|\psi; L^2(\Sigma^\varepsilon)\| \leqslant c\varepsilon^2\|\psi; H^1(\Omega^\varepsilon(R))\| \leqslant c\varepsilon^2\|\psi; W^1_{-\gamma}(\Omega^\varepsilon)\|. \tag{91}$$

In view of (84) the vector function (83) admits the representation

$$Z^{as} = w^{\varepsilon,in} + w^{\varepsilon,out}(S^0 + \varepsilon S') + \widetilde{Z}^{as}, \quad \widetilde{Z}^{as} \in W^1_\gamma(\Omega^\varepsilon).$$

Thus the difference $Z^\varepsilon - Z^{as}$ falls into $\mathfrak{W}^{1,\varepsilon}_\gamma(\Omega^\varepsilon)$ and fulfils the integral identity (28) where $\beta = -\gamma$. The right hand side $F^{as} \in W^1_{-\gamma}(\Omega^\varepsilon)^*$ is bounded as follows

$$\|F^{as}; W^1_{-\gamma}(\Omega^\varepsilon)^*\| \leqslant c\varepsilon^2$$

according to the estimates (87), (89) and (91).

Using Theorem 2.3 with $\beta = \gamma$ we conclude that $\|Z^\varepsilon - Z^{as}; \mathfrak{W}^{1,\varepsilon}_\gamma(\Omega^\varepsilon)\| \leqslant c\varepsilon^2$. Since the norm in the weighted space with detached asymptotics involves the coefficient matrix $\varepsilon\widetilde{S}^\varepsilon = S^\varepsilon - S^0 - \varepsilon S'$ of the decomposition $Z^\varepsilon - Z^{as}$ we achieve the desired estimate for the remainder in (57).

Theorem 3.1. *Let λ^ε be as in* (18) *and assume that $\Lambda \leqslant c_0$ for $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 > 0$. Then the augmented scattering matrix S^ε in the regularly perturbed domain Ω^ε, see* (13)*, takes the asymptotic form* (57)*, where S' and S^0 satisfy* (82) *and*

$$|\widetilde{S}^\varepsilon| \leqslant C_0\varepsilon, \tag{92}$$

where C_0 is independent on ε.

4. Detection of an eigenvalue

4.1. Reformulating the criterion. Since only the main correction term S' is constructed in the asymptotics (57) of the augmented scattering matrix S^ε, the inequality (48) cannot be satisfied directly. To avoid this difficulty, we are going to find an equivalent formulation of the criterion for the existence of a trapped mode.

In the meanwhile, we do not possess any information about the 3×3 - block $S^0_{\sharp\sharp}$ of the matrix S^0 in (53). But in any case we may choose a phase $\eta \in [0, 2\pi)$ such that the matrix

$$S^0_{\sharp\sharp} + e^{2i\eta}\mathbb{I}_3 \tag{93}$$

is non-singular and the norm of the inverse matrix is of order 1, that is, much smaller than ε^{-1}. This condition can be achieved because the eigenvalues of $S^0_{\sharp\sharp}$ are at a distance $O(\varepsilon)$ from the unit circle in the complex plane by the estimates (54).

Lemma 4.1. *There exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0]$, the system* (55), (56) *is equivalent with the following four relations for the blocks in* (53)

$$\operatorname{Re}(e^{i\eta}(S^\varepsilon_{\sharp\sharp})^* S^\varepsilon_{\sharp 4}) = 0 \in \mathbb{R}^3, \tag{94}$$

$$\operatorname{Im}(S^\varepsilon_{44}) = 0. \tag{95}$$

Proof. Since S^ε is unitary, the condition (55) implies (56). Therefore, (94) and (95) hold true.

Assume then that the relations (94) and (95) are attested. Taking into account the mutual orthogonality of columns in a unitary matrix we can write

$$\begin{aligned}0 &= \mathrm{e}^{\mathrm{i}\eta}(S^\varepsilon_{\sharp\sharp})^* S^\varepsilon_{\sharp 4} + \mathrm{e}^{\mathrm{i}\eta}(S^\varepsilon_{4\sharp})^* S^\varepsilon_{44} \\ &= -\mathrm{e}^{-\mathrm{i}\eta}(S^\varepsilon_{\sharp\sharp})^\top \overline{S^\varepsilon_{\sharp 4}} + \mathrm{e}^{\mathrm{i}\eta} S^\varepsilon_{44}\overline{(S^\varepsilon_{\sharp 4})} \\ &= -\mathrm{e}^{-\mathrm{i}\eta}\left(S^\varepsilon_{\sharp\sharp} - \mathrm{e}^{2\mathrm{i}\eta}S^\varepsilon_{44}\mathbb{I}_3\right)\overline{S^\varepsilon_{\sharp 4}} \\ &= -\mathrm{e}^{-\mathrm{i}\eta}\left(S^0_{\sharp\sharp} + \mathrm{e}^{2\mathrm{i}\eta}\mathbb{I}_3 + O(\varepsilon)\right) S^\varepsilon_{\sharp 4}.\end{aligned} \tag{96}$$

Here the first equality follows from (94) stating that $\mathrm{e}^{\mathrm{i}\eta}(S^\varepsilon_{\sharp\sharp})^* S^\varepsilon_{\sharp 4}$ is purely imaginary. In the second equality we have applied the symmetry of $S^\varepsilon_{\sharp\sharp}$, which follows trivially from the symmetry of S^ε. Finally, the last equality is a consequence of the asymptotic formulae (54). In (96), $O(\varepsilon)$ stands for a 3×3-matrix with the elements of order ε. For small ε, the matrix $S^0_{\sharp\sharp} + \mathrm{e}^{2\mathrm{i}\eta}\mathbb{I}_3 + O(\varepsilon)$ is non-singular according to our assumption on (93). Thus the vector $S^\varepsilon_{\sharp 4} = 0$ and $|S^\varepsilon_{44}| = 1$, which together with (94) and (95) ensure that S^ε_{44} is a real number situated close to -1. That is $S^\varepsilon_{44} = -1$. □

4.2. Refining the asymptotics of the augmented scattering matrix. To satisfy (94) and (95), we need to know the fourth column S^ε_4 of the augmented scattering matrix S^ε generated by the coefficients in the solution $Z^\varepsilon_4 = Z^\varepsilon_{j-}$, see (25) and (26). By (82) and (54) we have

$$S'_4 = \mathrm{i}J_4 + O(\varepsilon) = \mathrm{i}\Big((F_1, Z^0_4)_\Omega + (G, Z^0_4)_\Sigma + \Lambda(\widetilde{Z}^0, Z^0_4)_\Gamma\Big) + O(\varepsilon).$$

Formulae (49) and (51) show that

$$\|Z^0_4 - \mathrm{i}\varphi^0; \mathfrak{W}^{1,0}_\beta(\Omega^0)\| \leqslant c\varepsilon.$$

Hence, remembering that F_1 and G have compact supports and that $\widetilde{Z}^0$ has a fast decay, we obtain

$$S'_4 = (F_1, \varphi^0)_\Omega + (G, \varphi^0)_\Sigma + \Lambda(\widetilde{Z}^0, \varphi^0)_\Gamma + O(\varepsilon). \tag{97}$$

First of all, we specify an explicit formula for S'_{44}. Namely, according to (50), (73), (78) and (97), we write

$$S'_{44} = \Big([\Delta, \chi](\mathbf{w}'^{;in}_{1,-} - \mathbf{w}'^{;out}_{1,-}), \varphi^0\Big)_\Omega - \mathrm{i}\Lambda(\widetilde{\varphi}^0, \varphi^0)_\Gamma + \mathrm{i}\mathcal{B}(h; \varphi^0, \varphi^0) + O(\varepsilon), \tag{98}$$

where

$$\mathcal{B}(h; \varphi, \psi) = (h\partial_s\varphi, \partial_s\psi)_\Upsilon + k^2(h\varphi, \psi)_\Upsilon. \tag{99}$$

To get $-\mathcal{B}(h;\varphi^0,\varphi^0)$ in (98), we have integrated by parts in $(\partial_s(h\partial_s\varphi^0) - k^2h\varphi^0,\varphi^0)_\Upsilon$ using the knowledge that $\varphi^0|_\Upsilon$ is smooth and $h \in C_0^\infty(\Upsilon)$. Moreover, the first term on right hand side of (98) can be transformed into

$$\begin{aligned}
&\Big([\Delta,\chi](\mathbf{w}_{1,-}^{\prime,in} - \mathbf{w}_{1,-}^{\prime,out}),\varphi^0\Big)_\Omega\\
&= \Big((\Delta - k^2)(\chi\mathbf{w}_{1,-}^{\prime,in} - \chi\mathbf{w}_{1,-}^{\prime,out})\varphi^0\Big)_\Omega - \Big(\chi(\Delta - k^2)(\mathbf{w}_{1,-}^{\prime,in} - \mathbf{w}_{1,-}^{\prime,out}),\varphi^0\Big)_\Omega\\
&= \Big((\partial_z - \lambda^0)(\chi\mathbf{w}_{1,-}^{\prime,in} - \chi\mathbf{w}_{1,-}^{\prime,out}),\varphi^0\Big)_\Gamma\\
&= \Lambda(\chi(\mathbf{w}_{1,-}^{0,in} - \mathbf{w}_{1,-}^{0,out}),\varphi^0\Big)_\Gamma.
\end{aligned}$$

In the end we arrive at

$$S'_{44} = \mathrm{i}\Lambda\|\varphi^0; L^2(\Gamma)\|^2 - \mathrm{i}\mathcal{B}(h;\varphi^0,\varphi^0) + O(\varepsilon). \tag{100}$$

Setting

$$\Lambda = \lambda' + \widetilde{\lambda}^\varepsilon, \tag{101}$$

$$\lambda' = \|\varphi^0; L^2(\Gamma)\|^{-2}\mathcal{B}(h;\varphi^0,\varphi^0), \tag{102}$$

we further obtain

$$S'_{44} = \mathrm{i}\widetilde{\lambda}^\varepsilon\|\varphi^0; L^2(\Gamma)\|^2 + \widetilde{S}'_{44},\ |\widetilde{S}'_{44}| \leqslant c\varepsilon. \tag{103}$$

Let us then consider the column vector $S'_{\sharp 4} \in \mathbb{C}^3$ which, according to (54) and (81), can be solved from the equation

$$\begin{aligned}
&(S^0_{\sharp\sharp})^* S'_{\sharp 4}\\
&= -\mathrm{i}J_{\sharp 4} + O(\varepsilon) \qquad\qquad (104)\\
&= -\big([\Delta,\chi](\mathbf{w}_\sharp^{\prime,in} + \mathbf{w}_\sharp^{\prime,out}S^0_{\sharp\sharp} + \mathbf{w}_4^{\prime,out}S^0_{4\sharp}),\varphi^0\big)_\Omega - \Lambda\big(\widetilde{Z}^0_\sharp,\varphi^0\big)_\Gamma + \mathcal{B}(h;Z^0_\sharp,\varphi^0) + O(\varepsilon).
\end{aligned}$$

We proceed in the same manner as in the calculation of S'_{44}. First of all, we get rid of the wave $w_4^{0.out} = w_{1,-}^{0.out}$ in the asymptotics of $Z^0_\sharp$ by setting

$$\phi^0_\sharp = Z^0_\sharp - S^0_{\sharp 4}w_4^{0,out} = w_\sharp^{0,in} - w_\sharp^{0,out}S^0_{\sharp\sharp} + \widetilde{Z}^0_\sharp.$$

From the second inequality in (54), we derive the estimate

$$\|Z^0_\sharp - \phi^0_\sharp; \mathfrak{W}^{1,0}_\beta(\Omega^0)\| \leqslant c\varepsilon,$$

so that we may replace $Z^0_\sharp$ by $\phi^0_\sharp$ in (104). Repeating the calculations above we then obtain

$$(S^0_{\sharp\sharp})^* S'_{\sharp 4} = -\Lambda(\phi^0_\sharp,\varphi^0)_\Gamma + \mathcal{B}(h;\phi^0_\sharp,\varphi^0) + O(\varepsilon). \tag{105}$$

A similar expression has appeared in (100). However, the first integral on the right hand side of (105) must be understood in the Cauchy sense:

$$(\phi_\sharp^0, \varphi^0)_\Gamma = \lim_{\rho\to\infty} \int_{\Gamma(\rho)} \phi_\sharp^0(y,0)\varphi^0(y,0)\mathrm{d}y, \tag{106}$$

where $\Gamma(\rho) = \{(y,0) \in \Gamma : |y| \leqslant \rho\}$. The absence of the exponential wave $w_{1,-}^{0,out}$ in $\phi_\sharp^0$ ensures the existence of the limit in (106). Indeed, we get

$$\lim_{\rho\to\infty} \int_{\Gamma(\rho_0+\rho)\setminus\Gamma(\rho)} \phi_\sharp^0(y,0)\varphi^0(y,0)\mathrm{d}y = 0, \quad \rho_0 > 0,$$

because φ^0 decays exponentially, the waves $w_{0,\pm}^{0,in/out}$ and the remainder are at least bounded while the functions

$$y \mapsto w_{1,+}^{0,in/out}(y,0)\Big(K_+\chi_+(y)v_-^0(y,0) + K_-\chi_-(y)v_+^0(y,0)\Big)$$

are odd in view of (22).

Taking (99), (103) and (105) into account, we represent the profile function h in (13) as follows:

$$h(s) = \delta_1 h_1(s) + \delta_2 h_2(s) + \delta_3 h_3(s) = h_\sharp(s)\delta_\sharp, \tag{107}$$

where $\delta_\sharp = (\delta_1, \delta_2, \delta_3)^\top \in \mathbb{R}^3$ is a new parameter vector and the functions h_j compose the row vector $h_\sharp = (h_1, h_2, h_3)$ and satisfy the orthonormality conditions

$$\mathrm{Re}\Big(\mathrm{e}^{\mathrm{i}\eta}(\mathcal{B}(h_p;\phi_q^0,\varphi^0) - \frac{\mathcal{B}(h_p;\varphi^0,\varphi^0)}{\|\varphi^0;L^2(\Gamma)\|^{-2}}(\phi_q^0,\varphi^0)_\Gamma\Big) = \delta_{p,q}, \quad p,q = 1,2,3. \tag{108}$$

Then inserting (99), (102) and (108) into the equation (104), we obtain

$$\mathrm{Re}(\mathrm{e}^{\mathrm{i}\eta}(S_{\sharp\sharp}^0)^* S_{\sharp 4}') = \delta_\sharp - \widetilde{\lambda}^\varepsilon \mathrm{Re}(\mathrm{e}^{\mathrm{i}\eta}(\phi_\sharp^0,\varphi^0)_\Gamma) + \mathcal{S}_\sharp^\varepsilon,\ |\mathcal{S}_\sharp^\varepsilon| \leqslant c\varepsilon.$$

4.3. The desired profile function and the detection of the eigenvalue. Using (103), we rewrite the equation (95) as follows:

$$\widetilde{\lambda}^\varepsilon - t_0 = T_0^\varepsilon(\widetilde{\lambda}^\varepsilon, \delta_\sharp), \tag{109}$$

where

$$t_0 = \|\varphi^0; L^2(\Gamma)\|^{-2}\varepsilon^{-1}\mathrm{Im}(S_{44}^0),$$
$$T_0^\varepsilon(\widetilde{\lambda}^\varepsilon, \delta_\sharp) = -\|\varphi^0; L^2(\Gamma)\|^{-2}\mathrm{Im}(\varepsilon^{-1}\widetilde{S}_{44}^\varepsilon + \widetilde{S}_{44}').$$

Similarly, the equations (94) can be transformed into

$$\delta_\sharp - \widetilde{\lambda}^\varepsilon \mathrm{Re}(\mathrm{e}^{\mathrm{i}\eta}(\phi^0_\sharp, \varphi^0)_\Gamma) - t_\sharp = T^\varepsilon_\sharp(\widetilde{\lambda}^\varepsilon, \delta_\sharp), \tag{110}$$

where

$$t_\sharp = \varepsilon^{-1}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\eta}(S^0_{\sharp\sharp})^* S^0_{\sharp 4}),$$
$$T^\varepsilon_\sharp(\widetilde{\lambda}^\varepsilon, \delta_\sharp) = -\mathcal{S}^\varepsilon_\sharp - \varepsilon^{-1}\mathrm{Re}(\mathrm{e}^{\mathrm{i}\eta}((S^0_{\sharp\sharp})^* \widetilde{S}^\varepsilon_{\sharp 4} + (S^\varepsilon_{\sharp\sharp} - S^0_{\sharp\sharp})^* S^\varepsilon_{\sharp 4})).$$

Theorem 4.2. *There exist* $\varepsilon_0 > 0$ *and* $r_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$ *the non-linear system of equations* (109), (110) *has a unique solution* $(\widetilde{\lambda}^\varepsilon, \delta_\sharp)$ *in the ball*

$$\mathbb{B}_{\varepsilon r_0} = \{(\widetilde{\lambda}^\varepsilon, \delta_\sharp) \in \mathbb{R}^4 : |\widetilde{\lambda}^\varepsilon - t_0|^2 + |\delta_\sharp - t_\sharp - t_0 \mathrm{Re}(\mathrm{e}^{\mathrm{i}\eta}(\phi^0_\sharp, \varphi^0)_\Gamma)|^2 \leqslant \varepsilon r_0^2\}.$$

Proof. First, recall that the coordinate change $(y, z) \mapsto (y^\varepsilon, z^\varepsilon)$ in (39) transforms Ω^ε into the reference domain Ω^0 and Δ, $\partial_{\nu^\varepsilon}$ into the differential operators $\mathcal{L}^\varepsilon$, $\mathcal{N}^\varepsilon$ which depend smoothly (even analytically) on the parameters ε, $\delta_\sharp$ in (13), (107) and $\widetilde{\lambda}^\varepsilon$ in (18) and (101). By general results in the perturbation theory of linear operators, see e.g. [6], such dependence is inherited also by the augmented scattering matrix S^ε in the problem (9)–(11). Recall here that the norm in the weighted space $\mathfrak{W}^{1,0}_\beta(\Omega^\varepsilon)$ with detached asymptotics contains the coefficients $a^\varepsilon_{j\pm}$ of the expansion (34).

Secondly, by means of the asymptotic analysis presented in Section 3 and the transformations in Section 4.2 we obtain that

$$|t_0| + |t_\sharp| \leqslant c, \tag{111}$$
$$|T^\varepsilon_0(\widetilde{\lambda}^\varepsilon, \delta_\sharp)| + |T^\varepsilon_\sharp(\widetilde{\lambda}^\varepsilon, \delta_\sharp)| \leqslant c\varepsilon(1 + |\widetilde{\lambda}^\varepsilon| + |\delta_\sharp|). \tag{112}$$

The inequality (111) follows from (54), whereas (112) is the consequence of (57), (92), (103) and (107).

The statement now follows from the contraction principle in view of the previous estimates (111) and (112). □

The system (109), (110) is equivalent with the relations (94), (95) which, by Lemma 4.1, ensure the equality (55) and the criterion (44) for the existence of trapped modes. In this way, Theorem 4.2 gives the main result of our paper.

Theorem 4.3. *There exist positive* ε_0, r_0 *and* c_0 *such that, for any* $\varepsilon \in (0, \varepsilon_0]$, *conditions* (6)–(7), (33) *and* (108) *provide the profile function* h *of the form* (107) *in* (13). *In the corresponding water domain* Ω^ε, *the water wave problem* (9)–(11) *has an eigenvalue* $\lambda^\varepsilon = \lambda^0 + \varepsilon\lambda' + \varepsilon\widetilde{\lambda}^\varepsilon$, *where* λ' *is given by* (99) *and* $|\widetilde{\lambda}^\varepsilon| \leqslant \varepsilon r_0$ *so that the inequality* (12) *is valid.*

5. Final comments on the assumptions

5.1. Perturbation profile function. To fulfil the orthonormality conditions (106), it suffices to verify that the functions

$$Y_q = \partial_s \phi_q^0 \partial_s \varphi^0 + k^2 \phi_q^0 \varphi^0 - \|\varphi^0; L^2(\Gamma)\|^{-2} (\phi_q^0, \varphi^0)_\Upsilon (|\partial_s \varphi^0|^2 + k^2 |\varphi^0|^2), \quad q = 1,2,3, \quad (113)$$

are linearly independent in $L^2(\Upsilon)$ and then to fix the phase $\eta \in [0, 2\pi)$ properly. Furthermore, the perturbation of the arc Υ, in principle, may be chosen arbitrarily. This can be done by a simple numerical scheme. However, in many situations it is possible to show the necessary property theoretically. Let us present an example.

We assume that $k = 0$ and set

$$\Upsilon = \{(y, z) : \pm y \in (l, l + l_0), \, z = -d\}, \quad (114)$$

where l_0 is fixed and $l > 0$ is a large number, cf. Figure 2b. Then the last factor $|\partial_y \varphi^0(y, -d)|^2$ in (113) gets the order $e^{-2l\theta_1^0}$, cf. (14). Observing that under assumption $S_{33}^0 \neq -1$ (again we suppose that there is no trapped mode in the reference domain Ω^0), the functions $\partial_y \phi_q^0(y, -d)$ involving the oscillating waves $w_{0\pm}^{0,in/out}$ and one exponential wave $w_{1,+}^{0,in/out}$ cannot get the same order everywhere on the intervals (114), we see that it remains to check the linear independence of the products

$$\partial_y \phi_q^0(y, -d) \partial_y \varphi^0(y, -d), \quad q = 1, 2, 3, \; y \in \Upsilon. \quad (115)$$

Since $\partial_y \varphi^0(y, -d) \neq 0$ on Υ for a big l, the linear dependence of (115) means that the gradient of a linear combination $\phi_\sharp^0 c_\sharp$ with some column vector $c_\sharp \in \mathbb{C}^3 \setminus \{0\}$ vanishes everywhere on Υ. This is impossible for the nontrivial harmonic function $\phi_\sharp^0 c_\sharp$ due to the theorem on unique continuation, see [12, Chapter 4]. In other words, conditions (108) can be achieved by perturbation of the bottom at a distance from the obstacle. As mentioned above, to locate the perturbation (13) near the obstacle, numerical experiments are needed.

5.2. Trapped mode with too fast decay. If the assumption (33) is violated, a trapped mode at the frequency $\omega_0 = \sqrt{g\lambda^0}$ exists, but the computation of φ^0 in (6)–(7) has no relation to this particular trapped mode. However, one may enlarge the weight index β and again provide the trivial kernel $\ker \mathcal{A}_\beta^0$ of the problem operator (29) at $\varepsilon = 0$, but this time in the Kondratiev space of functions with "very fast" decay at infinity. This change of weighted space does not prevent our scheme for detecting an eigenvalue λ^ε and a trapped mode close to φ^0. However, in this case much many exponential packets of type (22) must be involved so that the size of the augmented matrix grows. Correspondingly, a larger number of orthonormality conditions on the ingredients of the profile function h must be imposed. This makes our scheme much more cumbersome, but does not affect its main conclusions.

A similar modification must be done in the case when condition (21) is not fulfilled. Then the coefficients $K_\pm$ in the asymptotic form (14) become either zero, or small comparable with the discrepancy of order ε in (7). In view of (6) and (7) all coefficients in the Fourier decompositions of φ^0 in $\Pi_\pm(R)$ cannot get the order ε. Thus it suffices to choose a root t_J of the transcendental equation (16) such that the coefficients on the exponential waves $e^{\mp\theta_j y}\cos(t_j(d^{-1}z+1))$ in $\Pi_\pm(R)$, where $j = 1, \ldots, J-1$ and $\theta_j = \sqrt{k^2 + d^{-2}t_j}$, are as small as $O(\varepsilon)$, but the coefficients $K^J_\pm$ on $e^{\mp\theta_J y}\cos(t_J(d^{-1}z+1))$ satisfy the condition (21). Then taking $\beta \in (\theta_J, \theta_{J+1})$ we widen the family of exponential wave packets and enlarge the size of the augmented scattering matrix, but still we can follow the scheme presented in Sections 3 and 4.

5.3. Shape of the water domain and the obstacle. If the channel $\Pi = \mathbb{R} \times \varpi$ and the obstacle in it are three-dimensional, where ϖ is a bounded domain in the plane, our scheme in Sections 3 and 4 is still applicable. In this case, instead of two oscillating waves (17), the number N of propagating waves in $\Pi \subset \mathbb{R}^3$ increases to infinity when $\omega^0 \to \infty$. In this way the size of the traditional scattering matrix becomes $N \times N$. However, the size of the augmented scattering matrix S^0 can be $(N+2) \times (N+2)$, if the coefficients $K_\pm$ of the first exponentially decaying waves satisfy condition (21).

Based on the general results in the asymptotic theory of elliptic operators in singularly perturbed domains, cf. [14], it is possible to consider non-smooth profiles, in particular, perturbations with corners or bumps of small diameter. The latter, however, requires a totally different techniques than those applied in Section 2.3.

A reduction of the John problem ([8]) to the abstract spectral equation with a continuous self-adjoint operator in Hilbert space, see e.g. [5], [27], can help in derivation of similar result to Theorem 4.3 in the presence of freely floating objects. This, however, is an open question up to now.

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