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Topological Structure of the Spaces of Composition Operators on Hilbert Spaces of Dirichlet Series

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Abstract. In this paper we study some topological properties of the space of bounded composition operators on some Hilbert spaces of Dirichlet series. We first obtain formulas for the norms and essential norms of composition operators and differences of composition operators on Hilbert spaces of Dirichlet series. Then we give a characterization of the isolated points in the topological space of bounded composition operators on some Hilbert spaces of Dirichlet series. Finally we obtain sufficient conditions such that two composition operators are in the same path component. We show, among other results, that all compact composition operators are in the same path component. For a certain class of frequencies we give complete description of path components.

Keywords. Hilbert space, entire Dirichlet series, composition operators, isolated point, path component

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1. Introduction

In this paper we study topological structure of the space of bounded composition operators acting on some Hilbert spaces of Dirichlet series. Let $\mathcal{H}$ be a space of holomorphic functions on a set $G \subseteq \mathbb{C}$. Let φ be an analytic map of G into itself. The *composition operator* C_φ is a linear operator defined as follows:

$$(C_\varphi f)(z) = f \circ \varphi(z), \quad z \in G, \ f \in H.$$

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Because of their natural appearance and their wide connections with many other mathematical fields, composition operators have been extensively studied in recent decades. One of the main themes for studying composition operators is to relate the operator theoretical properties of the C_φ to the functional properties of the symbol function φ. We refer to the monographs [2, 14, 17] for the general information on composition operators. In particular, an extensive study of composition operators has been carried out on spaces of classical Dirichlet series (see, e.g., [4, 6]).

In [8, 9], a study was carried out of composition operators on some classes of entire Dirichlet series. Various properties, such as boundedness, compactness and compact difference have been studied.

Let us first recall the basic properties of a Hilbert space of Dirichlet series. Let $0 \le (\lambda_n) \uparrow \infty$ be a given sequence of real non-negative numbers satisfying condition

$$L = \limsup_{n\to\infty} \frac{\log n}{\lambda_n} < \infty.$$

It is well known that the Dirichlet series with frequencies (λ_n),

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, \quad a_n, z \in \mathbb{C},$$

converges and hence represents an entire function in $\mathbb{C}$ if and only if

$$D = \limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty.$$

Denote by E the sequence space

$$E = \{(a_n) : D = -\infty\}.$$

Let $\beta = (\beta_n)$ be a sequence of real numbers, and let

$$l_\beta^2 = \left\{ a = (a_n) : \|a\| = \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

It is well known that l_β^2 is a Hilbert space with the inner product

$$\langle a, b \rangle = \sum_{n=1}^{\infty} a_n \bar{b}_n \beta_n^2, \quad \forall (a_n), (b_n) \in l_\beta^2,$$

(see, e.g., [16]). In [9] it was proved that $l_\beta^2 \subset E$ if and only if

$$\liminf_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty, \quad \text{or equivalently,} \quad \lim_{n\to\infty} \frac{\log \beta_n}{\lambda_n} = \infty. \tag{E}$$

In this case, by $\mathcal{H}(E,\beta)$ we denote the space of entire Dirichlet series with coefficients in l^2_β, that is

$$\mathcal{H}(E,\beta)=\left\{f(z)=\sum_{n=1}^{\infty}a_ne^{-\lambda_n z}:(a_n)\in l^2_\beta\right\}.$$

This is a Hilbert space with the inner product

$$\langle f,g\rangle=\sum_{n=1}^{\infty}a_n\bar{b}_n\beta_n^2,$$

where $f(z)=\sum_{n=1}^{\infty}a_ne^{-\lambda_n z}$ and $g(z)=\sum_{n=1}^{\infty}b_ne^{-\lambda_n z}$ are any two functions in $\mathcal{H}(E,\beta)$.

In [9], a study of composition operators on Hilbert spaces of Dirichlet series, three notions of order were considered: ordinary, Ritt, and logarithmic orders. In this context, a result of Polya [13] played a pivotal role. We may assume that the Ritt order ρ_R of our Dirichlet series equals 0, because in case the Ritt order is positive, as noted in [8], the logarithmic orders are infinite, and so there is not much information we can get from logarithmic orders.

In [9], it is proved that for every function $f\in\mathcal{H}(E,\beta)$, $\rho_R=0$ if and only if

$$\liminf_{n\to\infty}\frac{\log\beta_n}{\lambda_n\log\lambda_n}=\infty. \tag{R}$$

Based on the above result, the following Hilbert space of Dirichlet series was introduced in [9]: for a fixed sequence $\beta\in$ (R),

$$\mathcal{H}(E,\beta_R):=\left\{\sum_{n=1}^{\infty}a_ne^{-\lambda_n z}\,:\,(a_n)\in l^2_\beta\right\}.$$

It is also proved in [9] that any $f\in\mathcal{H}(E,\beta_R)$ has a finite logarithmic order if and only if the following condition is satisfied

$$\liminf_{n\to\infty}\frac{\log\beta_n}{\lambda_n^{1+\alpha}}=\infty,\quad\text{for some }\alpha>0. \tag{S}$$

Following [9], for a fixed sequence $\beta\in$ (S) we consider the following Hilbert space of Dirichlet series:

$$\mathcal{H}(E,\beta_S):=\left\{\sum_{n=1}^{\infty}a_ne^{-\lambda_n z}\,:\,(a_n)\in l^2_\beta\right\}.$$

Boundedness, compactness and compact differences of composition operators on $\mathcal{H}(E,\beta_S)$ have been characterized in [9]. Here we cite the results on boundedness and compactness from [9].

Theorem 1.1. *Let φ be an entire function, and let β be a sequence satisfying condition* (S). *Then the composition operator C_φ*

(i) *is bounded on $\mathcal{H}(E, \beta_S)$ if and only if $\varphi(z) = z + b$ with $\operatorname{Re} b \geq 0$;*

(ii) *is compact on $\mathcal{H}(E, \beta_S)$ if and only if $\varphi(z) = z + b$ with $\operatorname{Re} b > 0$.*

Let $C(\mathcal{H}(E, \beta_S))$ be the topological space of all bounded composition operators on $\mathcal{H}(E, \beta_S)$, equipped with the topology induced by the operator norm. In this paper we study certain topological properties, including isolated points and path components of the space $C(\mathcal{H}(E, \beta_S))$. Such properties for the spaces of composition operators on other spaces, including H^∞, Hardy spaces and Bergman spaces have been extensively studied by many authors. See [1, 3, 5, 7, 10–12, 15] for a few examples.

To our knowledge, this paper is the first attempt to study topological properties of the spaces of composition operators on Hilbert spaces of Dirichlet series. We will first give formulas of the norms and essential norms of composition operators and the differences of composition operators on $\mathcal{H}(E, \beta_S)$ in Section 2. In Section 3 we will give a complete characterization of isolated points of $C(\mathcal{H}(E, \beta_S))$, and in Section 4 we will study components of $C(\mathcal{H}(E, \beta_S))$. We will show that all compact composition operators are in the same path component. We will also show that a non-compact composition operator and a compact composition operator cannot be in the same path component. For a certain class of frequencies we give complete description of path components. Finally, we give a conjecture that two distinct non-compact composition operators on $\mathcal{H}(E, \beta_S)$ are not in the same path component.

2. Norms and essential norms of composition operators and their differences

2.1. Norms of composition operators. By Theorem 1.1, for an entire function φ on $\mathbb{C}$, the composition operator $C_\varphi(f) = f \circ \varphi$ is bounded on $\mathcal{H}(E, \beta_S)$ if and only if $\varphi(z) = z + b$, with $\operatorname{Re} b \geq 0$. We give the following formula for the norm of a bounded composition operator.

Theorem 2.1. *Let $\varphi(z) = z + b$ with $\operatorname{Re} b \geq 0$ (so that C_φ is bounded on $\mathcal{H}(E, \beta_S)$). Then its norm $\|C_\varphi\| = e^{-\lambda_1 \operatorname{Re} b}$.*

Proof. On one hand, for any $f(z) = \sum_{n=1}^\infty a_n e^{-\lambda_n z} \in \mathcal{H}(E, \beta_S)$ with $\|f\| = 1$, we have

$$\|C_\varphi(f)\| = \left\| \sum_{n=1}^\infty a_n e^{-\lambda_n b} e^{-\lambda_n z} \right\| = \left(\sum_{n=1}^\infty |a_n|^2 e^{-2\lambda_n \operatorname{Re} b} \beta_n^2 \right)^{\frac{1}{2}}$$

and hence

$$\|C_\varphi(f)\| \le e^{-\lambda_1 \mathrm{Re}\, b} \left(\sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \right)^{\frac{1}{2}} = e^{-\lambda_1 \mathrm{Re}\, b} \|f\| = e^{-\lambda_1 \mathrm{Re}\, b}.$$

On the other hand, for the unit vector $\delta_1 = \frac{e^{-\lambda_1 z}}{\beta_1}$ from the orthonormal basis $(\delta_n)_{n=1}^{\infty}$, it is obvious that $\|C_\varphi(\delta_1)\| = e^{-\lambda_1 \mathrm{Re}\, b}$. Combining these facts yields $\|C_\varphi\| = e^{-\lambda_1 \mathrm{Re}\, b}$. □

Remark 2.2. The result above also solves the isometry problem. Indeed, since a linear operator C_φ is an isometry if and only if $\|C_\varphi(f)\| = \|f\|$, by the calculation above, it is easy to check that C_φ is an isometry on $\mathcal{H}(E, \beta_S)$ if and only if $\|C_\varphi\| = 1$, and hence if and only if either it is bounded but not compact or $\lambda_1 = 0$.

2.2. Norms of the differences of composition operators. Now we give a formula for the norm of the difference of composition operators.

Theorem 2.3. *Let $\varphi_1 = z + b_1$ and $\varphi_2 = z + b_2$, with $Re\, b_i \ge 0$, $i = 1, 2$ (so that C_{φ_1} and C_{φ_2} are bounded on $\mathcal{H}(E, \beta_S)$). Then*

$$\|C_{\varphi_1} - C_{\varphi_2}\| = \sup_{n \in \mathbb{N}} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|.$$

Proof. Take any $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(E, \beta_S)$. Then

$$\begin{aligned}
\|(C_{\varphi_1} - C_{\varphi_2}) f\|^2 &= \|f \circ \varphi_1 - f \circ \varphi_2\|^2 \\
&= \left\| \sum_{n=1}^{\infty} a_n e^{-\lambda_n (z + b_1)} - \sum_{n=1}^{\infty} a_n e^{-\lambda_n (z + b_2)} \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} a_n (e^{-\lambda_n b_1} - e^{-\lambda_n b_2}) e^{-\lambda_n z} \right\|^2 \\
&= \sum_{n=1}^{\infty} |a_n|^2 |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|^2 \beta_n^2 \\
&\le \sup_{n \in \mathbb{N}} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|^2 \sum_{n=1}^{\infty} |a_n|^2 \beta_n^2 \\
&= \sup_{n \in \mathbb{N}} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|^2 \|f\|^2.
\end{aligned}$$

Hence $\|C_{\varphi_1} - C_{\varphi_2}\| \le \sup_{n \in \mathbb{N}} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|$.

On the other hand, for any $n \in \mathbb{N}$, let $q_n(z) = \frac{1}{\beta_n} e^{-\lambda_n z}$. It is obvious that $\|q_n\| = 1$.

$$\begin{aligned} \|(C_{\varphi_1} - C_{\varphi_2})q_n\|^2 &= \frac{1}{\beta_n} \left\| \left(e^{-\lambda_n b_1} - e^{-\lambda_n b_2}\right) e^{-\lambda_n z} \right\| \\ &= \frac{1}{\beta_n} \left|e^{-\lambda_n b_1} - e^{-\lambda_n b_2}\right| \left\|e^{-\lambda_n z}\right\| \\ &= \left|e^{-\lambda_n b_1} - e^{-\lambda_n b_2}\right|, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Hence, $\|C_{\varphi_1} - C_{\varphi_2}\| \geq \sup_{n\in\mathbb{N}} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|$.

Combining the last two inequalities yields

$$\|C_{\varphi_1} - C_{\varphi_2}\| = \sup_{n\in\mathbb{N}} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|. \qquad \square$$

Unlike composition operators on Hardy spaces, for the space $\mathcal{H}(E, \beta_S)$, different symbols may induce the same composition operator, as shown in the next corollary.

Corollary 2.4. *Let $\varphi_1 = z + b_1$ and $\varphi_2 = z + b_2$, with $\operatorname{Re} b_i \geq 0$, $i = 1, 2$ (so that C_{φ_1} and C_{φ_2} are bounded on $\mathcal{H}(E, \beta_S)$). Then $C_{\varphi_1} = C_{\varphi_2}$ if and only if, for any $n \in \mathbb{N}$,*

$$\lambda_n(b_2 - b_1) = 2k_n\pi i$$

for some integer k_n.

Proof. By Theorem 2.3, $C_{\varphi_1} = C_{\varphi_2}$ if and only if $\sup_{n\in\mathbb{N}} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}| = 0$, or

$$|e^{-\lambda_n b_1} - e^{-\lambda_n b_2}| = 0$$

for any $n \in \mathbb{N}$. Since

$$|e^{-\lambda_n b_1} - e^{-\lambda_n b_2}| = |e^{-\lambda_n b_1}||1 - e^{-\lambda_n(b_2-b_1)}| = e^{-\lambda_n \operatorname{Re} b_1}|1 - e^{-\lambda_n(b_2-b_1)}|,$$

we know that the above condition is equivalent to $e^{-\lambda_n(b_2-b_1)} = 1$ for any $n \in \mathbb{N}$, which is the same as

$$\lambda_n(b_2 - b_1) = 2k_n\pi i, \quad \text{for some integer } k_n. \tag{1}$$

The proof is complete. $\square$

Remark 2.5. Actually, the statement in Corollary 2.4 can be derived directly from the uniqueness property of Dirichlet series. Indeed, from $C_{\varphi_1} = C_{\varphi_2}$ it follows that $e^{-\lambda_n b_1} = e^{-\lambda_n b_2}$, $\forall n \in \mathbb{N}$, which gives (1). Note also that the converse holds as well.

2.3. Essential norms of composition operators. We recall that the essential norm of an operator C_φ on the space $\mathcal{H}(E,\beta_S)$ is defined as follows

$$\|C_\varphi\|_e = \inf\{\|C_\varphi - K\| : K \text{ is a compact operator on } \mathcal{H}(E,\beta_S)\}.$$

It is well known that the essential norm of a compact operator is zero. For the non-compact case, by Theorem 2.1, it is clear that $\|C_\varphi\| = 1$. Moreover, by Remark 2.2, C_φ is an isometry on $\mathcal{H}(E,\beta_S)$ and hence its essential norm is 1 (this is a basic property of the isometry in a Hilbert space). Then we have the following result.

Theorem 2.6. *Let C_φ be bounded but not compact in $\mathcal{H}(E,\beta_S)$. Then*

$$\|C_\varphi\|_e = \|C_\varphi\| = 1.$$

2.4. Essential norms of the differences of composition operators.

Theorem 2.7. *Let $\varphi_1 = z + b_1$ and $\varphi_2 = z + b_2$, with $Re\, b_i \geq 0$, $i = 1,2$ (so that C_{φ_1} and C_{φ_2} are bounded on $\mathcal{H}(E,\beta_S)$). Then*

$$\|C_{\varphi_1} - C_{\varphi_2}\|_e = \limsup_{n\to\infty} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|.$$

Proof. For any $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(E,\beta_S)$, and any $N \in \mathbb{N}$, define the partial sum operator

$$K_N f(z) = \sum_{n=1}^{N} a_n e^{-\lambda_n z}.$$

Then K_N is compact on $\mathcal{H}(E,\beta_S)$, since it is a finite-rank operator. Let $R_N = I - K_N$, where I is the identity operator. Since $C_{\varphi_1} - C_{\varphi_2}$ is bounded on $\mathcal{H}(E,\beta_S)$, we see that $(C_{\varphi_1} - C_{\varphi_2})K_N$ is compact on $\mathcal{H}(E,\beta_S)$ for any $N \in \mathbb{N}$. Hence

$$\begin{aligned}\|C_{\varphi_1} - C_{\varphi_2}\|_e &\leq \|(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_1} - C_{\varphi_2})K_N\| \\ &= \|(C_{\varphi_1} - C_{\varphi_2})(I - K_N)\| \\ &= \|(C_{\varphi_1} - C_{\varphi_2})R_N\|.\end{aligned}$$

Now, for any $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} \in \mathcal{H}(E,\beta_S)$ and any $N \in \mathbb{N}$,

$$\begin{aligned}\|(C_{\varphi_1} - C_{\varphi_2})R_N f\|^2 &\leq \left\|(C_{\varphi_1} - C_{\varphi_2}) \sum_{n=N+1}^{\infty} a_n e^{-\lambda_n z}\right\|^2 \\ &= \left\|\sum_{n=N+1}^{\infty} a_n e^{-\lambda_n (z+b_1)} - \sum_{n=N+1}^{\infty} a_n e^{-\lambda_n (z+b_2)}\right\|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=N+1}^{\infty} |a_n|^2 |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|^2 \beta_n^2 \\
&\le \sup_{n \ge N+1} \left| e^{-\lambda_n b_1} - e^{-\lambda_n b_2} \right|^2 \sum_{n=N+1}^{\infty} |a_n|^2 \beta_n^2 \\
&\le \sup_{n \ge N+1} \left| e^{-\lambda_n b_1} - e^{-\lambda_n b_2} \right|^2 \|f\|^2.
\end{aligned}
$$

Hence $\|(C_{\varphi_1} - C_{\varphi_2})R_N\| \le \sup_{n \ge N+1} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|$. Therefore

$$
\begin{aligned}
\|C_{\varphi_1} - C_{\varphi_2}\|_e &\le \lim_{N \to \infty} \|(C_{\varphi_1} - C_{\varphi_2})R_N\| \\
&= \lim_{N \to \infty} \sup_{n \ge N+1} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}| \\
&= \limsup_{n \to \infty} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|.
\end{aligned}
$$

On the other hand, for any $n \in \mathbb{N}$, let $q_n(z) = \frac{1}{\beta_n} e^{-\lambda_n z}$. Then $\|q_n\| = 1$ and $q_n \to 0$ weakly in $\mathcal{H}(E, \beta_S)$ as $n \to \infty$. Hence, for any compact operator K on $\mathcal{H}(E, \beta_S)$, $\|Kq_n\| \to 0$ as $n \to \infty$. Thus, from the proof of previous theorem, we get

$$
\begin{aligned}
\|C_{\varphi_1} - C_{\varphi_2} - K\| &\ge \limsup_{n \to \infty} \|(C_{\varphi_1} - C_{\varphi_2} - K)q_n\| \\
&\ge \limsup_{n \to \infty} (\|(C_{\varphi_1} - C_{\varphi_2})q_n\| - \|Kq_n\|) \\
&= \limsup_{n \to \infty} \|(C_{\varphi_1} - C_{\varphi_2})q_n\| \\
&= \limsup_{n \to \infty} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|.
\end{aligned}
$$

Hence, $\|C_{\varphi_1} - C_{\varphi_2}\|_e = \inf_K \{\|C_{\varphi_1} - C_{\varphi_2} - K\|\} \ge \limsup_{n \to \infty} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|$. Therefore

$$
\|C_{\varphi_1} - C_{\varphi_2}\|_e = \limsup_{n \to \infty} |e^{-\lambda_n b_1} - e^{-\lambda_n b_2}|. \qquad \square
$$

Remark 2.8. [9, Theorem 4.12] essentially follows from the above result by letting $\|C_{\varphi_1} - C_{\varphi_2}\|_e = 0$.

3. Isolated composition operators

Denote by $C(\mathcal{H}(E, \beta_S))$ the topological space of all bounded composition operators on $\mathcal{H}(E, \beta_S)$, equipped with the topology induced by the operator norms. In this section we give a characterization of isolated points in the space $C(\mathcal{H}(E, \beta_S))$.

Recall that, for any real number s, the integer part $[s]$ is the largest integer not greater than s, and the fractional part is defined by $\{s\} = s - [s]$. Obviously, $0 \le \{s\} < 1$ for any real number s.

Theorem 3.1. *Let* $0 \le (\lambda_n) \uparrow \infty$ *be a sequence of real numbers. Then* C_φ *is an isolated point in* $C(\mathcal{H}(E,\beta_S))$ *if and only if the following conditions hold*

(i) C_φ *is bounded but not compact on* $\mathcal{H}(E,\beta_S)$*;*

(ii) *There exist* $0<A<B<1$ *such that for each* $r\in\mathbb{R}$*, either* $\{r\lambda_{n_r}\}\in[A,B]$ *for some* $n_r\in\mathbb{N}$*, or* $\{r\lambda_n\}=0$ *for any* $n\in\mathbb{N}$*.*

Proof. Sufficiency. Let C_φ be a bounded composition operator on $\mathcal{H}(E,\beta_S)$ that is not compact. Then, by Theorem 1.1, $\varphi(z) = z + b$ with $\operatorname{Re} b = 0$. Let $b = id$, where d is a real number. Take any other bounded composition operator C_ψ on $\mathcal{H}(E,\beta_S)$. Then $\psi(z) = z + b_1, \operatorname{Re} b_1 \ge 0, \operatorname{Im} b_1 = d_1$.

We first suppose that $\operatorname{Re} b_1 > 0$. Then, C_ψ is compact, and by Theorem 2.6,

$$\|C_\varphi - C_\psi\| \ge \|C_\varphi\|_e = 1.$$

Next, suppose that $\operatorname{Re} b_1 = 0$. Let $r = \frac{d_1-d}{2\pi}$. First, suppose that $\{r\lambda_n\} = 0$ for any $n \in \mathbb{N}$, then $r\lambda_n = k_n$ for some integer k_n, and so $d_1 - d = 2k_n\pi$. Thus, by Corollary 2.4, C_ψ is just the same operator as C_φ.

Now suppose that $\{r\lambda_{n_r}\} \in [A,B]$ for some $n_r \in \mathbb{N}$. Then, by Theorem 2.3, $\|C_\varphi - C_\psi\| = \sup_{n\in\mathbb{N}} |e^{-i\lambda_n d} - e^{-i\lambda_n d_1}| = \sup_{n\in\mathbb{N}} |1 - e^{-i\lambda_n(d_1-d)}| = \sup_{n\in\mathbb{N}} |1 - e^{-2\pi i\lambda_n \frac{d_1-d}{2\pi}}| = \sup_{n\in\mathbb{N}} |1 - e^{-2\pi i\{r\lambda_n\}}| = \sup_{n\in\mathbb{N}} 2|\sin(\pi\{r\lambda_n\})| \ge 2|\sin(\pi\{r\lambda_{n_r}\})| \ge 2\min\{\sin(A\pi), \sin(B\pi)\}$, where A and B are independent of C_ψ. Hence, if we let

$$\eta = \min\{1,\ 2\sin(A\pi), 2\sin(B\pi)\},$$

then, in any cases, either C_ψ is the same as C_φ, or $\|C_\varphi - C_\psi\| \ge \eta$. Hence C_φ is an isolated point in $C(\mathcal{H}(E,\beta_S))$.

Necessity. First, if C_φ is an isolated point in $C(\mathcal{H}(E,\beta_S))$, then by Theorem 4.2 below, C_φ can never be compact, and hence (i) holds.

Next we prove that if C_φ is an isolated point in $C(\mathcal{H}(E,\beta_S))$ then (ii) holds. Assume that (ii) is not true. This means that for any pair $0 < A < B < 1$, there exists a $r_0 \in \mathbb{R}, r_0 \ne 0$ such that, for any $n \in \mathbb{N}$, $\{r_0\lambda_n\} \notin [A,B]$ and $\{r_0\lambda_{n_0}\} \ne 0$ for some $n_0 \in \mathbb{N}$. Because C_φ is an isolated point in $C(\mathcal{H}(E,\beta_S))$, according to the first part of necessity, we can assume $\varphi(z) = z + id, d \in \mathbb{R}$. Moreover, there exists $\varepsilon_0 > 0$ such that $\|C_\varphi - C_\psi\| \ge \varepsilon_0$, where C_ψ is any other bounded composition operator on $\mathcal{H}(E,\beta_S)$.

Now we construct a composition operator $C_\psi \ne C_\varphi$ such that $\|C_\varphi - C_\psi\| < \varepsilon_0$. Consider the function $g(x) = 1 - e^{-2\pi ix}$, $x \in \mathbb{R}$. It is clear that $g(x)$ is uniformly continuous on $\mathbb{R}$. That is, $\forall\varepsilon > 0$, $\exists\delta = \delta(\varepsilon) > 0$, when $|x - y| < \delta$, $x, y \in \mathbb{R}$, $|g(x) - g(y)| < \varepsilon$. In particular, we take $\varepsilon = \frac{\varepsilon_0}{2}$.

We can choose A, B satisfying the following conditions: $0 < A < \delta(\frac{\varepsilon_0}{2})$ and $1 - \delta(\frac{\varepsilon_0}{2}) < B < 1$. By assumption, there exists a r_0 such that, for any $n \in \mathbb{N}$, $\{r_0\lambda_n\} \notin [A,B]$. Then $\{r_0\lambda_n\} \in [0,A) \cup (B,1)$. Let $d_1 = d + 2\pi r_0$, and let

$\psi(z) = z + id_1$. Notice that $\frac{d_1-d}{2\pi} = r_0$, by Corollary 2.4, the condition that $\{r_0\lambda_{n_0}\} \neq 0$ for some $n_0 \in \mathbb{N}$ implies that $C_\psi \neq C_\varphi$.

For those $n \in \mathbb{N}$, such that $\{r_0\lambda_n\} \in [0, A)$, we have $\left|e^{-i\lambda_n d} - e^{-i\lambda_n d_1}\right| = \left|1 - e^{-i\lambda_n(d_1-d)}\right| = \left|1 - e^{-2\pi\lambda_n \frac{d_1-d}{2\pi}}\right| = \left|1 - e^{-2\pi i\left\{\lambda_n \frac{d_1-d}{2\pi}\right\}}\right| = \left|1 - e^{-2\pi i\{r_0\lambda_n\}}\right| = |g(\{r_0\lambda_n\}) - g(0)| \leq \frac{\varepsilon_0}{2} < \varepsilon_0$, for those $n \in \mathbb{N}$, such that $\{r_0\lambda_n\} \in (B, 1)$, we have

$$|e^{-i\lambda_n d} - e^{-i\lambda_n d_1}| = |g(\{r_0\lambda_n\}) - g(1)| \leq \frac{\varepsilon_0}{2} < \varepsilon_0.$$

Combining this two facts yields

$$\|C_\varphi - C_\psi\| \leq \sup_{n\in\mathbb{N}} |e^{-i\lambda_n d} - e^{-i\lambda_n d_1}| \leq \frac{\varepsilon_0}{2} < \varepsilon_0,$$

which is a contradiction. Thus (ii) must hold, and the proof is complete. □

The following corollaries are obvious.

Corollary 3.2. *Any compact composition operator C_φ on $\mathcal{H}(E, \beta_S)$ is a limit point in $C(\mathcal{H}(E, \beta_S))$.*

Corollary 3.3. *If the condition* (ii) *in Theorem* 3.1 *does not hold, then every bounded operator on $\mathcal{H}(E, \beta_S)$ is a limit point of $C(\mathcal{H}(E, \beta_S))$.*

Clearly, if $(\{r\lambda_n\})$ is dense in the interval $(0, 1)$ for any real number $r \neq 0$, then for any $0 < A < B < 1$, $(\{r\lambda_n\})$ has a limit point $p \in [A, B]$, and hence the condition in Theorem 3.1 is satisfied. Thus we have obtained the following result.

Corollary 3.4. *Let $0 \leq (\lambda_n) \uparrow \infty$ be a sequence of real numbers. Suppose that, for any real number $r \neq 0$, the sequence $(\{r\lambda_n\})$ is dense in the interval $(0, 1)$. Then C_φ is an isolated point in $C(H(E, \beta_S))$ if and only if it is bounded but not compact on $\mathcal{H}(E, \beta_S)$.*

Finally, we are going to apply our result to the classical Dirichlet series, that is, when $\lambda_n = \log n$. We need the following lemma.

Lemma 3.5. *Let r be any non-zero real number in $\mathbb{R}$, then $\{r(\log n)\}$ is dense in $(0, 1)$.*

Proof. Let $a \in (0, 1)$, for any $\varepsilon > 0$, we denote $I := (a - \varepsilon, a + \varepsilon)$, $I_n = n + I$. Consider $f(x) = r \log x, x \in \mathbb{R}$, it is easy to see $f(x)$ is continuous and increasing. Denote the preimage of I_n by J_n, which is $(e^{\frac{n+a-\varepsilon}{r}}, e^{\frac{n+a+\varepsilon}{r}})$. For each fixed ε, we have

$$\lim_{n\to\infty} (e^{\frac{n+a+\varepsilon}{r}} - e^{\frac{n+a-\varepsilon}{r}}) = \lim_{n\to\infty} e^{\frac{n+a-\varepsilon}{r}} (e^{\frac{2\varepsilon}{r}} - 1) = \infty.$$

Therefore, for any $\varepsilon > 0$, we have found an $n_\varepsilon \in \mathbb{N}$ such that $e^{\frac{n_\varepsilon + a + \varepsilon}{r}} - e^{\frac{n_\varepsilon + a - \varepsilon}{r}} > 1$, that is, we can find a natural number $N_\varepsilon \in J_{n_\varepsilon}$ such that $f(N_\varepsilon) \in I_{n_\varepsilon}$, or $\{r \log N_\varepsilon\} \in I$. Because of the arbitrariness of a and ε, the desired result follows. □

Corollary 3.6. *Let $\lambda_n = \log n$, i.e. $\mathcal{H}(E, \beta_S)$ consists of entire classical Dirichlet series. Then C_φ is an isolated point in $C(\mathcal{H}(E, \beta_S))$ if and only if it is bounded but not compact on $\mathcal{H}(E, \beta_S)$.*

Proof. This is a direct consequence of Lemma 3.5 and Corollary 3.4. □

4. Components of the spaces of composition operators on $\mathcal{H}(E, \beta_S)$

In this section we determine when two composition operators C_φ and C_ψ are in the same path component in $C(\mathcal{H}(E, \beta_S))$. There are three possibilities: both are compact, both are non-compact, or the two differ with regard to compactness.

4.1. Two composition operators differ with regard to compactness. In this case we have the following result.

Theorem 4.1. *Let C_φ be a non-compact composition operator and C_ψ a compact composition operator on $\mathcal{H}(E, \beta_S)$, then C_φ and C_ψ are not in the same path component in $C(\mathcal{H}(E, \beta_S))$.*

Proof. We prove the theorem by contradiction. Assume C_φ and C_ψ are in the same path component in $C(\mathcal{H}(E, \beta_S))$. This means that there exists a continuous function $C_{X(t)} : [0, 1] \to C(\mathcal{H}(E, \beta_S))$, such that $C_{X(0)} = C_\varphi$, $C_{X(1)} = C_\psi$. Let

$$U = \{t \in [0, 1] : C_{X(t)} \text{ is bounded but not compact}\},$$

and

$$V = \{t \in [0, 1] : C_{X(t)} \text{ is compact}\}.$$

Since for any $t_1 \in U$ and $t_2 \in V$, by Theorem 2.6, $\|C_{X(t_1)} - C_{X(t_2)}\| \geq \|C_{X(t_1)}\|_e = 1$, we know that $C_{X(t)}$ cannot be continuous on $[0, 1]$, which contradicts to our assumption. Hence C_φ and C_ψ are not in the same path component in $C(\mathcal{H}(E, \beta_S))$. □

4.2. Both composition operators are compact. In this case we have the following result.

Theorem 4.2. *Any two compact composition operators on $\mathcal{H}(E,\beta_S)$ are in the same path component in $C(\mathcal{H}(E,\beta_S))$.*

Proof. Let C_φ and C_ψ be two compact composition operators on $\mathcal{H}(E,\beta_S)$, with $\varphi(z) = z + b_1, \psi(z) = z + b_2, b_1 = c_1 + id_1, b_2 = c_2 + id_2$ and $c_1 > 0, c_2 > 0$. Without loss of generality, we can assume $c_2 \geq c_1$. Let $\varphi_t(z) = z+(1-t)b_1+tb_2$ $(0 \leq t \leq 1)$. We are proving that $\{C_{\varphi_t} : 0 \leq t \leq 1\}$ forms a continuous path from C_φ to C_ψ.

Fix an arbitrary $t \in [0,1)$ and let $\delta > 0$ be a real number with $t+\delta \in [0,1]$. In this case, by Theorem 2.3, we have

$$\begin{aligned}\left\|C_{\varphi_{t+\delta}} - C_{\varphi_t}\right\| &= \sup_{n\in\mathbb{N}} \left|e^{-\lambda_n[(1-t)b_1+tb_2]} - e^{-\lambda_n[(1-t-\delta)b_1+(t+\delta)b_2]}\right| \\ &= \sup_{n\in\mathbb{N}} \left\{\left|e^{-\lambda_n[(1-t)b_1+tb_2]}\right| \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right|\right\} \\ &= \sup_{n\in\mathbb{N}} \left\{e^{-\lambda_n \operatorname{Re}[(1-t)b_1+tb_2]} \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right|\right\} \\ &= \sup_{n\in\mathbb{N}} \left\{e^{-\lambda_n[(1-t)c_1+tc_2]} \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right|\right\}.\end{aligned}$$

Now let $\varepsilon > 0$. Since $\operatorname{Re} b_2 \geq \operatorname{Re} b_1$, we have $\forall n \in \mathbb{N}, \left|e^{-\lambda_n\delta(b_2-b_1)}\right| \leq 1$, and hence $\left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right| \leq 2$.

On one hand, as $0 \leq t < 1$, we have $\operatorname{Re}[(1-t)b_1+tb_2] > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n > N$, $\left|e^{-\lambda_n \operatorname{Re}[(1-t)b_1+tb_2]}\right| = e^{-\lambda_n[(1-t)c_1+tc_2]} < \frac{\varepsilon}{3}$, which implies that $e^{-\lambda_n[(1-t)c_1+tc_2]} \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right| < \frac{2\varepsilon}{3}$, for all $n > N$. Thus

$$\sup_{n>N} \left\{e^{-\lambda_n[(1-t)c_1+tc_2]} \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right|\right\} \leq \frac{2\varepsilon}{3} < \varepsilon. \tag{2}$$

On other hand, for each $n \in \{1,2,\ldots,N\}$, we have

$$e^{-\lambda_n[(1-t)c_1+tc_2]} \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right| \leq \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right|.$$

It is clear that $f_n(z) = 1 - e^{-\lambda_n(b_2-b_1)z}$ is a continuous function. Then for each $n \in \{1,2,\ldots,N\}$, there exists a $\delta_n > 0$ such that for $|z| < \delta_n$, $|f_n(z)| < \frac{\varepsilon}{2}$. Take $\delta < \min\{\delta_1,\delta_2,\ldots,\delta_N\}$, we have $\left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right| < \frac{\varepsilon}{2}, \forall n \in \{1,2,\ldots,N\}$. Consequently,

$$\sup_{1\leq n\leq N} \left\{e^{-\lambda_n[(1-t)c_1+tc_2]} \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right|\right\} \leq \frac{\varepsilon}{2} < \varepsilon. \tag{3}$$

Combining inequalities (2), (3) yields

$$\left\|C_{\varphi_{t+\delta}} - C_{\varphi_t}\right\| = \sup_{n\in\mathbb{N}} \left\{e^{-\lambda_n[(1-t)c_1+tc_2]} \left|1 - e^{-\lambda_n\delta(b_2-b_1)}\right|\right\} < \varepsilon,$$

which means that $\lim_{\delta\to 0}\left\|C_{\varphi_{t+\delta}} - C_{\varphi_t}\right\| = 0,\ \forall t \in [0,1)$.

For the case $t = 1$, considering $\left\|C_{\varphi_1} - C_{\varphi_{1-\delta}}\right\|$ and following the similar argument above, we can get the continuity of C_{φ_t} at $t = 1$.

Since $C_{\varphi_0} = C_\varphi$ and $C_{\varphi_1} = C_\psi$, we see that $\{C_{\varphi_t} : 0 \le t \le 1\}$ forms a continuous path from C_φ to C_ψ, and so C_φ and C_ψ are in the same path component. □

4.3. Both composition operators are non-compact. First we consider the consequences of two distinct non-compact composition operators belonging to the same path component. In view of Corollary 2.4, note that the word "distinct" applies to action of the operators themselves, rather than their symbols.

Proposition 4.3. *Let C_φ and C_ψ be two distinct non-compact composition operators on $\mathcal{H}(E,\beta_S)$, with $\varphi(z) = z + id_1, \psi(z) = z + id_2$, $d_1, d_2 \in \mathbb{R}$. Suppose that C_φ and C_ψ are in the same path component, that is, there exists a continuous mapping $C_{X(t)}$ acting from $[0,1] \to C(\mathcal{H}(E,\beta_S))$ with $C_{X(0)} = C_\varphi$, $C_{X(1)} = C_\psi$. Then the following assertions hold:*

(i) *$X(t) = z + id(t)$, where $d(t)$ is a function which maps $[0,1]$ into $\mathbb{R}$, with $d(0) = d_1$, $d(1) = d_2$.*
(ii) *$d(t)$ is not continuous on $[0,1]$ (the set of discontinuity points in $[0,1]$ is non-empty).*
(iii) *If $d(t)$ is continuous on a proper closed subinterval $[a,b]$ of $[0,1]$, then $d(t)$ can only be a constant on $[a,b]$.*

Proof. (i) By Theorem 4.1, each point on any path connecting C_φ and C_ψ must be a composition operator on $\mathcal{H}(E,\beta_S)$ which is not compact. Hence any path is given by $C_{X(t)}$, where $X(t) = z + id(t)$, $d(t)$ is a function $[0,1] \to \mathbb{R}$ satisfying $d(0) = d_1$ and $d(1) = d_2$.

(ii) Note that for an arbitrary $t \in [0,1)$ given, we can choose a real number $\delta > 0$ such that $t + \delta \in [0,1]$. In this case, recalling the notation $\{s\} = s - [s]$, for $s \in \mathbb{R}$, by Theorem 2.3, we have

$$\begin{aligned}\|C_{X(t+\delta)} - C_{X(t)}\| &= \sup_{n\in\mathbb{N}}\left|e^{-i\lambda_n d(t+\delta)} - e^{-i\lambda_n d(t)}\right| = \sup_{n\in\mathbb{N}}\left|1 - e^{i\lambda_n(d(t+\delta)-d(t))}\right| \\ &= \sup_{n\in\mathbb{N}}\left|1 - e^{2\pi i\lambda_n\frac{d(t+\delta)-d(t)}{2\pi}}\right| = \sup_{n\in\mathbb{N}}\left|1 - e^{2\pi i\left\{\lambda_n\frac{d(t+\delta)-d(t)}{2\pi}\right\}}\right| \\ &= \sup_{n\in\mathbb{N}} 2\left|\sin\left(\left\{\lambda_n\frac{d(t+\delta)-d(t)}{2\pi}\right\}\pi\right)\right| \\ &= 2\sup_{n\in\mathbb{N}}\sin\left(\left\{\lambda_n\frac{d(t+\delta)-d(t)}{2\pi}\right\}\pi\right).\end{aligned}$$

Now, expecting the contrary, we assume that $d(t)$ is continuous at every point of $[0,1]$. Define the following set

$$E_n = \left\{ x \in [0,1] : d(t) \text{ is not constant for } t \in \left[x - \frac{1}{n}, x + \frac{1}{n} \right] \cap [0,1] \right\}.$$

Obviously $\cap_{n=1}^{\infty} E_n$ is not empty, since $C_\varphi \neq C_\psi$, and hence we can take $t_0 \in \cap_{n=1}^{\infty} E_n$.

We consider the case $t_0 \in (0,1)$ (the situation for $t_0 = 0$ or $t_0 = 1$ is similar). In this case, there is a $\delta_0 > 0$, such that $d(t)$ is continuous on $[t_0 - \delta_0, t_0 + \delta_0] \subset [0,1]$ and $d(t)$ is not constant. Then $F(x) = d(t_0 + x) - d(t_0)$ is continuous on $[-\delta_0, \delta_0]$. Since $C_{X(t)}$ is continuous on $[0,1]$, there exists $\delta_1 > 0$, when $|t - t_0| < \delta_1$, $\|C_{X(t)} - C_{X(t_0)}\| < \frac{1}{2}$.

For those $0 < \delta' < \min\{\delta_0, \delta_1\}$, we have

$$\|C_{X(t_0+\delta')} - C_{X(t_0)}\| = 2 \sup_{n \in \mathbb{N}} \sin\left(\left\{ \lambda_n \frac{d(t_0 + \delta') - d(t_0)}{2\pi} \right\} \pi \right) \leq \frac{1}{2},$$

that is, $\forall n \in \mathbb{N}$, when $0 < \delta' < \min\{\delta_0, \delta_1\}$,

$$\sin\left(\left\{ \lambda_n \frac{d(t_0 + \delta') - d(t_0)}{2\pi} \right\} \pi \right) = \sin\left(\left\{ \lambda_n \frac{F(\delta')}{2\pi} \right\} \pi \right) \leq \frac{1}{4}.$$

Fix some $0 < \delta_0' < \min\{\delta_0, \delta_1\}$. Since $t_0 \in \cap_{n=1}^{\infty} E_n$, we can assume that $F(\delta_0') \neq 0$. Put $p = \frac{F(\delta_0')}{2\pi}$, we have $\forall n \in \mathbb{N}$, $\sin(\{p\lambda_n\}\pi) \leq \frac{1}{4}$.

Without loss of generality, we can assume that $F(\delta_0') > 0$. It is also clear that $F(0) = 0$. Then for any $M > 1$, we know that

$$0 < \frac{F(\delta_0')}{M} < F(\delta_0').$$

Applying the intermediate value theorem, it is clear that there exists a $\delta'' \in (0, \delta_0')$, such that $F(\delta'') = \frac{F(\delta_0')}{M}$. Then

$$\begin{aligned}
\|C_{X(t_0+\delta'')} - C_{X(t_0)}\| &= 2 \sup_{n \in \mathbb{N}} \sin\left(\left\{ \lambda_n \frac{d(t_0 + \delta'') - d(t_0)}{2\pi} \right\} \pi \right) \\
&= 2 \sup_{n \in \mathbb{N}} \sin\left(\left\{ \lambda_n \frac{F(\delta'')}{2\pi} \right\} \pi \right) \\
&= 2 \sup_{n \in \mathbb{N}} \sin\left(\left\{ \lambda_n \frac{F(\delta')}{2\pi M} \right\} \pi \right) \\
&\leq \frac{1}{2},
\end{aligned}$$

that is, $\forall n \in \mathbb{N}$, $\forall M > 1$, we have

$$\sin\left(\left\{ \lambda_n \frac{F(\delta_0')}{2\pi M} \right\} \pi \right) = \sin\left(\left\{ \frac{p\lambda_n}{M} \right\} \pi \right) \leq \frac{1}{4}. \tag{4}$$

But inequality (4) cannot be true. We can choose a natural number n_0 large enough so that $p\lambda_{n_0} > 1$, and for $M = 2p\lambda_{n_0}$ we have $\sin(\{\frac{p\lambda_{n_0}}{M}\}\pi) = 1$, which is a contradiction. Hence (ii) is true.

(iii) The proof of this result is similar as (ii). Indeed, in the proof of (ii), we only need to replace $[0,1]$ (or $(0,1)$) by $[a,b]$ (or (a,b)). $\square$

Proposition 4.3 states that a continuous path must be induced by a noncontinuous map. The counterintuitive nature of this result may suggest that no distinct non-compact composition operators can be in the same path component. Our next result shows that this is the case for a wide class of frequencies (λ_n). We first need the following lemma.

Lemma 4.4. *Let $a_n = aq^{n-1}$, where $a > 0, q \neq 1$ is a positive integer. Then there are only countably many real solutions satisfying the following inequality system*

$$\forall n \in \mathbb{N}, \quad \{a_n x\} < K \quad \textit{or} \quad 1 - \{a_n x\} < K \tag{P}$$

where $0 < K < \frac{1}{q^2}$.

Proof. Let $\mathcal{S} = \{\frac{m}{aq^t} : m \in \mathbb{Z}, t \in \mathbb{N}\}$. It suffices to show that if $x \notin \mathcal{S}$, then x is not a solution of (P). Assume x is a solution of (P), that is

$$\forall n \in \mathbb{N}, \quad \{x \cdot aq^{n-1}\} < K \quad \text{or} \quad 1 - \{x \cdot aq^{n-1}\} < K.$$

If $x \notin \mathcal{S}$ (which implies that $\exists n \in \mathbb{N}$, such that $\{x \cdot aq^{n-1}\} \neq 0$), we consider the following two cases.

Case I: There exists some $i_1 \in \mathbb{N}$ such that $0 < \{x \cdot aq^{i_1-1}\} < K$. Then we can see that there exists a $N_1 \in \mathbb{N}, N_1 \geq 1$, such that $\frac{1}{q^2} \leq q^{N_1} \cdot \{x \cdot aq^{i_1-1}\} < \frac{1}{q}$. Consider $aq^{i_1+N_1-1} \in \{a_n\}$, we have

$$x \cdot aq^{i_1+N_1-1} = q^{N_1}[x \cdot aq^{i_1-1}] + q^{N_1}\{x \cdot aq^{i_1-1}\},$$

which implies that $K < \frac{1}{q^2} \leq \{x \cdot aq^{i_1+N_1-1}\}$. Moreover, $1 - \{x \cdot aq^{i_1+N_1-1}\} > 1 - \frac{1}{q} \geq \frac{1}{2} > K$. Combining these two inequalities, we get a contradiction to (P).

Case II: There exists some $i_2 \in \mathbb{N}$ such that $1 - \{x \cdot aq^{i_2-1}\} < K$. Since x is a solution of (P) and $x \notin \mathcal{S}$, we have

$$x \cdot aq^{i_2-1} = [x \cdot aq^{i_2-1}] + 1 + (\{x \cdot aq^{i_2-1}\} - 1),$$

where $[x \cdot aq^{i_2-1}]$ is the integer part of $x \cdot aq^{i_2-1}$ and $-\frac{1}{q^2} < -K < \{x \cdot aq^{i_2-1}\} - 1 < 0$. Thus there exists a $N_2 \in \mathbb{N}$, $N_2 \geq 1$, such that $-\frac{1}{q} < q^{N_2} \cdot (\{r \cdot aq^{i_2-1}\} - 1) \leq -\frac{1}{q^2}$. Consider $aq^{i_2+N_2-1} \in \{a_n\}$, we have

$$x \cdot aq^{i_2+N_2-1} = q^{N_2}([x \cdot aq^{i_2-1}] + 1) + q^{N_2}(\{x \cdot aq^{i_2-1}\} - 1),$$

which implies that $K < \frac{1}{q^2} \leq 1 - \{x \cdot aq^{i_2+N_2-1}\}$. Moreover, $K < \frac{1}{2} \leq 1 - \frac{1}{q} < \{x \cdot aq^{i_2+N_2-1}\}$. Combining these two inequalities, we get a contradiction to (P). $\square$

Theorem 4.5. *If there is a subsequence* $\{\lambda_{n_k}\}$ *of* $\{\lambda_n\}$ *satisfying the condition* $\lambda_{n_k} = aq^{k-1}, k = 1, 2, \ldots,$ *where* $a > 0$ *and* $q \neq 1$ *is a positive integer, then there is no path between any distinct two non-compact composition operators on* $C(\mathcal{H}(E, \beta_S))$.

Proof. We prove this proposition by contradiction. Suppose there is a path $C_{X(t)}$ between two distinct non-compact composition operators $C_{X(0)}$ and $C_{X(1)}$ in $C(\mathcal{H}(E, \beta_S))$, that is, $C_{X(t)}$ is a continuous function from $[0, 1]$ to $C(\mathcal{H}(E, \beta_S))$, where $X(t) = z + id(t)$, $d(t)$ is a function from $[0, 1]$ into $\mathbb{R}$.

First note that since $C_{X(0)}$ and $C_{X(1)}$ are distinct non-compact composition operators on $C(\mathcal{H}(E, \beta_S))$, there exists a point $t_0 \in (0, 1)$ such that for any open neighborhood U_{t_0} of t_0, there is some $t \in U_{t_0} \cap [0, 1]$ for which $\|C_{X(t)} - C_{X(t_0)}\| > 0$.

For $0 < K < \frac{1}{q^2}$, since $\sin(K\pi) > 0$, there exists a $\delta > 0$ such that $t_0 + \delta$, $t_0 - \delta \in [0, 1]$ and when $t \in [0, \delta]$, we have

$$\|C_{X(t_0+t)} - C_{X(t_0)}\| = 2 \sup_{n\in\mathbb{N}} \sin\left(\left\{\lambda_n \frac{d(t_0 + t) - d(t_0)}{2\pi}\right\}\pi\right) < 2\sin(K\pi). \quad (5)$$

Obviously, we can take $\delta > 0$ small enough, so that $\|C_{X(t_0+\delta)} - C_{X(t_0)}\| > 0$. The inequality (5) is equivalent, due to Lemma 4.4 and the assumption of the theorem, to

$$\forall k \in \mathbb{N}, \quad \left\{\lambda_{n_k} \cdot \frac{d(t_0+t) - d(t_0)}{2\pi}\right\} < K \quad \text{or} \quad 1 - \left\{\lambda_{n_k} \cdot \frac{d(t_0+t) - d(t_0)}{2\pi}\right\} < K.$$

Again by Lemma 4.4, $\frac{d(t_0+t)-d(t_0)}{2\pi}$ can only take on values in the countable set $\mathcal{S} = \{\frac{m}{aq^t} : m \in \mathbb{Z}, t \in \mathbb{N}\}$.

Now we consider a function

$$G(t) = \|C_{X(t_0+t)} - C_{X(t_0)}\|, \quad t \in [-t_0, 1 - t_0].$$

By continuity of $C_{X(t)}$, $G(t)$ is a continuous function on $[-t_0, 1 - t_0]$. Moreover, since $\|C_{X(t_0+\delta)} - C_{X(t_0)}\| > 0$, that is, $G(\delta) > G(0) = 0$, the range of $G(t)$ on $[0, \delta]$ is some interval $[0, \Delta]$, where $\Delta > 0$.

However, as noted above, the range of $\frac{d(t_0+t)-d(t_0)}{2\pi}$ over $[0, \delta]$ is a countable set $\mathcal{S}$, and hence the range of $G(t)$ over this interval $[0, \delta] \subset [-t_0, 1 - t_0]$ is the set

$$G([0, \delta]) = \left\{2 \sup_{n\in\mathbb{N}} \sin\left(\left\{\lambda_n \frac{d(t_0 + t) - d(t_0)}{2\pi}\right\}\pi\right) : t \in [0, \delta]\right\},$$

which is also countable. This contradicts the fact that this range is the interval $[0, \Delta]$. □

Corollary 4.6. *Let* $\lambda_n = \log n$, *i.e.* $\mathcal{H}(E, \beta_S)$ *consists of entire classical Dirichlet series. Then there is no path between any non-compact composition operators in* $C(\mathcal{H}(E, \beta_S))$.

Proof. Note that $\lambda_{2^{2^n}} = \log 2^{2^n} = 2^n \log 2$ is a subsequence of $\{\lambda_n\}$ that satisfies the condition of Theorem 4.5, the result clearly follows from Theorem 4.5. □

In conclusion, based on Proposition 4.3 and Theorem 4.5, we would like to suggest the following conjecture:

Conjecture 4.7. Two distinct non-compact composition operators on $\mathcal{H}(E, \beta_S)$ are not in the same path component.

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