Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 35 (2016), 285–308
DOI: 10.4171/ZAA/1566

On Superposition Operators in Spaces of Regular and of Bounded Variation Functions

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Abstract. For a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ we define the superposition operator $\Psi_f : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ by the formula $\Psi_f(\varphi)(t) = f(t, \varphi(t))$. First we provide necessary and sufficient conditions for f under which the operator Ψ_f maps the space $R(0,1)$, of all real regular functions on $[0,1]$, into itself. Next we show that if an operator Ψ_f maps the space $BV(0,1)$, of all real functions of bounded variation on $[0,1]$, into itself, then

(1) it maps bounded subsets of $BV(0,1)$ into bounded sets if additionally f is locally bounded,
(2) $f = f_{cr} + f_{dr}$ where the operator $\Psi_{f_{cr}}$ maps the space $D(0,1) \cap BV(0,1)$, of all right-continuous functions in $BV(0,1)$, into itself and the operator $\Psi_{f_{dr}}$ maps the space $BV(0,1)$ into its subset consisting of functions with countable support,
(3) $\limsup_{n\to\infty} n^{\frac{1}{2}} |f(t_n, x_n) - f(s_n, x_n)| < \infty$ for every bounded sequence $(x_n) \subset \mathbb{R}$ and for every sequence $([s_n, t_n))$ of pairwise disjoint intervals in $[0,1]$ such that the sequence $(|f(t_n, x_n) - f(s_n, x_n)|)$ is decreasing.

Moreover we show that if an operator Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself, then f is locally Lipschitz in the second variable uniformly with respect to the first variable.
Keywords. Nemytskii superposition operators, regular functions, functions of bounded variation

Mathematics Subject Classification (2010). Primary 47H30, secondary 26A45

1. Introduction

For a given function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ we define the superposition operator $\Psi_f : \mathbb{R}^{[0,1]} \to \mathbb{R}^{[0,1]}$ by the formula

$$\Psi_f(\varphi)(t) = f(t, \varphi(t)).$$

This operator is called the Nemytskii (or nonautonomous) superposition operator. It plays an important rule in the theory of differential and integral

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equations. Properties of Nemytskii operators have been studied in various function spaces by many authors (see [1, 2]). The majority of the investigations deal with the autonomous Nemytskii operators, i.e. $f(t,x) = f(x)$ for every $(t,x) \in [0,1] \times \mathbb{R}$. About the nonautonomus case is not too much known. This paper contains studies of properties of nonautonomous Nemytskii operators in the space $R(0,1)$ of all real regular functions on $[0,1]$ (with no discontinuities of the second kind) and in the space $BV(0,1)$ of all real functions of bounded variation on $[0,1]$. The relation between these spaces is simple, every function of bounded variation is regular and $BV(0,1)$ forms a dense subset of $R(0,1)$ in the sup norm.

Bugajewska in [4] showed sufficient conditions for a function f which guarantee that the operator Ψ_f maps the space $BV(0,1)$ into itself and is bounded, i.e. it maps bounded subsets of $BV(0,1)$ into bounded sets. There exist examples of functions f for which the operator Ψ_f maps the space $BV(0,1)$ into itself but the function f does not satisfy the assumptions of the Bugajewska theorem. Such examples exist even in the class of functions on $[0,1] \times \mathbb{R}$ which are locally Lipschitz in the second variable uniformly with respect to the first variable (see Remark 4.9). We should not expect that there exist easy to be verified necessary and sufficient conditions for Nemytskii operators mapping the space $BV(0,1)$ into itself. The main purpose of the paper is to find conditions that possesses each function f for which the operator Ψ_f maps the space $BV(0,1)$ into itself. First observation we made is the fact that if a Nemytskii operator maps the space $BV(0,1)$ into itself then it maps also the space $R(0,1)$ into itself. We provide necessary and sufficient conditions for a function f under which the operator Ψ_f maps the space $R(0,1)$ into itself. The conditions we find are easy to verify and they remain necessary conditions for all Nemytskii operators mapping the space $BV(0,1)$ into itself. Each function f satisfying these conditions is the sum of its *right-continuous* part f_{cr} and of its *right-discrete* part f_{dr}, the operator $\Psi_{f_{cr}}$ maps the space $D(0,1)$, of all right-continuous functions in $R(0,1)$, into itself and the operator $\Psi_{f_{dr}}$ maps the space $R(0,1)$ into its subset consisting of functions with countable support. One may consider also the decomposition of f into the *left-continuous* part and the *left-discrete* part, the both decompositions have similar properties but usually do not coincide. Autonomous Nemytskii operators as well these mapping the space $R(0,1)$ into itself (see [3], the main result of [3] is a consequence of Theorem 3.1) as these mapping the space $BV(0,1)$ into itself (see [7]) do not possess the discrete part. All Nemytskii operators mapping the space $D(0,1)$ into itself have the same property. The formulation of the main result in [9] shows that the decomposition is useful also for the space $BV(0,1)$. We show in Theorem 4.7 that an operator Ψ_f maps the space $BV(0,1)$ into itself if and only if the operators $\Psi_{f_{dr}}$ and $\Psi_{f_{cr}}$ map also the space $BV(0,1)$ into itself. This result generalizes the decomposition theorem in [1]; Theorem 6.10. For continuous functions the both decomposition theorems provide the same decomposition. Our decomposition theorem do not require the assumption that the operator Ψ_f is bounded. We show in Theorem 4.3 that an operator Ψ_f which maps the space $BV(0,1)$ into itself is bounded if f is locally bounded. Moreover we show that every operator Ψ_f which maps the space $D(0,1) \cap BV(0,1)$ into itself is bounded. The proof of the decomposition theorem applies also the fact that if an operator Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself, then f is locally Lipschitz in the second variable uniformly with respect to the first variable. This generalizes the Josephy result (see [7]) onto nonautonomous case but in contrast to the autonomous case as we see in Example 4.11 the inverse theorem is not true. Finally we show that if an operator Ψ_f maps the space $BV(0,1)$ into itself, then $\limsup_{n\to\infty} n^{\frac{1}{2}}|f(t_n,x_n) - f(s_n,x_n)| < \infty$ for every bounded sequence $(x_n) \subset \mathbb{R}$ and for every sequence $([s_n,t_n))$ of pairwise disjoint intervals contained in $[0,1]$ such that the sequence $(|f(t_n,x_n) - f(s_n,x_n)|)$ is decreasing. We show that the estimation is "the best" as well for the discrete as for the continuous case.

The paper is divided into four sections. The second section is devoted to study properties of Nemytskii operators mapping the space $D(0,1)$ into itself. These operators play an important rule in the description of all Nemytskii operators in spaces $R(0,1)$ and $BV(0,1)$. Properties of Nemytskii operators mapping the space $R(0,1)$ into itself are studied in the third section. The last section contains investigations of Nemytskii operators in the space $BV(0,1)$.

2. Nemytskii operators in the space $D(0,1)$

The Banach space $D(0,1)$ consists of all real functions on the interval $[0,1]$ that are right continuous at each point of $[0,1)$ and left continuous at 1 with a left-hand limit at each point of $(0,1]$; it is equipped with the sup norm. The space $D(0,1)$ has many interesting properties (see [5,11]) and applications (see [10]).

Theorem 2.1. *For a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the following assertions are equivalent:*

(a) *the operator Ψ_f maps the space $D(0,1)$ into itself,*

(b) *f has the following properties:*

  (1) *the limit $\lim_{[0,s)\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ exists for every $(s,x) \in (0,1]\times\mathbb{R}$,*

  (2) *$\lim_{[0,1)\times\mathbb{R}\ni(u,y)\to(1,x)} f(u,y) = f(1,x)$ for every $x \in \mathbb{R}$ and*

  (3) *$\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f(u,y) = f(t,x)$ for every $(t,x) \in [0,1) \times \mathbb{R}$,*

(c) *the function $\tilde{f} : \mathbb{R} \to D(0,1)$ given by the formula*

$$\tilde{f}(x)(t) = f(t,x) \quad \text{for every } (t,x) \in [0,1] \times \mathbb{R}$$

is continuous,

(d) *the operator Ψ_f maps the space $D(0,1)$ into itself and it is continuous on $D(0,1)$.*

Proof. The implication (d) $\Longrightarrow$ (a) is obvious.

The proof of the implication (a) $\Longrightarrow$ (b) is a straightforward consequence of the following two claims.

Claim 1. *If an operator Ψ_f maps the space $D(0,1)$ into itself, then for every $x \in \mathbb{R}$*

(i) *the limit* $\lim_{u\to s-} f(u,x)$ *exists for every* $s \in (0,1]$ *and*
(ii) $\lim_{u\to 1-} f(u,x) = f(1,x)$ *and*
(iii) $\lim_{u\to t+} f(u,x) = f(t,x)$ *for every* $t \in [0,1)$.

Proof of Claim 1. For $a \in \mathbb{R}$, let $\phi_a : [0,1] \to \mathbb{R}$ be given by the formula $\phi_a(t) = a$. Then the function $t \to \Psi_f(\phi_a)(t) = f(t,a)$ is a member of the space $D(0,1)$ and it verifies conditions (i), (ii) and (iii). □

Claim 2. *If an operator Ψ_f maps the space $D(0,1)$ into itself, then for every $x \in \mathbb{R}$*

(i) *the limit* $\lim_{[0,s)\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ *exists for every* $s \in (0,1]$ *and*
(ii) $\lim_{[0,1)\times\mathbb{R}\ni(u,y)\to(1,x)} f(u,y) = f(1,x)$ *and*
(iii) $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f(u,y) = f(t,x)$ *for every* $t \in [0,1)$.

Proof of Claim 2. We show only the last fact, the other facts have similar proofs. Suppose that the limit does not exist. Then there exist $\varepsilon > 0$ and sequences $(t_n) \subset [0,1]$, $(x_n) \subset \mathbb{R}$ such that $t_n > t_{n+1} > t$ and $\lim_{n\to\infty} t_n = t$ and $\lim_{n\to\infty} x_n = x$ and $\sum_{n=1}^{\infty} |x_{n+1} - x_n| < \infty$ and $|f(t_n,x_n) - f(t_{n+1},x_{n+1})| > \varepsilon$ for each n. Let $\varphi : [0,1] \to \mathbb{R}$ be the function that is affine on each segment $[t_{n+1},t_n]$ and $\varphi(t_n) = x_n$ for every n and $\varphi(s) = x$ for $s \in [0,t]$ and $\varphi(s) = x_1$ for $s \in [t_1,1]$. It is easy to see that φ is a continuous function and a member of $BV(0,1)$ (it is clear that $\operatorname{var}_{[t_{n+1},t_n]}(\varphi) = |x_{n+1} - x_n|$ and $\operatorname{Var}(\varphi) = |x| + \sum_{n=1}^{\infty} |x_{n+1} - x_n|$ (see definitions in section 4)). But the limit $\lim_{n\to\infty} f(t_n,x_n) = \lim_{n\to\infty} \Psi_f(\varphi)(t_n)$ does not exist. This contradicts the fact that $\Psi_f(\varphi)$ is a member of $R(0,1)$. The equality $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f(u,y) = f(t,x)$ holds by Claim 1. □

(b) $\Longrightarrow$ (c). It follows from (b) that the function $t \to f(t,x) = \tilde{f}(x)(t)$ is a member of $D(0,1)$ for each $x \in \mathbb{R}$.

Claim 3. *If a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ fulfills the condition* (b), *then for every $x \in \mathbb{R}$ we have the equality*

$$\lim_{n\to\infty} \sup_{|y-x|<\frac{1}{n}} \sup_{t\in[0,1]} |f(t,y) - f(t,x)| = 0.$$

Proof of Claim 3. Suppose that the equality does not hold. Then there exist sequences $(x_n) \subset \mathbb{R}$ and $(t_n) \subset [0,1]$ and $\varepsilon > 0$ such that (t_n) is monotonic, $\lim_{n\to\infty} x_n = x$, and $|f(t_n, x_n) - f(t_n, x)| > \varepsilon$ for each n. But this contradicts (b) if (t_n) possesses a strictly monotonic subsequence. It remains to consider the case when the sequence (t_n) is constant. Suppose that $t_n = t < 1$ for each n. By the condition (b) for each n there exists $t < s_n < t + \frac{1-t}{n}$ such that $|f(s_n, x_n) - f(t, x_n)| < \frac{1}{n}$. Hence $\limsup_{n\to\infty} |f(s_n, x_n) - f(t, x)| \geqslant \varepsilon$. This contradicts the condition (b). If $t_n = 1$ for each n, then by the condition (b) for each n there exists $\frac{n-1}{n} < s_n < 1$ such that $|f(s_n, x_n) - f(1, x_n)| < \frac{1}{n}$. Hence $\limsup_{n\to\infty} |f(s_n, x_n) - f(1, x)| \geqslant \varepsilon$. This contradicts the condition (b). □

Now it is clear that the map $\tilde{f}$ is continuous.

(c) $\Longrightarrow$ (d). Let $\varphi \in D(0,1)$. Suppose that $(t_n) \subset [0,1]$ is a strictly monotonic sequence with a limit t. Let $x = \lim_{n\to\infty} \varphi(t_n)$. Since φ is a member of $D(0,1)$, the limit exists. Since $\tilde{f}$ is a continuous function, we have

$$\lim_{n\to\infty} \sup_{s\in[0,1]} |f(s, \varphi(t_n)) - f(s, x)| = 0.$$

Hence $\lim_{n\to\infty} |f(t_n, \varphi(t_n)) - f(t_n, x)| = 0$. Since the function $\tilde{f}(x)$ is a member of $D(0,1)$, we have

$$\lim_{n\to\infty} f(t_n, x) = \begin{cases} \lim_{s\to t-} \tilde{f}(x)(s) & \text{if } (t_n) \text{ is increasing} \\ \tilde{f}(x)(t) & \text{if } (t_n) \text{ is decreasing} \\ \tilde{f}(x)(1) & \text{if } \lim_{n\to\infty} t_n = 1. \end{cases}$$

Thus we have shown that the the function $\Psi_f(\varphi)$ has a left-hand limit at each point of $(0,1]$ and it is right continuous at each point of $[0,1)$ and it is left continuous at 1. Therefore $\Psi_f(\varphi)$ is an element of $D(0,1)$.

Let (φ_n) be a sequence in $D(0,1)$ that converges to φ uniformly on $[0,1]$. Suppose that there exist a sequence $(t_n) \subset [0,1]$ and $\varepsilon > 0$ such that

$$|\Psi_f(\varphi_n)(t_n) - \Psi_f(\varphi)(t_n)| = |f(t_n, \varphi_n(t_n)) - f(t_n, \varphi(t_n))| > 2\varepsilon$$

Since $\tilde{f}$ is a continuous function on $\mathbb{R}$, it is uniformly continuous on the interval $[-\sup_n \|\varphi_n\|, \sup_n \|\varphi_n\|]$. Therefore there exists N such that for every $n \geqslant N$ we have

$$\sup_{s\in[0,1]} |\tilde{f}(\varphi_n(t_n))(s) - \tilde{f}(\varphi(t_n))(s)| < \varepsilon.$$

Consequently $|f(t_n, \varphi_n(t_n)) - f(t_n, \varphi(t_n))| < \varepsilon$ for every $n \geqslant N$. But this contradicts the inequality above. Thus we have shown that the sequence $(\Psi_f(\varphi_n))$ converges to $\Psi_f(\varphi)$ uniformly on $[0,1]$. □

Let (M,ρ) be a metric space. For any function $f : M \to \mathbb{R}$ the oscillation function $d_f : M \to \mathbb{R} \cup \{\infty\}$ is defined by

$$d_f(t) = \inf_{\delta>0} \sup\{|f(s) - f(u)| : s, u \in M, \rho(s,t) \leqslant \delta, \rho(u,t) \leqslant \delta\}.$$

It is clear that f is continuous at t if and only if $d_f(t) = 0$.

Corollary 2.2. *If for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $D(0,1)$ into itself, then*

(a) *for every $a > 0$*

$$\sup\{|f(t,x)| : (t,x) \in [0,1] \times [-a,a]\} < \infty,$$

(b) *the operator Ψ_f maps bounded subsets of $D(0,1)$ into bounded sets,*

(c) *for every $\varepsilon > 0$ and $a > 0$ the set*

$$\{t \in [0,1] : \exists_{x\in[-a,a]} d_f(t,x) > \varepsilon\}$$

is finite,

(d) *for every bounded sequence $(x_n) \subset \mathbb{R}$ and every sequence $([s_n,t_n))$ of pairwise disjoint intervals contained in $[0,1]$ we have*

$$\lim_{n\to\infty} \big(f(t_n,x_n) - f(s_n,x_n)\big) = 0.$$

Proof. (a). Let $a > 0$. According to Theorem 2.1(c) the function $\tilde{f}$ is continuous on $\mathbb{R}$. Hence $f([-a,a])$ is a bounded set in $D(0,1)$. Moreover the following equality

$$\sup\{|f(t,x)| : (t,x) \in [0,1] \times [-a,a]\} = \sup_{x\in[-a,a]} \|\tilde{f}(x)\|$$

holds. Therefore the function f is bounded on $[0,1] \times [-a,a]$.

Part (b) is a straightforward consequence of (a).

(c). Suppose that the set is infinite. Then there exist a strictly monotonic sequence $(t_n) \subset [0,1]$ and a convergent sequence $(x_n) \subset [-a,a]$ and $\varepsilon > 0$ such that $d_f(t_n,x_n) > \varepsilon$ for each n. Without loss of generality we may assume that the sequence (t_n) is increasing. Then for each n there exist $s_n, u_n \in [0,1]$ and $y_n, z_n \in [-a-1, a+1]$ such that $s_n, u_n \in (\frac{t_{n-1}+t_n}{2}, \frac{t_n+t_{n+1}}{2})$ and $|y_n - x_n| < \frac{1}{n}$, $|z_n - x_n| < \frac{1}{n}$ and $|f(s_n,y_n) - f(u_n,z_n)| > \varepsilon$. This contradicts Theorem 2.1(b).

(d). Suppose that the above sequence does not converge to zero. Then there exists a strictly increasing sequence $(n_k) \subset \mathbb{N}$ such that the sequence (s_{n_k}) is strictly monotonic, the sequence (x_{n_k}) is convergent and $|f(t_{n_k},x_{n_k}) - f(s_{n_k},x_{n_k})| > \varepsilon$ for each k. This contradicts Theorem 2.1(b). □

3. Nemytskii operators in the space $R(0,1)$

The Banach space $R(0,1)$ consists of all bounded real functions on the interval $[0,1]$ that have a left-hand limit at each point of $(0,1]$ and right-hand limit at each point of $[0,1)$; it is equipped with the sup norm.

Theorem 3.1. *For a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the following assertions are equivalent:*

(a) *the operator Ψ_f maps the space $R(0,1)$ into itself,*

(b) *f has the following properties:*

  (1) *the limit $\lim_{[0,s)\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ exists for every $(s,x) \in (0,1]\times\mathbb{R}$ and*

  (2) *the limit $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f(u,y)$ exists for every $(t,x) \in [0,1)\times\mathbb{R}$,*

(c) *functions $f_{cr}, f_{dr} : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by the formulas*

$$f_{cr}(t,x) = \begin{cases} \lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f(u,y) & \text{if } (t,x) \in [0,1) \times \mathbb{R} \\ \lim_{[0,t)\times\mathbb{R}\ni(u,y)\to(t,x)} f(u,y) & \text{if } (t,x) \in \{1\} \times \mathbb{R} \end{cases}$$

and $f_{dr} = f - f_{cr}$ are well defined and they have the following properties:

  (1) *for every $a > 0$ and $\varepsilon > 0$ the set*

$$\{t \in [0,1] : \sup_{|x|\leqslant a} |f_{dr}(t,x)| > \varepsilon\} \quad \text{is finite,}$$

  (2) *the operator $\Psi_{f_{cr}}$ maps the space $D(0,1)$ into itself.*

Moreover, the decomposition described in (c) *is unique in the following sense: if functions $g, h : [0,1] \times \mathbb{R} \to \mathbb{R}$ have the following properties:*

(i) *$f = g + h$,*

(ii) *for every $a > 0$ and $\varepsilon > 0$ the set*

$$\{t \in [0,1] : \sup_{|x|\leqslant a} |h(t,x)| > \varepsilon\} \quad \text{is finite,}$$

(iii) *the operator Ψ_g maps the space $D(0,1)$ into itself,*

then $g = f_{cr}$ and $h = f_{dr}$.

Proof. The implication (a) $\Longrightarrow$ (b) follows immediately from the following claim, whose proof is essentially the same as the proof of Claim 2 in Theorem 2.1.

Claim 1. *If an operator Ψ_f maps the space $R(0,1)$ into itself, then for every $x \in \mathbb{R}$*

(i) *the limit $\lim_{[0,s)\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ exists for every $s \in (0,1]$ and*

(ii) *the limit $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f(u,y)$ exists for every $t \in [0,1)$.*

(b) $\implies$ (c). It is clear that the function f_{cr} is well defined and it fulfills the condition (3) of Theorem 2.1(b). Suppose that the function f_{cr} does not satisfy the condition (1) of Theorem 2.1(b). Then there exist a strictly increasing sequence $(t_n) \subset [0,1]$ and a convergent sequence $(x_n) \subset \mathbb{R}$ and $\varepsilon > 0$ such that

$$|f_{cr}(t_n, x_n) - f_{cr}(t_{n+1}, x_{n+1})| > 3\varepsilon.$$

By the definition of f_{cr} for each n there exists $t_n < s_n < t_{n+1}$ such that

$$|f_{cr}(t_n, x_n) - f(s_n, x_n)| < \varepsilon.$$

Hence $|f(s_n, x_n) - f(s_{n+1}, x_{n+1})| > \varepsilon$. But this contradicts the condition (1) of (b).

The same consideration shows that

$$\lim_{[0,1)\times\mathbb{R}\ni(u,y)\to(1,x)} f_{cr}(u,y) = \lim_{[0,1)\times\mathbb{R}\ni(u,y)\to(1,x)} f(u,y).$$

Thus we have shown that the operator $\Psi_{f_{cr}}$ maps the space $D(0,1)$ into itself.

It follows from the definitions of the functions f_{dr} and f_{cr} that for every $t \in [0,1)$ and $x \in \mathbb{R}$ we have the following equalities:

$$\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f_{dr}(u,y) = 0 \quad \text{and} \quad \lim_{[0,1)\times\mathbb{R}\ni(u,y)\to(1,x)} f_{dr}(u,y) = 0.$$

Claim 2. *The set* $\{t \in [0,1] : \sup_{x\in\mathbb{R}} |f_{dr}(t,x)| > 0\}$ *is countable.*

Proof of Claim 2. Suppose that for some $a > 0$ the set above is uncountable. Then there exists $\varepsilon > 0$ such that the set

$$\{t \in [0,1] : \sup_{|x|\leqslant a} |f_{dr}(t,x)| > \varepsilon\}$$

is uncountable. Let $t_1 = 1$. We find $0 < t_2 < t_1$ such that $\sup_{|x|\leqslant a} |f_{dr}(t_2,x)| > \varepsilon$ and the set $\{t \in [0,t_2] : \sup_{|x|\leqslant a} |f_{dr}(t,x)| > \varepsilon\}$ is uncountable. Otherwise the set

$$\bigcup_{n=1}^{\infty} \{t \in [0, \tfrac{n}{n+1}] : \sup_{|x|\leqslant a} |f_{dr}(t,x)| > \varepsilon\} = \{t \in [0,1) : \sup_{|x|\leqslant a} |f_{dr}(t,x)| > \varepsilon\}$$

is countable. Continuing the procedure we are able to find a strictly decreasing sequence $(t_n) \subset [0,1]$ such that $\sup_{|x|\leqslant a} |f_{dr}(t_n,x)| > \varepsilon$ for each $n > 1$. Let $t = \lim_{n\to\infty} t_n$. It is clear that for every n, we find $x_n \in [-a,a]$ such that $|f(t_n,x_n)| > \varepsilon$. Let x be an accumulation point of the sequence (x_n). Then

$$\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f_{dr}(u,y) \neq 0.$$

This contradicts the equality above. □

Claim 3. *For every $a > 0$ and for every $\varepsilon > 0$ the set*

$$\{t \in [0,1] : \sup_{|x|\leqslant a} |f_{dr}(t,x)| > \varepsilon\} \quad \textit{is finite,}$$

Proof of Claim 3. Suppose that for some $a > 0$ and some $\varepsilon > 0$ the set above is infinite. Then we are able to find a strictly monotonic sequence $(t_n) \subset [0,1]$ such that $\sup_{|x|\leqslant a} |f_{dr}(t_n,x)| > \varepsilon$ for each n. Since the set $[-a,a]$ is compact, we are able to find a convergent sequence $(x_n) \subset [-a,a]$ and a subsequence (s_n) of (t_n) such that $|f_{dr}(s_n,x_n)| > \varepsilon$ for each n. Let $t = \lim_{n\to\infty} t_n$ and $x = \lim_{n\to\infty} x_n$. If the sequence (t_n) is decreasing, then $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(t,x)} f_{dr}(u,y) \neq 0$. This contradicts the equality above. If the sequence (t_n) is increasing, then by Claim 2 for each n we find $s_n < u_n < s_{n+1}$ such that $f_{dr}(u_n,x_n) = 0$. It means that $\lim_{[0,t)\times\mathbb{R}\ni(u,y)\to(t,x)} f_{dr}(u,y)$ does not exist. Gathering together the property (1) of the function f and the property (1) of Theorem 2.1(b) of the function f_{cr} we obtain that $\lim_{[0,t)\times\mathbb{R}\ni(u,y)\to(t,x)}\big(f(u,y) - f_{cr}(u,y)\big)$ exists. We have arrived at a contradiction. □

(c) $\Longrightarrow$ (a).

Claim 4. *The operator $\Psi_{f_{cr}}$ maps the space $R(0,1)$ into itself.*

Proof of Claim 4. Let $\varphi \in R(0,1)$. Then there exists $\psi \in D(0,1)$ such that φ and ψ have the same right-hand limit at each point of $[0,1)$ and they have the same left-hand limit at each point of $(0,1]$. Suppose that there exist a strictly monotonic sequence $(t_n) \subset [0,1]$ and $\varepsilon > 0$ such that

$$|f_{cr}(t_n,\varphi(t_n)) - f_{cr}(t_n,\psi(t_n))| > \varepsilon.$$

But we have the equality $\lim_{n\to\infty} \varphi(t_n) = \lim_{n\to\infty} \psi(t_n)$. This contradicts Theorem 2.1(b). Thus we have shown that the functions $\Psi_{f_{cr}}(\varphi)$ and $\Psi_{f_{cr}}(\psi)$ have the same left-hand as well as the right-hand limit at every point of $[0,1]$ where they exist. Since $\Psi_{f_{cr}}(\psi)$ is a regular function, so it is $\Psi_{f_{cr}}(\varphi)$. □

For every bounded function $\varphi : [0,1] \to \mathbb{R}$ and $\varepsilon > 0$ the set $\{t \in [0,1] : |\Psi_{f_{dr}}(\varphi)(t)| \geqslant \varepsilon\}$ is finite. Consequently $\Psi_{f_{dr}}(\varphi)$ is a member of $R(0,1)$.

The uniqueness of the pair f_{cr} and f_{dr} is obvious. □

Corollary 3.2. *If for a function $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $R(0,1)$ into itself, then for every bounded sequence $(x_n) \subset \mathbb{R}$ and every sequence $([s_n,t_n))$ of pairwise disjoint intervals contained in $[0,1]$ we have*

$$\lim_{n\to\infty}\big(f(t_n,x_n) - f(s_n,x_n)\big) = 0.$$

Proof. By Corollary 2.2(d) we have $\lim_{n\to\infty}\big(f_{cr}(t_n,x_n) - f_{cr}(s_n,x_n)\big) = 0$. It follows from Theorem 3.1(c) that $\lim_{n\to\infty}\big(f_{dr}(t_n,x_n) - f_{dr}(s_n,x_n)\big) = 0$. □

For a given set Ω and $A \subset \Omega$ by $\chi_A : \Omega \to \mathbb{R}$ is denoted the characteristic function of the subset A, i.e. $\chi_A|A = 1$ and $\chi_A|_{\Omega\setminus A} = 0$. In constructions of functions we will apply the following convention: if $A \subset B$ are subsets of $[0,1]$ and $f : B \to \mathbb{R}$ is a function, then $f\chi_A$ denotes the function from $[0,1]$ to $\mathbb{R}$ such that $f\chi_A|_A = f|_A$ and $f\chi_A|_{[0,1]\setminus A} = 0$.

A function f satisfying Theorem 3.1 need not be locally bounded. According to Corollary 2.2 only the discrete part of f may not be locally bounded.

Example 3.3. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be given by the formula

$$f = \sum_{n=1}^{\infty} n\chi_{\{(0,\frac{1}{n})\}}.$$

Then for every $\varphi : [0,1] \to \mathbb{R}$ we have

$$\Psi_f(\varphi) = \Big(\sum_{n=1}^{\infty} n\chi_{\{\frac{1}{n}\}}(\varphi(0))\Big)\chi_{\{0\}},$$

where at most one of the summands is not zero. Therefore the operator Ψ_f maps the space $\mathbb{R}^{[0,1]}$ into $BV(0,1)$. It is easy to see that it maps the unit ball of $R(0,1)$ into an unbounded set in $R(0,1)$.

Our next purpose is the answer to the question when the operator $\Psi_f : R(0,1) \to R(0,1)$ is continuous.

Corollary 3.4. *If for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $D(0,1)$ into itself, then the operator $\Psi_f : R(0,1) \to R(0,1)$ is continuous.*

Proof. Claim 4 of Theorem 3.1 shows that the operator Ψ_f maps the space $R(0,1)$ into itself. Let (φ_n) be a sequence in $R(0,1)$ that converges to φ uniformly on $[0,1]$. Suppose that there exist a monotonic sequence $(t_n) \subset [0,1]$ and $\varepsilon > 0$ such that

$$|\tilde{f}(\varphi_n(t_n))(t_n) - \tilde{f}(\varphi(t_n))(t_n)| = |f(t_n, \varphi_n(t_n)) - f(t_n, \varphi(t_n))| > \varepsilon$$

for each n where $\tilde{f} : \mathbb{R} \to D(0,1)$ is the function defined in Theorem 2.1(c). Since the function φ is regular, we have the equality

$$\lim_{n\to\infty} \varphi_n(t_n) = \lim_{n\to\infty} \varphi(t_n).$$

This shows that the function $\tilde{f}$ is not uniformly continuous on the interval $[-\sup_n \|\varphi_n\|, \sup_n \|\varphi_n\|]$. But this contradicts Theorem 2.1(c). Thus we have shown that the sequence of functions $(\Psi_f(\varphi_n))$ converges uniformly to the function $\Psi_f(\varphi)$. □

Corollary 3.5. *For a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the following assertions are equivalent:*

(a) *the operator Ψ_f maps the space $R(0,1)$ into itself and it is continuous,*

(b) *there exists a pair of functions $f_{cr}, f_{dr} : [0,1] \times \mathbb{R} \to \mathbb{R}$ that fulfills the condition* (c) *of Theorem* 3.1 *and additionally it has the following property:*

  (3) *for every $t \in [0,1]$ the function $x \to f_{dr}(t,x)$ is continuous.*

Proof. Suppose first that Ψ_f is continuous. By Corollary 3.4 the operator $\Psi_{f_{dr}}$ is also continuous. Let $(x_n) \subset \mathbb{R}$ be a sequence that converges to x. Let $\varphi_n = x_n$ and $\varphi = x$. Then $\Psi_{f_{dr}}(\varphi_n)$ converges uniformly to $\Psi_{f_{dr}}(\varphi)$. Consequently the function f_{dr} has the property (3).

Suppose now that the function f_{dr} verifies the condition (3). Let (φ_n) be a sequence in $R(0,1)$ that converges to φ uniformly on $[0,1]$. Then there exists $a > 0$ such that $|\varphi_n| \leqslant a$ for every n. Let $\varepsilon > 0$. By Theorem 3.1 (c) the set $E = \{t \in [0,1] : \sup_{|x| \leqslant a} |f_{dr}(t,x)| > \varepsilon\}$ is finite. It is clear that

$$|\Psi_{f_{dr}}(\varphi_n)(t) - \Psi_{f_{dr}}(\varphi)(t)| \leqslant 2\varepsilon$$

for every $t \notin E$. It is clear that there exists N such that for every $n \geqslant N$ and $t \in E$

$$|f_{dr}(t, \varphi_n(t)) - f_{dr}(t, \varphi(t))| < \varepsilon.$$

Consequently for every $n \geqslant N$ we have

$$\sup_{t \in [0,1]} |\Psi_{f_{dr}}(\varphi_n)(t) - \Psi_{f_{dr}}(\varphi)(t)| \leqslant 2\varepsilon.$$

Thus we have shown that the operator $\Psi_{f_{dr}}$ is continuous. Gathering together this fact, Corollary 3.4 and Theorem 3.1(c) we obtain that the operator Ψ_f is continuous. □

We finish this section with an application of Theorem 3.1 to autonomous Niemytskii operators in the space $R(0,1)$ (see [3]).

Corollary 3.6. *For a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that $f(t,x) = f(0,x)$ for every $(t,x) \in [0,1] \times \mathbb{R}$ the following assertions are equivalent:*

(a) *the operator Ψ_f maps the space $R(0,1)$ into itself,*

(b) *f is a continuous function.*

Proof. (a) $\Longrightarrow$ (b). It follows from Theorem 3.1(b) that for every $x \in \mathbb{R}$ the limit $\lim_{y \to x} f(0,y)$ exists and is equal to $f(0,x)$.

The implication (b) $\Longrightarrow$ (a) follows from the fact that every continuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ verifies the condition (b) of Theorem 3.1. □

4. Nemytskii operators in the space $BV(0,1)$

For a function $f:[0,1]\to\mathbb{R}$ we define its variation $\mathrm{var}_{[a,b]}(f)$ on an interval $[a,b]\subset[0,1]$ in the usual way, i.e.

$$\mathrm{var}_{[a,b]}(f)=\sup\Big\{\sum_{k=0}^{n-1}|f(t_{k+1})-f(t_k)| : a\leqslant t_0<t_1<\cdots<t_n\leqslant b,\ n\in\mathbb{N}\Big\}.$$

Moreover for a function $f:[0,1]\to\mathbb{R}$ and an interval $[a,b]\subset[0,1]$ we define $\mathrm{var}_{[a,b)}(f)$ and $\mathrm{var}_{(a,b]}(f)$ in the following way:

$$\mathrm{var}_{[a,b)}(f)=\lim_{t\to b-}\mathrm{var}_{[a,t]}(f)\quad\text{and}\quad \mathrm{var}_{(a,b]}(f)=\lim_{t\to a+}\mathrm{var}_{[t,b]}(f).$$

We put $\mathrm{Var}(f)=|f(0)|+\mathrm{var}_{[0,1]}(f)$. The Banach space $BV(0,1)$ consists of all real functions f on $[0,1]$ such that $\mathrm{var}_{[0,1]}(f)<\infty$; it is equipped with the norm $\mathrm{Var}(f)$. The set $D(0,1)\cap BV(0,1)$ is a closed subspace of $BV(0,1)$.

The inspection of arguments used in the proofs of Theorem 2.1 and Theorem 3.1 give us the following result.

Proposition 4.1. (a) *If for a function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ the operator Ψ_f maps the space $D(0,1)\cap BV(0,1)$ into $D(0,1)$, then it maps $D(0,1)$ into itself.*

(b) *If for a function $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ the operator Ψ_f maps the space $BV(0,1)$ into $R(0,1)$, then it maps the space $R(0,1)$ into itself.*

Proof. (a). It is clear that Claims 1 and 2 in the proof of Theorem 2.1 remain valid in our case. Therefore the function f fulfills the condition (b) of Theorem 2.1.

(b). It is clear that Claim 1 in the proof of Theorem 3.1 remains valid in our case. Therefore the function f fulfills the condition (b) of Theorem 3.1. □

The part (a) of the following corollary is a straightforward consequence of the localized version of the Bugajewska result (see [1, Theorem 6.11]) and Theorem 2.1. The part (b) is obvious.

Corollary 4.2. (a) *If a function f: $[0,1]\times\mathbb{R}\to\mathbb{R}$ has the following properties:*

(1) *f fulfills the condition* (b) *of Theorem* 2.1*, and additionally*

(2) *for every $a>0$ we have*

$$\sup\Big\{\frac{|f(t,y)-f(t,x)|}{|y-x|} : t\in[0,1], x,y\in[-a,a], x\neq y\Big\}<\infty,$$

(3) *for every $a>0$ we have*

$$\sup\left\{\sum_{k=0}^{n-1}|f(t_{k+1},x_k)-f(t_k,x_k)|\ \middle|\ \begin{array}{l}0\leqslant t_0<\cdots<t_n\leqslant 1,\\ x_0,\ldots,x_{n-1}\in[-a,a],\ n\in\mathbb{N}\end{array}\right\}<\infty,$$

then the operator Ψ_f *maps the space* $D(0,1)\cap BV(0,1)$ *into itself.*

(b) *If* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is a function such that for every* $a>0$ *we have*

$$\sup\Big\{\sum_{t\in A}\sup_{|x|\leqslant a}|f(t,x)| : A\subset[0,1], A \text{ finite}\Big\}<\infty,$$

then the operator Ψ_f *maps the space* $BV(0,1)$ *into itself.*

Theorem 4.3. (a) *If a function* f: $[0,1]\times\mathbb{R}\to\mathbb{R}$ *has the following properties:*

(1) $\sup\{|f(t,x)| : (t,x)\in[0,1]\times[-a,a]\}<\infty$ *for every* $a>0$ *and*

(2) *the operator* Ψ_f *maps the space* $BV(0,1)$ *into itself,*

then the operator Ψ_f *maps bounded subsets of* $BV(0,1)$ *into bounded sets in* $BV(0,1)$.

(b) *If for a function* $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *the operator* Ψ_f *maps the space* $D(0,1)\cap BV(0,1)$ *into itself, then the operator* Ψ_f *maps bounded subsets of* $D(0,1)\cap BV(0,1)$ *into bounded sets in* $D(0,1)\cap BV(0,1)$.

Proof. (a). Suppose that for some $a>0$ there exists a sequence $(\varphi_k)\subset BV(0,1)$ such that $\mathrm{Var}(\varphi_k)\leqslant a$ for every k and $\sup_k \mathrm{Var}(\Psi_f(\varphi_k))=\infty$. Since f is bounded on $[0,1]\times[-a,a]$, we have $\sup_k \mathrm{var}_{[0,1]}(\Psi_f(\varphi_k))=\infty$. We put $s_1=0$ and $t_1=1$. If for every interval $[c,d]$ of length less then 2^{-1} we have $\sup_k \mathrm{var}_{[c,d]}(\Psi_f(\varphi_k))<\infty$, then also $\sup_k \mathrm{var}_{[0,1]}(\Psi_f(\varphi_k))<\infty$. Consequently there exists $s_1\leqslant s_2<t_2\leqslant t_1$ such that $\sup_k \mathrm{var}_{[s_2,t_2]}(\Psi_f(\varphi_k))=\infty$ and $t_2-s_2<2^{-1}$. Continuing the procedure we obtain an increasing sequence (s_n) and a decreasing sequence (t_n) such that $\sup_k \mathrm{var}_{[s_n,t_n]}(\Psi_f(\varphi_k))=\infty$ and $0<t_n-s_n<2^{-n}$ for each n. Let $u=\lim_n s_n=\lim_n t_n$.

It is clear that for each n

$$\sup_k \mathrm{var}_{[s_n,u]}(\Psi_f(\varphi_k))=\infty \quad \text{or} \quad \sup_k \mathrm{var}_{[u,t_n]}(\Psi_f(\varphi_k))=\infty.$$

Since the function f is bounded on $\{u\}\times[-a,a]$, without loss of generality $(*)$ we may assume that $\sup_k \mathrm{var}_{[s_n,u)}(\Psi_f(\varphi_k))=\infty$ for each n. Since $\lim_{n\to\infty}\mathrm{var}_{[s_n,u)}(\varphi)=0$ for every $\varphi\in BV(0,1)$, we are able to select increasing sequences $(n_k),(m_k)\subset\mathbb{N}$ such that

(1) $\mathrm{var}_{[s_{n_k},u)}\big(\Psi_f(\varphi_{m_k})\big)>2^k+1$ and

(2) $\mathrm{var}_{[s_{n_{k+1}},u)}\big(\Psi_f(\varphi_{m_k})\big)<1$.

It is clear that for every k there exist j_k and $s_{n_k}=u_{k,0}<\cdots<u_{k,j_k}<s_{n_{k+1}}$ such that

$$\sum_{j=0}^{j_k-1}|f(u_{k,j+1},\varphi_{m_k}(u_{k,j+1}))-f(u_{k,j},\varphi_{m_k}(u_{k,j}))|>2^k.$$

Let $\psi_k:[s_{n_k},s_{n_{k+1}})\to\mathbb{R}$ be an affine function on each segment $[u_{k,j},u_{k,j+1}]$ and $\psi_k(u_{k,j})=\varphi_{m_k}(u_{k,j})$ for every $0\leqslant j\leqslant j_k-1$ and $\psi_k(t)=\varphi_{m_k}(u_{k,j_k})$ for every $t \in [u_{k,j_k}, s_{n_{k+1}})$. It is clear that $\mathrm{var}_{[s_{n_k}, s_{n_{k+1}})}(\psi_k) \leqslant a$. Since the function $t \to \mathrm{var}_{[s_{n_k}, t]}(\psi_k)$ is continuous and increasing we find $s_{n_k} = v_{k,0} < v_{k,1} < \cdots < v_{k,2^k} = u_{k,j_k}$ such that

$$\mathrm{var}_{[v_{k,j}, v_{k,j+1}]}(\psi_k) \leqslant \frac{a}{2^k}$$

for every $0 \leqslant j \leqslant 2^k - 1$. Let $A_{k,l} = \{j : v_{k,l} \leqslant u_{k,j} \leqslant v_{k,l+1}\}$ for $0 \leqslant l \leqslant 2^k - 1$. By the Pigeonhole principle for each k there exists $0 \leqslant l_k \leqslant 2^k - 1$ such that

$$\begin{aligned} b_k &= |f(u_{k,\min A_{k,l_k}}, \psi_k(u_{k,\min A_{k,l_k}})) - f(v_{k,l_k}, \psi_k(v_{k,l_k}))| \\ &\quad + |f(v_{k,l_k+1}, \psi_k(v_{k,l_k+1})) - f(u_{k,\max A_{k,l_k}}, \psi_k(u_{k,\max A_{k,l_k}})| \\ &\quad + \sum_{j \in A_{l_k} \setminus \{\max A_{l_k}\}} |f(u_{k,j+1}, \psi_k(u_{k,j+1})) - f(u_{k,j}, \psi_k(u_{k,j}))| \\ &> 1 \end{aligned}$$

if $A_{k,l_k} \neq \emptyset$ (in the case $A_{l_k} = \{\max A_{l_k}\}$ the last sum is 0) and

$$b_k = |f(v_{k,l_k+1}, \psi_k(v_{k,l_k+1})) - f(v_{k,l_k}, \psi_k(v_{k,l_k}))| > 1$$

if $A_{k,l_k} = \emptyset$. It is clear that we are able to select an increasing sequence $(k_p) \subset \mathbb{N}$ such that

$$\sum_{p=1}^{\infty} |\psi_{k_{p+1}}(v_{k_{p+1}, l_{k_{p+1}}}) - \psi_{k_p}(v_{k_p, l_{k_p}})| \leqslant a.$$

Let $\eta_1 : [0, v_{k_1, l_{k_1}}] \to \mathbb{R}$ be given by the formula $\eta_1 = \psi_{k_1}(v_{k_1, l_{k_1}})$. For every $p \geqslant 2$, let $\eta_p : [v_{k_{p-1}, l_{k_{p-1}}+1}, v_{k_p, l_{k_p}}] \to \mathbb{R}$ be the affine function such that $\eta_p(v_{k_{p-1}, l_{k_{p-1}}+1}) = \psi_{k_{p-1}}(v_{k_{p-1}, l_{k_{p-1}}+1})$ and $\eta_p(v_{k_p, l_{k_p}}) = \psi_{k_p}(v_{k_p, l_{k_p}})$. Let

$$\psi = \lim_{p\to\infty} \psi_{k_p}(v_{k_p, l_{k_p}}) \chi_{[u,1]} + \sum_{p=1}^{\infty} \big(\eta_p \chi_{[v_{k_{p-1}, l_{k_{p-1}}+1}, v_{k_p, l_{k_p}})} + \psi_{k_p} \chi_{[v_{k_p, l_{k_p}}, v_{k_p, l_{k_p}+1})}\big)$$

where $v_{k_0, l_{k_0}+1} = 0$. It is clear that ψ is a continuous function on the interval $[s_{n_{k_p}}, s_{n_{k_{p+1}}}]$ for each $p \in \mathbb{N}$. Then

$$\begin{aligned} &\mathrm{Var}(\psi) \\ &= |\psi_{k_1}(v_{k_1, l_{k_1}})| + \sum_{p=1}^{\infty} \big(\mathrm{var}_{[v_{k_p, l_{k_p}}, v_{k_p, l_{k_p}+1}]}(\psi_{k_p}) + |\psi_{k_{p+1}}(v_{k_{p+1}, l_{k_{p+1}}}) - \psi_{k_p}(v_{k_p, l_{k_p}+1})|\big) \\ &\leqslant a + a + a + \sum_{p=1}^{\infty} |\psi_{k_p}(v_{k_p, l_{k_p}+1}) - \psi_{k_p}(v_{k_p, l_{k_p}})| \\ &\leqslant 3a + a \\ &= 4a \end{aligned}$$

and $\mathrm{Var}(\Psi_f(\psi)) \geqslant \sum_{p=1}^{\infty} b_{k_p} = \infty$. This contradicts our assumptions.

(b). Suppose that Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself. By Proposition 4.1 and Corollary 2.2 the function f satisfies the condition (1) of part (a). Suppose that for some $a > 0$ there exists a sequence $(\varphi_k) \subset D(0,1) \cap BV(0,1)$ such that $\mathrm{Var}(\varphi_k) \leqslant a$ for every k and $\sup_k \mathrm{Var}(\Psi_f(\varphi_k)) = \infty$. It is clear that we may repeat the consideration above for the operator Ψ_f and the sequence (φ_k) to construct a function ψ that possesses all properties above. It remains to show that ψ is a continuous function (if we considered in $(*)$ two cases, then it would be enough to show that ψ is a member of $D(0,1)$). It is clear that ψ is continuous at each point of the set $[0,1] \setminus \{u\}$ and it is right continuous at u if $u \neq 1$. Moreover ψ is a member of $BV(0,1)$ and $\psi(u) = \lim_{p\to\infty} \psi(v_{k_p, l_{k_p}})$. Since every function in $BV(0,1)$ has a left-hand limit at each point $(0,1]$, we have the equality $\psi(u) = \lim_{t\to u-} \psi(t)$. □

Corollary 4.4. *If for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself, then for every $a > 0$*

$$\sup_{|x|\leqslant a} \mathrm{Var}(f(\cdot, x)) < \infty.$$

Theorem 4.5. *If for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself, then for every $a > 0$*

$$\sup\Big\{ \frac{|f(t,y) - f(t,x)|}{|y - x|} : t \in [0,1], x, y \in [-a,a], x \neq y \Big\} < \infty.$$

Proof. Suppose that for some $a > 0$ the supremum is infinite. Then there exist sequences $(t_n) \subset [0,1]$ and $(x_n), (y_n) \subset [-a,a]$ such that

$$|f(t_n, y_n) - f(t_n, x_n)| > n|y_n - x_n|$$

By Proposition 4.1 and Corollary 2.2 the function f is bounded on $[0,1] \times [-a,a]$. Therefore $\lim_{n\to\infty} |x_n - y_n| = 0$. It is clear that we are able to select a sequence $(n_k) \subset \mathbb{N}$ such that the sequence (t_{n_k}) is monotonic and the series $\sum_{k=1}^{\infty} |y_{n_k} - x_{n_k}|$ and $\sum_{k=1}^{\infty} |y_{n_{k+1}} - y_{n_k}|$ are convergent and

$$|f(t_{n_k}, y_{n_k}) - f(t_{n_k}, x_{n_k})| > 2^{k+1}|y_{n_k} - x_{n_k}|$$

for each k. It is clear that there exists a sequence $(j_n) \subset \mathbb{N}$ such that the series $\sum_{k=1}^{\infty} j_k|y_{n_k} - x_{n_k}|$ is convergent but the series $\sum_{k=1}^{\infty} j_k 2^k |y_{n_k} - x_{n_k}|$ is divergent.

It is enough to consider the following two cases:

(1) the sequence (t_{n_k}) is strictly monotonic,
(2) the sequence (t_{n_k}) is constant.

Consider first the case (1). Without loss of generality we may assume that the sequence (t_{n_k}) is increasing. By Proposition 4.1 and Theorem 2.1(b) for every $k \in \mathbb{N}$ there exist $s_{k,1}, \ldots, s_{k,j_k}, u_{k,0}, \ldots, u_{k,j_k} \in [t_{n_k}, t_{n_{k+1}})$ such that $t_{n_k} = u_{k,0} < s_{k,1} < u_{k,1} < \cdots < s_{k,j_k} < u_{k,j_k} < t_{n_{k+1}}$ and

$$|f(u_{k,m}, y_{n_k}) - f(s_{k,m}, x_{n_k})| > 2^k |y_{n_k} - x_{n_k}|$$

for every $m = 1, \ldots, j_k$. Let $\psi_k : [t_{n_k}, t_{n_{k+1}}) \to \mathbb{R}$ be an affine function on each segment $[s_{k,j}, u_{k,j}]$ and on each segment $[u_{k,j-1}, s_{k,j}]$ such that $\psi_k(s_{k,j}) = x_{n_k}$ for every $1 \leqslant j \leqslant j_k$ and $\psi_k(u_{k,j}) = y_{n_k}$ for every $0 \leqslant j \leqslant j_k$ and $\psi_k(t) = y_{n_k}$ for every $t \in [u_{k,j_k}, t_{n_{k+1}})$. Let

$$\varphi = y_{n_1} \chi_{[0,t_{n_1})} + \lim_{k\to\infty} y_{n_k} \chi_{[\lim_{k\to\infty} t_{n_k}, 1]} + \sum_{k=1}^{\infty} \psi_k \chi_{[t_{n_k}, t_{n_{k+1}})}.$$

Then $\mathrm{Var}(\varphi) = |y_{n_1}| + \sum_{k=1}^{\infty} \big(|y_{n_{k+1}} - y_{n_k}| + 2j_k |y_{n_k} - x_{n_k}|\big) < \infty$ and

$$\mathrm{Var}(\Psi_f(\varphi)) \geqslant \sum_{k=1}^{\infty} \sum_{m=1}^{j_k} |f(u_{k,m}, y_{n_k}) - f(s_{k,m}, x_{n_k})| \geqslant \sum_{k=1}^{\infty} j_k 2^k |y_{n_k} - x_{n_k}| = \infty.$$

This contradicts our assumptions.

Suppose now that the sequence (t_{n_k}) is constant. If $t_{n_1} < 1$, then by Theorem 2.1(b) there exists a strictly decreasing sequence (s_k) converging to t_{n_1} such that $|f(s_k, y_{n_k}) - f(s_k, x_{n_k})| > 2^{k+1} |y_{n_k} - x_{n_k}|$. If $t_{n_1} = 1$, then by Theorem 2.1(b) there exists a strictly increasing sequence (s_k) converging to t_{n_1} such that $|f(s_k, y_{n_k}) - f(s_k, x_{n_k})| > 2^{k+1} |y_{n_k} - x_{n_k}|$. It is clear that the above consideration remains valid also in these cases. □

Theorem 4.6. *If for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself, then the operator Ψ_f maps the space $BV(0,1)$ into itself.*

Proof. Suppose that $\varphi \in BV(0,1)$. Then there exists $\psi \in D(0,1) \cap BV(0,1)$ such that φ and ψ have the same right-hand limit at each point of $[0,1)$ and they have the same left-hand limit at each point of $(0,1]$. It is clear that $\mathrm{var}_{[0,1]}(\psi) \leqslant \mathrm{var}_{[0,1]}(\varphi)$ and the set $A = \{t \in [0,1] : |(\varphi - \psi)(t)| > 0\}$ is countable. Moreover the following inequalities

$$\mathrm{var}_{[0,1]}(\varphi - \psi) \leqslant 2 \sum_{t \in A} |(\varphi - \psi)(t)| \leqslant 2\mathrm{var}_{[0,1]}(\varphi - \psi) \leqslant 4\mathrm{var}_{[0,1]}(\varphi)$$

hold. The last inequality follows from the fact that

$$\sup\Big\{|\eta(1) - \lim_{s\to 1-} \eta(s)| + \sum_{t\in B} |\eta(t) - \lim_{s\to t+} \eta(s)| : B \subset [0,1), B \text{ finite}\Big\} \leqslant \mathrm{var}_{[0,1]}(\eta)$$

for every $\eta \in BV(0,1)$. According to Theorem 4.5 there exists $L > 0$ such that $|f(t,x) - f(t,y)| \leqslant L|y-x|$ for every $t \in [0,1]$ and $x, y \in [-\mathrm{Var}(\varphi), \mathrm{Var}(\varphi)]$. For every $0 \leqslant t_0 < t_1 < \cdots < t_n \leqslant 1$ we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} \big|\Psi_f(\varphi)(t_{j+1}) - \Psi_f(\varphi)(t_j)\big| \\
&= \sum_{j=0}^{n-1} \big|f(t_{j+1}, \varphi(t_{j+1})) - f(t_j, \varphi(t_j))\big| \\
&\leqslant \sum_{j=0}^{n-1} \Big(\big|f(t_{j+1}, \psi(t_{j+1})) - f(t_j, \psi(t_j))\big| + L|\psi(t_{j+1}) - \varphi(t_{j+1})| + L|\psi(t_j) - \varphi(t_j)|\Big) \\
&\leqslant \mathrm{var}_{[0,1]}(\Psi_f(\psi)) + 4L\,\mathrm{var}_{[0,1]}(\varphi).
\end{aligned}$$

This shows that $\Psi_f(\varphi)$ is a member of $BV(0,1)$. □

Now we are ready to show the decomposition theorem.

Theorem 4.7. *For a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the following assertions are equivalent:*

(a) *the operator Ψ_f maps the space $BV(0,1)$ into itself,*

(b) *there exists a unique pair of functions $f_{cr}, f_{dr} : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that*

  (1) *$f = f_{cr} + f_{dr}$,*

  (2) *the operator $\Psi_{f_{dr}}$ maps the space $BV(0,1)$ into itself and for every $a > 0$ and every $\varepsilon > 0$ the set*

$$\{t \in [0,1] : \sup_{|x| \leqslant a} |f_{dr}(t,x)| > \varepsilon\} \quad \textit{is finite,}$$

  (3) *the operator $\Psi_{f_{cr}}$ maps the space $D(0,1) \cap BV(0,1)$ into itself.*

Proof. (a) $\Longrightarrow$ (b). By Proposition 4.1 and Theorem 3.1 the pair of functions f_{cr} and f_{dr} satisfies the condition (1) and the second part of the condition (2). Moreover the operator $\Psi_{f_{cr}}$ maps the space $D(0,1)$ into itself and the operator $\Psi_{f_{dr}}$ maps the space $R(0,1)$ into its subset consisting of functions with countable support.

Suppose that $\varphi \in D(0,1) \cap BV(0,1)$. Then $\Psi_f(\varphi)$ is a member of $BV(0,1)$ and $\Psi_{f_{cr}}(\varphi)$ is a member of $D(0,1)$. By the second part of the property (2) the functions $\Psi_f(\varphi)$ and $\Psi_{f_{cr}}(\varphi)$ have the same right-hand limit at each point of $[0,1)$ and they have the same left-hand limit at each point of $(0,1]$. Since the function $\Psi_{f_{cr}}(\varphi)$ is right continuous on $[0,1]$, we have the inequality $\mathrm{var}_{[0,1]}\Psi_f(\varphi) \geqslant \mathrm{var}_{[0,1]}\Psi_{f_{cr}}(\varphi)$. Therefore $\Psi_{f_{cr}}(\varphi)$ is a members of $D(0,1) \cap BV(0,1)$. Thus we have shown that $\Psi_{f_{cr}}$ maps the space $D(0,1) \cap BV(0,1)$ into itself. In view of Theorem 4.6 the operator $\Psi_{f_{cr}}$ maps the space $BV(0,1)$

into itself. Consequently also the operator $\Psi_{f_{dr}}$ maps the space $BV(0,1)$ into itself.

The implication (b) $\Longrightarrow$ (a) immediately follows from Theorem 4.6. □

Theorem 4.8. *If for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $BV(0,1)$ into itself, then*

(a) *for every bounded sequence $(x_n) \subset \mathbb{R}$ and every sequence $([s_n, t_n))$ of pairwise disjoint intervals contained in the interval $[0,1]$ such that the sequence $(|f(t_n, x_n) - f(s_n, x_n)|)$ is decreasing we have*

$$\limsup_{n\to\infty} n^{\frac{1}{2}} |f(t_n, x_n) - f(s_n, x_n)| < \infty,$$

(b) *for every bounded sequence $(x_n) \subset \mathbb{R}$ and every sequence (t_n) of distinct points of $[0,1]$ such that the sequence $(|f_{dr}(t_n, x_n)|)$ is decreasing we have*

$$\limsup_{n\to\infty} n^{\frac{1}{2}} |f_{dr}(t_n, x_n)| < \infty,$$

(c) *For every $\alpha < -\frac{1}{2}$ there exists a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and a sequence $(s_n) \subset [0,1]$ such that*

  (1) *the operator Ψ_f maps the space $BV(0,1)$ into itself,*

  (2) $\sup\{|f(s_n, x)| : x \in \mathbb{R}\} = n^\alpha$ *and* $f(t,x) = 0$ *for every* $(t,x) \notin \{s_k : k \in \mathbb{N}\} \times \mathbb{R}$.

(d) *For every $\alpha < -\frac{1}{2}$ there exists a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that*

  (1) *the operator Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself,*

  (2) *there exist a sequence $(x_n) \subset [0,1]$ and a sequence $([s_n, t_n))$ of pairwise disjoint intervals contained in $[0,1]$ such that*

$$|f(t_n, x_n) - f(s_n, x_n)| = n^\alpha .$$

Proof. (a). Let (x_n) be a bounded sequence in $\mathbb{R}$. Let $([s_n, t_n))$ be a sequence of pairwise disjoint intervals contained in $[0,1]$ such that the sequence $(|f(t_n, x_n) - f(s_n, x_n)|)$ is decreasing. Let $a > 0$ be such that $(x_n) \subset [-a, a]$. Let $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ be given by the formula

$$h = \sum_{n=1}^{\infty} f_{dr} \chi_{\{t\in[0,1]:\sup_{|x|\leqslant n} |f_{dr}(t,x)|>n\}\times\{x\in\mathbb{R}:n-1\leqslant|x|<n\}} .$$

It is clear that for every bounded function $\varphi : [0,1] \to \mathbb{R}$ the support of the function $\Psi_h(\varphi)$ is a finite set. Consequently the operator Ψ_h maps the space $BV(0,1)$ into itself. By Theorem 4.7 the operator $\Psi_{f_{dr}-h}$ maps the space $BV(0,1)$ into itself. The function $f_{dr} - h$ is bounded on the product $[0,1] \times [-b, b]$ for each $b > 0$. By Theorem 4.3 the operator $\Psi_{f_{dr}-h}$ maps bounded subsets of

$BV(0,1)$ into bounded sets. According to Theorem 4.7(b) and the definition of the function h the sequence $(h(t_n,x_n)-h(s_n,x_n))$ contains only finite nonzero elements. Let $C=\sum_{n=1}^{\infty}|h(t_n,x_n)-h(s_n,x_n)|$.

By the Erdös-Szekeres theorem (see [6]) any sequence of $(n-1)^2+1$ elements of the interval $[-a,a]$ contains a monotonic subsequence of length n. Consequently any sequence of $(n-1)^2+k$ elements of the interval $[-a,a]$, where $1\leqslant k\leqslant 2n-1$, may be covered by $2n$ monotonic subsequences. Therefore the set of $(n-1)^2+k$ points, where $1\leqslant k\leqslant 2n-1$, of the product $[0,1]\times[-a,a]$ may be covered by graphs of $2n$ monotonic functions. By the Pigeonhole principle for every n and $1\leqslant k\leqslant 2n-1$ there exist monotonic functions $\varphi_{1,n,k},\varphi_{2,n,k}:[0,1]\to[-a,a]$ such that

$$\operatorname{var}_{[0,1]}(\Psi_{f_{dr}-h}(\varphi_{1,n,k}))\geqslant\frac{\sum_{j=1}^{(n-1)^2+k}|(f_{dr}-h)(t_j,x_j)|}{2n}$$

and

$$\operatorname{var}_{[0,1]}(\Psi_{f_{dr}-h}(\varphi_{2,n,k}))\geqslant\frac{\sum_{j=1}^{(n-1)^2+k}|(f_{dr}-h)(s_j,x_j)|}{2n}.$$

Hence

$$\begin{aligned}&\operatorname{var}_{[0,1]}(\Psi_{f_{dr}-h}(\varphi_{1,n,k}))+\operatorname{var}_{[0,1]}(\Psi_{f_{dr}-h}(\varphi_{2,n,k}))\\&\geqslant\frac{\sum_{j=1}^{(n-1)^2+k}|(f_{dr}-h)(t_j,x_j)-(f_{dr}-h)(s_j,x_j)|}{2n}.\end{aligned}$$

By the Pigeonhole principle for every n and $1\leqslant k\leqslant 2n-1$ there exists a monotonic function $\varphi_{3,n,k}:[0,1]\to[-a,a]$ which is constant on each interval $(s_j,t_j]$ for $1\leqslant j\leqslant(n-1)^2+k$ such that

$$\begin{aligned}\operatorname{var}_{[0,1]}(\Psi_{f_{cr}}(\varphi_{3,n,k}))&\geqslant\frac{\sum_{j=1}^{(n-1)^2+k}|f_{cr}(t_j,x_j)-\lim_{s\to s_j+}f_{cr}(s,x_j)|}{2n}\\&=\frac{\sum_{j=1}^{(n-1)^2+k}|f_{cr}(t_j,x_j)-f_{cr}(s_j,x_j)|}{2n}.\end{aligned}$$

Therefore

$$\begin{aligned}&\operatorname{var}_{[0,1]}(\Psi_{f_{dr}-h}(\varphi_{1,n,k}))+\operatorname{var}_{[0,1]}(\Psi_{f_{dr}-h}(\varphi_{2,n,k}))+\operatorname{var}_{[0,1]}(\Psi_{f_{cr}}(\varphi_{3,n,k}))\\&\geqslant\frac{\sum_{j=1}^{(n-1)^2+k}|(f-h)(t_j,x_j)-(f-h)(s_j,x_j)|}{2n}\\&\geqslant\frac{((n-1)^2+k)|f(t_{(n-1)^2+k},x_{(n-1)^2+k})-f(s_{(n-1)^2+k},x_{(n-1)^2+k})|}{2n}-\frac{C}{2n}.\end{aligned}$$

It is clear that the following inequalities $\operatorname{Var}(\varphi_{1,n,k})\leqslant 3a$, $\operatorname{Var}(\varphi_{2,n,k})\leqslant 3a$ and $\operatorname{Var}(\varphi_{3,n,k})\leqslant 3a$ hold. According to Theorem 4.3 the sum above is bounded by a constant independent from n and k.

(b). The consideration above shows also part (b).

(c). For every n, let $1-\frac{1}{n}\leqslant t_{n,1}<\cdots<t_{n,4^n}<1-\frac{1}{n+1}$. We put

$$f=\sum_{n=1}^{\infty}\sum_{k=1}^{4^n}(k+\sum_{j=1}^{n-1}4^j)^{\alpha}\chi_{\{(t_{n,k},\frac{k-[\frac{k}{2^n}]2^n}{2^n})\}}$$

where $[x]$ denotes the integer part of x and for $n=1$ the sum $\sum_{j=1}^{n-1}4^j$ is 0. For every $\varphi\in BV(0,1)$ and every $t\in[0,1]\setminus\bigcup_{n=1}^{\infty}\{t_{n,k}:1\leqslant k\leqslant 4^n\}$ we have $\Psi_f(\varphi)(t)=0$. Hence

$$\begin{aligned}
\operatorname{var}_{[1-\frac{1}{n},1-\frac{1}{n+1})}(\Psi_f(\varphi)) &\leqslant 2(4^{n-1})^{\alpha}\sum_{j=0}^{2^n-1}\sum_{k=1}^{2^n}\chi_{\{\frac{k}{2^n}-[\frac{k}{2^n}]\}}(\varphi(t_{n,j2^n+k}))\\
&\leqslant 2^{n+1}(4^{n-1})^{\alpha}\sum_{j=0}^{2^n-1}\operatorname{var}_{[t_{n,j2^n+1},t_{n,(j+1)2^n}]}(\varphi)\\
&\leqslant 2^{1+n(1+2\alpha)-2\alpha}\operatorname{var}_{[1-\frac{1}{n},1-\frac{1}{n+1})}(\varphi).
\end{aligned}$$

Therefore $\operatorname{var}_{[0,1]}(\Psi_f(\varphi))\leqslant\operatorname{var}_{[0,1]}(\varphi)\sum_{n=1}^{\infty}2^{1+n(1+2\alpha)-2\alpha}<\infty$. For every n we put

$$s_n=t_{k,n-\sum_{j=1}^{k-1}4^j}$$

where k is such that $\sum_{j=1}^{k-1}4^j<n\leqslant\sum_{j=1}^{k}4^j$. Then

$$\sup\{|f(s_n,x)|:x\in\mathbb{R}\}=\left|f\left(s_n,\frac{n-\sum_{j=1}^{k-1}4^j-[\frac{n-\sum_{j=1}^{k-1}4^j}{2^n}]2^n}{2^n}\right)\right|=n^{\alpha}.$$

(d). For every n and $1\leqslant k\leqslant 4^n$, let

$$\begin{aligned}
A_{n,k}&=[\tfrac{k-[\frac{k}{2^n}]2^n}{2^n}-\tfrac{1}{2^{n+2}},\tfrac{k-[\frac{k}{2^n}]2^n}{2^n}+\tfrac{1}{2^{n+2}}], & B_{n,k}&=[\tfrac{1}{n+1}+\tfrac{2k-1}{4^n6n^2},\tfrac{1}{n+1}+\tfrac{2k+1}{4^n6n^2}],\\
A_{n,k-}&=[\tfrac{k-[\frac{k}{2^n}]2^n}{2^n}-\tfrac{1}{2^{n+2}},\tfrac{k-[\frac{k}{2^n}]2^n}{2^n}], & B_{n,k-}&=[\tfrac{1}{n+1}+\tfrac{2k-1}{4^n6n^2},\tfrac{1}{n+1}+\tfrac{k}{4^n3n^2}],\\
A_{n,k+}&=[\tfrac{k-[\frac{k}{2^n}]2^n}{2^n},\tfrac{k-[\frac{k}{2^n}]2^n}{2^n}+\tfrac{1}{2^{n+2}}], & B_{n,k+}&=[\tfrac{1}{n+1}+\tfrac{k}{4^n3n^2},\tfrac{1}{n+1}+\tfrac{2k+1}{4^n6n^2}],
\end{aligned}$$

and

$$g_{n,k}(x)=\begin{cases}2^{n+2}\big(x-\frac{k-[\frac{k}{2^n}]2^n}{2^n}+\frac{1}{2^{n+2}}\big) & \text{if } x\in A_{n,k-}\\ -2^{n+2}\big(x-\frac{k-[\frac{k}{2^n}]2^n}{2^n}-\frac{1}{2^{n+2}}\big) & \text{if } x\in A_{n,k+}\\ 0 & \text{if } x\notin A_{n,k},\end{cases}$$

$$h_{n,k}(t)=\begin{cases}4^n6n^2\big(t-\big(\frac{1}{n+1}+\frac{2k-1}{4^n6n^2}\big)\big) & \text{if } t\in B_{n,k-}\\ -4^n6n^2\big(t-\big(\frac{1}{n+1}+\frac{2k+1}{4^n6n^2}\big)\big) & \text{if } t\in B_{n,k+}\\ 0 & \text{if } t\notin B_{n,k}.\end{cases}$$

For any subsets A, B of $\mathbb{R}$, let $\xi_A(B) = \begin{cases} 1 & \text{if } A \cap B \neq \emptyset \\ 0 & \text{if } A \cap B = \emptyset. \end{cases}$ We put

$$f(t,x) = \sum_{n=1}^{\infty} \sum_{k=1}^{4^n} (k + \sum_{j=1}^{n-1} 4^j)^\alpha h_{n,k}(t) g_{n,k}(x)$$

for every $(t,x) \in [0,1] \times \mathbb{R}$. The family $\{B_{n,k} : 1 \leqslant k \leqslant 4^n, n \in \mathbb{N}\}$ consists of pairwise disjoint intervals. For every $\varphi \in BV(0,1)$ and for every

$$\tfrac{1}{n+1} + \tfrac{2k-1}{4^n 6n^2} \leqslant t_0 < \cdots < t_m \leqslant \tfrac{1}{n+1} + \tfrac{2k+1}{4^n 6n^2}$$

we have

$$\begin{aligned}
&\sum_{j=0}^{m-1} |f(t_{j+1}, \varphi(t_{j+1})) - f(t_j, \varphi(t_j))| \\
&\leqslant \left(k + \sum_{j=1}^{n-1} 4^j\right)^\alpha \xi_{A_{n,k}}(\varphi(B_{n,k})) \sum_{j=0}^{m-1} \big(|h_{n,k}(t_{j+1}) - h_{n,k}(t_j)| g_{n,k}(\varphi(t_{j+1})) \\
&\quad + h_{n,k}(t_j) |g_{n,k}(\varphi(t_{j+1})) - g_{n,k}(\varphi(t_j))|\big) \\
&\leqslant 4^{(n-1)\alpha} \xi_{A_{n,k}}(\varphi(B_{n,k})) \sum_{j=0}^{m-1} \big(|h_{n,k}(t_{j+1}) - h_{n,k}(t_j)| + |g_{n,k}(\varphi(t_{j+1})) - g_{n,k}(\varphi(t_j))|\big) \\
&\leqslant 4^{(n-1)\alpha} \xi_{A_{n,k}}(\varphi(B_{n,k})) \big(2 + 2^{n+2} \mathrm{var}_{B_{n,k}}(\varphi)\big)
\end{aligned}$$

where the last inequality follows from the fact that the Lipschitz constant of $g_{n,k}$ is equal to 2^{n+2}. Thus we have shown that

$$\mathrm{var}_{B_{n,k}}(\Psi_f(\varphi)) \leqslant 4^{(n-1)\alpha} \xi_{A_{n,k}}(\varphi(B_{n,k})) \big(2 + 2^{n+2} \mathrm{var}_{B_{n,k}}(\varphi)\big)$$

for each n and $1 \leqslant k \leqslant 4^n$. Since $f(\frac{1}{n+1} + \frac{2k-1}{4^n 6n^2}, x) = f(\frac{1}{n+1} + \frac{2k+1}{4^n 6n^2}, x) = 0$ for every $1 \leqslant k \leqslant 4^n$, we have

$$\begin{aligned}
&\mathrm{var}_{[\frac{1}{n+1}, \frac{1}{n})}(\Psi_f(\varphi)) \\
&\leqslant \sum_{k=0}^{2^n-1} \sum_{j=1}^{2^n} \mathrm{var}_{B_{n,j+2^n k}}(\Psi_f(\varphi)) \\
&\leqslant \sum_{k=0}^{2^n-1} 4^{(n-1)\alpha} \Big(\sum_{j=1}^{2^n} 2\xi_{A_{n,j+2^n k}}(\varphi(B_{n,j+2^n k})) + \sum_{j=1}^{2^n} 2^{n+2} \mathrm{var}_{B_{n,j+2^n k}}(\varphi)\Big) \\
&\leqslant \sum_{k=0}^{2^n-1} 4^{(n-1)\alpha} \big(\mathrm{var}_{[\frac{1}{n+1} + \frac{1+2^n k}{4^n 3n^2} - \frac{1}{4^n 6n^2}, \frac{1}{n+1} + \frac{2^n(k+1)}{4^n 3n^2} + \frac{1}{4^n 6n^2}]}(\varphi)(2^{n+2} + 2^{n+2}\big) \\
&\leqslant 2^{3+n(1+2\alpha)-2\alpha} \mathrm{var}_{[\frac{1}{n+1}, \frac{1}{n})}(\varphi).
\end{aligned}$$

Therefore

$$\mathrm{var}_{[0,1]}(\Psi_f(\varphi)) \leqslant \mathrm{var}_{[0,1]}(\varphi) \sum_{n=1}^{\infty} 2^{3+n(1+2\alpha)-2\alpha} < \infty.$$

This shows that the operator Ψ_f maps the space $BV(0,1)$ into itself. It is easy to see that f is a continuous function. By Theorem 2.1 the operator Ψ_f maps the space $D(0,1)$ into itself. For every n we put

$$s_n = \frac{1}{k+1} + \frac{n - \sum_{j=1}^{k-1} 4^j}{4^k 3k^2}, \qquad t_n = \frac{1}{k+1} + \frac{2n+1-2\sum_{j=1}^{k-1} 4^j}{4^k 6k^2},$$

$$x_n = \frac{\left(n - \sum_{j=1}^{k-1} 4^j\right) - \left[\frac{n-\sum_{j=1}^{k-1} 4^j}{2^n}\right] 2^n}{2^n}$$

where k is such that $\sum_{j=1}^{k-1} 4^j < n \leqslant \sum_{j=1}^{k} 4^j$. It is easy to check that $f(t_n, x_n) = 0$ and $f(s_n, x_n) = n^\alpha$ and the sequence $([s_n, t_n))$ consists of pairwise disjoint intervals. □

Remark 4.9. For every $-1 \leqslant \alpha < -\frac{1}{2}$ the function f constructed in the proof of Theorem 4.8(d) does not satisfy the assumptions of the Bugajewska theorem;

$$\sup\left\{ \sum_{k=0}^{n-1} |f(u_{k+1}, x_k) - f(u_k, x_k)| \;\middle|\; \begin{array}{l} 0 \leqslant u_0 < \cdots < u_n \leqslant 1, \\ x_0, \ldots, x_{n-1} \in [0,1],\ n \in \mathbb{N} \end{array} \right\}$$
$$\geqslant \sup\left\{ \sum_{k=1}^{n} |f(t_k, x_k) - f(s_k, x_k)| : n \in \mathbb{N} \right\} = \infty$$

where sequences (s_n) and (t_n) are defined in the proof above. In view of Theorem 4.5 the function f is locally Lipschitz in the second variable uniformly with respect to the first variable.

The above result raise the following question: *it is true that if for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $BV(0,1)$ into itself, then for every bounded sequence $(x_n) \subset \mathbb{R}$ and for every sequence $([s_n, t_n))$ of pairwise disjoint intervals contained in $[0,1]$ we have*

$$\sum_{n=1}^{\infty} \left| f(t_n, x_n) - f(s_n, x_n) \right|^2 < \infty.$$

Corollary 4.10. *If for a function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ the operator Ψ_f maps the space $D(0,1) \cap BV(0,1)$ into itself, then for every $a > 0$ and for every sequence (t_n) of distinct points of $[0,1]$ such that the sequence $(\sup\{d_f(t_n, x) : x \in [-a, a]\})$ is decreasing we have*

$$\lim_{n\to\infty} n^{\frac{1}{2}} \sup\{d_f(t_n, x) : x \in [-a, a]\} < \infty.$$

Proof. Let $\mathfrak{j}(x) = 1 - x$ and $g(t,x) = f(1-t,x)$. It is clear that the map $\varphi \to \varphi \circ \mathfrak{j}$ is an isomorphism of $BV(0,1)$. Since $\Psi_g(\varphi \circ \mathfrak{j})(t) = \Psi_f(\varphi)(\mathfrak{j}(t))$ the operator Ψ_g maps $BV(0,1)$ into itself. By Theorem 4.7 the operator $\Psi_{g_{dr}}$ maps the space $BV(0,1)$ into itself. Moreover by Theorem 3.1 and Theorem 2.1 we have

$$g_{dr}(t,x) = g(t,x) - \lim_{s\to t+} g(s,x) = f(1-t,x) - \lim_{s\to 1-t-} f(s,x) = d_f(1-t,x).$$

An appeal to Theorem 4.8(b) completes the proof. □

Our next example shows that the inverse theorem to Theorem 4.5 does not hold. This example shows also that the Ljamin theorem is false. The Ljamin theorem was regarded as a correct theorem for a long time (see [2]). Bugajewska noted in [4] that there is no correct proof of the theorem and suggested that it is false. Maćkowiak in [8] gave the first example that confirmed the suggestion. The Maćkowiak example is not good for us, the function f in this example is discrete (i.e. $f = f_{dr}$).

Example 4.11. For every n, let

$$g_n(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{n}] \\ \frac{2}{n} - x & \text{if } x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \notin [0, \frac{2}{n}], \end{cases}$$

$$h_n(t) = \begin{cases} 3n^2(t - (1 - \frac{1}{n} - \frac{1}{3n^2})) & \text{if } t \in [1 - \frac{1}{n} - \frac{1}{3n^2}, 1 - \frac{1}{n}] \\ -3n^2(t - (1 - \frac{1}{n} + \frac{1}{3n^2})) & \text{if } t \in [1 - \frac{1}{n}, 1 - \frac{1}{n} + \frac{1}{3n^2}] \\ 0 & \text{if } t \notin [1 - \frac{1}{n} - \frac{1}{3n^2}, 1 - \frac{1}{n} + \frac{1}{3n^2}]. \end{cases}$$

We put

$$f(t,x) = \sum_{n=2}^{\infty} h_n(t) g_n(x).$$

for every $(t,x) \in [0,1] \times \mathbb{R}$. It is clear that f is a continuous function. In view of Theorem 2.1 the operator Ψ_f maps the space $D(0,1)$ into itself. Let $\varphi(t) = 1-t$ for every $t \in [0,1]$. It is clear that $\mathrm{Var}(\varphi) \leqslant 2$ and

$$f\big(1 - \tfrac{1}{n}, \varphi(1 - \tfrac{1}{n})\big) = \tfrac{1}{n} \quad \text{and} \quad f\big(1 - \tfrac{1}{n} - \tfrac{1}{3n^2}, \varphi(1 - \tfrac{1}{n} - \tfrac{1}{3n^2})\big) = 0.$$

Consequently $\Psi_f(\varphi)$ is not a member of $BV(0,1)$. For every $\frac{2}{n+1} < x \leqslant \frac{2}{n}$ we have $f(t,x) = \sum_{j=2}^{n} h_j(t) g_j(x)$ and

$$\mathrm{Var}(f(\cdot,x)) \leqslant \sum_{j=2}^{n} \mathrm{Var}(h_j) g_j(x) \leqslant \sum_{j=2}^{n} \mathrm{Var}(h_j) x = 2nx \leqslant 4.$$

It is clear that for every $x, y \in \mathbb{R}$ and $t \in [0,1]$ we have

$$|f(t,y) - f(t,x)| \leqslant \sum_{n=2}^{\infty} |h_n(t)||g_n(y) - g_n(x)| \leqslant \sum_{n=2}^{\infty} |h_n(t)||y - x| \leqslant |y - x|.$$

The last inequality follows from the fact that supports of functions h_n are pairwise disjoint.

Acknowledgement. I would like to thank Professor Mieczysław Cichoń for bringing the problems of Nemytskii operators in spaces of regular functions to my attention.

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Received October 29, 2014; revised October 5, 2015