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Stability for Semilinear Parabolic Problems in L_2 and $W^{1,2}$

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Abstract. Asymptotic stability is studied for semilinear parabolic problems in $L_2(\Omega)$ and interpolation spaces. Some known results about stability in $W^{1,2}(\Omega)$ are improved for semilinear parabolic systems with mixed boundary conditions. The approach is based on Amann's power extrapolation scales. In the Hilbert space setting, a better understanding of this approach is provided for operators satisfying Kato's square root problem.

Keywords. Asymptotic stability, existence, uniqueness, parabolic PDE, strongly accretive operator, sesquilinear form, fractional power, Kato's square root problem

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1. Introduction

Consider a semilinear parabolic system, for instance the reaction-diffusion system

$$\frac{\partial u}{\partial t} = D\Delta u + \widetilde{f}(u)$$

in a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, subject e.g. to Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Although our results will be more general, we consider for simplicity for the moment only this system, assuming that D is a positive definite matrix, and $\widetilde{f}\colon \mathbb{C}^n \to \mathbb{C}^n$ is differentiable at 0 with $\widetilde{f}(0) = 0$ and subject to certain growth conditions specified later on. In chemical reaction-diffusion systems the reaction term $\widetilde{f}$ is typically (termwise) a polynomial, the degree of the polynomial being the maximal number of molecules reacting simulteneously. Depending on the substances, some damping of the growth due

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to saturation may occur for large u. Thus, summarizing, the growth of $\widetilde{f}$ is often at most polynomial, and mathematical results apply to a richer class of systems if the growth condition admits a larger power.

One could expect that if $\widetilde{B} = \widetilde{f}'(0) \in \mathbb{C}^{n\times n}$ is such that all the eigenvalues λ of the problem

$$D\Delta u + \widetilde{B}u = \lambda u$$

with Neumann boundary conditions belong to the left half-plane, then 0 should be a stable equilibrium of the system in $H := L_2(\Omega)$ and $V := W^{1,2}(\Omega)$. However, this is not so obvious because the superposition operator $f(u)(x) = \widetilde{f}(u(x))$ is not differentiable in H unless it is affine (even if f is also dependent on x), see e.g. [15].

In fact, to the authors' knowledge no stability result of the above type is currently known in the space $H = L_2(\Omega)$. In the space $V = W^{1,2}(\Omega)$ the stability can be analysed only under the assumption $f\colon V \to H$, more precisely, by assuming in space dimension $d \geq 3$ a growth condition of type $|\widetilde{f}(u)| \leq C(1+|u|^p)$ with $p \leq \frac{d}{d-2}$ (see e.g. [11] or [23] for a particular example). Note that this growth condition is more restrictive than the subcritical growth condition $p < \frac{d+2}{d-2}$, which is usually only necessary to obtain results in the space V.

In this paper, by using spaces larger than H as an auxiliary tool, we obtain, for the first time, stability results in H. Furthermore, we prove stability results in V under the more natural subcritical growth condition $p < \frac{d+2}{d-2}$.

In some applications, the growth condition can be dropped by restricting admissible perturbations of the initial data to $W^{1,q}(\Omega)$ with $q > d$ (instead of V). Then $f\colon W^{1,q}(\Omega) \to L_q(\Omega)$ is automatically differentiable and the stability in $W^{1,q}(\Omega)$ follows from the classical results [11]. However, taking perturbations from $L_q(\Omega)$, even with arbitrarily large q, is not in the classical framework anymore because of the lack of differentiability of $f\colon L_q(\Omega) \to L_q(\Omega)$. Furthermore, these non-Hilbert spaces are not always natural. For example, models with obstacles (unilateral sources or sinks), e.g. described by variational inequalities, naturally give rise to Hilbert spaces V and H. In order to compare these models with the corresponding models without obstacles and to see e.g. that a change of stability is caused solely by obstacles, one needs to work in both models in the *same* space. The instability of equilibria in V for certain problems with obstacles was proved in [14, 23], and the instability in H is work in progress. The same equilibria for analogous problems without obstacles were shown to be stable in V only under the growth condition $p \leq \frac{d}{d-2}$ (see [23]), and, as mentioned above, we are not aware of any stability results in H at all. In Example 3.17 of the present paper, we obtain these missing stability results in H and relax the growth condition up to $p < \frac{d+2}{d-2}$ in case of V.

2. Abstract results

2.1. Stability in Banach spaces. Throughout this section, let H be a complex Banach space, and A be a densely defined operator in H which is sectorial in the sense of [11] and satisfies $\sigma(A) \cap (-\infty, 0] = \emptyset$.

In particular, A is of positive type in the sense of [5], and so one can define A^α for all $\alpha \in \mathbb{R}$ on its domain $D(A^\alpha)$ in a standard way. We equip $D(A^\alpha)$ with the norm $\|u\|_{H_\alpha} := |A^\alpha u|$. In case $\alpha \geq 0$, we define $H_\alpha := D(A^\alpha)$, and in case $\alpha < 0$, we define H_α as the completion of H with respect to the norm $\|\cdot\|_{H_\alpha}$.

Let us recall some properties (see [4] or [5, Chapter V]). The spaces H_α are densely embedded into each other. Either all of these embeddings is compact or none. The embeddings are compact if and only if A has a compact resolvent. The operator A induces by restriction (in case $\alpha > 0$) or graph completion (in case $\alpha < 0$) isomorphisms $A_\alpha \colon H_{\alpha+1} \to H_\alpha$

For $\beta > \alpha$, A_β is the H_β-realization of A_α, that is, $A_\beta = A_\alpha|_{D(A_\beta)}$ with $D(A_\beta) = A_\alpha^{-1}(H_\beta) = H_{\beta+1}$. All A_α are thus densely defined operators in H_α. They have the same spectrum as A and are sectorial in H_α (hence of positive type). In particular, $-A_\alpha$ generates an analytic semigroup in H_α. The corresponding semigroups correspond to each other by restriction or (unique) continuous extension, respectively.

We need to apply Amann's theory in different scales of spaces. The crucial observation for us is that there is a relation between these different scales. It follows from the previous remarks that all our hypotheses which we needed for that theory for (H, A) are also satisfied with the choice $(H_{-\gamma}, A_{-\gamma})$. Starting with this couple instead, we obtain by the above definitions a corresponding family of spaces $(H_{-\gamma})_\alpha$. For instance, we have $(H_{-\gamma})_0 = H_{-\gamma}$. The following lemma states that these spaces are related to our original spaces H_α.

Lemma 2.1. *If* $\alpha, \gamma \in \mathbb{R}$ *then* $H_\alpha = (H_{-\gamma})_{\alpha+\gamma}$.

Proof. Set $\beta := \alpha + \gamma$. In case $\beta \geq 0$, it follows from the theory in [5] that the operator $A_{-\gamma}^\beta$ is a norm-preserving isomorphism from H_α onto $H_{-\gamma}$. Hence, by the definition of $(H_{-\gamma})_\beta$, we obtain

$$u \in H_\alpha \iff A_{-\gamma}^\beta u \in H_{-\gamma} \iff u \in (H_{-\gamma})_\beta,$$

and the norm equality

$$\|u\|_{H_\alpha} = \|A_{-\gamma}^\beta u\|_{H_{-\gamma}} = \|u\|_{(H_{-\gamma})_\beta}.$$

In case $\beta \leq 0$, the operator A_α^β is a norm-preserving isomorphism from H_α onto $H_{-\gamma}$. Hence,

$$\|u\|_{H_\alpha} = \|A_\alpha^\beta u\|_{H_{-\gamma}} = \|A_{-\gamma}^\beta u\|_{H_{-\gamma}} = \|u\|_{(H_{-\gamma})_\beta}$$

for all $u \in H_{-\gamma}$. Since $H_{-\gamma}$ is densely embedded into H_α as well as into $(H_{-\gamma})_\beta$, the assertion follows. □

Remark 2.2. For Lemma 2.1, we used only that A is a densely defined operator of positive type in H.

For the rest of the section, we fix

$$\alpha \in [0,1), \quad \gamma \in [0, 1-\alpha). \tag{2.1}$$

Given a subset $U \subseteq \mathbb{R} \times H_\alpha$ and a function $f\colon U \to 2^{H_{-\gamma}}$, we consider the problem

$$u'(t) + Au(t) \in f(t, u(t)). \tag{2.2}$$

Definition 2.3. We call $u \in C([t_0, t_1), H_{-\gamma})$ a *γ-weak/mild solution* of (2.2) if there is some $f_0\colon (t_0, t_1) \to H_{-\gamma}$ with $f_0 \in L_1((t_0, \tau), H_{-\gamma})$ for every $\tau \in (t_0, t_1)$ such that the following holds for every $t \in (t_0, t_1)$: $(t, u(t)) \in U$; $f_0(t) \in f(t, u(t))$, and

(γ-*weak solution*) $u'(t) \in H_{-\gamma}$ exists in the sense of the norm of $H_{-\gamma}$, $u(t) \in D(A_{-\gamma})$, and $u'(t) + A_{-\gamma}u(t) = f_0(t)$.

(γ-*mild solution*)

$$u(t) = e^{-(t-t_0)A_{-\gamma}} u(t_0) + \int_{t_0}^{t} e^{-(t-s)A_{-\gamma}} f_0(s)\, ds. \tag{2.3}$$

Remark 2.4. Each γ-mild solution is a γ-weak solution.

We say that f satisfies a *right local Hölder-Lipschitz* condition if f is single-valued and for each $(t_0, u_0) \in U$ there is a (relative) neighborhood $U_0 \subseteq [t_0, \infty) \times H_\alpha$ of (t_0, u_0) with $U_0 \subseteq U$ such that there are constants $L < \infty$ and $\sigma > 0$ with

$$\|f(t,u) - f(s,v)\|_{H_{-\gamma}} \le L \cdot (|t-s|^\sigma + \|u - v\|_{H_\alpha}) \quad \text{for all } (t,u) \in U_0. \tag{2.4}$$

Similarly, we call f *left-locally bounded into $H_{-\gamma}$* if for each $t_1 > t_0$ and each bounded $M \subseteq H_\alpha$ there is some $\varepsilon > 0$ such that $f\left(U \cap ([t_1 - \varepsilon, t_1) \times M)\right)$ is bounded in $H_{-\gamma}$.

Definition 2.5. An element $u_0 \in H_{1-\gamma}$ is called a *γ-weak equilibrium* of (2.2) if $A_{-\gamma}u_0 \in f(t, u_0)$ for every $t > 0$.

Since the operators are extensions of each other, we have:

Remark 2.6. If $0 \le \widetilde{\gamma} \le \gamma$, then each $\widetilde{\gamma}$-weak equilibrium is a γ-weak equilibrium. Conversely, if u_0 is a γ-weak equilibrium with $A_{-\gamma}u_0 \in H_{-\widetilde{\gamma}}$, that is, if $u_0 \in H_{1-\widetilde{\gamma}}$, then u_0 is a $\widetilde{\gamma}$-weak equilibrium. Moreover, "0-weak equilibrium" means the same as "equilibrium". In particular, each equilibrium u_0 is a γ-weak equilibrium, and the converse holds if $A_{-\gamma}u_0 \in H_{-0} = H$, that is, if $u_0 \in H_1 = D(A)$.

We make the following hypothesis.

(B_γ) Let u_0 be a γ-weak equilibrium, $U_1 \subseteq H_\alpha$ an open neighborhood of u_0, and $[0,\infty) \times U_1 \subseteq U$. Assume that there is a bounded linear map $B\colon H_\alpha \to H_{-\gamma}$ such that the function $g(t,u) \coloneqq f(t, u_0 + u) - A_{-\gamma} u_0 - Bu$ satisfies

$$\lim_{\|u\|_{H_\alpha} \to 0} \frac{\sup\left\{\|v\|_{H_{-\gamma}} : v \in g\left((0,\infty) \times \{u\}\right)\right\}}{\|u\|_{H_\alpha}} = 0.$$

If f is single-valued, condition (B_γ) means $B = \frac{\partial f}{\partial u}(t, u_0)$ (in the Fréchet sense) uniformly for $t \in (0,\infty)$. In particular, if f is single-valued and independent of t, it is equivalent to $B = f'(u_0)$.

Theorem 2.7 (Asymptotic stability)**.** *Assume* (2.1). *Let hypothesis* (B_γ) *be satisfied. Suppose that*

$$0 < \omega < \min\{\operatorname{Re} \lambda : \lambda \in \sigma(A_{-\gamma} - B)\}. \tag{2.5}$$

Then there exist $M_1, M_2 > 0$ such that if $t_1 > t_0 \geq 0$ and $u \in C([t_0, t_1), H_\alpha)$ is a γ-mild solution of (2.2) *with $\|u(t_0) - u_0\|_{H_\alpha} \leq M_1$, then u satisfies the asymptotic stability estimate*

$$\|u(t) - u_0\|_{H_\alpha} \leq M_2 e^{-\omega(t-t_0)} \|u(t_0) - u_0\|_{H_\alpha} \quad \textit{for all } t \in [t_0, t_1). \tag{2.6}$$

If $f\colon U \to H_{-\gamma}$ satisfies a right local Hölder-Lipschitz condition in the sense (2.4)*, then additionally for every $t_0 \geq 0$ and every $u_1 \in H_\alpha$ with $\|u_1 - u_0\|_{H_\alpha} \leq M_1$ there is a unique γ-weak solution $u \in C([t_0,\infty), H_\alpha)$ with $u(t_0) = u_1$. This solution satisfies* (2.6) *with $t_1 = \infty$.*

Remark 2.8. Since $B\colon H_\alpha \to H_{-\gamma}$ is bounded, and $A_{-\gamma}$ is sectorial, it follows that $A_{-\gamma} - B$ is sectorial, see e.g. [7, Remark 3.2]. Hence, the minimum in (2.5) indeed exists.

Remark 2.9. In the case $\gamma = 0$, that is, if (A_γ, H_γ) is replaced by (A, H), the assertion of Theorem 2.7 is classical and almost mathematical folklore. A special case of this is proved in [11], and the general case is obtained by straightforward extensions of that proof.

Proof of Theorem 2.7. It is sufficient to apply the special case of Remark 2.9 with (A, H, α) replaced by $(A_{-\gamma}, H_{-\gamma}, \beta)$ with $\beta \coloneqq \alpha + \gamma$. Note that the semigroup generated by $A_{-\gamma}$ is indeed an extension of the semigroup generated by A. Moreover, by Lemma 2.1, the space $(H_{-\gamma})_\beta$ in the corresponding assertion obtained by Remark 2.9 is indeed the same as the space H_α in the assertion of Theorem 2.7. □

In the same way as in the above proof of Theorem 2.7, one can also obtain generalizations of most other classical assertions about (2.2) like instability, existence, uniqueness, or regularity results as in [11] or [19]: Most results carry over to the case when (A, H) is replaced by $(H_{-\gamma}, A_{-\gamma})$.

Theorem 2.7 is not yet very convenient for applications, because the spectrum $\sigma(A_{-\gamma} - B)$ corresponds to a very abstract operator. One way to deal with this problem is the following result which enables us to replace this spectrum by $\sigma(A - B)$ under an additional hypothesis.

Theorem 2.10. *Assume* (2.1)*. Let hypothesis* (B_γ) *be satisfied. Suppose that at least one of*

$$B(H_{1-\gamma}) \subseteq H \tag{2.7}$$

$$\text{or} \quad B(H_1) \subseteq H, \tag{2.8}$$

$$A_{-\gamma}u - Bu \in H \implies u \in H_1 \tag{2.9}$$

holds. Then $\sigma(A_{-\gamma} - B) = \sigma(A - B) \neq \mathbb{C}$*. In particular, the conclusion of Theorem* 2.7 *holds if*

$$0 < \omega < \min\{\operatorname{Re}\lambda : \lambda \in \sigma(A - B)\}.$$

Proof. We first note that (2.7) implies (2.8) and (2.9), because $A\colon H_1 \to H$ is the H-realization of $A_{-\gamma}\colon H_{1-\gamma} \to H_{-\gamma}$. Moreover, (2.8) and (2.9) are equivalent to the assertion that $C_H := A - B\colon H_1 \to H$ is the H-realization of $C := A_{-\gamma} - B\colon H_{1-\gamma} \to H_{-\gamma}$.

Putting $\beta := \alpha + \gamma \in [0, 1)$, we have by Lemma 2.1 that $D(A^{\beta}_{-\gamma}) = (H_{-\gamma})_\beta = H_\alpha$. From Remark 2.8, we obtain trivially $\sigma(A_{-\gamma} - B) \neq \mathbb{C}$.

Considering C as an operator in $H_{-\gamma}$ with domain $D(C) = H_{1-\gamma}$, we find in particular that there is $\mu > 0$ such that $\mu I + C$ has a bounded inverse R, and $R(H) \subseteq R(H_{-\gamma}) = D(C) \subseteq H$. Hence [5, Lemma V.1.1.1] implies that the spectra of C and of its H-realization C_H coincide. □

Another way to solve the calculation of the spectrum is to reduce it to the calculation of eigenvalues under some compactness assumptions.

We call $\lambda \in \mathbb{C}$ a *γ-weak eigenvalue of* $A - B$ if it is an eigenvalue of $A_{-\gamma} - B$. Analogously to Remark 2.6, we obtain:

Remark 2.11. If $0 \le \widetilde{\gamma} \le \gamma$ and λ is a $\widetilde{\gamma}$-weak eigenvalue of $A - B$, then λ is a γ-weak eigenvalue of $A - B$.

Conversely, if λ is a γ-weak eigenvalue of A–B with eigenvector $u \in H_{1-\gamma} \subseteq H$ (recall that $\gamma \le 1$) satisfying $Bu \in H_{-\widetilde{\gamma}}$ or $u \in H_{1-\widetilde{\gamma}}$, then λ is a $\widetilde{\gamma}$-weak eigenvalue of $A - B$ with eigenvector $u \in H_{1-\widetilde{\gamma}}$.

Moreover, "0-weak eigenvalue" means the same as "eigenvalue". In particular, each eigenvalue λ of $A - B$ is a γ-weak eigenvalue of $A - B$; conversely, if λ

is a γ-weak eigenvalue of $A - B$ with eigenvector $u \in H_{1-\gamma}$ satisfying $Bu \in H$ or $u \in H_1$, then λ is an eigenvalue of $A - B$ with eigenvector $u \in H_1$.

Remark 2.11 implies in particular:

Proposition 2.12. *If at least one of* (2.7) *or* (2.9) *holds, then λ is a γ-weak eigenvalue of $A-B$ with eigenspace E if and only if λ is an eigenvalue of $A-B$ with the same eigenspace E, and automatically $E \subseteq D(A) = H_1$.*

Theorem 2.13 (Asymptotic stability with eigenvalues)**.** *Assume that one of the embeddings $H_\beta \to H_\delta$ is compact for $\beta > \delta$, that is, A has a compact resolvent. Then the spectrum $\sigma(A_{-\gamma} - B)$ of Theorem* 2.7 *consists only of isolated γ-weak eigenvalues. In particular, if there is $\omega > 0$ such that every γ-weak eigenvalue λ of $A - B$ satisfies* $\operatorname{Re}\lambda > \omega$*, then the conclusion of Theorem* 2.7 *holds.*

Proof. Recall that the first hypothesis implies that all of the embeddings $H_\beta \to H_\delta$ are compact if $\beta > \delta$. In particular, the embedding $H_{1-\gamma} \to H_\alpha$ is compact.

Putting $C := B + \lambda I \colon H_\alpha \to H_{-\gamma}$, we are to show now that $A_{-\gamma} - C$ is a Fredholm operator of index 0 (in the space $H_{-\gamma}$). Since $A_{-\gamma} \colon H_{1-\gamma} \to H_{-\gamma}$ is such a Fredholm operator, it suffices to show by [13, Theorem IV.5.26] that $C := B + \lambda I \colon H_\alpha \to H_{-\gamma}$ is relatively compact with respect to $A_{-\gamma}$. Thus, let u_n and $A_{-\gamma}u_n$ be bounded in $H_{-\gamma}$. Then u_n is bounded in $H_{1-\gamma}$, and thus u_n contains a subsequence convergent in H_α. Hence, Cu_n contains a subsequence convergent in $H_{-\gamma}$, as required.

Since I is bounded in $H_{-\gamma}$ and $A_{-\gamma} - B - \lambda I$ is Fredholm for all λ, [13, Theorem IV.5.31] implies that the dimension of the null space of $A_{-\gamma} - B - \lambda I$ is constant for all λ except for a set of isolated points. Since $A_{-\gamma} - B$ is sectorial and thus has a nonempty resolvent set, it follows that this constant is zero. □

A further difficulty in the application of Theorem 2.7 is that the operator $A_{-\gamma}$ as well as the space $H_{-\gamma}$ and semigroups $e^{-tA_{-\gamma}}$ are rather abstract objects so that e.g. the meaning of γ-weak and γ-mild solutions or of γ-weak eigenvalues is somewhat obscure. We can understand these objects much better when we restrict ourselves to the Hilbert space setting.

2.2. Hilbert spaces. For the rest of the paper, we assume that $(H, (\cdot,\cdot), |\cdot|)$ is a complex Hilbert space. We assume that $(V, \|\cdot\|)$ is a complex Banach space (actually isomorphic to a Hilbert space if a form a as required below exists) which is densely embedded into H. Identifying H with a subspace of the antidual V' by means of the scalar product $(\cdot,\cdot)$, we have a customary Gel'fand triple $V \subseteq H \subseteq V'$ of densely embedded spaces. Let $a \colon V \times V \to \mathbb{C}$ be a sesquilinear form on V which is continuous, that is, there is $C \in [0, \infty)$ with

$$|a(u, v)| \le C\|u\|\|v\|,$$

and which is strongly accretive in the sense that there is $c > 0$ with

$$\operatorname{Re} a(u,u) \geq c\|u\|^2 \quad \text{for all } u \in V. \tag{2.10}$$

The operator $A\colon D(A) \to H$ with its natural domain $D(A) \subseteq H$ is defined in the obvious way by means of the duality

$$(Au, v) = a(u, v) \quad \text{for all } v \in H.$$

Similarly, $\mathscr{A}\colon V \to V'$ is defined by means of the duality

$$(\mathscr{A}u, v) = a(u, v) \quad \text{for all } v \in V,$$

where now the brace on the left denotes the antidual pairing of V' and V.

It is well-known that $\mathscr{A}$ is an isomorphism and that A is a densely defined sectorial operator of positive type, so that all previous results apply for A.

Definition 2.14. We call A a *Kato operator* if $H_{\frac{1}{2}} \cong V$.

Examples of Kato operators are when a is symmetric (see also Corollary 2.19 and the remarks thereafter) or when we are in the context of differential equations: In the latter case, one usually obtains Kato operators under very mild hypotheses. This is the famous Kato square root problem, and we will use such results later on.

It turns out that each of the operators A_α ($\alpha \in \mathbb{R}$) is a Kato operator if and only if A is. The latter is also equivalent to the equality $\mathscr{A} = A_{-\frac{1}{2}}$. For the completeness of exposition, we will prove these assertions. We first discuss the spaces $H_{-\gamma}$ in more detail. To this end, we first define H_α^* analogously to H_α with respect to the norm $\|u\|_{H_\alpha^*} := |(A^\alpha)^* u|$, where A^* denotes the Hilbert space adjoint of A, noting that $(A^*)^\alpha = (A^\alpha)^*$.

We denote by $[\cdot,\cdot]_\theta$ the complex interpolation functor of order $\theta \in (0,1)$, see e.g. [20]. For convenience, we include $\theta = 0$ and $\theta = 1$ in the obvious manner by putting $[X,Y]_0 \coloneqq X$ and $[X,Y]_1 \coloneqq Y$. We start by collecting some results from [5] and [12]:

Proposition 2.15. *We have the reiteration formulas*

$$H_{(1-\theta)\alpha+\theta\beta} \cong [H_\alpha, H_\beta]_\theta \quad \text{if } \alpha,\beta \in \mathbb{R},\ 0 \leq \theta \leq 1, \tag{2.11}$$

and the duality representation

$$H_{-\gamma} \cong (H_\gamma^*)' \quad \text{if } -1 \leq \gamma \leq 1. \tag{2.12}$$

Additionally,

$$H_\gamma^* \cong H_\gamma \quad \text{if } \gamma \in \left[0, \frac{1}{2}\right), \tag{2.13}$$

and (2.13) *holds with* $\gamma = \frac{1}{2}$ *if and only if* A *is a Kato operator.*

Using Proposition 2.15, we obtain now several equivalent characterizations of the space $H_{-\gamma}$. For the case that A is a Kato operator, we have more such characterizations.

Corollary 2.16. *There hold the formulas*

$$H_{-\gamma} \cong (H_\gamma^*)' \cong [H, D(A^*)']_\gamma \cong [H, D(A^*)]'_\gamma \quad \text{for all } \gamma \in (0,1). \tag{2.14}$$

If A is a Kato operator, then also

$$H_{-\gamma} \cong (H_\gamma^*)' \cong H'_\gamma \cong [H,V]'_{2\gamma} \cong [V',V]'_{\frac{1}{2}+\gamma} \cong [V',V]_{\frac{1}{2}-\gamma} \cong [V',H]_{1-2\gamma} \tag{2.15}$$

for all $\gamma \in [0, \frac{1}{2}]$ and

$$H_{-\gamma} \cong (H_\gamma^*)' \cong [D(A^*)', V']_{2-2\gamma} \cong [V, D(A^*)]'_{2\gamma-1} \quad \text{for all } \gamma \in \left[\frac{1}{2}, 1\right]. \tag{2.16}$$

Proof. The formula (2.14) is shown in a straightforward manner with (2.11) and (2.12) by inserting $D(A^*) = H_1^*$ and $H = H_0 = H'_0 = H_0^* = (H_0^*)'$. The formulas (2.15) and (2.16) are shown similarly, by using also (2.13) with $\gamma \in [0, \frac{1}{2}]$ and $V \cong H_{\frac{1}{2}} \cong H^*_{\frac{1}{2}}$. □

Remark 2.17. The last three equalities in (2.15) are independent of A and consequently also valid if A fails to be a Kato operator. (Here we use that there *exists* a Kato operator, e.g. associated to the symmetrization of a.)

Note that the existence of the form a implies that V is actually a Hilbert space with an equivalent norm; for instance, the symmetrization of a defines a corresponding scalar product. Recall that $\mathscr{A} : V \to V'$ is an isomorphism. Hence, any scalar product b on V induces a scalar product b_a on V' and vice versa by means of the formula

$$b_a(u,v) := b(\mathscr{A}^{-1}u, \mathscr{A}^{-1}v) \quad \text{for all } u, v \in V', \tag{2.17}$$

where the corresponding norms are equivalent to the original norms each. We call a scalar product b on V an *A-Kato scalar product* if the corresponding norm is equivalent to the original norm on V, and if there are $c_1, c_2 > 0$ satisfying

$$\operatorname{Re} b(u, A^{-1}u) \geq c_1|u|^2 \quad \text{and} \quad |b(u, A^{-1}v)| \leq c_2|u||v| \quad \text{for all } u, v \in H. \tag{2.18}$$

By density of $D(A)$ and since the Hilbert space is complex, the estimates (2.18) follow if one can show that there are $c_1, c_3 > 0$ such that

$$\operatorname{Re} b(u, A^{-1}u) \geq c_1|u|^2 \quad \text{and} \quad |b(u, A^{-1}u)| \leq c_3|u|^2 \quad \text{for all } u \in D(A).$$

By this approach of working with scalar product on V, one can now obtain an *elementary* proof of the following result.

Theorem 2.18. *The following assertions are equivalent:*

(1) *A is a Kato operator.*
(2) *There is an A-Kato scalar product on V.*
(3) *There is a scalar product on V' generating an equivalent norm and a sesquilinear form $\mathfrak{a}\colon H \times H \to \mathbb{C}$ satisfying*

$$\operatorname{Re} \mathfrak{a}(u,u) \geq c_1 |u|^2, \quad |\mathfrak{a}(u,v)| \leq c_2 |u||v|$$

with $c_1, c_2 > 0$ such that $\mathscr{A}$ is associated to $\mathfrak{a}$.
(4) *$D(\mathscr{A}^{\frac{1}{2}}) \cong H$.*
(5) *$\mathscr{A} = A_{-\frac{1}{2}}$ (under a canonical identification).*
(6) *For some $\alpha \in \mathbb{R}$ the space H_α can be equipped with a scalar product generating an equivalent norm such that the operator A_α is a Kato operator with a form defined on $H_{\alpha+\frac{1}{2}}$.*
(7) *For every $\alpha \in \mathbb{R}$ the space H_α can be equipped with a scalar product generating an equivalent norm such that the operator A_α is a Kato operator with a form defined on $H_{\alpha+\frac{1}{2}}$.*

In each case,

$$b(u,v) := (A^{\frac{1}{2}} u, A^{\frac{1}{2}} v) \tag{2.19}$$

defines an A-Kato scalar product on V.

We sketch the proof of Theorem 2.18; full details can be found in [9]. For the implication (1) $\Longrightarrow$ (2), one calculates that the scalar product (2.19) (which was also used e.g. in [3]) is indeed A-Kato under hypothesis (1). The proof of the implication (2) $\Longrightarrow$ (3) is a straightfoward calculation by using the scalar product (2.17) of V' and the form $\mathfrak{a}(u,v) = b_a(\mathscr{A}u, v)$; the only technical step here consists in the verification that the domain of the operator associated to $\mathfrak{a}$ is indeed $D(\mathscr{A}) = V$. For the proof of the implication (3) $\Longrightarrow$ (4), we apply (2.11) to $(\mathscr{A}, \mathfrak{a}, V, V')$ in place of (A, a, H_1, H_0) and obtain $D(\mathscr{A}^{\frac{1}{2}}) \cong [V', V]_{\frac{1}{2}}$; according to Remark 2.17 and (2.15) with $\gamma = 0$, the latter space is $[V', V]_{\frac{1}{2}} \cong [V', H]_1 = H$, and so (4) holds. To see (4) $\Longrightarrow$ (1), note that if (4) holds then $\mathscr{A}^{\frac{1}{2}}\colon H \to V'$ and $\mathscr{A}^{\frac{1}{2}}\mathscr{A}^{\frac{1}{2}} = \mathscr{A}\colon V \to V'$ are isomorphisms, and so the H-realization of $\mathscr{A}^{\frac{1}{2}}$ must be an isomorphism $V \to H$. This H-realization is $A^{\frac{1}{2}}\colon V \to H$, and so $D(A^{\frac{1}{2}}) \cong V$ which implies (1). For the implication (1) $\Longrightarrow$ (5), we use Corollary 2.16 and obtain that the operators $A_{-\frac{1}{2}}\colon V \cong H_{\frac{1}{2}} \to H_{-\frac{1}{2}} \cong V'$ and $\mathscr{A}\colon V \to V'$ are both continuous extensions of $A\colon D(A) \to H \subseteq V'$. Since $D(A) = H_1$ is dense in $H_{\frac{1}{2}} \cong V$, it follows that these extensions must coincide, and so (5) holds. The implications (5) $\Longrightarrow$ (1) $\Longrightarrow$ (6) and (7) $\Longrightarrow$ (1) are trivial. From the already established equivalence (1) $\Longleftrightarrow$ (2), one finally obtains easily the remaining implication (6) $\Longrightarrow$ (7) by "transporting" the scalar product and A_α-Kato scalar product between H_{α_1} and H_{α_2} with appropriate isomorphisms $A_{\beta_1}^{\beta_2}$ (essentially by replacing $\mathscr{A}$ in (2.17) with such isomorphisms); it is only a technical matter to calculate that one obtains indeed A_β-Kato scalar products.

We note that some of the equivalences of Theorem 2.18 are mathematical folklore, see e.g. [6, Section 5.5.2], and some can be obtained easily by using deep results [10, Section 7.3.3], but our elementary approach by observing the equivalence (1) $\iff$ (2) (and actually our introduction of an A-Kato scalar product) appears to be new. This equivalence can also be used to establish e.g. the following side result.

Corollary 2.19. *A is Kato if there are $\alpha > -1$ and $\beta, M \geq 0$ with*

$$\operatorname{Re}((A^*)^{-1}(Au + Mu), u) \geq \alpha |u|^2 \quad \textit{and} \quad |(A^*)^{-1}Au| \leq \beta |u| \tag{2.20}$$

for all $u \in D(A)$.

Indeed, a straightforward calculation shows that condition (2.20) is equivalent to the assertion that

$$b(u,v) := \frac{1}{2}\Big(a(u,v) + \overline{a(v,u)} + M \cdot (u,v)\Big)$$

defines an A-Kato scalar product. Note that (2.20) is trivially satisfied for selfadjoint operators, so that we can conclude from Corollary 2.19 and thus indirectly from Theorem 2.18 also Kato's famous result [12] that for symmetric a the operator A is a Kato opertor.

Assertion (5) of Theorem 2.18 not only helps us to understand $A_{-\frac{1}{2}}$ but actually $A_{-\gamma}$ in case $\gamma \leq \frac{1}{2}$, because that operator is just a restriction of $A_{-\frac{1}{2}}$. Since also the corresponding semigroups are restrictions of each other, we obtain for instance a characterization of γ-mild and γ-weak solutions which is easier to understand from an analytic point of view:

Corollary 2.20. *If A is a Kato operator and $\gamma \leq \frac{1}{2}$, one can replace* (2.3) *in Definition* 2.3 *equivalently by*

$$u(t) = e^{-(t-t_0)\mathscr{A}} u(t_0) + \int_{t_0}^{t} e^{-(t-s)\mathscr{A}} f_0(s)\, ds,$$

and $u'(t) + A_{-\gamma} u(t) = f_0(t)$ equivalently by

$$u'(t) + a(u(t), v) = (f_0(t), v) \quad \textit{for all } v \in V.$$

If A is a Kato operator, then Theorem 2.18(5) implies that $\mathscr{A}$ is sectorial so that the above semigroup $e^{-t\mathscr{A}}$ in V' is indeed well defined and analytic. The latter can be calculated even if A fails to be a Kato operator, see e.g. [18, Theorem 1.55], but then the space V' may differ from $H_{-\frac{1}{2}}$.

In a similar way, one obtains a more analytic characterization of γ-weak eigenvalues.

Corollary 2.21. *Let $\gamma \in [0, \frac{1}{2}]$ and A be a Kato operator. Then λ is a γ-weak eigenvalue of $A-B$ with corresponding eigenvector $u \neq 0$ if and only if $u \in H_{1-\gamma}$ and*

$$a(u, \varphi) - (Bu, \varphi) = \lambda(u, \varphi) \quad \text{for all } \varphi \in V.$$

Moreover, the hypothesis (2.7) *of Proposition* 2.12 *is satisfied if $B(V) \subseteq H$.*

3. Superposition operators in L_2 and Sobolev spaces

Let $\Omega \subseteq \mathbb{R}^d$ be open and $H := L_2(\Omega, \mathbb{C}^n)$. In the following, we use the scalar product (and respective dual pairing)

$$(u, v) := \int_\Omega u(x) \cdot \overline{v(x)}\, dx.$$

In case $d \geq 3$, we put $p_* := \frac{2d}{d-2}$; in case $d \leq 2$, we fix an arbitrary $p_* \in (2, \infty)$. Let $V \subseteq W^{1,2}(\Omega, \mathbb{C}^n)$ be a closed subspace which is dense in H. We assume that Ω is such that Sobolev's embedding theorem is valid in the sense that there is a continuous embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$.

Remark 3.1. For the case that Ω is such that the dense embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$ holds only with some smaller power $p_* \in (2, \infty)$, all subsequent considerations hold as well with this choice of p_*.

Lemma 3.2. *Let A be a Kato operator.*

(1) *Let $\gamma \in [0, \frac{1}{2}]$.*

(a) *We have a continuous embedding $L_{q_\gamma}(\Omega, \mathbb{C}^n) \subseteq H'_\gamma \cong (H^*_\gamma)' \cong H_{-\gamma}$ with*

$$q_\gamma := \left(\frac{1}{2} + \gamma - \frac{2\gamma}{p_*}\right)^{-1} \quad \left(= \frac{2d}{d+4\gamma} \in \left[\frac{2d}{d+2}, 2\right] \quad \text{if } d \geq 3\right). \tag{3.1}$$

(b) *If we have a continuous embedding $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$ $(1 \leq p < \infty)$, then we also have a continuous embedding $H_{1-\gamma} \subseteq L_{p_\gamma}(\Omega, \mathbb{C})$ with*

$$p_\gamma := \left(\frac{2\gamma}{p_*} + \frac{1 - 2\gamma}{p}\right)^{-1}. \tag{3.2}$$

(2) *Let $\gamma \in [\frac{1}{2}, 1]$.*

(a) *We have a continuous embedding $H_{1-\gamma} \subseteq L_{p_\gamma}(\Omega, \mathbb{C})$ with*

$$p_\gamma := \left(\gamma - \frac{1}{2} + \frac{2 - 2\gamma}{p_*}\right)^{-1} \left(= \frac{2d}{4\gamma + d - 2} \in \left[\frac{2d}{d+2}, 2\right] \quad \text{if } d \geq 3\right). \tag{3.3}$$

(b) *If we have a continuous embedding* $D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ $(1 \le p < \infty)$, *then we also have a continuous embedding* $L_{q_\gamma}(\Omega, \mathbb{C}^n) \subseteq (H_\gamma^*)' \cong H_{-\gamma}$ *with*

$$q_\gamma := \left(1 - \frac{2\gamma - 1}{p} - \frac{2 - 2\gamma}{p_*}\right)^{-1}. \tag{3.4}$$

Proof. By hypothesis, we have a continuous dense embedding $V \subseteq L_{p_*}(\Omega, \mathbb{C}^n)$. Hence, with $\frac{1}{p_*'} + \frac{1}{p_*} = 1$ also the (Banach space) adjoint embedding $L_{p_*'}(\Omega, \mathbb{C}^n) \subseteq V'$ is continous and dense. In case $\gamma = \frac{1}{2}$ we have $q_\gamma = p_\gamma = p_*'$, and thus the assertion (2) follows. In case $\gamma = 0$ we have $q_\gamma = 2$ and $p_\gamma = p$, and the assertion (1) is trivial. In case $\gamma \in (0, \frac{1}{2})$, we use [20, Theorem 1.18.4], the fact that $[\cdot, \cdot]_\theta$ is an interpolation functor of order θ (see e.g. [20, Theorem 1.9.3(a)]), and (2.15). Then we have a continuous embedding

$$L_{q_\gamma}(\Omega, \mathbb{C}^n) \cong [L_{p_*'}(\Omega, \mathbb{C}^n), L_2(\Omega, \mathbb{C}^n)]_{1-2\gamma} \subseteq [V', H]_{1-2\gamma} \cong H_\gamma',$$

which proves (1a).

Defining p' by $\frac{1}{p'} + \frac{1}{p} = 1$, we find in view of $H_1^* = D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ that $L_p'(\Omega, \mathbb{C}^n) = L_p(\Omega, \mathbb{C}^n)' \subseteq (H_1^*)'$. This shows (2) for $\gamma = 1$, since $q_\gamma = p'$ and $p_\gamma = 2$. Moreover, for $\gamma \in (\frac{1}{2}, 1)$, we find similarly as above with (2.16) the continuous embedding

$$L_{q_\gamma}(\Omega, \mathbb{C}^n) = [L_{p'}(\Omega), L_{p_*'}(\Omega, \mathbb{C}^n)]_{2-2\gamma} \subseteq [D(A^*)', V']_{2-2\gamma} \cong (H_\gamma^*)',$$

which implies (2b). A similar argument shows with Proposition 2.15 that in case $\gamma \in (0, \frac{1}{2})$

$$H_{1-\gamma} \cong [H_{\frac{1}{2}}, H_1]_{1-2\gamma} \subseteq [L_{p_*}(\Omega, \mathbb{C}^n), L_p(\Omega, \mathbb{C}^n)]_{1-2\gamma} \cong L_{p_\gamma}(\Omega, \mathbb{C}^n),$$

proving (1b), while in case $\gamma \in (\frac{1}{2}, 1)$

$$H_{1-\gamma} \cong [H_0, H_{\frac{1}{2}}]_{2-2\gamma} \subseteq [L_2(\Omega, \mathbb{C}^n), L_{p_*}(\Omega, \mathbb{C}^n)]_{2-2\gamma} \cong L_{p_\gamma}(\Omega, \mathbb{C}^n),$$

proving (2a) (all embeddings being countinuous). □

We assume also that the nonlinearity $f(t, \cdot)$ is given by a superposition operator induced by a function $\widetilde{f}\colon [0, \infty) \times \Omega \times \mathbb{C}^n \to 2^{\mathbb{C}^n}$, that is, for each $t \in [0, \infty)$

$$f(t, u) := \left\{ v\colon \Omega \to \mathbb{C}^n \;\middle|\; \begin{array}{l} v \text{ measurable and} \\ v(x) \in \widetilde{f}(t, x, u(x)) \text{ for almost all } x \in \Omega \end{array} \right\} \tag{3.5}$$

For the stability result, without loss of generality, we will consider only the case $u_0 = 0$ and assume that $\widetilde{f}$ is uniformly linearizable at $u = 0$ in the following sense. There are $r \in (1,\infty]$, a measurable $\widetilde{B}\colon \Omega \to \mathbb{C}^{n\times n}$, and a function $\widetilde{g}\colon (0,\infty)\times\Omega\times\mathbb{C}^n \to 2^{\mathbb{C}^n}$ with

$$\widetilde{f}(t,x,u) = \widetilde{B}(x)u + \widetilde{g}(t,x,u) \quad \text{for all } (t,x,u) \in (0,\infty)\times\Omega\times\mathbb{C}^n$$

such that

$$\lim_{|u|\to 0} \frac{\sup\{|v| : v \in \widetilde{g}((0,\infty)\times\{x\}\times\{u\})\}}{|u|} = 0 \tag{3.6}$$

for almost all $x \in \Omega$. Moreover, we assume that there is $C_0 \in (0,\infty)$ such that

$$\sup\{|v| : v \in \widetilde{g}((0,\infty)\times\{x\}\times\{u\})\} \le C_0 \cdot (|u| + |u|^\sigma) \quad \text{for all } u \in \mathbb{C}^n \tag{3.7}$$

for almost all $x \in \Omega$ and some $\sigma \in (1,\infty)$. We define a corresponding multiplication operator B by

$$Bu(x) \coloneqq \widetilde{B}(x)u(x) \quad \text{for all } x \in \Omega. \tag{3.8}$$

With this notation, the following holds.

Proposition 3.3. *Let A be a Kato operator and $u_0 = 0$. Suppose that*

$$\Omega \textit{ has finite (Lebesgue) measure} \tag{3.9}$$

and that $r \in [1,\infty]$ and $\sigma \in (0,\infty)$ are such that $\widetilde{B} \in L_r(\Omega,\mathbb{C}^{n\times n})$ and (3.6) and (3.7) hold.

(1) *Let $\alpha = 0$. Assume*

$$\begin{cases} r = 2 & \text{if } d = 1 \\ r > 2 & \text{if } d = 2 \\ r = \dfrac{2p_*}{p_*-2} \quad (= d) & \text{if } d \ge 3, \end{cases} \tag{3.10}$$

and

$$\begin{cases} \sigma = 2 & \text{if } d = 1 \\ \sigma < 2 & \text{if } d = 2 \\ \sigma = 2 - \dfrac{2}{p_*} \quad \left(= 1 + \dfrac{2}{d}\right) & \text{if } d \ge 3. \end{cases} \tag{3.11}$$

Then for every $\gamma \in [\frac{1}{2},1)$ there holds $f\colon [0,\infty)\times H_\alpha \to 2^{(H_\gamma^*)'}$, and the hypothesis (B_γ) of Theorems 2.7 and 2.13 is satisfied with $H_\alpha = L_2(\Omega,\mathbb{C}^n)$.

(2) *Let $\alpha = 0$. Assume that the embedding $D(A^*) \subseteq L_p(\Omega,\mathbb{C}^n)$ is continuous for some $p \in (p_*,\infty)$, and*

$$r > \frac{2p}{p-2} \quad \textit{and} \quad \sigma < 2 - \frac{2}{p}. \tag{3.12}$$

Then

$$\gamma_0 := \frac{(\sigma-2)p_* p - 2p_* + 4p}{4(p-p_*)} < 1, \quad \gamma_1 := \frac{2pp_* - r(pp_* + 2p_* - 4p)}{4(p-p_*)r} < 1, \quad (3.13)$$

and for every $\gamma \in [\max\{\gamma_0, \gamma_1, \frac{1}{2}\}, 1)$ *the same conclusion as in* (1) *is valid.*

(3) *Let* $\alpha = \frac{1}{2}$. *Suppose that* $\widetilde{B} \in L_r(\Omega, \mathbb{C}^{n\times n})$ *with some*

$$r > \frac{p_*}{p_* - 2} \quad \left(= \frac{d}{2} \quad \text{if } d \geq 3\right), \quad (3.14)$$

and that (3.6) *and* (3.7) *hold with some*

$$\sigma < p_* - 1 \quad \left(= \frac{d+2}{d-2} \quad \text{if } d \geq 3\right). \quad (3.15)$$

Then

$$\gamma_0 := \frac{2\sigma - p_*}{2p_* - 4} < \frac{1}{2} \quad \text{and} \quad \gamma_1 := \frac{p_*}{r(p_* - 2)} - \frac{1}{2} < \frac{1}{2}, \quad (3.16)$$

and for every $\gamma \in [\max\{0, \gamma_0, \gamma_1\}, \frac{1}{2})$ *we have* $f\colon [0,\infty) \times H_\alpha \to 2^{H'_\gamma}$, *and the hypothesis* ($\mathrm{B}_\gamma$) *of Theorems* 2.7 *and* 2.13 *is satisfied with* $H_\alpha = V$.

Proof. In case (1), it is no loss of generality to assume $\gamma = \frac{1}{2}$, and we assume first $d \geq 3$. In cases (1) and (2), we put $\widetilde{p} = 2$ and define q_γ by (3.4), while in case (3), we put $\widetilde{p} = p_*$ and define q_γ by (3.1). Then we put $U := L_{\widetilde{p}}(\Omega, \mathbb{C}^n)$ and $V_\gamma := L_{q_\gamma}(\Omega, \mathbb{C}^n)$. Letting r satisfy (3.10), (3.12), or (3.14), and requiring $\gamma \geq \gamma_1$ with γ_1 as in (3.13) or (3.16) in the respective cases, we find

$$\frac{1}{q_\gamma} \geq \frac{1}{\widetilde{p}} + \frac{1}{r},$$

and so we obtain from the (generalized) Hölder inequality that $B\colon U \to V_\gamma$ is bounded. Since we have a bounded embedding $H_\alpha \subseteq U$, we obtain from Lemma 3.2 that $B\colon H_\alpha \to (H_\gamma^*)'$ is bounded.

Moreover, letting σ satisfy (3.11), (3.12), or (3.15), and requiring $\gamma \geq \gamma_0$ with γ_0 as in (3.13) or (3.16) in the respective cases, we find $\sigma \leq \frac{\widetilde{p}}{q_\gamma}$. Hence, the superposition operator g generated by $\widetilde{g}$ satisfies $g\colon [0,\infty) \times U \to 2^{V_\gamma}$ and

$$\lim_{\|u\|_U \to 0} \frac{\sup\left\{\|v\|_{V_\gamma} : v \in g\left((0,\infty) \times \{u\}\right)\right\}}{\|u\|_U} = 0,$$

see [22, Theorem 4.14]. Since we have continuous embeddings $H_\alpha \subseteq U$ and $V_\gamma \subseteq (H_\gamma^*)'$ (Lemma 3.2), the condition (B_γ) is proved.

Case (1) with $d = 2$ is treated in a similar way (with a sufficiently large p_*), and for $d = 1$ we can put $q_\gamma = 1$ in the above calculation, since in this case we have still a continuous embedding $V_\gamma \subseteq (H_\gamma^*)'$ by the continuity of the embedding $(H_\gamma^*)' \subseteq H_{\frac{1}{2}} \subseteq L_\infty(\Omega, \mathbb{C}^n)$. □

Remark 3.4. The last observation in the proof extends to a more general situation: If $\gamma \in [0, \frac{1}{2}]$ is such that the embedding $H_\gamma \subseteq L_\infty(\Omega, \mathbb{C}^n)$ is continuous, then the conclusion of Proposition 3.3(1) is valid with $r = \sigma = 2$ (we put $q_\gamma = 1$ in the proof).

Remark 3.5. For $d \geq 3$ assertion (2) of Proposition 3.3 requires strictly less about r and σ than assertion (1), because in view of $p > p_*$ there holds

$$\frac{2p}{p-2} < \frac{2p_*}{p_*-2} \quad \text{and} \quad 2 - \frac{2}{p} > 2 - \frac{2}{p^*}.$$

Remark 3.6. In case $d \geq 3$ the quantities γ_0 and γ_1 in (3.16) have the form

$$\gamma_0 = \frac{(d-2)\sigma - d}{4} \quad \text{and} \quad \gamma_1 = \frac{1}{2}\left(\frac{d}{r} - 1\right). \tag{3.17}$$

Remark 3.7. Proposition 3.3(3) holds also with $\gamma = \frac{1}{2}$. Moreover, for $\gamma = \frac{1}{2}$ one does not have to require that the inequalities in (3.14) or (3.15) are strict. However, the choice $\gamma = \frac{1}{2}$ violates the hypothesis (2.1) of Theorems 2.7 and 2.13 if $\alpha = \frac{1}{2}$.

Remark 3.8. Hypothesis (3.9) is obviously needed for the assertion (3) of Proposition 3.3. However, we used this hypothesis also for the assertion (1) when we applied [22, Theorem 4.14]. If hypothesis (3.9) fails, one can apply other criteria for the differentiability of superposition operators like e.g. [22, Theorem 4.9], but we do not formulate corresponding results here.

While Proposition 3.3 gives a sufficient condition for the hypothesis (B_γ), this is not sufficient to apply Theorem 2.13. For the latter, one also has to estimate all γ-weak eigenvalues of $A - B$, and the latter in turn is usually simpler if one knows that all γ-weak eigenvalues of $A - B$ are eigenvalues of $A - B$. For the operator B from (3.8), this is the content of the following result.

Proposition 3.9. *Suppose* (3.9). *Let B have the form* (3.8) *with some* $\widetilde{B} \in L_r(\Omega, \mathbb{C}^n)$, $r \in [1, \infty]$.

(1) *If r satisfies* (3.10), *then $B|_V \colon V \to H$ is bounded.*
(2) *If A is a Kato operator, $\gamma \in [\frac{1}{2}, 1)$, and*

$$\gamma \leq \widetilde{\gamma}_0 := \begin{cases} 1 - \dfrac{p_*}{(p_*-2)r} & \textit{if } r < \infty \\ 1 & \textit{if } r = \infty, \end{cases} \tag{3.18}$$

then $B|_{H_{1-\gamma}} \colon H_{1-\gamma} \to H$.

(3) *If A is a Kato operator,*

$$2 < r < \frac{2p_*}{p_* - 2} \quad (= d \quad \text{if } d \geq 3), \tag{3.19}$$

and if there is $p \geq \frac{2r}{r-2}$ ($> p_*$) with $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$, then

$$\widetilde{\gamma}_p := \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r} - \frac{1}{p}\right) \cdot \left(\frac{1}{p_*} - \frac{1}{p}\right)^{-1} \in \left[0, \frac{1}{2}\right), \tag{3.20}$$

and for all $\gamma \leq \widetilde{\gamma}_p$ the operator $B|_{H_{1-\gamma}} \colon H_{1-\gamma} \to H$ is bounded.

If A is a Kato operator, the hypotheses of (1), (2), *or* (3) *are satisfied and $\gamma \leq \frac{1}{2}$, $\gamma \leq \widetilde{\gamma}_0$, or $\gamma \leq \widetilde{\gamma}_p$, respectively, then λ is a γ-weak eigenvalue of $A - B$ if and only if λ is an eigenvalue of $A - B$.*

Proof. In case (1) with $d \geq 3$, we apply in view of $\frac{1}{2} = \frac{1}{p_*} + \frac{1}{r}$ the (generalized) Hölder inequality to obtain that $B \colon L_{p_*}(\Omega, \mathbb{C}^n) \to L_2(\Omega, \mathbb{C}^n)$ is bounded and thus $B \colon V \to H$ is bounded. Case (1) with $d \geq 2$ is similar (with sufficiently large p_*), and for $d = 1$ one can formally put $p_* = \infty$ by the continuity of the embedding $H_{\frac{1}{2}} \subset C(\overline{\Omega}, \mathbb{C}^n)$.

In case (2) and (3), we define p_γ by (3.3) or (3.2), respectively, and observe that, due to (3.18) or (3.20), respectively, we have the estimate $\frac{1}{2} \geq \frac{1}{p_\gamma} + \frac{1}{r}$. Hence, by the (generalized) Hölder inequality, $B \colon L_{p_\gamma}(\Omega, \mathbb{C}^n) \to L_2(\Omega, \mathbb{C}^n)$ is bounded, and thus also $B \colon H_{1-\gamma} \to H$ is bounded by Lemma 3.2. The last assertion follows from Proposition 2.12 and Corollary 2.21. □

If one is interested in stability in H (the case $\alpha = 0$), one should consider Proposition 3.3 part (1) or (2). In the former case, Proposition 3.9(1) is automatically satisfied, and in the latter case one would like to apply Proposition 3.9(2). In the latter case, $\gamma \in [\frac{1}{2}, 1)$ has to satisfy $\gamma_i \leq \gamma \leq \widetilde{\gamma}_0$ for $i = 0, 1$ with γ_i from (3.13). Obviously, γ_1 and $\widetilde{\gamma}_0$ depend monotonically on r, and $\gamma_1 < \widetilde{\gamma}_0$ if r is sufficiently large, and then $\gamma_0 < \widetilde{\gamma}_0$ if σ is sufficiently small, so that Proposition 3.3(2) and Proposition 3.9(2) apply simultaneously for all γ from some proper interval (if r is sufficiently large).

If one is interested in stability in V (the case $\alpha = \frac{1}{2}$), one should consider Proposition 3.3(3). In this case, the hypothesis of Proposition 3.9(1) means an additional requirement for r. The purpose of Proposition 3.9(3) is to relax this requirement. However, it is not immediately clear whether this relaxed requirement applies in the situation of Proposition 3.3(3), since then $\gamma \in [0, \frac{1}{2}]$ needs to satisfy $\gamma_i \leq \gamma \leq \widetilde{\gamma}_p$ for $i = 0, 1$ with γ_i from (3.16). Although γ_1 and $\widetilde{\gamma}_p$ depend monotonically on r and satisfy $\gamma_1 < \widetilde{\gamma}_p$ if r is sufficiently large, one cannot choose r arbitrarily large in view of (3.19): Otherwise already the additional requirement of Proposition 3.9(1) is satisfied. In fact, the following observation may be somewhat discouraging at a first glance.

Remark 3.10. If (3.19) holds, then the term $\widetilde{\gamma}_p$ in (3.20) is strictly increasing with respect to $p \geq \frac{2r}{r-2}$. In particular,

$$\widetilde{\gamma}_\infty := \sup_{p \in [\frac{2r}{r-2}, \infty)} \widetilde{\gamma}_p = \lim_{p \to \infty} \widetilde{\gamma}_p = \frac{r-2}{4r} p_*.$$

Thus, even if we know that $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$ for every $p \in (1, \infty)$, we still have $\gamma < \widetilde{\gamma}_\infty$, and the latter can be arbitrarily small if r is sufficiently close to 2.

Nevertheless we will show in the following remark that Proposition 3.3(3) and Proposition 3.9(3) apply simultaneously with the same γ provided that r is not "too" small and σ is not "too" large.

Remark 3.11. Suppose that Sobolev's embedding theorem holds in the sense described earlier and, moreover, that we have a continuous embedding $D(A) \subseteq L_p(\Omega, \mathbb{C}^n)$ with $p = \frac{2d}{d-4}$ in case $d \geq 5$ and any $p \in (p_*, \infty)$ in case $d \leq 4$. For instance, by standard Sobolev embedding theorems (see [16, Theorem 1.4.5]), this is the case if $D(A) \subseteq W^{2,2}(\Omega, \mathbb{C}^n)$. Proposition 3.9(3) applies with

$$\begin{cases} r \in [\frac{d}{2}, d] \text{ and } \gamma \leq \widetilde{\gamma}_{2d/(d-4)} = 1 - \frac{d}{2r} & \text{if } d \geq 5 \\ r \in (2, d) \text{ and } \gamma < \widetilde{\gamma}_\infty = \frac{r-2}{4r} p_* & \text{if } d = 3, 4. \end{cases}$$

In view of (3.17) it follows that if

$$\begin{cases} r \in [\frac{2}{3}d, d) \text{ and } \sigma \leq \frac{(d+4)r - 2d}{(d-2)r} & \text{if } d \geq 5 \\ r \in (\frac{d^2}{2d-2}, d) \text{ and } \sigma < \frac{d^2 r - 4d}{(d-2)^2 r} & \text{if } d = 3, 4, \end{cases}$$

then Proposition 3.3(3) applies with

$$\begin{cases} \max\{\gamma_0, \gamma_1\} \leq \widetilde{\gamma}_{\frac{2d}{d-4}} & \text{if } d \geq 5 \\ \max\{\gamma_0, \gamma_1\} < \widetilde{\gamma}_\infty & \text{if } d = 3, 4. \end{cases}$$

Hence, in these cases there exists $\gamma \in [0, \frac{1}{2})$ for which Proposition 3.3(3) and Proposition 3.9(3) apply simultaneously.

A result similar to Proposition 3.3 holds for a Lipschitz condition. We assume that $\widetilde{f}\colon [0, \infty) \times \Omega \times \mathbb{C}^n \to \mathbb{C}^n$ is single-valued. Let $\widetilde{p} \geq 1$, $\sigma > 0$, and $\gamma \in [0, \frac{1}{2}]$. We define q_γ by (3.1). We assume that for each $t_0 \in [0, \infty)$ there are $L_{t_0} \geq 0$, $\sigma_{t_0} > 0$, and a neighborhood $I \subseteq [0, \infty)$ of t_0 such that for each $t \in I$ there are measurable $a_t, b_t\colon \Omega \to [0, \infty)$ with

$$\int_\Omega a_t(x)^{\widetilde{p}}\, dx \leq 1 \quad \text{and} \quad \int_\Omega b_t(x)^{q_\gamma}\, dx \leq 1$$

such that for almost all $x \in \Omega$ the uniform (for all $u, v \in \mathbb{C}^n$) estimate

$$|\widetilde{f}(t,x,u) - \widetilde{f}(t,x,v)| \leq L_{t_0} \cdot (a_t(x) + |u| + |v|)^{\sigma-1} |u - v| \tag{3.21}$$

holds and such that for each $t, s \in I$ we have for almost all $x \in \Omega$ the uniform (for all $u \in \mathbb{C}^n$) estimate

$$|\widetilde{f}(t,x,u) - \widetilde{f}(s,x,u)| \leq L_{t_0} (b_t(x) + b_s(x) + |u|^\sigma) |t - s|^{\sigma_{t_0}}. \tag{3.22}$$

Finally, we assume that

$$\widetilde{f}(t,\cdot,u) \text{ is measurable for all } (t,u) \in [0,\infty) \times \mathbb{C}^n,\ \widetilde{f}(0,\cdot,0) \in L_{q_\gamma}(\Omega,\mathbb{C}^n). \tag{3.23}$$

Proposition 3.12. *Let A be a Kato operator, and assume* (3.9). *Assume one of the following:*

(1) *Let $\alpha = 0$ and $\gamma \in [\frac{1}{2}, 1)$. Suppose that conditions* (3.21)–(3.23) *hold with $\widetilde{p} = 2$ and with σ from* (3.11).

(2) *Let $\alpha = 0$, and assume that the embedding $D(A^*) \subseteq L_p(\Omega, \mathbb{C}^n)$ is continuous for some $p \in (p_*, \infty)$, Let σ satisfy* (3.12), *and thus γ_0 from* (3.13) *satisfies $\gamma_0 < 1$. Let $\gamma \in [\max\{\gamma_0, 0\}, 1)$, and suppose that* (3.21)–(3.23) *hold with $\widetilde{p} = 2$.*

(3) *Let $\alpha = \frac{1}{2}$. Let σ satisfy* (3.15), *and thus γ_0 from* (3.16) *satisfies $\gamma_0 < \frac{1}{2}$. Let $\gamma \in [\max\{\gamma_0, 0\}, \frac{1}{2})$, and suppose that conditions* (3.21)–(3.23) *hold with $\widetilde{p} = p_*$.*

Then f maps $[0,\infty) \times H_\alpha$ into $H_{-\gamma}$ and satisfies a right local Hölder-Lipschitz condition (2.4) *and is left-locally bounded into $H_{-\gamma}$.*

Proof. We use the notation of the proof of Proposition 3.3. Note that (3.23) implies in view of (3.22) by a straightforward estimate that $f(t, 0) \in V_\gamma$ for every $t > 0$. From [14, Appendix] we obtain together with (3.21) that for each $t \in I$ the function $f(t, \cdot)$ maps U into V_γ and satisfies a Lipschitz condition on every bounded set $M \subseteq U$ with Lipschitz constant being independent of $t \in I$. Using (3.22), we find by a straighforward estimate that

$$\|f(t,u) - f(s,u)\|_{V_\gamma} \leq C_{M,t_0} |t - s|^{\sigma_{t_0}} \quad \text{for all } t, s \in I,\ u \in M,$$

where C_{M,t_0} is independent of $t, s \in I$ and $u \in M$. Combining both assertions and the triangle inequality, we obtain that $f\colon [0,\infty) \times U \to V_\gamma$ satisfies a right Hölder-Lipschitz condition and is left-locally bounded into V_γ. Since we have bounded embeddings $H_\alpha \subseteq U$ and $V_\gamma \subseteq (H_\gamma^*)' \cong H_{-\gamma}$ by (2.12), the assertion follows. □

Remark 3.13. If $\alpha = 0$ and $\gamma \in [0, 1)$ is such that the embedding $H_\gamma \subseteq L_\infty(\Omega, \mathbb{C}^n)$ is continuous, then the conclusion of Proposition 3.12 is also valid (with the same proof and $q_\gamma = 1$, cf. Remark 3.4).

3.1. Semilinear parabolic PDEs. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $\Gamma_D, \Gamma_N \subseteq \partial\Omega$ be disjoint and measurable (with respect to the $(d-1)$-dimensional Hausdorff measure) and such that

$$(\partial\Omega) \setminus (\Gamma_D \cup \Gamma_N) \text{ is a null set.} \tag{3.24}$$

It is explicitly admissible that $\Gamma_D = \emptyset$ or $\Gamma_N = \emptyset$. Given $a_{j,k}, b_j \in L_\infty(\Omega, \mathbb{C}^{n\times n})$ $(j,k = 1,\dots,d)$ and $\widetilde{f}\colon [0,\infty) \times \Omega \times \mathbb{C}^n \to 2^{\mathbb{C}^n}$, we consider the semilinear PDE

$$\frac{\partial u}{\partial t} + Pu = \widetilde{f}_0(t,x,u) \quad \text{on } \Omega, \tag{3.25}$$

where

$$Pw := -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\left(a_{j,k}(x)\frac{\partial w(x)}{\partial x_k}\right) + \sum_{j=1}^{d} b_j(x)\frac{\partial w(x)}{\partial x_j}.$$

We impose the mixed boundary condition

$$\begin{cases} u = 0 & \text{on } \Gamma_D \\ \displaystyle\sum_{j,k=1}^{d} \nu_j a_{j,k}\frac{\partial u}{\partial x_k} = 0 & \text{on } \Gamma_N, \end{cases} \tag{3.26}$$

where $\nu(x) = (\nu_1(x),\dots,\nu_n(x))$ denotes the outer normal at $x \in \partial\Omega$.

We put $H := L_2(\Omega, \mathbb{C}^n)$ and

$$V := \{u \in W^{1,2}(\Omega,\mathbb{C}^n) : u|_{\Gamma_D} = 0 \text{ in the sense of traces}\},$$

equipping V with the norm of $W^{1,2}(\Omega,\mathbb{C}^n)$.

Our main assumption is as follows.

(C) Gårding's inequality holds, that is, there are $c, \widetilde{c} > 0$ with

$$\operatorname{Re}\sum_{j,k=1}^{d}\int_\Omega\left(a_{j,k}(x)\frac{\partial u(x)}{\partial x_k}\right)\cdot\overline{\frac{\partial u(x)}{\partial x_j}}\,dx \geq c\|\nabla u\|^2_{L_2(\Omega,\mathbb{C}^{dn})} - \widetilde{c}\|u\|^2_{L_2(\Omega,\mathbb{C}^n)} \tag{3.27}$$

for all $u \in V$. Moreover, at least one of the following holds:

(1) $a_{j,k}(x) = (a_{k,j}(x))^*$ for almost all $x \in \Omega$ and all $j,k = 1,\dots,d$.

(2) Gårding's inequality (3.27) holds even with $\widetilde{c} = 0$. Moreover, Γ_D satisfies the geometric hypotheses described in [8, Assumption 9.1].

(3) The matrices $\operatorname{Re}\left(\sum_{j,k=1}^{d} a_{j,k}(x)\xi_j\xi_k\right)$ are positive definite, uniformly with respect to all $x \in \Omega$ and $\xi \in \mathbb{R}^d$ with $|\xi| = 1$, $a_{j,k} \in C^1(\overline{\Omega}, \mathbb{C}^{n\times n})$, $b_j \in \operatorname{Lip}(\overline{\Omega}, \mathbb{C}^{n\times n})$ for all $j,k = 1,\dots,d$, Γ_D and Γ_N are open in $\partial\Omega$ domains, and the set (3.24) is a Lipschitz manifold of dimension $d-2$.

For a discussion of various algebraic conditions sufficient for Gårding's inequality (3.27), we refer the reader to e.g. [2, 17].

By a standard estimate, we obtain from Gårding's inequality (3.27) that the form

$$a(u,v):=\int_\Omega \left(\sum_{j,k=1}^{d} \left(a_{j,k}(x) \frac{\partial u(x)}{\partial x_k} \right) \cdot \overline{\frac{\partial v(x)}{\partial x_j}} + \sum_{j=1}^{d} \left(b_j(x) \frac{\partial u(x)}{\partial x_j} + Mu(x) \right) \cdot \overline{v(x)} \right) dx$$

satisfies (2.10) if $M \geq 0$ is sufficiently large. Keeping such an M fixed, we now introduce the function

$$\widetilde{f}(t,x,u) := \widetilde{f}_0(t,x,u) + Mu$$

and define strong (weak, mild) solutions of (3.25), (3.26) as strong (weak, mild) solutions of (2.2) with the superposition operator (3.5). A connection between the solutions of (3.25), (3.26) and (2.2) is described in e.g. [21, Theorem 4.4.4].

Theorem 3.14. *Assume that hypothesis* (C) *holds. Then the operator A associated with a is a Kato operator.*

Moreover, let $(\widetilde{f}, \alpha, \gamma)$ satisfy the hypotheses of Proposition 3.3 *part* (1) *or* (2) (*or* (3)), *and suppose that there is some $\omega > 0$ such that every γ-weak eigenvalue λ of $A - B$ satisfies* $\operatorname{Re} \lambda \geq \omega$. *Then $u_0 = 0$ is asymptotically stable in $H_\alpha = L_2(\Omega, \mathbb{C}^n)$ (or $H_\alpha = W^{1,2}(\Omega, \mathbb{C}^n)$) in the sense that for every $\varepsilon > 0$ there is $\delta > 0$ such that any γ-mild solution $u \in C([0,\infty), H_\alpha)$ of* (3.25), (3.26) *with $\|u(0,\cdot)\|_{H_\alpha} \leq \delta$ satisfies $\|u(t,\cdot)\|_{H_\alpha} \leq \varepsilon$ for all $t \geq 0$, and $\|u(t,\cdot)\|_{H_\alpha} \to 0$ exponentially fast as $t \to \infty$.*

If in addition $\widetilde{f}$ satisfies the hypothesis of Proposition 3.12 *part* (1) (*or* (3)), *then for every $u_0 \in H_\alpha$ there is a unique γ-mild solution $u \in C([0,\infty), H_\alpha)$ of* (3.25), (3.26) *with $u(0,\cdot) = u_0$.*

Remark 3.15. We emphasize that under the additional assumptions mentioned in Proposition 3.9, it suffices to consider eigenvalues of $A - B$ instead of γ-weak eigenvalues. Note that $A - B$ is actually independent of M (because the terms with M cancel).

Proof of Theorem 3.14. Assume first that $b_1 = \cdots = b_d = 0$. Then A is a Kato operator. Indeed, in case (C)(1), this follows from Proposition 2.15, because a is symmetric. In case (C)(2), this follows from the main result of [8], and in case (C)(3) this follows from the main result of [3] in view of [1].

Since neither the space H_1 nor its topology depends on M or b_j, we obtain from Proposition 2.15 that also the space $H_{\frac{1}{2}} \cong [H, H_1]_{\frac{1}{2}}$ does not depend on M or b_j, and so we obtain from the special case $b_1 = \cdots = b_d = 0$ also in the general case that A is a Kato operator.

Note that if the hypothesis of Proposition 3.3(1) is satisfied, then also the hypothesis of Proposition 3.9(1) is satisfied. Hence, the assertion follows from Theorem 2.13. □

Remark 3.16. In Theorem 3.14, the hypotheses of Proposition 3.3 part (1) or (2) can also be replaced by the hypothesis of Remark 3.4.

Example 3.17. Let $\Omega \subseteq \mathbb{R}^d$ be bounded with a Lipschitz boundary, $\Gamma_D, \Gamma_N \subseteq \partial\Omega$ be measurable with (3.24). Let $f_1, f_2 \colon \mathbb{C}^2 \to \mathbb{C}$ be continuous with $f_i(0) = 0$, and suppose that there are $L \geq 0$ and $\rho > 0$ with

$$|f_i(u) - f_i(v)| \leq L\,(1 + |u| + |v|)^\rho\,|u - v| \tag{3.28}$$

for all $u \in \mathbb{C}^2$. Assume that $(b_{i1}, b_{i2}) = f_i'(0)$ exist for $i = 1, 2$, are real, and satisfy the sign conditions

$$b_{11} > 0, \quad b_{11} + b_{22} < 0, \quad b_{11}b_{22} - b_{12}b_{21} > 0.$$

For $d_1, d_2 > 0$, we consider the reaction-diffusion system

$$\frac{\partial u_j}{\partial t} = d_j \Delta u_j + f_j(u_1, u_2) \quad \text{on } \Omega, \text{ for } j = 1, 2, \tag{3.29}$$

with mixed boundary conditions (for $u = (u_1, u_2)$)

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N. \tag{3.30}$$

Let $\kappa_k > 0$ $(k = 1, 2, \dots)$ denote the nonzero eigenvalues of Δ with boundary conditions (3.30); if Γ_D is a null set, do *not* include the trivial eigenvalue $\kappa_0 = 0$ into this sequence. Suppose (d_1, d_2) lies to the right/under the envelope of the hyperbolas

$$C_k = \{(d_1, d_2) : d_1, d_2 > 0 \text{ and } (\kappa_k d_1 - b_{11})(\kappa_k d_2 - b_{22}) = b_{12}b_{21}\},$$

that is, (d_1, d_2) belongs to

$$\bigcap_{k=1}^{\infty} \left\{ (d_1, d_2) : d_1 \geq \kappa_k^{-1} b_{11} \text{ or } d_2 < \frac{\kappa_k^{-2} b_{12} b_{21}}{d_1 - \kappa_k^{-1} b_{11}} + \frac{b_{22}}{\kappa_k} \right\}, \tag{3.31}$$

see Figure 3.1.

Consider the following two cases.

(1) $H_\alpha = L_2(\Omega, \mathbb{C}^2)$ and one of the following holds:
- (a) $\gamma \in [\frac{1}{2}, 1)$ and either $d = 1$, $\rho \leq 1$, or $d = 2$, $\rho < 1$, or $d \geq 3$, $\rho \leq \frac{2}{d}$;
- (b) $D(A)$ is continuously embedded into $L_p(\Omega, \mathbb{C})$, $\rho < 1 - \frac{2}{p}$, and $\gamma \in (0, 1)$ is sufficiently large;
- (c) $\rho \leq 1$, $\gamma \in (\frac{1}{2}, 1)$, and H_γ is continuously embedded into $L_\infty(\Omega, \mathbb{C})$;
- (d) $D(A)$ is continuously embedded into $W^{2,2}(\Omega, \mathbb{C})$, and either $d \leq 3$, $\rho \leq 1$, $\gamma \in (\frac{d}{4}, 1)$, or $d \geq 4$, $\rho < \frac{4}{d}$, and $\gamma \in (0, 1)$ is sufficiently large.

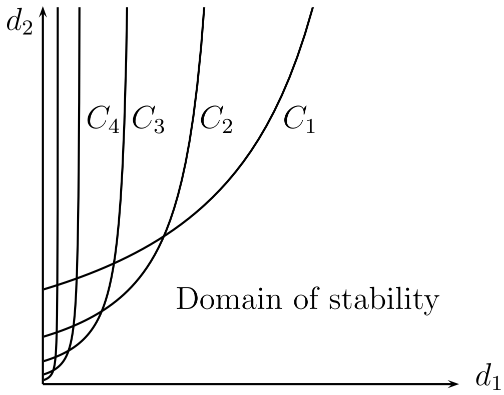

Figure 3.1: The hyperbolas C_k

(2) $H_\alpha = W^{1,2}(\Omega, \mathbb{C}^2)$ and one of the following holds:

(a) $d \le 2$, $\rho > 0$, $\gamma \in [0, \frac{1}{2})$;

(b) $d \ge 3$, $\rho < \frac{4}{d-2}$, $\gamma \in [\max\{0, \gamma_0\}, \frac{1}{2})$, where γ_0 is defined in (3.16) with $\sigma = \rho + 1$.

In cases (1) and (2) the following holds. For each $\varepsilon > 0$ there is $\delta > 0$ such that for each $u_0 \in H_\alpha$ with $\|u_0\|_{H_\alpha} \le \delta$ there is a unique γ-mild solution $u \in C([0,\infty), H_\alpha)$ of (3.29), (3.30) with $u(0,\cdot) = u_0$, $\|u(t,\cdot)\|_{H_\alpha} \le \varepsilon$ for all $t > 0$ and $\|u(t,\cdot)\|_{H_\alpha} \to 0$ exponentially as $t \to \infty$.

We first note that (1d) is actually a special case of (1b) and (1c) by the Sobolev embedding theorems and [11, Theorem 1.6.1], respectively. Since f_i is independent of x and t, hypothesis (3.7) follows with $\sigma = \rho + 1$ from (3.28) and from the definition of f_i'. Note also that the symmetry of A implies $D(A) = D(A^*)$ and $H_\gamma = H_\gamma^*$. The existence and uniqueness assertion follows from Proposition 3.12 or from Remark 3.13 in case (1c). For the stability assertion, we apply Theorem 3.14 or Remark 3.16 in case (1c) with $r = \infty$ and $\sigma = 1 + r$. In view of Proposition 3.9, it thus suffices to verify that there is $\omega > 0$ such that every eigenvalue λ of $A - B$ satisfies $\operatorname{Re}\lambda \ge \omega$. Under condition (3.31) the latter was verified in [23]. It can be shown by a similar calculation that if $d_i > 0$ violate (3.31) then there is an eigenvalue λ of $A - B$ with $\operatorname{Re}\lambda \le 0$ ($\lambda = 0$ if $(d_1, d_2) \in \bigcup_{k=1}^\infty C_k$). In this sense, the domain of stability sketched in Figure 3.1 is maximal.

Note that (1d) involves a strictly weaker requirement concerning ρ than (1a) for every $d \ge 2$. The embedding required for (1d) holds in case $\Gamma_D = \emptyset$ or $\Gamma_N = \emptyset$ if $\partial\Omega$ is sufficiently smooth.

In the space $H_\alpha = W^{1,2}(\Omega, \mathbb{C}^2)$ and $d \ge 3$, the result in [23] essentially needed the more restrictive hypothesis $\rho \le \frac{2}{d-2}$ which is (almost) by the factor 2 worse than our above requirement for that case. For instance, in case $\Gamma_D = \emptyset$ or

$\Gamma_N = \emptyset$ and smooth $\partial\Omega$, we can now treat nonlinearities like $f_i(u,v) = (|u|+|v|)^p$ or $f_i(u,v) = |v|^{p-1}v$ under the subcritical growth condition $p < \frac{d+2}{d-2}$ while previously this was possible only if $p \le \frac{d}{d-2}$. In particular, in space dimension $d = 3$, we can now treat polynomial nonlinearities of degree 4 (even "almost" of degree 5) while previously only degree 3 could be handled for that space.

Moreover, for the space $H_\alpha = L_2(\Omega, \mathbb{C}^2)$, we are not aware of any similar previous results of such a type while now we can treat at least polynomials of degree 2 in space dimension $d \le 3$.

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