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Harnack's Inequality for Stokes Graph

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Abstract. A metric graph model is suggested for the Stokes flow concentrated in a neighborhood of a network. Conditions of δ'-type are assumed at the graph vertices. Sign preserving solutions are studied. The analogue of the Harnack's inequality is proved.

Keywords. Stokes flow, metric graph, Harnack's inequality

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1. Introduction

Flows concentrated near a network represent a highly interesting class of fluid flows. It may be flows through a system of coupled narrow tubes (channels). Examples of such flows are presented in nanofluidics (flows through nanotubes [2, 18, 20]) and biophysics (blood flow [23–25]). Another situation is when we have flow concentration near lines or surfaces due to specific variation of viscosity. It is typical, for instance, for geophysical flows [7, 9]. Of course, spatial scales for these problems differ considerably (in many orders), "narrow" means small ratio of width and length. The problems are complicated and it is of a high relevance to construct models that allow one to simplify it.

At the same time, we observe rapid development of the quantum graph model, which is very effective method of complex quantum system studying. It was first introduced more than a half of century ago but the intensive research (especially, mathematical) started in 1980th ([4,6]). As for the state of the art in the field, see, e.g., [5]. The model has some relevant features. On the one hand, it allows one to get explicit solutions of the model problem, on the other hand, it gives one good approximation in many particular physical problems (see, e.g., [12, 13, 21]). This is a reason for seeking of a new field of application for such

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effective instrument. In this paper we make first steps to the development of a model analogues of the quantum graph for fluid mechanics.

There are several ways to consider the hydrodynamic equations on a network (metric graph). One can deal with the 1D Navier-Stokes equation. For example, for compressible fluid of constant viscosity $\widetilde{\eta}$ it takes the form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \widetilde{\eta}\frac{\partial^2 v_x}{\partial x^2} = 0.$$

It is a nonlinear equation. To get the corresponding linear one we can linearize it in a neighborhood of some solution or simply neglect the nonlinear inertial terms. It is possible to obtain the 1D Navier-Stokes equation by the asymptotic procedure for 3D or 2D Navier-Stokes equations in thin tubes (see, e.g., [15, 16]). It is also possible to study linear 2D or 3D Stokes equations in a system of narrow tubes and to consider the limiting procedure for the tubes widths tending to zero. This approach is similar to that for waveguides and thick quantum graphs (see, e.g., [14, 19, 22]). As a result of this procedure the most popular model for the Stokes flow in a network occurs. In the model it is assumed that there is the Poiseuille flow in the tubes. It allows one to reduce the problem to the description of the pressure distribution over the metric graph. The pressure is linear at the edges, and the continuity condition for the pressure at vertices occurs (see, e.g., good review in [10]). In our case the situation is more complicated (the flow is not localized inside tubes, the viscosity and the density varies essentially), and this model is not appropriate.

We will follow a method that is used in diffraction theory to describe solutions concentrated near some lines. New spatial variables with different scales (using multiplication by small parameter) are introduced and formal asymptotic expansion in this small parameter are considered in [1, 11]. As a result, we get 1D problem on a metric graph for the main term of the asymptotic expansion. We call it the Stokes graph. Let us briefly describe this procedure.

Consider the plane flow. 2D Stokes equations for the case of varying viscosity. In Cartesian coordinates $(\widetilde{x}, \widetilde{y})$ it has the form

$$2\widetilde{\eta}\frac{\partial^2 v_x}{\partial \widetilde{x}^2} + 2\frac{\partial \widetilde{\eta}}{\partial \widetilde{x}}\frac{\partial v_x}{\partial \widetilde{x}} + \widetilde{\eta}\frac{\partial^2 v_x}{\partial \widetilde{y}^2} + \widetilde{\eta}\frac{\partial^2 v_y}{\partial \widetilde{y}\partial \widetilde{x}} + \frac{\partial \widetilde{\eta}}{\partial \widetilde{y}}\frac{\partial v_x}{\partial \widetilde{y}} + \frac{\partial \widetilde{\eta}}{\partial \widetilde{y}}\frac{\partial v_y}{\partial \widetilde{x}} - \frac{\partial P}{\partial \widetilde{x}} = -\widetilde{\rho}G_x, \quad (1)$$

$$\widetilde{\eta}\frac{\partial^2 v_y}{\partial \widetilde{x}^2} + \widetilde{\eta}\frac{\partial^2 v_x}{\partial \widetilde{y}\partial \widetilde{x}} + \frac{\partial \widetilde{\eta}}{\partial \widetilde{x}}\frac{\partial v_x}{\partial \widetilde{y}} + \frac{\partial \widetilde{\eta}}{\partial \widetilde{x}}\frac{\partial v_y}{\partial \widetilde{x}} + 2\frac{\partial \widetilde{\eta}}{\partial \widetilde{y}}\frac{\partial v_y}{\partial \widetilde{y}} + 2\widetilde{\eta}\frac{\partial^2 v_y}{\partial \widetilde{y}^2} - \frac{\partial P}{\partial \widetilde{y}} = -\widetilde{\rho}G_y, \quad (2)$$

$$\frac{\partial(\widetilde{\rho}v_x)}{\partial \widetilde{x}} + \frac{\partial(\widetilde{\rho}v_y)}{\partial \widetilde{y}} = 0. \quad (3)$$

Here (v_x, v_y) is the flow velocity, $\widetilde{\eta} = \widetilde{\eta}(\widetilde{x}, \widetilde{y})$ is the viscosity, P is the pressure, $\widetilde{\rho} = \widetilde{\rho}(\widetilde{x}, \widetilde{y})$ is the density, (G_x, G_y) is the gravitational force. Note that (3) is the continuity equation.

At first, we will consider the flow in a narrow 2D channel. More precisely, we will deal with the case when the values of $\widetilde{\eta}$ and $\widetilde{\rho}$ are smooth and outside the strip $\{(\widetilde{x},\widetilde{y})\colon \widetilde{x}\in(-\infty,\infty), \widetilde{y}\in(-\varepsilon,\varepsilon)\}$ are essentially greater than inside it:

$$\widetilde{\eta}(\widetilde{x},\widetilde{y}) = \eta(\widetilde{x})\widetilde{\eta}_2(\widetilde{y}), \widetilde{\rho}(\widetilde{x},\widetilde{y}) = \rho(\widetilde{x})\widetilde{\rho}_2(\widetilde{y}),$$

$$\widetilde{\eta}_2(\widetilde{y}) = \begin{cases} \eta_c, & |\widetilde{y}| \le \varepsilon \\ \eta_w, & |\widetilde{y}| > 2\varepsilon \end{cases}, \quad \eta_w \gg \eta_c,$$

$$\widetilde{\rho}_2(\widetilde{y}) = \begin{cases} \rho_c, & |\widetilde{y}| \le \varepsilon \\ \rho_w, & |\widetilde{y}| > 2\varepsilon \end{cases}, \quad \rho_w \gg \rho_c.$$

Here ε is a small parameter. It means that the flow is indeed only inside the strip. We will consider it in the strip only. We introduce new coordinates $x = \widetilde{x}$, $y = \frac{\widetilde{y}}{\varepsilon}$. Equations (1)–(3) take the following form after the coordinate replacement:

$$2\widetilde{\eta}\frac{\partial^2 v_x}{\partial x^2} + 2\frac{\partial\widetilde{\eta}}{\partial x}\frac{\partial v_x}{\partial x} + \varepsilon^{-2}\widetilde{\eta}\frac{\partial^2 v_x}{\partial y^2} + \varepsilon^{-1}\widetilde{\eta}\frac{\partial^2 v_y}{\partial y\partial x} + \varepsilon^{-2}\frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_x}{\partial y} + \varepsilon^{-1}\frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_y}{\partial x} - \frac{\partial P}{\partial x} = -\widetilde{\rho}G_x, \tag{4}$$

$$\widetilde{\eta}\frac{\partial^2 v_y}{\partial x^2} + \varepsilon^{-1}\widetilde{\eta}\frac{\partial^2 v_x}{\partial y\partial x} + \varepsilon^{-1}\frac{\partial\widetilde{\eta}}{\partial x}\frac{\partial v_x}{\partial y} + \frac{\partial\widetilde{\eta}}{\partial x}\frac{\partial v_y}{\partial x} + 2\varepsilon^{-2}\frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_y}{\partial y} + 2\varepsilon^{-2}\widetilde{\eta}\frac{\partial^2 v_y}{\partial y^2} - \varepsilon^{-1}\frac{\partial P}{\partial y} = -\widetilde{\rho}G_y, \tag{5}$$

$$\frac{\partial(\widetilde{\rho}v_x)}{\partial x} + \varepsilon^{-1}\frac{\partial(\widetilde{\rho}v_y)}{\partial y} = 0. \tag{6}$$

We will seek the solutions in the form of series in ε:

$$v_x = v_x^0 + v_x^1\varepsilon + \cdots, \quad v_y = v_y^0 + v_y^1\varepsilon + \cdots, \quad P = P^0 + P^1\varepsilon + \cdots.$$

Let us insert the series into equations (4), (5), (6) and collect terms of the same powers of ε. We get the chain of equations for the series coefficients. Terms of order ε^{-2} are in the Stokes equations (in new variables) only:

$$\widetilde{\eta}\frac{\partial^2 v_x^0}{\partial y^2} + \frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_x^0}{\partial y} = 0, \quad \widetilde{\eta}\frac{\partial^2 v_y^0}{\partial y^2} + \frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_y^0}{\partial y} = 0.$$

These simple equations give us

$$v_x^0 = \frac{f_x(x)}{\eta(x)}\int\frac{dy}{\widetilde{\eta}_2(y)} = u(x)\int\frac{dy}{\widetilde{\eta}_2(y)}, \quad v_y^0 = \frac{f_y(x)}{\eta(x)}\int\frac{dy}{\widetilde{\eta}_2(y)}. \tag{7}$$

Here $u(x), f_y(x)$ are several functions of one variable. They should be determined later. Terms of order ε^{-1} are in three equations. The continuity equation gives us: $v_y^0 = \frac{g_y(x)}{\widetilde{\rho}}$. In order not to have contradiction with (7) we conclude that $v_y^0 = 0$, i.e., the series for v_y starts from another power than for v_x. It is a conventional situation (see, e.g., [26]). The Stokes equations give us the following relations for the terms of this order:

$$\widetilde{\eta}\frac{\partial^2 v_x^1}{\partial y^2} + \frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_x^1}{\partial y} + \widetilde{\eta}\frac{\partial^2 v_y^0}{\partial y\partial x} + \frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_y^0}{\partial x} = 0,$$
$$2\widetilde{\eta}\frac{\partial^2 v_y^1}{\partial y^2} + 2\frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_y^1}{\partial y} + \widetilde{\eta}\frac{\partial^2 v_x^0}{\partial y\partial x} + \frac{\partial\widetilde{\eta}}{\partial x}\frac{\partial v_x^0}{\partial y} - \frac{\partial P^0}{\partial y} = 0.$$

These equations are solvable in respect to v_x^1, v_y^1. The pressure term P^0 should be determined later.

Consider the terms of zero order. We have the following three equations:

$$\frac{\partial\widetilde{\rho}}{\partial x}v_x^0 + \widetilde{\rho}\frac{\partial v_x^0}{\partial x} + \frac{\partial\widetilde{\rho}}{\partial y}v_y^1 + \widetilde{\rho}\frac{\partial v_y^1}{\partial y} = 0,$$
$$2\widetilde{\eta}\frac{\partial^2 v_x^0}{\partial x^2} + 2\frac{\partial\widetilde{\eta}}{\partial x}\frac{\partial v_x^0}{\partial x} + \widetilde{\eta}\frac{\partial^2 v_x^2}{\partial y^2} + \widetilde{\eta}\frac{\partial^2 v_y^1}{\partial y\partial x} + \frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_x^2}{\partial y} + \frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_y^1}{\partial x} - \frac{\partial P^0}{\partial x} = -\widetilde{\rho}G_x,$$
$$\widetilde{\eta}\frac{\partial^2 v_x^1}{\partial y\partial x} + \frac{\partial\widetilde{\eta}}{\partial x}\frac{\partial v_x^1}{\partial y} + 2\frac{\partial\widetilde{\eta}}{\partial y}\frac{\partial v_y^2}{\partial y} + 2\widetilde{\eta}\frac{\partial^2 v_y^2}{\partial y^2} - \frac{\partial P^1}{\partial y} = -\widetilde{\rho}G_y.$$

At present, we confine our attention by the first term of the asymptotic series for the velocity. The pressure is determined at the next stage (by taking into account the solvability condition ($\frac{\partial^2 P^0}{\partial y\partial x} = \frac{\partial P^0}{\partial x\partial y}$)). Here we insert $v_y^0 = 0$. The last equation does not contain information about v^0. But the first two equations give us an interesting correlation. Taking into account the expressions for $\widetilde{\eta}, \widetilde{\rho}, v_x^0$, one can see that it leads to the following equation for the function $u(x)$:

$$u - \frac{\eta'}{\eta}\frac{\rho'}{\rho}u = -\frac{\rho}{\eta}G_x. \tag{8}$$

Particularly, in the homogeneous case ($G_x = 0$)

$$u" - \frac{\eta'}{\eta}\frac{\rho'}{\rho}u = 0. \tag{9}$$

we have the 1D Schrödinger equation with a specific potential for the function $u(x)$ corresponding to zero energy (the similar problem for the quantum graph is known as the threshold resonance, see, e.g., [3]).

Thus, for the flow concentrated near a line we have one-dimensional model. If our flow is concentrated near a network (system of coupled segments) we can consider the corresponding metric graph with the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + \frac{\eta'}{\eta}\frac{\rho'}{\rho}$$

on the edges for the description of the flow. We call it the Stokes graph (Γ). It is necessary to determine boundary conditions at the graph vertices. Consider a vertex (let it be zero point) with n output edges. From physical conditions one has

$$\rho_1(0) = \rho_2(0) = \cdots = \rho_n(0) = \rho(0)$$

and

$$u_1{}'(+0) = u_2{}'(+0) = \cdots = u_n{}'(+0) = u'(0).$$

The last condition is related with the pressure continuity (see, e.g., [8, 10]). Here $u_j{}'(+0)$ is the derivative in the outgoing direction at the vertex 0. The continuity equation gives us for this vertex:

$$\sum_{j=1}^{n} u_j = -\left(\frac{\rho(0)}{\sum_{j=1}^{n} \rho_j'(+0)}\right) u'(0). \tag{10}$$

It is similar to well-known δ'-coupling condition for the quantum graph [12]. The coupling constant is related with the density derivative.

We will consider the following case: $q = \frac{\eta'}{\eta}\frac{\rho'}{\rho}$ is uniformly continuous on every edge of the graph Γ (we mark the set of its edges as E and the set of its vertices as V),

$$\theta = \frac{\rho(0)}{\sum_{j=1}^{n} \rho_j'(+0)}$$

is positive. We will deal with the following homogeneous equation:

$$(Hu)(x) = 0. \tag{11}$$

Here operator H is defined as follows. At each edge it acts as

$$(Hu)(x) = -\frac{d^2}{dx^2} u(x) + q(x)\, u(x), \quad x \in E(\Gamma). \tag{12}$$

Elements from its domain belong to the Sobolev space H^2 at each edge. As for vertices, the following condition takes place (for a vertex x having n output edges):

$$u_1{}'(+x) = u_2{}'(+x) = \cdots = u_n{}'(+x) = u'(x), \quad x \in V(\Gamma), \tag{13}$$

$$\sum_{j=1}^{n} u_j(x) + \theta u'(x) = 0, \qquad x \in V(\Gamma), \ \ \theta > 0. \tag{14}$$

$$u(a) = 0, \qquad a \in \partial\Gamma. \tag{15}$$

Definition. Operator H is denoted as sign preserving on the graph Γ if the inequality $u(x)\, u(x') > 0$ takes place for any nontrivial solution of equation (11). Here x, x' are arbitrary points of $\Gamma \setminus \partial\Gamma$.

The main result of the paper is the following inequality (16) that is analogous to the Harnack's inequality for an elliptic operator on a manifold.

Theorem (Main Theorem)**.** *Let H be sign preserving on graph Γ. Then there exists a constant γ, determined by the operator H and the structure of the graph Γ only, such that each non-negative on Γ solution $u(x)$ of inequality $Hu \geq 0$ satisfies the following inequality:*

$$\max_{x\in\Gamma_0} u(x) \leq \gamma \min_{x\in\Gamma_0} u(x), \tag{16}$$

on any locally compact (in respect to Γ) subgraph Γ_0, $\Gamma_0 \subset \Gamma$.

2. Sign preserving solution

Theorem 2.1. *Any sign preserving solution $u(x)$ of the homogeneous equation (11), satisfying conditions (13)–(15), belongs to one of the following two types:*

(i) *it is trivial ($\equiv 0$), or*

(ii) *it doesn't have zeros in Γ.*

In the last case if $u(a) = 0$ where $a \in \partial\Gamma$ then $u'(a) \neq 0$.

Proof. Consider u, $u(x) \geq 0$, and let Q be a set of zeros of $u(x)$ in Γ. If Q is not empty then it is relatively closed in Γ. It remains to show that if Q is not empty then it is an open set in Γ. Hence, we get that $Q = \Gamma$.

Let $\tilde{x}$ be a point of the set Q. If $\tilde{x}$ is an inner point of any edge e_{i0} then $\tilde{x}$ is a minimum point for $u_{i0}(x)$ on e_{i0}. Consequently, $u_{i0}(\tilde{x}) = 0$, $u'_{i0}(\tilde{x}) = 0$, $u(x) \equiv 0$. Thus $e_{i0} \subseteq Q$.

Let $\tilde{x}$ be an inner vertex. One has $\sum_{j=1}^{n} u_j(\tilde{x}) + \theta u'(\tilde{x}) = 0$, $\theta > 0$ and $u_1(\tilde{x}) = 0$, $u_k(\tilde{x}) \geq 0$, where $k = 2, \ldots, n$. Consequently, $u_k(\tilde{x}) = 0$, and $u'(\tilde{x}) = 0$, correspondingly, $u(x) \equiv 0$ on the whole adjoining to $\tilde{x}$ edges. Thus $\tilde{x}$ is an inner point of Q. Then Q is open in Γ.

If $u(a) = 0$ where $a \in \partial\Gamma$ and $u'(a) = 0$ then on the edge adjoined to a one has $u(x) \equiv 0$ on this edge and it leads to the situation described above. Hence, $u'(a) \neq 0$.

Thus, the theorem is proved. □

Theorem 2.2. *Let $u(x)$ be a nontrivial solution of equation (3) and Γ_0 is a subgraph, where the solution is sign preserving. Then any sign preserving solution $\nu(x)$ of equation (3) is proportional to $u(x)$ on Γ_0.*

The proof of the theorem is based on Theorem 2.1 and is similar to the corresponding statement in [17].

Corollary 2.3. *If $u(x)$ is a solution of equation $Hu = 0$ having no zeros inside Γ then for any linearly independent solution v of the same equation, the ratio $\frac{v}{u}$ has no global maximum or global minimum inside the Γ.*

One can see that if H is sign preserving on Γ then the problem:

$$H(u) = f, \quad u(a) = 0, \quad a \in \partial\Gamma \tag{17}$$

is nonsingular.

Let problem (17) be nonsingular. For every boundary vertex $\tau \in \partial\Gamma$ we consider the following problem:

$$H(u) = 0, \quad u(\tau) = 1, \quad u(x) = 0 \quad (x \in \partial\Gamma,\ x \neq \tau). \tag{18}$$

Denote the solution of the problem (18) as u_τ.

Theorem 2.4. *If $\partial\Gamma \neq \emptyset$ then the following properties are equivalent:*

a) *Each of the problem* (18) *has nonnegative on Γ solution.*
b) *There exists a positive on $\Gamma \cup \partial\Gamma$ solution w of equation* (11) *and* $\inf_\Gamma w > 0$
c) *There exists a nonnegative on Γ solution that does not vanish at any points of $\partial\Gamma$.*
d) *Equation* (11) *(with conditions* (13)–(15)*) is sign preserving on Γ.*
e) *There exists a function $h(x)$ such that $\inf_\Gamma h > 0$. For every u:*

$$(Hu)(x) = -\frac{1}{h(x)}\frac{d}{dx}\left(h^2(x)\frac{d}{dx}\frac{u(x)}{h(x)}\right), \quad x \in E(\Gamma). \tag{19}$$

Proof. Under the assumption of a), every solution u_τ is strictly positive on Γ according to the Theorem 2.1. Consequently, the corresponding sum over all τ, $\tau \in \partial\Gamma$ has no zeros in $\partial\Gamma$. So a) $\Rightarrow$ b). It is obvious that b) $\Rightarrow$ c). According to Theorem 2.2, c) $\Rightarrow$ d). Under the condition of d) every $u_\tau(x)$ exists due to the fact that the problem is not singular, and can't have subgraphs with negative sign preserving solution, therefore can't have negative values. So d) $\Rightarrow$ a).

The representation (19) from e) is correct if one takes $h = w$, where w is from b). Consequently, b) $\Rightarrow$ e). Vice versa if H is represented as (19) with any $h(x) \in C^2(\Gamma)$ uniformly positive on Γ, then $(Hu)(x) \equiv 0$, it fulfills condition b). Thus, the theorem is proved. □

Theorem 2.5. *Let H be sign preserving inside Γ. If for non-zero u, the following inequalities take place:*

$$(Hu)(x) \geq 0, \quad u(x) \geq 0 \quad (x > 0) \quad in\ \Gamma,$$

then $u(x) > 0$ in Γ. If $u(a) = 0$, where $a \in \partial\Gamma$, then $u'(a) \neq 0$.

Proof. Let there exist $u(x)$ (u is not identically zero) that isn't positive on the whole Γ. Then $\Omega = \{x : u(x) > 0\}$ doesn't coincide with Γ. Let Ω_0 be any connected component of Ω. As Ω_0 is a subgraph of Γ, $\Omega_0 \neq \Gamma$, then $\partial\Omega_0$ contains a point $\tilde{x} \notin \partial\Gamma$ and so $\tilde{x} \in \Gamma$, at this point one has $u(\tilde{x}) = 0$.

The point $\tilde{x}$ is a minimum point of u in Γ. If $\tilde{x}$ belongs to any edge of Γ, then $u'(\tilde{x}) = 0$ (independently on the orientation chosen in neighborhood of $\tilde{x}$).

If $\tilde{x}$ coincides with a vertex, then condition (14) takes place:

$$\sum_{j=1}^{n} u_j(\tilde{x}) + \theta u'(\tilde{x}) = 0,$$

Hence, $u'(\tilde{x}) \geq 0$ on all of the edges, $u_1(\tilde{x}) = 0$, $u'_k(\tilde{x}) \geq 0$ (where $k = 2, \ldots, n$). Consequently,

$$u_k(\tilde{x}) = 0, \quad u'(\tilde{x}) = 0,$$

Hence, $u(x) \equiv 0$ on the whole edge adjoining to $\tilde{x}$. As Ω_0 does not coincide with Γ, then H is sign preserving on Ω_0. The solution $w(x)$ of the equation (11) exists according to Theorem 2.4b) and it is uniformly positive on Ω_0. It is obvious that $w(\tilde{x}) > 0$.

For any edge $e_0 \in \Omega_0 \subseteq \Gamma$ adjoining to $\tilde{x}$ ($\in \partial\Omega_0$) we set orientation in the line out of $\tilde{x}$ and take representation (19) from Theorem 2.4e). Inequality $Hu \geq 0$ means that the function $\phi(x) \equiv h^2\left(\frac{u_{e_0}}{h(x)}\right)$ satisfies the inequality: $\phi'(x) \leq 0$, that's why $\phi(x)$ is non-increasing on e_0.

Since $u_k(\tilde{x}) = 0$, $u'(\tilde{x}) = u'_k(\tilde{x}) = 0$ on e_0 then $\phi(\tilde{x}) = 0$. Consequently, $\phi(\tilde{x}) \geq 0$ on e_0. Then, $\frac{u_{e_0}}{h(x)}$ is non-increasing on e_0. Thus, Ω_0 can't differ from Γ and $u(x) > 0$ on Γ.

If $u(\tilde{z}) = 0$ for $\tilde{z} \in \partial\Gamma$ and $u'(\tilde{z}) = 0$ then in the neighborhood of $\tilde{z}$ (on the edge adjoint to $\tilde{z}$) one can use the previous considerations that don't lead to a contradiction with the inequality $Hu \geq 0$. The theorem is proved. □

Theorem 2.6. *Let H be sign preserving inside Γ. Then any nontrivial solution $u(x)$ of the inequality $Hu \geq 0$, (i.e., the solution that has nonnegative values on the boundary $\partial\Gamma$ and $u(a) \geq 0$ for any $a \in \partial\Gamma$) is strictly positive inside Γ.*

The proof of the theorem is based on Theorem 2.5 and is similar to the corresponding statement in [17].

Consider the properties of the Green function. It is evident that the Green function exists for sign preserving operator H. Let a function u, $u \in C^2(\Gamma)$ (u isn't identically zero) satisfy the conditions: $(Hu)(x) \geq 0$, $u(x) \geq 0$ ($x \in \Gamma$). If $u(a) = 0$ for $a \in \partial\Gamma$, then $u'(a) \neq 0$.

Theorem 2.7. *Let H be sign preserving inside Γ then the Green function $G(x,s)$ obeys the following properties:*

(i) $G(x,s) > 0$.

(ii) $G(x,s)$ *is continuous function on* $\Gamma \setminus V(\Gamma)$.

Proof. According to the Green function's definition and Theorem 2.5 we can represent each solution u of the differential inequality (see above) in the form:

$$u(x) = \int_\Gamma G(x,\xi) f(\xi)\, d\xi > 0,$$

where $f(x)$ in nonnegative function, $f \in C[E(G)]$.

Consider $G(x,\xi)$, $\xi \in e_i$, $x \in e_i$ (where e_i is the edge of the graph Γ). According to the definition, $G(x,\xi)$ is continuous in respect to x and ξ on all edges of the graph Γ. Let us fix a point $\xi_0 \in e_i$, where $\xi_0 \notin V(G)$ ($G(x,\xi_0)$ is naturally continuous on e_i). Let us assume that there exists such point x_0 that: $G(x_0,\xi_0) < 0$. Then $G(x_0,\xi) < 0$ in some neighborhood of the point ξ_0 (due to the continuity property): $(\xi_0 - \varepsilon, \xi_0 + \varepsilon)$, hence there exists a function f such that $f(\xi) > 0$ if $\xi \in (\xi_0 - \varepsilon, \xi_0 + \varepsilon)$ and $f(\xi) = 0$ if $\xi \notin (\xi_0 - \varepsilon, \xi_0 + \varepsilon)$. So

$$u(x) = \int_\Gamma G(x,\xi) f(\xi)\, d\xi < 0,$$

hence, we came to a contradiction, and, consequently, $G(x,s) > 0$.

Assume that $G(x_0,\xi_0) = 0$. One can present the Green function $G(x_0,\xi_0)$ in the following well known form:

$$G(x_0,\xi_0) = \begin{cases} \dfrac{y_1(x_0)\, y_2(\xi_0)}{W(\xi_0)}, & 0 \le x_0 \le \xi_0, \\ \dfrac{y_2(x_0)\, y_1(\xi_0)}{W(\xi_0)}, & \xi_0 \le x_0 \le l, \end{cases} \tag{20}$$

where y_1, y_2 are linearly independent solutions that satisfy the boundary conditions at different ends of the edge, W is its Wronskian. As the solution is sign preserving then there are no zeros of the function $G(x_0,\xi_0)$ inside the graph Γ. That is why $G(x,\xi) \neq 0$ inside the graph.

By a similar way, one can show that at the vertices of the graph Γ the Green function is positive: $G(x_0,\xi_0) > 0$. Let $\xi_0 \in V(G)$ and $G(x,\xi_0) = 0$ where $x \in e_i$, $x \to \xi_0$. Then, taking into account formula (20) for $G(x_0,\xi_0)$, one obtains that $u_j(\xi_0) = 0$ (where $j = 1,\ldots,n$) on this edge and it gives us a contradiction.

Thus, the Green function $G(x,\xi)$ is continuous on the edges of graph Γ and positive on Γ. The theorem is proved. $\square$

3. Harnack's inequality

In this section we will prove the Main Theorem. To obtain inequality (16) the following lemmas and theorems are needed.

Lemma 3.1. *Let H be sign preserving on Γ and let $w(x)$ be a uniformly positive solution of equation $Hu = 0$. Then, for any point $\xi \in \Gamma \cup \partial\Gamma$ function*

$$\frac{1}{w(x)} \frac{G(x,\xi)}{G(\xi,\xi)}$$

has the maximum value only if $x = \xi$.

Proof. Note that such function w exists according to the Theorem 2.4. For every connected component Γ_0 of the set $\Gamma \setminus \{\xi\}$, the point ξ is a boundary point: $\xi \in \partial\Gamma_0$. According to Corollary 2.3 of Theorem 2.2, the function $\frac{1}{w(x)} \frac{G(x,\xi)}{G(\xi,\xi)}$ can't have extremes inside Γ_0.

As $\frac{G(x,\xi)}{G(\xi,\xi)} = 0$ for every $x \in \partial\Gamma_0$ except $x = \xi$ then ξ is the only maximum point in $\Gamma_0 \cup \partial\Gamma_0$. □

Lemma 3.2. *Let H be sign preserving on Γ. Then*

$$\frac{G(x,\xi)}{G(\xi,\xi)} \quad \textit{is a majorant of} \quad \frac{G(x,\eta)}{G(\eta,\eta)}$$

if and only if

$$\frac{G(\eta,\eta)}{G(\eta,\eta)} \le \frac{G(\eta,\xi)}{G(\eta,\xi)}.$$

Lemma 3.3. *Let H be sign preserving on Γ and $e = (a,b)$ be an edge of Γ. Then, for any point $\xi \in e$ the following inequalities take place:*

$$\frac{G(x,a)}{G(a,a)} \ge \frac{G(\xi,a)}{G(a,a)} \frac{G(x,\xi)}{G(\xi,\xi)} \ge \frac{G(b,a)}{G(a,a)} \frac{G(x,b)}{G(b,b)}, \tag{21}$$

where $x \in \Gamma$, $a, b \in V(\Gamma)$.

Proof. The first inequality follows from Lemma 3.2. It is enough to compare values of

$$\frac{G(\cdot,a)}{G(a,a)} \quad \text{and} \quad \frac{G(\xi,a)}{G(a,a)} \frac{G(\cdot,\xi)}{G(\xi,\xi)}$$

at the point ξ.

To prove the next inequality, according to the same lemma, it is enough to show that

$$\frac{G(b,a)}{G(a,a)} \frac{G(b,\xi)}{G(\xi,\xi)} = \frac{G(b,a)}{G(a,a)} \frac{G(b,b)}{G(b,b)}.$$

It will be proved if

$$\frac{G(x,a)}{G(a,a)} = \frac{G(\xi,a)}{G(a,a)}\frac{G(x,\xi)}{G(\xi,\xi)}, \quad \text{where } x \in (\xi, b).$$

Consider the set $\Gamma \setminus \{\xi\}$ and its connected component Γ_0 that contains (ξ, b). The difference $\frac{G(\xi,a)}{G(a,a)}\frac{G(x,\xi)}{G(\xi,\xi)} - \frac{G(x,a)}{G(a,a)}$ satisfies (on Γ_0) the equation $Hu = 0$ at all points and has zero values on $\partial\Gamma_0$ including the point ξ. That's why it should be zero. □

Lemma 3.4. *Let H be sign preserving on Γ. Then there exists a strictly positive on Γ function $g_0(x) \in C(\Gamma)$ that gives a bound from below for all ratios*

$$\frac{G(x,\xi)}{G(\xi,\xi)} \quad \textit{on } \Gamma.$$

Proof. If e is any edge of Γ and $e = (a, b)$ then, according to the inequality (21)

$$\frac{G(x,\xi)}{G(\xi,\xi)} \geq \frac{1}{\frac{G(x,a)}{G(a,a)}}\frac{G(b,a)}{G(a,a)}\frac{G(x,b)}{G(b,b)},$$

where $\xi \in e, b \in V(\Gamma)$. Such choice provides the separateness from zero for $\frac{G(x,a)}{G(a,a)}$ when $\xi \in e$. Thereby for any edge e from Γ the ratio $\frac{G(x,b)}{G(b,b)}$ (where $b \in V(\Gamma)$) is such that for any $k = k(e)$ the following inequality takes place:

$$\frac{G(x,\xi)}{G(\xi,\xi)} \geq k(\gamma)\frac{G(x,b)}{G(b,b)}.$$

Due to the finiteness of the numbers of edges and vertices the lemma takes place for any $k_0 > 0$ and

$$g_0(x) = \min_{b\in V(\Gamma)} \frac{G(x,b)}{G(b,b)}.$$

Thus the proof is finished. □

Lemma 3.5. *Let H be sign preserving on Γ. Then for any nonnegative on Γ solution $u(x)$ of inequality $Hu \geq 0$ the following inequality is valid:*

$$u(x) \geq u(\xi)\frac{G(x,\xi)}{G(\xi,\xi)}, \quad \textit{where } x, \xi \in \Gamma. \tag{22}$$

Proof. On every connected component Γ_0 of the set $\Gamma \setminus \{\xi\}$ the function

$$h(x) = u(x) - u(\xi)\frac{G(x,\xi)}{G(\xi,\xi)}$$

is a solution of $Hu \geq 0$ and has nonnegative on $\partial\Gamma$ values. According to Theorem 2.6, function $h(x)$ is strictly positive inside Γ. □

Theorem 3.6. *Let H be sign preserving on Γ. Then, any nonnegative on $\partial\Gamma$ solution u of $Hu \geq 0$ satisfies the inequality:*

$$u(x) \geq ||u|| g_0(x), \tag{23}$$

where $x \in \Gamma$ and $||u|| = \sup_\Gamma u(x)$, g_0 is a positive on Γ function,

$$g_0(x) = \min_{b \in V(\Gamma)} \frac{G(x,b)}{G(b,b)}.$$

Proof. Under the conditions of the theorem, one has $u(x) > 0$ on Γ (according to Theorem 2.6). Let s be a maximum point of $u(x)$ on $\Gamma \cup \partial\Gamma$: $||u|| = u(s)$. Then, according to Lemma 3.5, we have:

$$u(x) \geq u(s) \frac{G(x,s)}{G(s,s)} = ||u|| \frac{G(x,s)}{G(s,s)} \geq ||u|| \min_{b \in V(\Gamma)} \frac{G(x,b)}{G(b,b)} = ||u|| g_0(x).$$

The proof is finished. □

Corollary 3.7. *Under the conditions of Theorem 3.6, for any locally compact (with respect to Γ) subgraph Γ_0, there exists a constant $\chi = \chi(\Gamma_0, H)$ such that for every nonnegative on $\partial\Gamma$ solution of the inequality $Hu \geq 0$ the Harnack's inequality takes place:*

$$\max_{\Gamma_0} u(x) \leq \chi \min_{\Gamma_0} u(x). \tag{24}$$

Proof. The proof is evident. It is enough to set $x = x_0$ in the inequality

$$u(x) \geq ||u|| g_0(x),$$

where x_0 is a minimum point of function u on Γ_0. □

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