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Partial Regularity of Polyharmonic Maps to Targets of Sufficiently Simple Topology

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Abstract. We prove that polyharmonic maps $\mathbb{R}^m \supset \Omega \to N$ locally minimizing $\int |D^k f|^2 \, dx$ are smooth on the interior of Ω outside a closed set Σ with $\mathcal{H}^{m-2k}(\Sigma) = 0$, provided that the target manifold $N \subset \mathbb{R}^n$ is smooth, closed, and fulfills

$$\pi_1(N) = \ldots = \pi_{2k-1}(N) = 0.$$

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1. Introduction

Polyharmonic maps are a natural generalization of harmonic maps. Given two Riemannian manifolds M and N, and a bounded domain $\Omega \subset M$, a harmonic map $f : \Omega \to N$ is a map which is stationary for the Dirichlet energy

$$E(f) := \frac{1}{2} \int_\Omega |Df|^2 \, dx.$$

Here dx means integration with respect to the Riemannian measure on M. In this paper, we will always assume that $M = \mathbb{R}^m$. Our methods carry over to the non-flat case, but that would make the paper considerably more difficult to read.

Assuming that $N \subset \mathbb{R}^n$ is a sufficiently smooth submanifold of $\mathbb{R}^n$, we define, for $k \in \mathbb{N}$ and $p > 1$, energies

$$E^{k,p}(f) := \frac{1}{p} \int_\Omega |D^k f|^p \, dx,$$

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where here $D^k f$ means the full k-th derivative of f regarded as a mapping $f : \Omega \to \mathbb{R}^n$. The problem of understanding maps stationary for $E^{k,p}$ among $W^{k,p}$-maps $f : \Omega \to N$ generalizes the harmonic map problem for $E = E^{1,2}$. The special case $k = 1$ is that of *p-harmonic maps*, while the case $p = 2$, $k \geq 2$ gives (*extrinsically*) *k-polyharmonic maps*; they are called *biharmonic* if $k = 2$.

Most of the regularity theory of harmonic maps has been carried over successfully to the biharmonic case. The usual assumptions that ensure regularity are that the maps are either *stationary*, i.e. stationary with respect to variations in the domain, or *locally minimizing* the corresponding energy. Here is an overview over the most important results. (We will not mention the sub-critical domain dimensions, where regularity is relatively easy to prove. Moreover, we restrict to interior regularity and smooth compact targets. We denote the domain dimension by m.)

- All harmonic maps are smooth if $m = 2$, see [11].
- All biharmonic maps are smooth if $m = 4$, see [18].
- If $m \geq 3$, all stationary harmonic maps are smooth outside a closed singular set Σ satisfying $\mathcal{H}^{m-2}(\Sigma) = 0$, see [3].
- If $m \geq 5$, all stationary biharmonic maps are smooth outside a closed singular set Σ satisfying $\mathcal{H}^{m-4}(\Sigma) = 0$, see [6, 17, 19].
- If $m \geq 3$, all locally minimizing harmonic maps are smooth outside a closed singular set Σ of Hausdorff dimension $\leq m - 3$, see [16].
- If $m \geq 5$, all locally minimizing biharmonic maps are smooth outside a closed singular set Σ of Hausdorff dimension $\leq m - 5$, see [15].

Surprisingly, most of this list cannot be continued for polyharmonic maps, i.e. for $E^{k,2}$ with $k \geq 3$. What we have is a result by Scheven and the author.

- All k-polyharmonic maps are smooth if $m = 2k$, see [7].

(A simpler proof for sphere targets has been given in [9], and a very different proof for minimizers in [14]. Boundary regularity can be found in [12].)

The reason for the lack of results in the supercritical dimensions $m > 2k$ is that no *monotonicity formula* is known. Such a monotonicity formula allows one to work in a Morrey space where the nonlinearity of the Euler-Lagrange equation for $E^{k,2}$ is critical in some sense. It is available for $k = 1$ and $k = 2$, the latter formulated in [6] and proven in [1]. The proofs rely on showing that $\frac{d}{dr}(r^{2k-n} \int_{B_r} |D^k f|^2 \, dx)$ has a sign, up to terms that can be controlled. But already for $k = 3$, there seems to be no obvious way in which this continues to hold. Blatt [4] shows that some monotonicity formula holds if $m \leq 20$, but there are terms that have the wrong sign in higher dimensions. Calculations show that for $k \geq 4$ things are getting even more difficult.

Without a monotonicity formula as a starting point, there are not many methods left to prove partial regularity. The aim of this paper is to do so at least under additional topological assumptions on the target manifold. Here is what we will prove in Theorem 3.3.

- If $m \geq 2k+1$, all locally minimizing k-polyharmonic maps are smooth outside a closed singular set Σ satisfying $\mathcal{H}^{m-2k}(\Sigma) = 0$, provided that N satisfies $\pi_1(N) = \ldots = \pi_{2k-1}(N) = 0$.

Note that this allows, for example, $N = S^\ell$ for any $\ell \geq 2k$. To the best of the author's knowledge, this is the first partial regularity result for k-polyharmonic maps ($k \geq 3$), except for the case $k = 3$, $m \leq 20$ discussed by Blatt [4]. An ε-regularity theorem under maybe slightly unnatural assumptions has been proven by Angelsberg and Pumberger [2].

The key observation that such a topological condition simplifies the regularity question for minimizers is taken from work by Hardt and Lin [10], where they prove partial regularity for minimizing p-harmonic maps. In the regularity proof, it is important to construct suitable comparison maps for minimizers, and this is much easier if there is a projection $\mathbb{R}^n \to N$ that is good enough not to destroy $W^{1,p}$-estimates. We adopt this technique, even though we cannot construct a projection that is well-behaved on $W^{k,2}$. But combinig the methods of [10] with rather recent ones by Bousquet, Ponce and Van Schaftingen [5], we find a projection that is at least well-behaved on $W^{k,2} \cap W^{1,2k}$, which is good enough for our purposes.

Once we have a good projection, we can use $\mathbb{R}^n$-valued comparison maps, and the partial regularity proof from here follows well-established lines similar to [13]. Note, however, that our result does not yield locally finite $(m-2k-1)$-dimensional Hausdorff measure of the singular set, as might have been expected from the harmonic and biharmonic cases. We get stuck at $\mathcal{H}^{m-2k}(\Sigma) = 0$, since the better result would require to perform "Federer's dimension reduction argument", which heavily relies on having a monotonicity formula.

2. Projections to submanifolds

We start our considerations with the question how to project mappings into $\mathbb{R}^n$ to some submanifold N without losing estimates in Sobolev spaces. This follows ideas by Hardt and Lin [10]. We occasionally write $\pi_0(N) = 0$ to express the fact that N is connected. The other $\pi_i(N)$ are the usual homotopy groups (with any base point in N).

Proposition 2.1. *Let N be a smooth compact Riemannian submanifold of $\mathbb{R}^n$ with*

$$\pi_0(N) = \pi_1(N) = \ldots = \pi_j(N) = 0$$

for some $j \in \{0, \ldots, n-2\}$. *Then there exists a finite union* X *of closed* $(n-j-2)$*-dimensional cubes in* $\mathbb{R}^n$ *and a smooth retraction* $P : \mathbb{R}^n \setminus X \to N$ *such that*

(i) *there is some* $R > 0$ *such that* $|D^i P|$ *is bounded on* $\mathbb{R}^n \setminus B_R(0)$*, and we have the estimate* $|D^i P(x)| \leq C(i)\mathrm{dist}(x, X)^{-i}$ *for all* $i \in \mathbb{N}$ *and all* $x \in B_R(0)$*;*

(ii) *there is some* $r > 0$ *such that the restriction of* P *to* $B_r(N)$ *is the nearest point retraction to* N.

In order to prove Proposition 2.1, we try to mimic an argument by Hardt and Lin from [10] for the $k = 1$ case, which, combined with some assertions from its proof, reads as follows.

Lemma 2.2 ([10, Lemma 6.1])**.** *If* N *is a compact smooth submanifold of* $\mathbb{R}^n$ *with*

$$\pi_0(N) = \pi_1(N) = \ldots = \pi_j(N) = 0,$$

then there exist a compact $(n-j-2)$*-dimensional Lipschitz polyhedron* Y *in* $\mathbb{R}^n$ *and a locally Lipschitz retraction* $\widetilde{P} : \mathbb{R}^n \setminus Y \to N$ *so that* $|DP(x)| \leq C\mathrm{dist}(x, Y)^{-1}$ *on some* $B_R(0)$ *and so that the restriction of* P *to some* $B_s(N)$ $(s > 0)$ *is the nearest point retraction to* N.

The construction of such a $\widetilde{P}$ uses several explicit topological arguments which are concerned with piecewise-linear constructions. We cannot immediately modify those to construct P for Proposition 2.1. We therefore use a technique called "thickening", invented by Bousquet, Ponce, and Van Schaftingen, to handle higher order Sobolev mappings.

Lemma 2.3 (Thickening; [5, Proposition 4.1])**.** *Let* $\ell \in \{0, \ldots, m-1\}$, $\eta > 0$, $0 < \rho < 1$, $\mathcal{S}^n$ *be a cubication of* $\mathbb{R}^n$ *of radius* η, $\mathcal{U}^n$ *be a subskeleton of* $\mathcal{S}^n$*, and* $\mathcal{T}^{n-\ell-1}$ *be the dual skeleton of* $\mathcal{U}^\ell$. *There exists a smooth map* $\Phi : \mathbb{R}^n \setminus \mathcal{T}^{n-\ell-1} \to \mathbb{R}^n$ *such that*

(a) Φ *is injective;*

(b) *for every* $\sigma^n \in \mathcal{S}^n$, $\Phi(\sigma^n \setminus \mathcal{T}^{n-\ell-1}) \subset \sigma^n \setminus \mathcal{T}^{n-\ell-1}$*;*

(c) $\Phi(x) = x$ *for every* x *outside* $\mathcal{U}^n + [-\rho\eta, \rho\eta]^n$*;*

(d) $\Phi(\mathcal{U}^n \setminus \mathcal{T}^{n-\ell-1}) \subset \mathcal{U}^\ell + [-\rho\eta, \rho\eta]^n$*;*

(e) *for every* $i \in \mathbb{N}$ *and every* $x \in \mathbb{R}^n \setminus \mathcal{T}^{n-\ell-1}$,

$$|D^i\Phi(x)| \leq C\eta \,\mathrm{dist}(x, \mathcal{T}^{n-\ell-1})^{-i}$$

for some constant $C > 0$ *depending on* i, n *and* ρ*;*

(f) *for every* $0 < \beta < \ell + 1$*, for every* $i \in \mathbb{N}$ *and for every* $x \in \mathbb{R}^n \setminus \mathcal{T}^{n-\ell-1}$,

$$\eta^{i-1}|D^i\Phi(x)| \leq C(\mathrm{jac}\,\Phi(x))^{\frac{i}{\beta}}$$

for some constant $C > 0$ *depending on* β, i, n *and* ρ.

Proof of Proposition 2.1. We start with the mapping $\widetilde{P}$ from Lemma 2.2 and smoothen it the following way. Let $\phi : \mathbb{R}^n \setminus Y \to \mathbb{R}$ be a smooth function whith $\phi \equiv 0$ on $B_{\frac{s}{2}}(N)$ and $0 < \phi(x) < \operatorname{dist}(x, Y)$ elsewhere, $\eta : \mathbb{R}^n \to \mathbb{R}$ a standard mollifier with support in B^n. For $0 < \tau < \frac{1}{2}$, we let

$$P_\tau(x) := \fint_{B_{\tau\phi(x)}(x)} \widetilde{P}(y)\eta\Big(\frac{x-y}{\tau\phi(x)}\Big)\, dy$$

and estimate

$$\operatorname{dist}(P_\tau(x), N) \le \operatorname{dist}(P_\tau(x), \widetilde{P}(x)) \le C \fint_{B_{\tau\phi(x)}(x)} |\widetilde{P}(y) - \widetilde{P}(x)|\, dy \le C\tau,$$

where C comes from the bounds on η, $|D\widetilde{P}|$ (from Lemma 2.2) and on ϕ. We choose $\tau > 0$ small enough that the nearest point retraction $\Pi : B_{C\tau}(N) \to N$ is well-defined and smooth. Then we let $\widehat{P} := \Pi \circ P_\tau : \mathbb{R}^n \setminus Y \to N$. This is a smooth map, but we do not get estimates on higher derivatives.

In order to obtain those, we are ready to modify $\widehat{P}$ further using Lemma 2.3. To this end observe that, by the compactness and Lipschitz regularity of Y, and the compactness and smoothness of N, we find a cubications $\mathcal{S}^n$ of $\mathbb{R}^n$ with arbitrarily small radius for which the skeleton $\mathcal{S}^{j+1}$ does not intersect Y and hence has positive distance to Y. By $\mathcal{U}^n$ we mean the subskeleton generated by the n-cells of $\mathcal{S}^n$ that intersect Y. By choosing the radius $\eta > 0$ of $\mathcal{S}^n$ and some $r \in (0, \frac{s}{2})$ sufficiently small, we can assume

$$\mathcal{U}^n + [-\eta, \eta]^n \cap B_r(N) = \emptyset. \tag{1}$$

Now we choose $\rho > 0$ small enough that

$$(\mathcal{U}^{j+1} + [-\rho\eta, \rho\eta]^n) \cap Y = \emptyset. \tag{2}$$

We apply Lemma 2.3 with $\ell = j + 1$, which gives us $\Phi : \mathbb{R}^n \setminus \mathcal{T}^{n-j-2} =: \mathbb{R}^n \setminus X \to \mathbb{R}^n$ with the properties (a)–(f). By (c), Φ is the identity outside a small neighbourhood of Y, and where it is not, by (d), it maps to the compact set $\subset \mathcal{U}^{j+1} + [-\rho\eta, \rho\eta]^n$ where P is smooth because of (2). Therefore, $P := \widehat{P} \circ \Phi$ is a smooth mapping $\overline{B_R(0)} \setminus X \to N$, for any $R > 0$. Because of (c) and (1), P is the nearest point retraction when retricted to $B_r(N)$. This proves part (ii) of the proposition.

In order to finish the proof of (i), we denote the interior of $\mathcal{U}^n + [-\rho\eta, \rho\eta]^n$ by Z and observe that $P = \widehat{P}$ on $\overline{B_R(0)} \setminus Z$ if we have chosen R large enough. Here $\overline{B_R(0)} \setminus Z$ is a compact set, which means that $|D^i P|$ is bounded there for every $i \in \mathbb{N}$. On Z, on the other hand, the bound $|D^i P(x)| \le C(i)\operatorname{dist}(x, X)^{-i}$ follows from the smoothness of $\widehat{P}$ on the compact set $\mathcal{U}^{j+1} + [-\rho\eta, \rho\eta]^n$ and the estimates in (e). Therefore, all that is left to do is to define P properly on

$\mathbb{R}^n \setminus \overline{B_R(0)}$. This is done by letting $P(x) := P(\frac{R}{|x|}\, x)$. This mapping P will most probably not be smooth along $\partial B_R(0)$, which we can, however, easily repair by smoothing around that sphere, by the same technique we used at the beginning of the proof. □

As long as it does not require any additional effort, we will allow for general $p > 1$, while for the proof of our main theorem we will only need $p = 2$.

Choosing j sufficiently large, we can use Proposition 2.1 to project mappings $f \in W^{k,p} \cap W^{1,kp}(U, \mathbb{R}^n)$ to mappings in $P \circ f \in W^{k,p} \cap W^{1,kp}(U, N)$, with controlled energy. This is not as straightforward as it seems, since the set X where P is singular, must be in general position with respect to f. The idea is taken from [10, Theorem 6.2].

Proposition 2.4 (Generic projection of $W^{k,p}$-mappings)**.** *Let $V \subset\subset U \subset \mathbb{R}^n$ be open and smooth subsets, and let $N \subset \mathbb{R}^n$ be a smooth closed submanifold with*

$$\pi_0(N) = \pi_1(N) = \ldots = \pi_{\lfloor kp \rfloor - 1}(N).$$

Let $k \in \mathbb{N}$, $p > 1$, and let a map $f \in W^{k,p} \cap W^{1,kp}(U, \mathbb{R}^n) \cap W^{k,p}(U \setminus V, N)$ be given. Then there is a map $g \in W^{k,p} \cap W^{1,kp}(U, N)$ with $g = f$ on $U \setminus V$ and

$$\sum_{i=1}^{k} \|D^i g\|_{L^{\frac{kp}{i}}(V)}^{\frac{kp}{i}} \leq C \sum_{i=1}^{k} \|D^i f\|_{L^{\frac{kp}{i}}(V)}^{\frac{kp}{i}}.$$

(Note that $W^{k,p} \cap W^{1,kp} \subset W^{i,\frac{kp}{i}}$ for $1 \leq i \leq k$.)

Proof. Let P be the projection from Proposition 2.1, then

$$C_P := \sup\{|D^i P(y)| : y \in \mathbb{R}^n \setminus B_R(0), 1 \leq i \leq k\} < \infty.$$

For $a \in \mathbb{R}^n$, we define the translates $X_a := \{y + a : y \in X\}$, and the mapping $P_a : \mathbb{R}^n \setminus X_a \to N$ by $P_a(y) := P(y - a)$.

We pick some small $r > 0$ which will be determined later. Iterating the chain rule and using Young's inequality, we have

$$|D^k(P_a \circ f)| \leq C \sum_{i,j=1}^{k} |(D^j P_a) \circ f|\, |D^i f|^{\frac{k}{i}}.$$

Therefore, $P_a \circ f \in W^{k,p}(U, N)$ for almost all $a \in B_r(0)$, and by Fubini's theorem

$$\begin{aligned}
\int_{B_r(0)}\int_U |D^k(P_a\circ f)|^p\,dx\,da &\le C\sum_{i,j=1}^k \int_U |D^i f|^{\frac{kp}{i}} \int_{B_r(0)} |(D^jP_a)(f(x))|^p\,da\,dx\\
&\le C\sum_{i,j=1}^k \int_U |D^i f|^{\frac{kp}{i}} \int_{B_r(0)} |(D^jP)(f(x)-a)|^p\,da\,dx\\
&\le C\Big(\sum_{i=1}^k \int_U |D^i f|^{\frac{kp}{i}}\,dx\Big)\Big(C_P^p r^m + \sum_{j=1}^k \int_{B_R(0)} |D^jP(y)|^p\,dy\Big).
\end{aligned}$$

By part (i) of Proposition 2.1 and the choice of $j = \lfloor kp \rfloor - 1$, the last integral is finite, and we therefore have

$$\int_{B_r(0)}\int_U |D^k(P_a\circ f)|^p\,dx\,da \le C\sum_{i=1}^k \int_U |D^i f|^{\frac{kp}{i}}\,dx.$$

This means that there exists a constant C and an $a \in B_r(0)$ such that

$$\int_U |D^k(P_a\circ f)|^p\,dx \le C\sum_{i=1}^k \int_U |D^i f|^{\frac{kp}{i}}\,dx. \tag{3}$$

Note that C depends on r, but r will be fixed soon. Now $P_a \circ f$ is almost the map we are looking for, but it does not coincide with f where f happens to take values in N. Therefore we define

$$g := (P_{a|N})^{-1} \circ P_a \circ f,$$

which has that additional property. Since $P = P_0$ has full rank near N, the inverse function theorem says that $(P_{a|N})^{-1} : N \to N$ exists and is smooth for all sufficiently small $a \in \mathbb{R}^n$. Therefore, if we choose $r > 0$ small enough, for all $a \in B_r(0)$ the $(P_{a|N})^{-1}$ are C^k-diffeomorphisms with uniformly bounded constants, hence g satisfies an estimate similar to (3), which is the assertion of the proposition for the highest-order term on the left-hand side. The other estimates follow from the same inequality with (k, p) replaced by $(i, \frac{kp}{i})$. □

In what follows, we consider $W^{k,p}$-mappings from B^m to N, $k \ge 1$. Their image must be (essentially) in one component of N, even if N is not connected. Therefore we will not have to bother whether or not N is connected and can skip the assumption $\pi_0(N) = 0$ in the theorems we are going to prove.

3. Partial regularity

The proofs in this section use ideas from Hardt and Lin [10] concerning projections and Luckhaus [13] for the overall strategy.

The first step towards partial regularity is a compactness theorem, stating that the limit of minimizers is a minimizer, even if we blow up the target manifold while approaching the limit.

Theorem 3.1 (Compactness). *Let $N \subseteq \mathbb{R}^n$ be a smooth closed submanifold with*

$$\pi_1(N) = \ldots = \pi_{\lfloor kp \rfloor - 1}(N) = 0.$$

Define submanifolds N_i ($i \in \mathbb{N} \cup \{\infty\}$) in one of the following ways.

(i) $N_i := N$ *for all* $i \in \mathbb{N} \cup \{\infty\}$.
or

(ii) $N_i := \delta_i^{-1}(N - y_i)$ ($i \in \mathbb{N}$) *for given sequences* $\delta_i \searrow 0$ *and* $y_i \in \mathbb{R}^n$, *which are chosen in such a way that all* N_i *intersect some compact set* $Z \subset \mathbb{R}^n$. *Then the* N_i *converge locally in Hausdorff distance to some affine subspace of* $\mathbb{R}^n$ *which we denote by* N_∞. *(Actually,* N_∞ *is the limit of the affine tangent spaces* $z_i + T_{z_i}N_i$ *for any sequence of* $z_i \in N_i \cap Z$.*)*

Now assume we are given a sequence $(u_i)_{i\in\mathbb{N}}$ of maps $u_i \in W^{k,p}(B^m, N_i)$ which are $E^{k,p}$-minimizing with respect to their boundary values. We also assume that the sequence u_i is bounded in $W^{j,\frac{kp}{j}}(B^m, N_i)$ for every $j \in \{1, \ldots, k\}$. If $u_i \to u_\infty$ weakly in $\bigcap_{j=1}^n W^{j,\frac{kp}{j}}(B^m, \mathbb{R}^n)$ for some $u_\infty \in W^{k,p}(B^m, N_\infty)$, then u_∞ is also $E^{k,p}$-minimizing with respect to its boundary values, and the u_i actually converge to u_∞ in $W^{k,p}$ on every compact subset of B^m.

Proof. In order to prove that u_∞ is $E^{k,p}$-minimizing, it is enough to show that $E^{k,p}(u_\infty) \le E^{k,p}(v)$ for any $v \in W^{k,p}(B, N_\infty)$ which coincides with u_∞ outside a compact subset of B. Let such a v be given, then there is $\rho_0 < 1$ such that $u_\infty = v$ on $B \setminus B_{\rho_0}$, and we can assume $\rho_0 \in (\frac{3}{4}, 1)$.

Every $v \in W^{k,p}(B^m, N_\infty)$ is the $W^{k,p}(B^m, \mathbb{R}^n)$-limit of a sequence $(v_i)_{i\in\mathbb{N}}$ of maps $v_i \in W^{k,p}(B^m, N_i)$. This is trivial under assumption (i). If we assume (ii) instead, N_∞ is an affine space, hence bounded functions are dense in $W^{k,p}(B^m, N_\infty)$. Moreover, every bounded function $b \in W^{k,p}(B^m, N_\infty)$ is the $W^{k,p}$-limit of $b_i \in W^{k,p}(B^m, N_i)$ which can simply be defined as the nearest-point projection of N_i applied to b. This proves the asserted approximability also under assumption (ii).

In what follows, all choices and constants are allowed to depend on v, and hence also on ρ_0. Weak convergence of the u_i in $\bigcap_{j=1}^k W^{j,\frac{kp}{j}}(B^m, \mathbb{R}^n)$ gives convergence in $\bigcap_{j=1}^k W^{j-1,\frac{kp}{j}}(B^m, \mathbb{R}^n)$. Hence, we can assume not only the existence of K such that

$$\int_{B^m} (|D^h u_i|^{\frac{kp}{j}} + |D^h v_i|^{\frac{kp}{j}})\, dx \le K^{\frac{kp}{j}}$$

for all $i \in \mathbb{N}$ and $h \le j \in \{1, \dots, k\}$, but also

$$\sum_{j=1}^{k} \int_{B^m \setminus B_{\rho_0}} |D^{h-1} u_i - D^{h-1} v_i|^{\frac{kp}{j}}\, dx \le \varepsilon_i^{\frac{kp}{j}} K^{\frac{kp}{j}}$$

for all $i \in \mathbb{N}$ and $h \le j \in \{1, \dots, k\}$ with a sequence $\varepsilon_i \searrow 0$. Let $\bar\rho := \frac{1}{2}(1+\rho_0)$. Fix a sequence $\lambda_i \searrow 0$, to be determined later; we may assume $\lambda_i < 1 - \bar\rho$. For every i, there is some $\rho_i \in (\rho_0, \bar\rho)$ such that

$$\int_{B_{\rho_i+\lambda_i} \setminus B_{\rho_i}} \left(|D^h u_i|^{\frac{kp}{j}} + |D^h v_i|^{\frac{kp}{j}} \right) dx \le C \lambda_i K^{\frac{kp}{j}}$$

for all $i \in \mathbb{N}$ and $h \le j \in \{1, \dots, k\}$, with c independent of h, i and j.

Fix a nondecreasing C^∞-function $\eta\colon [0,1] \to \mathbb{R}$ which is $\equiv 0$ near 0 and $\equiv 1$ near 1. Define $w_i : B_{\rho_i+\lambda_i} \setminus B_{\rho_i} \to \mathbb{R}^n$ by

$$w_i(x) := u_i(x) + \eta\left(\frac{|x| - \rho_i}{\lambda_i} \right) (v_i(x) - u_i(x)).$$

It is straightforward to check that

$$\int_{B_{\rho_i+\lambda_i} \setminus B_{\rho_i}} |D^j w_i|^{\frac{kp}{j}}\, dx \le C K^{\frac{kp}{j}} \left(\lambda_i + \varepsilon_i^{\frac{kp}{j}} \lambda_i^{-kp} \right)$$

for all $i \in \mathbb{N}$ and $j \in \{1, \dots, k\}$. Now we use Proposition 2.4 to project w_i back into N_i. Note that even under the assumptions (ii) this can be applied with constants independent on i, since the projections only get better if the target is enlarged. The projected maps $z_i : B_{\rho_i+\lambda_i} \setminus B_{\rho_i} \to N_i$ satisfy the estimate

$$\int_{B_{\rho_i+\lambda_i} \setminus B_{\rho_i}} |D^k z_i|^p\, dx \le C K^p \left(\lambda_i^{\frac{1}{k}} + \varepsilon_i^p \lambda_i^{-kp} \right).$$

This means that we can choose $\lambda_i := \varepsilon_i^{\frac{1}{2k}}$ (which is small enough for almost all i), then we have

$$\int_{B_{\rho_i+\lambda_i} \setminus B_{\rho_i}} |D^k z_i|^p\, dx \le C K^p \left(\varepsilon_i^{\frac{1}{2k^2}} + \varepsilon_i^{\frac{p}{2}} \right).$$

Note that

$$z_i(x) := \begin{cases} u_i(x) & \text{for } x \in B_1 \setminus B_{\rho_i+\lambda_i}, \\ \text{the } z_i(x) \text{ from above} & \text{for } x \in B_{\rho_i+\lambda_i} \setminus B_{\rho_i}, \\ v_i(x) & \text{for } x \in B_{\rho_i} \end{cases}$$

fits together as a $W^{k,p}(B^m, N_i)$-function. Remember that every u_i is minimizing on B^m with respect to its own boundary values, and that z_i is admissible for being compared with u_i, even on $B_{\rho_i+\lambda_i}$. Therefore we have

$$\begin{aligned}
\|D^k v\|^p_{L^p(B_1)} &= \lim_{i\to\infty}\Big(\|D^k u_\infty\|^p_{L^p(B_1\setminus B_{\rho_i+\lambda_i})} + \|D^k v\|^p_{L^p(B_{\rho_i+\lambda_i})}\Big)\\
&= \lim_{i\to\infty}\Big(\|D^k u_\infty\|^p_{L^p(B_1\setminus B_{\rho_i+\lambda_i})} + \|D^k v_i\|^p_{L^p(B_{\rho_i+\lambda_i})}\Big)\\
&\geq \lim_{i\to\infty}\Big(\|D^k u_\infty\|^p_{L^p(B_1\setminus B_{\rho_i+\lambda_i})} + \|D^k v_i\|^p_{L^p(B_{\rho_i})} + CK^p\Big(\varepsilon_i^{\frac{1}{2k^2}} + \varepsilon_i^{\frac{p}{2}}\Big)\Big)\\
&\geq \lim_{i\to\infty}\Big(\|D^k u_\infty\|^p_{L^p(B_1\setminus B_{\rho_i+\lambda_i})} + \|D^k z_i\|^p_{L^p(B_{\rho_i+\lambda_i})}\Big)\\
&\geq \lim_{i\to\infty}\Big(\|D^k u_\infty\|^p_{L^p(B_1\setminus B_{\rho_i+\lambda_i})} + \|D^k u_i\|^p_{L^p(B_{\rho_i+\lambda_i})}\Big)\\
&\geq \|D^k u_\infty\|^p_{L^p(B_1)},
\end{aligned} \tag{4}$$

the last "$\geq$" being the weak lower semicontinuity of $u \mapsto \|D^k u\|_{L^p}$. This proves that u_∞ is $E^{k,p}$-minimizing on B^m. Moreover, we may choose $v := u_\infty$ and ρ_0 arbitrarily close to 1 in (4), in which case all "$\geq$" must be "$=$". This proves $\|u_i\|_{W^{k,p}(B_{\rho_0})} \to \|u_\infty\|_{W^{k,p}(B_{\rho_0})}$, and thereby $u_i \to u_\infty$ in $W^{k,p}(B_{\rho_0}, \mathbb{R}^n)$, which is the final step in the proof of Theorem 3.1. ☐

Before we prove our partial regularity theorem, we need a version of the Gagliardo-Nirenberg interpolation inequality.

Lemma 3.2 (Gagliardo-Nirenberg interpolation). *For any $u \in W^{k,2} \cap L^\infty(\mathbb{R}^n)$, every $1 \leq j < k$, and every smooth positive η with compact support, we have*

$$\int \eta^{2k}|Du|^{2k}\,dx \leq C\|u\|_\infty^{2k-2}\Big(\int \eta^{2k}|D^k u|^2\,dx + \|\nabla\eta\|_\infty^{2-2k}\int \eta^2|Du|^2\,dx\Big).$$

Proof. By partial integration and Young's inequality, we have

$$\begin{aligned}
\int \eta^{2k}|Du|^{2k}\,dx &= \int \eta^{2k}|Du|^{2k-2}\langle Du, Du\rangle\,dx\\
&\leq C\int \eta^{2k}|Du|^{2k-2}|D^2u||u|\,dx + \int \eta^{2k-1}|Du|^{2k-1}|\nabla\eta||u|\,dx\\
&\leq \varepsilon\int \eta^{2k}|Du|^{2k}\,dx + C(\varepsilon)\|u\|_\infty^k\int \eta^{2k}|D^2u|^k\,dx\\
&\quad + C\|u\|_\infty\|\nabla\eta\|_\infty\int \eta^{2k-1}|Du|^{2k-1}\,dx\\
&\leq \varepsilon\int \eta^{2k}|Du|^{2k}\,dx + C(\varepsilon)\|u\|_\infty^k\int \eta^{2k}|D^2u|^k\,dx\\
&\quad + \varepsilon\int \eta^{2k}|Du|^{2k}\,dx + C(\varepsilon)\|u\|_\infty^{2k-2}\|\nabla\eta\|_\infty^{2k-2}\int \eta^2|Du|^2\,dx.
\end{aligned}$$

In the last line, we have used Young's inequality in the less usual form

$$ab^{2k-1} = b^{2k-1-\frac{1}{k-1}}\left(ab^{\frac{1}{k-1}}\right) \le \varepsilon b^{2k} + C(\varepsilon)a^{2k-2}b^2$$

with exponents $\frac{2k-2}{2k-3}$ and $2k-2$. Absorbing the terms of order ε into the left-hand side, we find

$$\int \eta^{2k}|Du|^{2k}\,dx \le C\|u\|_\infty^k \int \eta^{2k}|D^2u|^k\,dx + C\|u\|_\infty^{2k-2}\|\nabla\eta\|_\infty^{2k-2}\int \eta^2|Du|^2\,dx.$$

Iterating this argument, we inductively prove

$$\int \eta^{2k}|Du|^{2k}\,dx \le C\|u\|_\infty^{2k-\frac{2k}{j}} \int \eta^{2k}|D^ju|^{\frac{2k}{j}}\,dx + C\|u\|_\infty^{2k-2}\|\nabla\eta\|_\infty^{2k-2}\int \eta^2|Du|^2\,dx$$

for $j = 2\dots k$, and the $j = k$ case is the assertion of the lemma. □

Theorem 3.3 (Partial regularity). *Let $k \in \mathbb{N}$, $m > 2k$, and let N be a smooth closed submanifold of $\mathbb{R}^n$ satisfying*

$$\pi_1(N) = \ldots = \pi_{2k-1}(N) = 0.$$

Then every $u : B^m \to N$ which is locally minimizing $E^{k,2}$ is smooth in the interior of B^m outside a closed set Σ with $\mathcal{H}^{m-2k}(\Sigma) = 0$.

Proof. Since $k = 1$ is the well-known harmonic map case, we assume $k \ge 2$. Note that better statements are available for $k \in \{1,2\}$. Let us prove the following "discrete Morrey space estimate".

(M) *For every $\alpha \in (0,1)$, there exist $\theta \in (0,\frac{1}{2})$, $\delta_0 > 0$, and $r_0 \in (0,\frac{1}{4})$ such that for any $E^{k,2}$-minimizing $u \in W^{k,2}(B^m, N)$, the smallness condition*

$$\sum_{j=1}^k r^{2j-m}\int_{B_r(y)} |D^jf|^2\,dx \le \delta_0^2$$

for some $r \in (0, r_0)$ and $y \in B_{\frac{3}{4}}$ implies

$$\sum_{j=1}^k (\theta r)^{2j-m}\int_{B_{\theta r}(y)} |D^jf|^2\,dx \le \theta^{2\alpha}\sum_{j=1}^k r^{2j-m}\int_{B_r(y)} |D^jf|^2\,dx.$$

Assume that (M) does not hold for some $\alpha \in (0,1)$. Then, for all $\theta \in (0,1)$, we find sequences $x_i \in B_{\frac{3}{4}}$, $(0,\frac{1}{4}) \ni r_i \searrow 0$, and $E^{k,2}$-minimizing mappings $f_i \in W^{k,2}(B^m, N)$ such that

$$\sum_{j=1}^k r_i^{2j-m}\int_{B_{r_i}(x_i)} |D^jf_i|^2\,dx =: \delta_i^2 \searrow 0,$$

but

$$\sum_{j=1}^{k}(\theta r_i)^{2j-m}\int_{B_{\theta r_i}(x_i)}|D^j f_i|^2\,dx > \theta^{2\alpha}\delta_i^2$$

for all i. Let $\overline{f_i}$ be the mean value of f_i on $B_{r_i}(x_i)$. We define rescaled mappings

$$u_i(x) := \frac{f_i(x_i + r_i x) - \overline{f_i}}{\delta_i}\,.$$

and rescaled target manifolds $N_i := \delta_i^{-1}(N - \overline{f_i})$. Then every $u_i \in W^{k,2}(B^m, N_i)$ has mean value 0 and fulfills $\sum_{j=1}^{k}\int_{B^m}|D^j u_i|^2\,dx = 1$. By Lemma 3.2, the derivatives $D^j u_i$ are even controlled in $L^{\frac{2k}{j}}(B_{\frac{1}{2}})$,

$$\int_{B_{\frac{1}{2}}}|D^j u_i|^{\frac{2k}{j}}\,dx \le C \quad \text{for } j = 1, \dots, k.$$

A subsequence (again denoted by u_i) then converges weakly in $\bigcap_{j=1}^{k} W^{j,\frac{2k}{j}}(B_{\frac{1}{2}})$ and almost everywhere to some $u_\infty : B_{\frac{1}{2}} \to N_\infty$, where N_∞ is a linear subspace of $\mathbb{R}^n$. By Theorem 3.1, u_∞ is $E^{k,2}$-minimizing. Hence u_∞ is k-polyharmonic, i.e. $\Delta^k u_\infty = 0$. A standard estimate for polyharmonic maps (see [7, Lemma 6.2]) says that for all $\sigma \in (0, \frac{1}{4})$, we have

$$\int_{B_\sigma}|D^j u_\infty|^2\,dx \le C\sigma^m \int_{B_{\frac{1}{2}}}|D^j u_\infty|^2\,dx$$

for all $j \in \{1, \dots, k\}$, with C not depending on σ.

Since we have $W^{k,2}$-norm convergence $u_i \to u_\infty$ on B_θ for all $\theta \in (0, \frac{1}{2})$ and weak convergence on $B_{\frac{1}{2}}$, we have

$$\begin{aligned}
\lim_{i\to\infty}\delta_i^{-2}\sum_{j=1}^{k}(\theta r_i)^{2j-m}\int_{B_{\theta r_i}(x_i)}|D^j f_i|^2\,dx &= \lim_{i\to\infty}\sum_{j=1}^{k}\theta^{2j-m}\int_{B_\theta}|D^j u_i|^2\,dx \\
&= \sum_{j=1}^{k}\theta^{2j-m}\int_{B_\theta}|D^j u_\infty|^2\,dx \\
&\le C\sum_{j=1}^{k}\theta^{2j}\int_{B_{\frac{1}{2}}}|D^j u_\infty|^2\,dx \\
&\le C\theta^2\sum_{j=1}^{k}\int_{B_{\frac{1}{2}}}|D^j u_\infty|^2\,dx \\
&\le C\theta^2\lim_{i\to\infty}\sum_{j=1}^{k}\int_{B_{\frac{1}{2}}}|D^j u_i|^2\,dx \\
&\le C_0\theta^2.
\end{aligned}$$

Choose θ small enough to have $C_0\theta^2 \leq \frac{\theta^{2\alpha}}{2}$, then

$$\lim_{i\to\infty} \sum_{j=1}^{k} (\theta r_i)^{2j-m} \int_{B_{\theta r_i}(x_i)} |D^j f_i|^2 \, dx \leq \frac{\theta^{2\alpha}}{2} \, \delta_i^2,$$

which contradicts the assumption that (M) does not hold. Hence we have proven (M).

Now (M) is built in such a way that once its smallness condition is satisfied on $B_r(y)$, it also holds on $B_{\theta^i r(y)}$ for all $i \in \mathbb{N}$. Hence we can iterate the assertion of (M). Even better, there is $\delta_1 \in (0, \delta_0)$ such that if the smallness condition holds on $B_r(y)$ with δ_1 instead of δ_0, the original smallness condition is satisfied on every $B_{\frac{r}{2}}(z) \subset B_r(y)$, and we can iterate from there. This means that, once we have

$$\sum_{j=1}^{k} r^{2j-m} \int_{B_r(y)} |D^j f|^2 \, dx \leq \delta_1^2$$

for some $r > 0$, the function f will obey

$$\int_{B_s(z)} |Df|^2 \, dx \leq C\delta_0^2 s^{m-2+2\alpha}$$

for all $z \in B_{\frac{r}{2}}(y)$, $s \in (0, \frac{r}{2})$. Hence f is $C^{0,\alpha}$-Hölder continuous on $B_{\frac{r}{2}}(y)$ by Morrey's Dirichlet growth criterion, for any $\alpha \in (0,1)$. By [7, Proposition 7.1], any Hölder continuous weakly polyharmonic map is smooth.

Summarizing, we have proved that our polyharmonic map is smooth in the neighborhood of any point where

$$\inf_{r} \sum_{j=1}^{k} r^{2j-m} \int_{B_r(y)} |D^j f|^2 \, dx \leq \delta_1^2. \tag{5}$$

Let $\Sigma(f, \delta_1)$ be the set of all points where this does not hold. Then the singular set of f is contained in $\Sigma(f, \delta_1)$. And from a standard argument, see e.g. [8, Chapter IV] and note that $r^{2k-m} \int_{B_r(y)} |Du|^{2k} \, dx$ is controlled by the left-hand side of (5), we know that $\mathcal{H}^{m-2k}(\Sigma(f, \delta_1)) = 0$. This proves our partial regularity theorem. □

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