Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 35 (2016), 411–447
DOI: 10.4171/ZAA/1572

Global Bifurcation for Fractional p-Laplacian and an Application

Leandro M. Del Pezzo and Alexander Quaas

Abstract. We prove the existence of an unbounded branch of solutions to the non-linear non-local equation

$$(-\Delta)_p^s u = \lambda |u|^{p-2}u + f(x,u,\lambda) \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,$$

bifurcating from the first eigenvalue. Here $(-\Delta)_p^s$ denotes the fractional p-Laplacian and $\Omega \subset \mathbb{R}^n$ is a bounded regular domain. The proof of the bifurcation results relies in computing the Leray-Schauder degree by making an homotopy respect to s (the order of the fractional p-Laplacian) and then to use results of local case (that is $s = 1$) found in the paper of del Pino and Manásevich [J. Diff. Equ. 92 (1991)(2), 226–251]. Finally, we give some application to an existence result.

Keywords. Bifurcation, fractional p-Laplacian, existence results

Mathematics Subject Classification (2010). 35R11,35B32,47G20,45G05

1. Introduction

In this paper, we study Rabinowitz's global bifurcation type result from the first eigenvalue in a bounded domain of the non-linear non-local operator called the fractional p-Laplacian operator, that is

$$(-\Delta)_p^s u = 2\mathcal{K}(1-s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy,$$

where $\mathcal{K}$ is a constant depending on the dimension and p. Observe that, this operator extends the fractional Laplacian ($p = 2$).

L. M. Del Pezzo: CONICET and Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pabellon I, Ciudad Universitaria (1428), Buenos Aires, Argentina; ldpezzo@dm.uba.ar
A. Quaas: Departamento de Matemática, Universidad Técnica Federico Santa María Casilla V-110, Avda. España, 1680 – Valparaíso, Chile; alexander.quaas@usm.cl

More precisely, we prove the existence of an unbounded branch of solutions to the non-linear non-local equation

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u + f(x,u,\lambda) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.1}$$

bifurcating from the first eigenvalue of the fractional p-Laplacian assuming that f is $o(|u|^{p-2}u)$ near zero and $\Omega \subset \mathbb{R}^n$ is a bounded regular domain.

Bifurcation and global bifurcation are basic principles in mathematical analysis that can be established using, for example, implicit function theorem or degree theory and, in some simple situation, sub and super solution method, i.e. Perron's method. In particular, bifurcation is used as a starting point to prove existence of solution to ODE's and PDE's, see for example [27,37]. Some of the pioneer works related with our method can be found in [14,35,36]. Then many others generalization are established in different context of local operator, see for instance [4, 5, 11, 18, 19, 22, 26, 33, 34] and the reference therein.

Fractional equations are nowadays classical in analysis, see for example [40]. Fractional Laplacian have attracted much interest since they are connected with different applications and sometimes from the mathematical point of view the non-local character introduce difficulties that need some new approaches, see for instance [17, 39] and the reference therein.

In [12], the fractional p-Laplacian is studied through energy and test function methods and it is used to obtain Hölder extensions. See also [6, 7], where the authors consider a non-local "Tug-of-War" game, and [25].

Recently, existence and simplicity of the first eigenvalue in a bounded domain for the fractional p-Laplacian are obtained and also some regularity result are established in [16, 21, 24, 28]. Some results of these works extend the results of [3] to the non-local case.

In the process of writing this article, appearing the following work [24] where the authors, using barrier arguments, prove C^α-regularity up to the boundary for the weak solutions of a non-local non-linear problem driven by the fractional p-Laplacian operator. This result generalises the main result in [38], where the case $p = 2$ is studied.

Thus, there is natural to ask if bifurcation occurs, this is even not known, as far as we know, for the case $p = 2$, except for some related very recent results that can be found in [20, 30, 32]. More precisely, in [20], the authors prove a multiplicity and bifurcation result for the following problem

$$\begin{cases} -\mathscr{L}_K u = \lambda u + |u|^{2^\star-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

where $n > 2s$ and $2^\star = \frac{2n}{n-sp}$. Here $\mathscr{L}_K$ is the non-local operator

$$-\mathscr{L}_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy, \quad x \in \mathbb{R}^n$$

whose model is given by the fractional Laplacian. They show that, in a suitable left neighborhood of any Dirichlet eigenvalue of $-\mathscr{L}_K$, the number of trivial solution of (1.2) is at least twice the multiplicity of the eigenvalue. In [32], the authors extend the above bifurcation and multiplicity result to the fractional p-Laplacian operator. Finally in [30], using variational method, the authors prove that the next problem admits at least one non-trivial solution

$$\begin{cases} (-\Delta)^s u(x) - \lambda u = \mu f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $n > 2s$, f is a function satisfying suitable regularity and growth conditions and the parameters λ and μ lie in a suitable range.

In our approach, to establish the Rabinowitz's type of global bifurcation result, we use Leray-Schauder degree that can be computed by making an homotopy respect to s (the order of the fractional p-Laplacian operator) and then use the homotopy invariance of the Leray-Schauder degree to deduce that the degree is the same as in the local case ($s = 1$), i.e. the p-Laplacian, which is already computed in [34]. Notice that in [34] similar ideas are used, where the homotopy was done with respect to p and the result were deduced from the (by now) classical case of the Laplacian. To do this homotopy with respect to s, we need as a starting point different properties of the first eigenvalue in terms of s up to $s = 1$, analogous properties to the ones that were obtained in [34], but now with respect to s not respect to p.

Notice that one of our limiting procedures s to 1 are obtained in the weak formulation with the help of some limiting properties of the fractional Sobolev spaces already studied in [8]. Moreover, in [25] this limiting procedure is done by viscosity solution techniques for a very close related operator.

Before stated our main theorem we will give the precisely assumption of the function $f\colon \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

1. f satisfies a Carathéodory condition in the first two variables;
2. $f(x,t,\lambda) = o(|t|^{p-1})$ near $t = 0$, uniformly a.e. with respect to x and uniformly with respect to λ on bounded sets;
3. There exists $q \in (1, p_s^\star)$ such that

$$\lim_{|t|\to\infty} \frac{|f(x,t,\lambda)|}{|t|^{q-1}} = 0$$

uniformly a.e. with respect to x and uniformly with respect to λ on bonded sets.

Here $p_s^\star$ is the fractional critical Sobolev exponent, that is

$$p_s^\star := \begin{cases} \frac{np}{n-sp} & \text{if } sp < n, \\ \infty & \text{if } sp \geq n. \end{cases}$$

We denote by $\lambda_1(s,p)$ the first eigenvalue of following eigenvalue problem

$$\begin{cases} (-\Delta)_p^s u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.3}$$

Our main result:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, and $p \in (1,\infty)$. The pair $(\lambda_1(s,p),0)$ is a bifurcation point of* (1.1)*. Moreover, there is a connected component of the set of non-trivial weak solutions of* (1.1) *in $\mathbb{R}\times\widetilde{W}^{s,p}(\Omega)$ whose closure contains $(\lambda_1(s,p),0)$ and it is either unbounded or contains a pair $(\mu,0)$ for some eigenvalue μ of* (1.3) *with $\mu > \lambda_1(s,p)$.*

Notice that the ideas of the proof can be used for other problems. As for example, a very closely related problem such as bifurcation from infinity by the change of variable $v = \frac{u}{\|u\|^2_{\widetilde{W}^{s,p}(\Omega)}}$, for details see for example [19].

Then, we use the above theorem for some application, more precisely, we prove existence of a non-trivial weak solution of the following non-linear non-local problem

$$\begin{cases} (-\Delta)_p^s u = g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.4}$$

where $\frac{g(t)}{|t|^{p-2}t}$ is bounded and crosses the first eigenvalue.

Theorem 1.2. *Let $g\colon \Omega \to \mathbb{R}$ continuous such that $g(0) = 0$ and g satisfies*

A1. $\frac{g(t)}{|t|^{p-2}t}$ *is bounded;*

A2. $\underline{\lambda} := \lim_{t\to 0} \frac{g(t)}{|t|^{p-2}t} < \lambda_1(s,p) < \liminf_{|t|\to\infty} \frac{g(t)}{|t|^{p-2}t}$.

Then there exists a non-trivial weak solution u of (1.4) *such that u has constant-sign in Ω.*

For the prove of this existence result we need some extra qualitative properties of the branch of solutions in the above theorem. Some of these properties come in some cases from the study of the first eigenvalue of the fractional p-Laplacian with weights, see Section 4.

The paper is organized as follows. In Section 2, we review some results of fractional Sobolev spaces and some properties of the Leray-Schauder degree; in Section 3 we study the Dirichlet problem with special interest in proving continuity in terms of s (see Lemma 3.1 below); in Section 4 we study the eigenvalue problem with weights. In addition, we establish the continuity of the eigenvalue respect to s that will help us to make the homotopy and then to compute the degree. In Section 5 we prove our main theorem. Finally, in Section 6 we prove our existence results.

2. Preliminaries

2.1. Fractional Sobolev spaces. First, we briefly recall the definitions and some elementary properties of the fractional Sobolev spaces. We refer the reader to [1, 15, 17, 23] for further reference and for some of the proofs of the results in this subsection.

Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1,\infty)$. We define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) \colon \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dxdy < \infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\Omega)} := \int_\Omega |u(x)|^p\, dx \quad \text{and} \quad |u|^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dxdy.$$

A proof of the following proposition can be found in [1, 15].

Proposition 2.1. *Let Ω be an open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1,\infty)$. We have that*

- *$W^{s,p}(\Omega)$ is a separable Banach space;*
- *If $1 < p < \infty$ then $W^{s,p}(\Omega)$ is reflexive.*

We denote by $W^{s,p}_0(\Omega)$ the closure of the space $C^\infty_0(\Omega)$ of smooth functions with compact support in $W^{s,p}(\Omega)$. We denote by $\widetilde{W}^{s,p}(\Omega)$ the space of all $u \in W^{s,p}(\Omega)$ such that $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$, where $\tilde{u}$ is the extension by zero of u.

The proof of the next theorem is given in [1, Theorem 7.38].

Theorem 2.2. *For any $s \in (0,1)$ and $p \in (1,\infty)$, the space $C^\infty_0(\mathbb{R}^n)$ is dense in $W^{s,p}(\mathbb{R}^n)$, that is $W^{s,p}_0(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$.*

In the next result, we show the explicit dependence of the constant of [17, Proposition 2.1] on s, that is needed for our propose.

Lemma 2.3. *Let Ω be an open set in $\mathbb{R}^n$, $p \in [1,\infty)$ and $0 < s \le s' < 1$. Then*

$$|u|^p_{W^{s,p}(\Omega)} \le |u|^p_{W^{s',p}(\Omega)} + C(n,p)\left(\frac{1}{sp} - \frac{1}{s'p}\right) \|u\|^p_{L^p(\Omega)}$$

for any $u \in W^{s',p}(\Omega)$.

Proof. Let $u \in W^{s',p}(\Omega)$, then

$$
\begin{aligned}
|u|^p_{W^{s,p}(\Omega)} &= \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy \\
&= \int_\Omega \int_{A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy
\end{aligned}
\tag{2.1}
$$

where $A_y = \Omega \cap \{x \in \mathbb{R}^n \colon |x-y| < 1\}$.

Using that $s' \geq s$, we have that

$$
\int_\Omega \int_{A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy \leq \int_\Omega \int_{A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\,dxdy. \tag{2.2}
$$

On the other hand, we have that

$$
\begin{aligned}
&\int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy \\
&= \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}} |x-y|^{(s'-s)p}\,dxdy \\
&= \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}} \left(|x-y|^{(s'-s)p} - 1\right) dxdy + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\,dxdy \\
&\leq 2^{p-1} \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)|^p + |u(y)|^p}{|x-y|^{n+ps'}} \left(|x-y|^{(s'-s)p} - 1\right) dxdy \\
&\quad + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\,dxdy
\end{aligned}
$$

Observe that for any $x, y \in \Omega$ we have that $x \in \Omega \setminus A_y$ if only if $y \in \Omega \setminus A_x$, that is $\chi_{\Omega\setminus A_y}(x) = \chi_{\Omega\setminus A_x}(y)$. Therefore

$$
\begin{aligned}
&\int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)|^p}{|x-y|^{n+sp}} \left(|x-y|^{(s'-s)p} - 1\right) dxdy \\
&= \int_\Omega \int_\Omega \frac{|u(x)|^p}{|x-y|^{n+sp}} \chi_{\Omega\setminus A_y}(x) \left(|x-y|^{(s'-s)p} - 1\right) dxdy \\
&= \int_\Omega \int_\Omega \frac{|u(x)|^p}{|x-y|^{n+sp}} \chi_{\Omega\setminus A_x}(y) \left(|x-y|^{(s'-s)p} - 1\right) dxdy \\
&= \int_\Omega \int_{\Omega\setminus A_x} \frac{|u(x)|^p}{|x-y|^{n+sp}} \left(|x-y|^{(s'-s)p} - 1\right) dydx.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dxdy \\
&\le 2^p \int_\Omega \int_{\Omega\setminus A_x} \frac{|u(x)|^p}{|x-y|^{n+ps'}} \left(|x-y|^{(s'-s)p}-1\right) dxdy + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\, dxdy \\
&\le 2^p \int_\Omega |u(x)|^p \int_{\{|x-y|\ge 1\}} \frac{|x-y|^{(s'-s)p}-1}{|x-y|^{n+ps'}}\, dydx + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\, dxdy \\
&\le 2^p \int_\Omega |u(x)|^p \int_{\{|z|\ge 1\}} \frac{|z|^{(s'-s)p}-1}{|z|^{n+ps'}}\, dzdx + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\, dxdy \\
&\le 2^p \|u\|^p_{L^p(\Omega)} \int_{\{|z|\ge 1\}} \frac{|z|^{(s'-s)p}-1}{|z|^{n+ps'}}\, dz + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\, dxdy.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dydx \\
&\le C(n,p)\left(\frac{1}{ps}-\frac{1}{ps'}\right)\|u\|^p_{L^p(\Omega)} + \int_\Omega \int_{\Omega\setminus A_y} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps'}}\, dxdy.
\end{aligned}
\tag{2.3}
$$

Therefore, combining (2.1), (2.2) and (2.3), we get

$$
|u|^p_{W^{s,p}(\Omega)} \le |u|^p_{W^{s',p}(\Omega)} + C(n,p)\left(\frac{1}{sp}-\frac{1}{s'p}\right)\|u\|^p_{L^p(\Omega)}.
$$

The proof is now complete. □

Remark 2.4. The space $W^{s',p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$ for any $0 < s \le s' < 1$ and $1 \le p < \infty$.

Lemma 2.5. *Let Ω be an bounded open set in $\mathbb{R}^n$, $s \in (0,1)$ and $p \in [1,\infty)$. Then*

$$
\|u\|^p_{L^p(\Omega)} \le \frac{sp|\Omega|^{\frac{sp}{n}}}{2\omega_n^{\frac{sp}{n}+1}} |u|^p_{W^{s,p}(\mathbb{R}^n)}
$$

for any $u \in \widetilde{W}^{s,p}(\Omega)$. Here ω_n denotes n-dimensional measure of the unit sphere S^n.

Proof. Let $u \in \widetilde{W}^{s,p}(\Omega)$. Then

$$
\begin{aligned}
|u|^p_{W^{s,p}(\mathbb{R}^n)} &= \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\, dxdy + 2\int_\Omega \int_{\mathbb{R}^n\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{n+ps}}\, dxdy \\
&\ge 2\int_\Omega |u(x)|^p \int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+ps}}\, dydx.
\end{aligned}
$$

Let $r = \left(\frac{|\Omega|}{w_n}\right)^{\frac{1}{n}}$. Following the proof of [17, Lemma 6.1], we get

$$\int_{\mathbb{R}^n\setminus\Omega} \frac{1}{|x-y|^{n+ps}}\,dydx \geq \int_{\mathbb{R}^n\setminus B_r(x)} \frac{1}{|x-y|^{n+ps}}\,dydx = \omega_n \int_r^\infty \frac{d\rho}{\rho^{sp+1}} = \frac{\omega_n}{sp}\frac{1}{r^{sp}}$$

which proves the lemma. □

The proofs of the next two theorems are given in [17, Proposition 2.2], and [15, Proposition 4.43], respectively.

Theorem 2.6. *Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$ with bounded boundary $s \in (0,1)$, and $p \in (1,\infty)$. Then, there exists a positive constant $C = C(n,s,p)$ such that*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega).$$

In particular, $W^{1,p}(\Omega)$ is continuously embedded in $W^{s,p}(\Omega)$.

Theorem 2.7. *Let $s \in (0,1)$, $p \in [1,\infty)$, and $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Then $W^{s,p}(\Omega)$ is continuously embedded in $W^{s,p}(\mathbb{R}^n)$.*

The proof of the following embedding theorem can be found in [15, Theorem 4.47].

Theorem 2.8. *Let $s \in (0,1)$ and $p \in (1,\infty)$. Then we have the following continuous embeddings:*

$$\begin{array}{lll} W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) & \text{for all } 1 \leq q \leq p_s^\star & \text{if } sp < n; \\ W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n) & \text{for all } 1 \leq q < \infty & \text{if } sp = n; \\ W^{s,p}(\mathbb{R}^n) \hookrightarrow C_b^{0,\beta}(\mathbb{R}^n) & \text{where } \beta = s - \dfrac{n}{p}, & \text{if } sp > n. \end{array}$$

Here $p_s^\star$ is the fractional critical Sobolev exponent, that is

$$p_s^\star := \begin{cases} \frac{np}{n-sp} & \text{if } sp < n, \\ \infty & \text{if } sp \geq n. \end{cases}$$

Remark 2.9. Note that $p_s^\star$, as a function of s, is continuous in $(0,1]$ where $p_1^\star$ is the critical Sobolev exponent, i.e.

$$p_1^\star := \begin{cases} \frac{np}{n-p} & \text{if } p < n, \\ \infty & \text{if } p \geq n. \end{cases}$$

A proof of the next theorem can be found in [29, Theorem 1].

Theorem 2.10. *Let $s \in (0,1)$, $p \in [1,\infty)$ and $sp < n$. Then there exists a constant $C = C(n,p)$ such that*

$$\|u\|^p_{L^{p_s^\star}(\mathbb{R}^n)} \le C\frac{s(1-s)}{(n-sp)^{p-1}}|u|^p_{W^{s,p}(\mathbb{R}^n)} \quad \textit{for all } u \in W^{s,p}(\mathbb{R}^n).$$

By Theorem 2.7 and Theorem 2.8, we have the next result.

Corollary 2.11. *Let $s \in (0,1)$, $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. The conclusions of Theorem* 2.8 *remain true if $\mathbb{R}^n$ is replaced by Ω.*

The following embedding theorem is established in [15, Theorem 4.58]. See also [1].

Theorem 2.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $s \in (0,1)$ and $p \in [1,\infty)$. Then we have the following compact embeddings:*

$$\begin{aligned} &W^{s,p}(\Omega) \hookrightarrow L^q(\Omega) && \textit{for all } q \in [1, p_s^\star), && \textit{if } sp \le n; \\ &W^{s,p}(\Omega) \hookrightarrow C_b^{0,\lambda}(\Omega) && \textit{for all } \lambda < s - \frac{n}{p}, && \textit{if } sp > n. \end{aligned}$$

Remark 2.13. Let $\Omega \subset \mathbb{R}^n$ a bounded domain with Lipschitz boundary. By the above theorem, we have that the embedding of $W^{s,p}(\Omega)$ into $L^p(\Omega)$ is compact for every $s \in (0,1)$ and for every $p \in (1,\infty)$.

The next results are proven in [8, Corollaries 2 and 7].

Theorem 2.14. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p \in (1,\infty)$. Assume $u \in L^p(\Omega)$, then*

$$\lim_{s\to 1^-} \mathcal{K}(1-s)|u|^p_{W^{s,p}(\Omega)} = |u|^p_{W^{1,p}(\Omega)}$$

with

$$|u|^p_{W^{1,p}(\Omega)} = \begin{cases} \int_\Omega |\nabla u|^p\,dx & \textit{if } u \in W^{1,p}(\Omega), \\ \infty & \textit{if } u \notin W^{1,p}(\Omega). \end{cases}$$

Here $\mathcal{K}$ depends only the p and n.

Remark 2.15. Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $p \in (1,\infty)$ and $\phi \in C_0^\infty(\Omega)$. Then

$$\begin{aligned} |\phi|^p_{W^{s,p}(\Omega)} &\le |\phi|^p_{W^{s,p}(\mathbb{R}^n)} \\ &= |\phi|^p_{W^{s,p}(\Omega)} + 2\int_\Omega\int_{\mathbb{R}^n\setminus\Omega} \frac{|\phi(x)|^p}{|x-y|^{n+sp}}\,dx \\ &\le |\phi|^p_{W^{s,p}(\Omega)} + \frac{C}{sp}\frac{\|\phi\|^p_{L^p(\Omega)}}{\operatorname{dist}(K,\partial\Omega)^{sp}}, \end{aligned}$$

where K is the support of ϕ and C depends only of n. Then by Theorem 2.14 we have

$$\lim_{s\to 1^-} \mathcal{K}(1-s)|\phi|^p_{W^{s,p}(\mathbb{R}^n)} = |\phi|^p_{W^{1,p}(\Omega)}.$$

Theorem 2.16. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$, $p \in (1,\infty)$ and $u_s \in W^{s,p}(\Omega)$ for $s \in (0,1)$. Assume that*

$$\int_\Omega u_s dx = 0 \quad \text{and} \quad (1-s)|u_s|^p_{W^{s,p}(\Omega)} \le C$$

for all $s \in (0,1)$. Then, there exists $u \in W^{1,p}(\Omega)$ and a subsequence $\{u_{s_k}\}_{k\in\mathbb{N}}$ such that $s_k \to 1^-$ as $k \to \infty$,

$$\begin{aligned} &u_{s_k} \to u \quad \text{strongly in } L^p(\Omega),\\ &u_{s_k} \rightharpoonup u \quad \text{weakly in } W^{1-\varepsilon,p}(\Omega), \quad \text{for all } \varepsilon > 0. \end{aligned}$$

Note that in the previous theorem, the assumption

$$\int_\Omega u_s dx = 0 \quad \forall s \in (0,1)$$

can be replaced by $\{u_s\}_{s\in(0,1)}$ is bounded in $L^p(\Omega)$.

Remark 2.17. Let $0 < s < s' < 1$, and $1 < p < \infty$. From the proof of [8, Lemma 2] and [8, Corollary 7], it follows that

$$(1-s)|u|^p_{W^{s,p}(\Omega)} \le 2^{(1-s)p}\mathrm{diam}(\Omega)^{(s'-s)p}(1-s')|u|^p_{W^{s',p}(\mathbb{R}^n)} \tag{2.4}$$

for all $u \in W^{s',p}(\mathbb{R}^n)$. Here $\mathrm{diam}(\Omega)$ denotes the diameter of Ω. See also [8, Remark 6].

Observe that for any $u = \phi \in C_0^\infty(\Omega)$ passing to the limit in (2.4) as $s' \to 1$ and using Theorem 2.14, we get $(1-s)|\phi|^p_{W^{s,p}(\Omega)} \le \frac{2^{(1-s)p}\mathrm{diam}(\Omega)^{(1-s)p}}{\mathcal{K}}|\phi|^p_{W^{1,p}(\Omega)}$, that is

$$\mathcal{K}(1-s)|\phi|^p_{W^{s,p}(\Omega)} \le 2^{(1-s)p}\mathrm{diam}(\Omega)^{(1-s)p}|\phi|^p_{W^{1,p}(\Omega)}.$$

Remark 2.18. Let $s_0 \in (0, \min\{\frac{n}{p}, s\})$, $u \in \widetilde{W}^{s,p}(\Omega)$, $x \in \Omega$ and $B = B_r(x)$ with $r = \mathrm{diam}(\Omega)$. Then, by Theorem 2.10, there exists a constant $C = C(n,p)$ such that

$$\begin{aligned} \|u\|^p_{L^{p^\star_{s_0}}(\mathbb{R}^n)} &\le C\frac{s_0(1-s_0)}{(n-s_0p)^{p-1}}|u|^p_{W^{s_0,p}(\mathbb{R}^n)}\\ &= \frac{Cs_0}{(n-s_0p)^{p-1}}\left((1-s_0)|u|^p_{W^{s_0,p}(B)} + 2(1-s_0)\int_\Omega\int_{\mathbb{R}^n\setminus B}\frac{|u(x)|^p}{|x-y|^{n+s_0p}}dxdy\right). \end{aligned}$$

By Remark 2.17,

$$(1-s_0)|u|^p_{W^{s_0,p}(B)} \leq 2^{(1-s_0)p}(4\,\mathrm{diam}(\Omega))^{(s-s_0)p}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)}.$$

On the other hand

$$2(1-s_0)\int_\Omega\int_{\mathbb{R}^n\setminus B}\frac{|u(x)|^p}{|x-y|^{n+s_0p}}dxdy \leq \frac{2^{1-s_0p}\omega_n}{\mathrm{diam}(\Omega)^{s_0p}\,s_0p}\int_\Omega |u(x)|^p dx.$$

Then there exists a constant $C=C(n,p)$ such that

$$\|u\|^p_{L^{p^\star_{s_0}}(\mathbb{R}^n)} \leq \frac{C}{(n-s_0p)^{p-1}\mathrm{diam}(\Omega)^{s_0p}}\left(\mathrm{diam}(\Omega)^{sp}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)}+\frac{1}{s_0p}\int_\Omega|u(x)|^pdx\right).$$

Our last result gives a characterization of $W^{s,p}_0(\Omega)$. For the proof we refer the reader to [23, Corollary 1.4.4.5].

Theorem 2.19. *Let $\Omega\subset\mathbb{R}^n$ be bounded open set with Lipschitz boundary, $s\in(0,1]$ and $p\in(1,\infty)$. If $s\neq\frac{1}{p}$ then*

$$W^{s,p}_0(\Omega)=\widetilde{W}^{s,p}(\Omega),$$

Furthermore, when $0<s<\frac{1}{p}$ we have

$$W^{s,p}_0(\Omega)=W^{s,p}(\Omega).$$

Remark 2.20. $\widetilde{W}^{s,p}(\Omega)$ is a Banach space for the norm induced by $W^{s,p}(\mathbb{R}^n)$. Moreover, if $\Omega\subset\mathbb{R}^n$ is a bounded open set with Lipschitz boundary, $s\in(0,1]$ and $p\in(1,\infty)$, then $C^\infty_0(\Omega)$ is dense in $\widetilde{W}^{s,p}(\Omega)$ and $\widetilde{W}^{s,p}(\Omega)\subset W^{s,p}_0(\Omega)$. See [23, Theorem 1.4.2.2 and Corollary 1.4.4.10].

For $s\in(0,1)$ and $p\in(1,\infty)$, we define the space $W^{-s,p'}(\Omega)$ ($\widetilde{W}^{-s,p'}(\Omega)$) as the dual space of $W^{s,p}_0(\Omega)$ ($\widetilde{W}^{s,p}(\Omega)$) where $\frac{1}{p'}+\frac{1}{p}=1$.

2.2. Leray-Schauder degree. For the definition and some properties of Leray-Schauder degree see, for instance, [13, 37].

The proof of the next Leray-Schauder degree property is given in [34, Lemma 2.4].

Lemma 2.21. *Let X,Y be Banach spaces with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. Assume that $Y\subset X$ and that the inclusion $i\colon Y\to X$ is continuous. Let Ω_X, Ω_Y be bounded open sets in X and Y, respectively, both containing 0, let $T\colon X\to Y$ be a completely continuous operator such that*

$$x-Tx\neq 0\quad\forall x\in X\setminus\{0\}.$$

Then

$$\deg_X(I-i\circ T,\Omega_X,0)=\deg_Y(I-T\circ i,\Omega_Y,0).$$

3. The Dirichlet problem

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, and $p \in (1, \infty)$. We consider the operator

$$\mathcal{L}_{s,p}u \coloneqq \begin{cases} -\Delta_p u & \text{if } s = 1, \\ (-\Delta)_p^s u & \text{if } 0 < s < 1, \end{cases} \tag{3.1}$$

where Δ_p is the p-Laplace operator, that is

$$\Delta_p u \coloneqq \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

and $(-\Delta)_p^s$ is the fractional p-Laplace operator, that is

$$(-\Delta)_p^s u = 2\mathcal{K}(1-s)\text{P.V.} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}\, dy$$

with $\mathcal{K}$ is the constant of Theorem 2.14.

For further details on the fractional p-Laplace operator, we refer to [21, 28] and references therein.

It is well known that the Dirichlet problem

$$\begin{cases} -\Delta_p u = h & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique weak solution for each $h \in W^{-1,p'}(\Omega)$, i.e. there exists a unique $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla\phi(x)\, dx = \langle h, \phi\rangle \quad \forall \phi \in C_0^\infty(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

We also recall that the weak solution is the unique minimizer of the functional $J_{1,p} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$J_{1,p}(v) = \frac{1}{p}|v|^p_{W_0^{1,p}(\Omega)} - \langle h, v\rangle.$$

See, for instance, [41] and references therein.

Now, we study the Dirichlet problem for fractional p-Laplace equation.

Let $s \in (0,1)$, $p \in (1,\infty)$ and $h \in W^{-s,p'}(\Omega)$. We say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak solution of the Dirichlet problem

$$\begin{cases} (-\Delta)_p^s u = h & \text{in } \Omega, \\ \quad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.2}$$

if

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) = \langle h, v\rangle_s \quad \forall v \in \widetilde{W}^{s,p}(\Omega),$$

where

$$\mathcal{H}_{s,p}(u,v) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+sp}}(v(x)-v(y))\,dydx,$$

and $\langle\cdot,\cdot\rangle_s$ denotes the duality pairing between $\widetilde{W}^{s,p}(\Omega)$ and $\widetilde{W}^{-s,p'}(\Omega)$.

It is clear that, the weak solutions are critical points of the functional $J_{s,p} : \widetilde{W}^{s,p}(\Omega) \to \mathbb{R}$ given by

$$J_{s,p}(v) = \frac{1}{p}\mathcal{K}(1-s)|v|^p_{W^{s,p}(\mathbb{R}^n)} - \langle h, v\rangle_s.$$

Now, it is easy to see that $J_{s,p}$ is bounded below, coercive, strictly convex and sequentially weakly lower semi continuous. Then it has a unique critical point which is a global minimum. Therefore the Dirichlet problem (3.2) has a unique weak solution.

Thus, given $s \in (0,1]$ and $h \in \widetilde{W}^{-s,p'}(\Omega)$, the Dirichlet problem

$$\begin{cases} \mathcal{L}_{s,p}u = h & \text{in } \Omega, \\ \quad\;\; u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

has a unique weak solution $u_{s,p,h} \in \widetilde{W}^{s,p}(\Omega)$. Moreover, the operator

$$\begin{aligned} \mathcal{R}_{s,p} \colon \widetilde{W}^{-s,p'}(\Omega) &\to \widetilde{W}^{s,p}(\Omega) \\ h &\to u_{s,p,h} \end{aligned}$$

is continuous. By the Rellich-Kondrachov theorem (case $s = 1$) and Theorem 2.12 (case $s \in (0,1)$), the restriction of $\mathcal{R}_{s,p}$ to $L^{q'}(\Omega)$ with $q \in (1, p_s^\star)$ is a completely continuous operator, that is for every weakly convergent sequence $\{h_k\}_{k\in\mathbb{N}}$ from $L^{q'}(\Omega)$, the sequence $\{\mathcal{R}_{s,p}(h_k)\}_{k\in\mathbb{N}}$ is norm-convergent in $\widetilde{W}^{s,p}(\Omega)$.

Our next result show that the operator $\mathcal{R}_{s,p}$ is continuous with respect to s and h.

Lemma 3.1. *Let $p \in (1,\infty)$, $s_0 \in (0,1)$, and $1 < q < p^\star_{s_0}$. Then the operator*

$$\begin{aligned} \mathcal{R}_p \colon [s_0,1] \times L^{q'}(\Omega) &\to L^q(\Omega) \\ (s,h) &\to \mathcal{R}_{s,p}(h) \end{aligned}$$

is completely continuous.

Proof. We start by proving that $\mathcal{R}_p$ is compact.

Let $\{(s_k,h_k)\}_{k\in\mathbb{N}}$ be a bounded sequence in $[s_0,1]\times L^{q'}(\Omega)$. We want to prove that $u_k=\mathcal{R}_p(s_k,h_k)$ has a strongly convergent subsequence in $L^q(\Omega)$.

For all $k\in\mathbb{N}$, u_k satisfies

$$|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)}=\int_\Omega h_k(x)u_k(x)\,dx.$$

Then, by Hölder inequality and using $q<p^\star_{s_0}$, we have

$$|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)}\le\|h_k\|_{L^{q'}(\Omega)}\|u_k\|_{L^q(\Omega)}\le C\|u_k\|_{W^{s_0,p}(\Omega)} \tag{3.3}$$

where C is a constant independent of k. Thus, by Lemma 2.3, Lemma 2.5 and (3.3), we get $\|u_k\|_{W^{s_0,p}(\Omega)}\le C$ for some constant C independent of k. Hence $\{u_k\}_{k\in\mathbb{N}}$ has a strongly convergent subsequence in $L^q(\Omega)$ due to $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{s_0,p}(\Omega)$ and $1<q<p^\star_{s_0}$.

Finally, we show that $\mathcal{R}_p$ is continuous.

Let $(s_k,h_k)\to(s,h)$ in $[s_0,1]\times L^{q'}(\Omega)$ as $k\to\infty$, $u_k=\mathcal{R}_p(s_k,h_k)$ $k\in\mathbb{N}$, and $u=\mathcal{R}_p(s,h)$. We want to show that $u_k\to u$ strongly in $L^q(\Omega)$. In fact, we only need to show that u is the only accumulation point of $\{u_k\}_{k\in\mathbb{N}}$ due to $\mathcal{R}_p$ is compact.

Let $\{u_j\}_{j\in\mathbb{N}}$ be a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ converging to v in $L^q(\Omega)$. We have to prove that $v=u$.

Give $w\in\widetilde{W}^{s,p}(\Omega)$ we define

$$|w|^p_{s,p}=\begin{cases}|w|^p_{W^{1,p}(\Omega)} & \text{if } s=1,\\ \mathcal{K}(1-s)|w|^p_{W^{s,p}(\mathbb{R}^n)} & \text{if } s\in(0,1).\end{cases}$$

Let $\tilde{v}$ be the continuation of v by zero outside Ω. Then, it is enough to prove that

$$\frac{1}{p}|\tilde{v}|^p_{s,p}-\int_\Omega v(x)h(x)\,dx\le\frac{1}{p}|w|^p_{s,p}-\int_\Omega w(x)h(x)\,dx\quad\forall w\in\widetilde{W}^{s,p}(\Omega). \tag{3.4}$$

On the other hand, we know that

$$\frac{1}{p}|u_j|^p_{s_j,p}-\int_\Omega u_j(x)h_j(x)\,dx\le\frac{1}{p}|w|^p_{s_j,p}-\int_\Omega w(x)h_j(x)\,dx \tag{3.5}$$

for all $w\in\widetilde{W}^{s_j,p}(\Omega)$.

Now we need consider the following two cases.

Case $s\neq1$. Since $u_j\to v$ strongly in $L^q(\Omega)$, we have that $u_j\to\tilde{v}$ a.e. in $\mathbb{R}^n$. Then, using that $h_j\to h$ strongly in $L^{q'}(\Omega)$ and by Fatou's lemma, we have

$$\frac{1}{p}|\tilde{v}|^p_{s,p}-\int_\Omega v(x)h(x)\,dx\le\liminf_{j\to\infty}\frac{1}{p}|u_j|^p_{s_j,p}-\int_\Omega u_j(x)h_j(x)\,dx. \tag{3.6}$$

Thus, for any $\phi \in C_0^\infty(\Omega)$, by (3.6), (3.5) and dominate convergence theorem, we get

$$\frac{1}{p}|\tilde{v}|_{s,p}^p - \int_\Omega v(x)h(x)\,dx \le \frac{1}{p}|\phi|_{s,p}^p - \int_\Omega \phi(x)h(x)\,dx.$$

Therefore, $v \in \widetilde{W}^{s,p}(\Omega)$ and by density, (3.4) holds.

Case $s = 1$. Let $\phi \in C_0^\infty(\Omega)$. By (3.5) and Remark 2.15, we have

$$\begin{aligned}\limsup_{j\to\infty} \frac{1}{p}|u_j|_{s_j,p}^p - \int_\Omega v(x)h(x)\,dx &= \limsup_{j\to\infty} \frac{1}{p}|u_j|_{s_j,p}^p - \int_\Omega u_j(x)h_j(x)\,dx \\ &\le \frac{1}{p}|\phi|_{1,p}^p - \int_\Omega \phi(x)h(x)\,dx,\end{aligned}$$

due to $u_j \to v$ strongly in $L^q(\Omega)$ and $h_j \to h$ strongly in $L^{q'}(\Omega)$. Then

$$\limsup_{j\to\infty} \frac{1}{p}|u_j|_{s_j,p}^p \le \frac{1}{p}|\phi|_{1,p}^p - \int_\Omega \phi(x)h(x)\,dx + \int_\Omega v(x)h(x)\,dx. \tag{3.7}$$

Therefore $|u_j|_{s_j,p} \le C$ for some constant C independent of j.

Thus, by Theorem 2.16, there exist $w \in W_0^{1,p}(\Omega)$ and a subsequence of $\{u_j\}_{j\in\mathbb{N}}$, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, such that

$$\begin{aligned}u_j &\to w \quad \text{strongly in } L^p(\Omega)\\ u_j &\rightharpoonup w \quad \text{weakly in } W^{1-\varepsilon,p}(\Omega)\end{aligned}$$

for all $\varepsilon > 0$. Then $v = w$, and $v \in W_0^{1,p}(\Omega)$.

On the other hand, given $\varepsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that $1 - \varepsilon < s_j$ for all $j \ge j_0$ due to $s_j \to 1$. Then, by Remark 2.17

$$\mathcal{K}\varepsilon|u_j|_{W^{1-\varepsilon,p}(\Omega)}^p \le 2^{\varepsilon p}\mathrm{diam}(\Omega)^{(s_j-1+\varepsilon)p}|u_j|_{s_j,p}^p \quad \forall j \ge j_0. \tag{3.8}$$

Thus, using $u_j \rightharpoonup v$ weakly in $W^{1-\varepsilon,p}(\Omega)$ and by (3.8) and (3.7),

$$\begin{aligned}\frac{2^{-\varepsilon p}\mathrm{diam}(\Omega)^{-\varepsilon p}}{p}\mathcal{K}\varepsilon|v|_{W^{1-\varepsilon,p}(\Omega)}^p &\le \liminf_{j\to\infty}\frac{1}{p}|u_j|_{s_j,p}^p\\ &\le \frac{1}{p}|\phi|_{1,p}^p - \int_\Omega \phi(x)h(x)\,dx + \int_\Omega v(x)h(x)\,dx.\end{aligned}$$

Now, by Theorem 2.14, letting $\varepsilon \to 0^+$ we get

$$\frac{1}{p}|v|_{W^{1,p}(\Omega)}^p \le \frac{1}{p}|\phi|_{1,p}^p - \int_\Omega \phi(x)h(x)\,dx + \int_\Omega v(x)h(x)\,dx.$$

Thus, since ϕ is arbitrary, we have that

$$\frac{1}{p}|v|_{W^{1,p}(\Omega)}^p - \int_\Omega v(x)h(x)\,dx \le \frac{1}{p}|\phi|_{1,p}^p - \int_\Omega \phi(x)h(x)\,dx \quad \forall\phi \in C_0^\infty(\Omega).$$

Hence, by density, (3.4) holds. This completes the proof. $\square$

Remark 3.2. Let $s_0 \in (0,1)$, and $p \in (1,\infty)$. Then the operator

$$\begin{aligned} \mathcal{R}_p \colon [s_0,1] \times L^{p'}(\Omega) &\to L^p(\Omega) \\ (s,h) &\to \mathcal{R}_{s,p}(h) \end{aligned}$$

is completely continuous.

4. The eigenvalue problems with weight

In this section we show some results concerning the the following eigenvalue problems

$$\begin{cases} \mathcal{L}_{s,p}(u) = \lambda h(x)|u|^{p-2}u & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.1}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $s \in (0,1]$, $p \in (1,\infty)$ and $h \in \mathcal{A} = \{f \in L^\infty(\Omega) \colon |\{x \in \Omega \colon f(x) > 0\}| > 0\}$.

4.1. Case $s = 1$, the first p-eigenvalue. The first eigenvalue $\lambda_1(1,p,h)$ can be characterized as

$$\lambda_1(1,p,h) := \inf \left\{ |u|^p_{W^{1,p}(\Omega)} \colon u \in W^{1,p}_0(\Omega), \int_\Omega h(x)|u(x)|^p \, dx = 1 \right\},$$

and it is simple and isolated, see [3]. For simplicity, we omit mention of h when $h \equiv 1$, and thus we write $\lambda_1(1,p)$ in place of $\lambda_1(1,p,1)$.

4.2. Case $s \in (0,1)$, the first fractional p-eigenvalue. Now, we analyse the (non-linear non-local) eigenvalue problems

$$\begin{cases} (-\Delta)^s_p u = \lambda h(x)|u|^{p-2}u & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{4.2}$$

A function $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak solution of (4.2) if it satisfies

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) = \lambda \int_\Omega h(x)|u(x)|^{p-2}u(x)v(x)\, dx \quad \forall v \in \widetilde{W}^{s,p}(\Omega). \tag{4.3}$$

We say that $\lambda \in \mathbb{R}$ is a fractional p-eigenvalue provided there exists a non-trivial weak solution $u \in \widetilde{W}^{s,p}(\Omega)$ of (4.2). The function u is a corresponding eigenfunction.

The first fractional p-eigenvalue is

$$\lambda_1(s,p,h) := \mathcal{K}(1-s) \inf\left\{ |u|^p_{W^{s,p}(\mathbb{R}^n)} \colon u \in \widetilde{W}^{s,p}(\Omega), \int_\Omega h(x)|u(x)|^p dx = 1 \right\}. \tag{4.4}$$

As before, in the case $h \equiv 1$, for simplicity, we write $\lambda_1(s,p)$ in place of $\lambda_1(s,p,1)$.

First we want to mention that $\{u \in \widetilde{W}^{s,p}(\Omega)\colon \int_\Omega h(x)|u(x)|^p\,dx = 1\} \neq \emptyset$ due to $|\{x \in \Omega\colon h(x) > 0\}| > 0$. Therefore $\lambda_1(s,p,h)$ is well defined and is non-negative.

We also know that $\lambda_1(s,p) > 0$ and there exists a non-negative function $u \in \widetilde{W}^{s,p}(\Omega)$ such that

- $u > 0$ in Ω, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$;
- u is a minimizer of (4.4) with $h \equiv 1$;
- u is a weak solution of (4.2) with $\lambda = \lambda_1(s,p)$ and $h \equiv 1$, that is u is an eigenfunction of (3.1) with eigenvalue $\lambda_1(s,p)$.

Moreover $\lambda_1(s,p)$ is simple, and if $sp > n$ then $\lambda_1(s,p)$ is isolated. See [28, Theorems 5, 14 and 19], [9, Theorem A1] and [21, Theorem 4.2].

The rest of this section is devoted to generalize these results for the first eigenvalue of (4.2) with $h \equiv 1$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,\infty)$, and $h \in \mathcal{A}$. There exists a non-negative function $u \in \widetilde{W}^{s,p}(\Omega)$, such that*

- *$u \neq 0$ in Ω;*
- *u is a minimizer of (4.4);*
- *u is a weak solution of (4.2) with $\lambda = \lambda_1(s,p,h)$, that is u is an eigenfunction of (3.1) with eigenvalue $\lambda_1(s,p,h)$.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence, that is $u_j \in \widetilde{W}^{s,p}(\Omega)$,

$$\int_\Omega h(x)|u_j(x)|^p\,dx = 1 \text{ and } \lim_{j\to\infty} \mathcal{K}(1-s)|u_j|^p_{W^{s,p}(\mathbb{R}^n)} = \lambda(s,p,h).$$

Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $\widetilde{W}^{s,p}(\Omega)$. Therefore, there exit a subsequence, still denoted by $\{u_j\}_{j\in\mathbb{N}}$, and $u \in \widetilde{W}^{s,p}(\Omega)$ such that

$$\begin{aligned} &u_j \rightharpoonup u \text{ weakly in } \widetilde{W}^{s,p}(\Omega),\\ &u_j \to u \text{ strongly in } L^p(\Omega). \end{aligned}$$

Thus $\int_\Omega h(x)|u(x)|^p\,dx = 1$ and

$$\mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} \le \lim_{j\to\infty} \mathcal{K}(1-s)|u_j|^p_{W^{s,p}(\mathbb{R}^n)} = \lambda(s,p,h).$$

Then $\mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} = \lambda(s,p,h)$, that is u is a minimizer of (4.4). It is easy to see that $|u|$ is also a minimizer of (4.4), this shows that there exists a non-negative minimizer of (4.4).

Finally, by the Lagrange multiplier rule (see [31, Theorem 2.2.10]) there exist $a, b \in \mathbb{R}$ such that $a + b \neq 0$, and

$$a\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) + b\int_\Omega h(x)|u(x)|^{p-2}u(x)v(x)\,dx = 0 \quad \forall v \in \widetilde{W}^{s,p}(\Omega).$$

If $a = 0$, then $b \neq 0$ and taking $v = u$, we get $\int_\Omega h(x)|u(x)|^p\,dx = 0$ a contradiction because $\int_\Omega h(x)|u(x)|^p\,dx = 1$. Hence $a \neq 0$, and without any loss of generality, we can assume that $a = 1$. Then

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) + b\int_\Omega h(x)|u(x)|^{p-2}u(x)v(x)\,dx = 0 \quad \forall v \in \widetilde{W}^{s,p}(\Omega).$$

Again, taking $v = u$ and using that

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) = \mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} = \lambda_1(s,p,h)$$

and $\int_\Omega h(x)|u(x)|^p\,dx = 1$, we have $b = -\lambda_1(s,p,h)$. □

Our next aim is to show that a non-negative eigenfunction associated to $\lambda_1(s,p,h)$ is in really positive. For this we will need a strong minimum principle.

We start by a definition. Let $p \in (1,\infty)$, $s \in (0,1)$, $h \in \mathcal{A}$, and $\lambda \in \mathbb{R}$. We say that $u \in \widetilde{W}^{s,p}(\Omega)$ is a weak super-solution of (4.2) if

$$\mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) \geq \lambda\int_\Omega h(x)|u(x)|^{p-1}u(x)v(x)\,dx$$

for all $v \in \widetilde{W}^{s,p}(\Omega), v \geq 0$.

Following the proof of the Di Castro-Kussi-Palatucci logarithmic lemma (see [16, Lemma 1.3]) we have the following result.

Lemma 4.2. *Let Ω be a bounded domain, $s \in (0,1), p \in (1,\infty)$, $h \in \mathcal{A}$, $\lambda > 0$ and u be a weak super-solution of* (4.2) *such that $u \geq 0$ in $B_R(x_0) \subset\subset \Omega$. Then for any $B_r = B_r(x_0) \subset B_{\frac{R}{2}}(x_0)$ and $0 < \delta < 1$*

$$\begin{aligned}
&\int_{B_r}\int_{B_r}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p \frac{dxdy}{|x-y|^{n+sp}}\\
&\leq Cr^{n-sp}\left\{\delta^{1-p}r^{sp}\int_{\mathbb{R}^n\setminus B_{2r}} \frac{u^-(y)^{p-1}}{|y-x_0|^{n+sp}}\,dy + 1\right\} + C\lambda\|h\|_{L^1(B_{2r})},
\end{aligned}$$

where $u^- = \max\{-u, 0\}$ and C depends only on $n, s,$ and p.

Proof. Let $0 < r < \frac{R}{2}$, $0 < \delta$ and $\phi \in C_0^\infty(B_{\frac{3r}{2}})$ be such that

$$0 \leq \phi \leq 1, \quad \phi \equiv 1 \text{ in } B_r \quad \text{and} \quad |D\phi| < Cr^{-1} \text{ in } B_{\frac{3r}{2}} \subset B_R.$$

Taking $v=(u+\delta)^{1-p}\phi^p$ as test function in (4.3) we have that

$$\lambda\int_{B_{\frac{3r}{2}}} h(x)\frac{u(x)^{p-1}}{(u(x)+\delta)^{p-1}}\phi(x)^p\,dx \leq \mathcal{K}(1-s)\mathcal{H}_{s,p}(u,(u+\delta)^{1-p}\phi^p).$$

Then,using that $0\leq u^{p-1}(u+\delta)^{1-p}\phi^p\leq 1$ in $B_{\frac{3r}{2}}$,

$$0\leq \mathcal{K}(1-s)\mathcal{H}_{s,p}(u,(u+\delta)^{1-p}\phi^p)+\lambda\|h\|_{L^1(B_{2r})}. \tag{4.5}$$

In the proof of [16, Lemma 1.3], it is shown that

$$\begin{aligned}&\mathcal{H}_{s,p}(u,(u+\delta)^{1-p}\phi^p)\\ &\leq Cr^{n-sp}\left\{\delta^{1-p}r^{sp}\int_{\mathbb{R}^n\setminus B_{2r}}\frac{u_-(y)^{p-1}}{|y-x_0|^{n+sp}}\,dy+1\right\}-\int_{B_r}\int_{B_r}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p\frac{dxdy}{|x-y|^{n+sp}},\end{aligned}$$

where C depends only on n,s, and p. Then, by (4.5), the lemma holds. □

To prove of the next theorem, we adapt the proof of [9, Theorem A.1].

Theorem 4.3. *Let Ω be a bounded domain, $s\in(0,1),p\in(1,\infty)$, $h\in\mathcal{A}$, $\lambda>0$, and u be a weak super-solution of* (4.2) *such that $u\geq 0$ in Ω. If $u\neq 0$ in Ω then $u>0$ a.e. in Ω.*

Proof. We start proving that if $K\subset\subset\Omega$ is a compact connected set such that $u\not\equiv 0$ then $u>0$ a.e. in K.

Since $K\subset\subset\Omega$ and K is compact then there exist $r\in(0,1)$ and $x_1,\dots,x_k\in K$ such that

$$\begin{aligned}&K\subset\{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega)>2r\},\quad K\subset\bigcup_{j=1}^{k}B_{\frac{r}{2}}(x_i),\quad\text{and}\\ &|B_{\frac{r}{2}}(x_i)\cap B_{\frac{r}{2}}(x_i+1)|>0\quad\forall i\in\{1,\dots,k-1\}.\end{aligned}$$

Suppose, to the contrary, $|\{x\in K\colon u(x)=0\}|>0$. Then there exists $i\in\{1,\dots,k\}$ such that $Z=|\{x\in K\colon u(x)=0\}\cap B_{\frac{r}{2}}(x_i)|$ has positive measure.

Given $\delta>0$, in the proof of [9, Theorem A.1], it is shown that

$$\int_{B_{\frac{r}{2}}(x_i)}\left|\log\left(1+\frac{u(x)}{\delta}\right)\right|^p dx\leq\frac{r^{n+sp}}{|Z|}\int_{B_{\frac{r}{2}}(x_i)}\int_{B_{\frac{r}{2}}(x_i)}\left|\log\left(\frac{u(x)+\delta}{u(y)+\delta}\right)\right|^p\frac{dxdy}{|x-y|^{n+sp}}.$$

Then, by Lemma 4.2, $\int_{B_{\frac{r}{2}}(x_i)}\left|\left(1+\frac{u(x)}{\delta}\right)\right|^p dx\leq\frac{C}{|Z|}\max\{r^{2n},r^{n+sp}\}$ with C independent of δ. Then, passing to the limit as δ goes to 0, we have that $u\equiv 0$ in $B_{\frac{r}{2}}(x_i)$. Thus, proceeding as in the proof of [9, Theorem A1] we can conclude that $u\equiv 0$ in K, that is a contradiction.

Proceeding as in the proof of [9, Theorem A1] we can conclude the general case. □

Corollary 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $s \in (0,1)$, $p \in (1,\infty)$, $h \in \mathcal{A}$, and $u \in \widetilde{W}^{s,p}(\Omega)$ be a non-negative eigenfunction corresponding to $\lambda_1(s,p,h)$. Then $u > 0$ almost everywhere in Ω.*

Observe that, if u is an eigenfunction corresponding to $\lambda_1(s,p,h)$, then either $hu_+ \not\equiv 0$ or $hu_- \not\equiv 0$, where $u_+ = \max\{u,0\}$ and $u_- = \max\{u,0\}$.

On the other hand, for any function $v\colon \mathbb{R}^N \to \mathbb{R}$

$$\begin{aligned}|v_+(x)-v_+(y)|^p &\le |v(x)-v(y)|^{p-2}(v(x)-v(y))(v_+(x)-v_+(y)),\\ |v_-(x)-v_-(y)|^p &\le -|v(x)-v(y)|^{p-2}(v(x)-v(y))(v_-(x)-v_-(y)),\end{aligned} \tag{4.6}$$

for all $x,y \in \mathbb{R}^n$. Therefore, if $hu_+ \not\equiv 0$ then u_+ is an eigenfunction corresponding to $\lambda_1(s,p,h)$. Moreover, by Corollary 4.4, $u_+ > 0$ almost everywhere in Ω.

We similarly deduce that if $hu_- \not\equiv 0$ then $u_- > 0$ almost everywhere in Ω. Then the next result is proved.

Corollary 4.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,\infty)$, and $h \in \mathcal{A}$. If $u \in \widetilde{W}^{s,p}(\Omega)$ is an eigenfunction corresponding to $\lambda_1(s,p,h)$, then either $u > 0$ or $u < 0$ almost everywhere in Ω.*

The proof of the result given below follows from a careful reading of [21, proof of Theorem 3.2].

Theorem 4.6. *Let Ω be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,\infty)$, $h \in \mathcal{A}$, $\lambda > 0$, and $u \in \widetilde{W}^{s,p}(\Omega)$ be a weak solution to (4.2). Then $u \in L^\infty(\mathbb{R}^n)$.*

Now, we prove that $\lambda_1(s,p,h)$ is also simple when $h \not\equiv 1$. For this we need the following lemma. For the proof see [2, Lemma 6.2].

Lemma 4.7. *Let $p \in (1,\infty)$. For $v > 0$ and $u \ge 0$, we have*

$$L(u,v) \ge 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n$$

where

$$L(u,v)(x,y) = |u(y)-u(x)|^p - |v(y)-v(x)|^{p-2}(v(y)-v(x))\left(\frac{u(y)^p}{v(y)^{p-1}} - \frac{u(x)^p}{v(x)^{p-1}}\right)$$

The equality holds a.e in $\mathbb{R}^n \times \mathbb{R}^n$ if and only if $u = kv$ a.e. in $\mathbb{R}^n$ for some constant k.

Theorem 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,\infty)$, $h \in \mathcal{A}$, and u be a positive eigenfunction corresponding to $\lambda_1(s,p,h)$. If $\lambda > 0$ is such that there exists a non-negative eigenfunction v of (3.1) with eigenvalue λ, then $\lambda = \lambda_1(s,p,h)$ and there exists $k \in \mathbb{R}$ such that $v = ku$ a.e. in Ω.*

Proof. Since $\lambda_1(s,p,h)$ is the first eigenvalue we have that $\lambda_1(s,p,h)\leq\lambda$. Let $m\in\mathbb{N}$ and $v_m:=v+\frac{1}{m}$.

We begin by proving that $w_m:=\frac{u^p}{v_m^{p-1}}\in\widetilde{W}^{s,p}(\Omega)$. It is immediate that $w_m=0$ in $\mathbb{R}^n\setminus\Omega$ and $w_m\in L^p(\Omega)$, due to $u\in L^\infty(\Omega)$, see Theorem 4.6.

On the other hand

$$
\begin{aligned}
|w_m(y)-w_m(x)| &= \left|\frac{u(y)^p-u(x)^p}{v_m(y)^{p-1}}+\frac{u(x)^p\left(v_m(x)^{p-1}-v_m(y)^{p-1}\right)}{v_m(y)^{p-1}v_m(x)^{p-1}}\right| \\
&\leq m^{p-1}\left|u(y)^p-u(x)^p\right|+\|u\|^p_{L^\infty(\Omega)}\frac{|v_m(y)^{p-1}-v_m(x)^{p-1}|}{v_m(y)^{p-1}v_m(x)^{p-1}} \\
&\leq m^{p-1}p(u(y)^{p-1}+u(x)^{p-1})|u(y)-u(x)| \\
&\quad+\|u\|^p_{L^\infty(\Omega)}(p-1)\frac{|v_m(y)^{p-2}+v_m(x)^{p-2}|}{v_m(y)^{p-1}v_m(x)^{p-1}}|v_m(y)-v_m(x)| \\
&\leq 2\|u\|^{p-1}_{L^\infty(\Omega)}m^{p-1}p|u(y)-u(x)| \\
&\quad+\|u\|^p_{L^\infty(\Omega)}(p-1)m^{p-1}\left(\frac{1}{v_m(y)}+\frac{1}{v_m(x)}\right)|v(y)-v(x)| \\
&\leq C(m,p,\|u\|_{L^\infty(\Omega)})\left(|u(y)-u(x)|+|v(y)-v(x)|\right)
\end{aligned}
$$

for all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$. Hence $w_m\in\widetilde{W}^{s,p}(\Omega)$ for all $m\in\mathbb{N}$ due to $u,v\in\widetilde{W}^{s,p}(\Omega)$.

Then, by Lemma 4.7 and since $u,v\in\widetilde{W}^{s,p}(\Omega)$ are two positive eigenfunctions of problem (3.1) with eigenvalue $\lambda_1(s,p,h)$ and λ respectively, we have

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{L(u,v_m)(x,y)}{|x-y|^{n+sp}}dxdy \\
&\leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(y)-u(x)|^p}{|x-y|^{n+sp}}dxdy \\
&\quad-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|v(y)-v(x)|^{p-2}(v(y)-v(x))}{|x-y|^{n+sp}}\left(\frac{u(y)^p}{v_m(y)^{p-1}}-\frac{u(x)^p}{v_m(x)^{p-1}}\right)dxdy \\
&\leq \frac{1}{\mathcal{K}(1-s)}\left\{\lambda_1(s,p,h)\int_\Omega h(x)u(x)^p\,dx-\lambda\int_\Omega h(x)v(x)^{p-1}\frac{u(x)^p}{v_m(x)^{p-1}}\,dx\right\}.
\end{aligned}
$$

By Fatou's lemma and the dominated convergence theorem

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{L(u,v)(x,y)}{|x-y|^{n+sp}}\,dxdy=0$$

due to $\lambda_1(s,p,h)\leq\lambda$. Then $L(u,v)(x,y)=0$ a.e. in $\mathbb{R}^n\times\mathbb{R}^n$. Hence, by Lemma 4.7, $u=kv$ for some constant $k>0$. $\square$

By Corollary 4.5 and Theorem 4.8, we have that $\lambda_1(s,p,h)$ is simple.

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,\infty)$, and $h \in \mathcal{A}$. Then $\lambda_1(s,p,h)$ is simple.*

Now, we get a lower bound for the measure of the nodal sets.

Lemma 4.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $s \in (0,1)$, $p \in (1,\infty,)$, $s_0 \in (0, \min\{\frac{n}{p}, s\})$, and $h \in \mathcal{A}$. If $u \in \widetilde{W}^{s,p}(\Omega)$ is an eigenfunction of* (3.1) *with eigenvalue $\lambda > \lambda_1(s,p,h)$, then there exists a constant $C = C(n,p)$ such that*

$$\left(C \frac{s_0(n-s_0p)^{p-1}\mathrm{diam}(\Omega)^{s_0p}}{\mathrm{diam}(\Omega)^{sp}\lambda\|h\|_{L^\infty(\Omega)}s_0p + \mathcal{K}} \right)^{\frac{p^\star_{s_0}}{p^\star_{s_0}-p}} \le |\Omega^\pm|.$$

Here $\Omega^+ = \{x \in \Omega \colon u(x) > 0\}$ and $\Omega^- = \{x \in \Omega \colon u(x) < 0\}$.

Proof. By Theorem 4.8, u changes sign then $u^+ \not\equiv 0$. In addition, $u^+ \in \widetilde{W}^{s,p}(\Omega)$. It follows from (4.6) that

$$\begin{aligned} \mathcal{K}(1-s)|u^+|^p_{W^{s,p}(\mathbb{R}^n)} &\le \mathcal{K}(1-s)\mathcal{H}_{s,p}(u,u^+) \\ &\le \lambda \int_{\Omega^+} h(x)|u^+(x)|^p dx \\ &\le \lambda \|h\|_{L^\infty(\Omega)} \int_{\Omega^+} |u^+(x)|^p dx. \end{aligned} \tag{4.7}$$

On the other hand, by Remark 2.18, there exists a constant $C = C(n,p)$ such that

$$\|u^+\|^p_{L^{p^\star_{s_0}}(\mathbb{R}^n)} \le \frac{C}{(n-s_0p)^{p-1}\mathrm{diam}(\Omega)^{s_0p}} \left(\mathrm{diam}(\Omega)^{sp}(1-s)|u^+|^p_{W^{s,p}(\mathbb{R}^n)} + \frac{1}{s_0p}\|u^+\|^p_{L^p(\Omega)} \right).$$

Then, by (4.7) and Hölder's inequality, we get

$$\|u^+\|^p_{L^{p^\star_{s_0}}(\mathbb{R}^n)} \le \frac{C}{(n-s_0p)^{p-1}\mathrm{diam}(\Omega)^{s_0p}} \left(\frac{\mathrm{diam}(\Omega)^{sp}\lambda\|h\|_{L^\infty(\Omega)}}{\mathcal{K}} + \frac{1}{s_0p} \right) \|u^+\|^p_{L^{p^\star_{s_0}}(\Omega)} |\Omega^+|^{\frac{p^\star_{s_0}-p}{p^\star_{s_0}}}.$$

Hence

$$\left(C \frac{\mathcal{K}s_0p(n-s_0p)^{p-1}\mathrm{diam}(\Omega)^{s_0p}}{\mathrm{diam}(\Omega)^{sp}\lambda\|h\|_{L^\infty(\Omega)}s_0p + \mathcal{K}} \right)^{\frac{p^\star_{s_0}}{p^\star_{s_0}-p}} \le |\Omega^+|.$$

In order to prove the second inequality, it will suffice to proceed as above, using the function $u^-(x) = \max\{0, -u(x)\}$ instead of u^+. $\square$

Finally, we show that the first eigenvalue is isolated, see also [10].

Theorem 4.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,\infty)$, and $h \in \mathcal{A}$. Then the first eigenvalue is isolated.*

Proof. By the definition of $\lambda_1(s,p,h)$, we have that $\lambda_1(s,p,h)$ is left-isolated.

To prove that $\lambda_1(s,p,h)$ is right-isolated, we argue by contradiction. We assume that there exists a sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k > \lambda_1(s,p,h)$ and $\lambda_k \searrow \lambda_1(s,p,h)$ as $k \to \infty$. Let u_k be an eigenfunction associated to λ_k, we can assume that $\int_\Omega h(x)|u_k(x)|^p\,dx = 1$. Then $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $\widetilde{W}^{s,p}(\Omega)$ and therefore we can extract a subsequence (that we still denoted by $\{u_k\}_{k\in\mathbb{N}}$) such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } \widetilde{W}^{s,p}(\Omega),\\ u_k &\to u \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

Then $\int_\Omega h(x)|u(x)|^p\,dx = 1$ and

$$\begin{aligned} \mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} &\le \mathcal{K}(1-s)\liminf_{k\to\infty}|u_k|^p_{W^{s,p}(\mathbb{R}^n)}\\ &= \lim_{k\to\infty}\lambda_k\int_\Omega h(x)|u_k(x)|^p\,dx\\ &= \lambda_1(s,p,h)\int_\Omega h(x)|u(x)|^p\,dx. \end{aligned}$$

Hence, u is an eigenvalue of (3.1) with eigenvalue $\lambda_1(s,p,h)$. By Corollary 4.5, we can assume that $u > 0$.

On the other hand, by the Egorov's theorem, for any $\varepsilon > 0$ there exists a subset A_ε of Ω such that $|A_\varepsilon| < \varepsilon$ and $u_k \to u > 0$ uniformly in $\Omega \setminus A_\varepsilon$. This contradicts the fact that, by Lemma 4.10,

$$\left(C\frac{s_0(n-s_0p)^{p-1}\mathrm{diam}(\Omega)^{s_0p}}{\mathrm{diam}(\Omega)^{sp}\lambda_k\|h\|_{L^\infty(\Omega)}s_0p+\mathcal{K}}\right)^{\frac{p^\star_{s_0}}{p^\star_{s_0}-p}} \le |\{x \in \Omega\colon u_k(x) < 0\}|$$

where $s_0 \in (0, \min\{s, \frac{n}{p}\})$ and C depends on n and p. The theorem is proved. □

4.3. Global properties. In the rest of this section, for simplicity, we will take $h \equiv 1$.

Lemma 4.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and $p \in (1,\infty)$. The first eigenvalue function $\lambda_1(\cdot,p)\colon (0,1] \to \mathbb{R}$ is continuous.*

Proof. Let $\{s_j\}_{j\in\mathbb{N}}$ be a sequence in $(0,1]$ convergent to $s \in (0,1]$. We will show that

$$\lim_{j\to\infty}\lambda_1(s_j,p) = \lambda_1(s,p). \tag{4.8}$$

We need to consider two cases: $s \in (0,1)$ and $s = 1$.

Case $s \in (0,1)$. Let $\phi \in C_0^\infty(\Omega)$, $\phi \not\equiv 0$. Then

$$\lambda_1(s_j,p) \le \mathcal{K}(1-s_j)\,\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+s_jp}}\,dxdy}{\int_\Omega |\phi(x)|^p\,dx}$$

for all $j \in \mathbb{N}$. Therefore, by dominated convergence theorem,

$$\limsup_{j\to\infty}\lambda_1(s_j,p) \le \mathcal{K}(1-s)\,\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+sp}}\,dxdy}{\int_\Omega |\phi(x)|^p\,dx}.$$

As ϕ is arbitrary

$$\limsup_{j\to\infty}\lambda_1(s_j,p) \le \lambda_1(s,p)$$

due to (4.4).

Thus, to prove (4.8), we need to show that

$$\liminf_{j\to\infty}\lambda_1(s_j,p) \ge \lambda_1(s,p).$$

Let $\{s_k\}_{k\in\mathbb{N}}$ be a subsequence of $\{s_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}\lambda_1(s_k,p) = \liminf_{j\to\infty}\lambda_1(s_j,p). \tag{4.9}$$

Let u_k be an eigenfunction of (3.1) with eigenvalue $\lambda_1(s_k,p)$ such that $\|u_k\|_{L^p(\Omega)} = 1$. Since

$$\lim_{k\to\infty}\mathcal{K}(1-s_k)|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)} = \lim_{k\to\infty}\lambda_1(s_k,p) \le \limsup_{j\to\infty}\lambda_1(s_j,p) \le \lambda_1(s,p).$$

Then $\{\mathcal{K}(1-s_k)|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)}\}_{k\in\mathbb{N}}$ is bounded, therefore $\{|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)}\}_{k\in\mathbb{N}}$ is bounded.

On the other hand, given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $s - \varepsilon < s_k$ for all $k \ge k_0$ and, by Lemma 2.3, we have

$$|u_k|^p_{W^{s-\varepsilon,p}(\mathbb{R}^n)} \le |u_k|^p_{W^{s_k,p}(\mathbb{R}^n)} + C(n,p)\left(\frac{1}{(s-\varepsilon)p} - \frac{1}{s_kp}\right)\|u_k\|^p_{L^p(\Omega)} \tag{4.10}$$

for all $k \ge k_0$. Thus, using that $\|u_k\|_{L^p(\Omega)} = 1$ for all $k \in \mathbb{N}$ and $\{|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)}\}_{k\in\mathbb{N}}$ is bounded, we have that $\{u_k\}_{k\ge k_0}$ is bounded in $\widetilde{W}^{s-\varepsilon,p}(\Omega)$. Then there exists $u \in \widetilde{W}^{s-\varepsilon,p}(\Omega)$ such that, for a subsequence that we still call $\{u_k\}_{k\ge k_0}$,

$$\begin{aligned} &u_k \rightharpoonup u \quad \text{weakly in } W^{s-\varepsilon,p}(\mathbb{R}^n),\\ &u_k \to u \quad \text{strongly in } L^p(\Omega). \end{aligned} \tag{4.11}$$

Then $\|u\|_{L^p(\Omega)} = 1$ and by (4.11), (4.10) and (4.9), we get

$$\begin{aligned}
&\mathcal{K}(1-s)|u|^p_{W^{s-\varepsilon,p}(\mathbb{R}^n)} \\
&\le \liminf_{k\to\infty} \mathcal{K}(1-s_k)|u_k|^p_{W^{s-\varepsilon,p}(\mathbb{R}^n)} \\
&\le \liminf_{k\to\infty} \left\{ \mathcal{K}(1-s_k)|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)} + C(n,p)\mathcal{K}(1-s_k)\left(\frac{1}{(s-\varepsilon)p} - \frac{1}{s_k p}\right)\right\} \\
&= \liminf_{k\to\infty} \left\{ \lambda_1(s_k,p) + C(n,p)\mathcal{K}(1-s_k)\left(\frac{1}{(s-\varepsilon)p} - \frac{1}{s_k p}\right)\right\} \\
&= \liminf_{j\to\infty} \lambda_1(s_j,p) + C(n,p)\mathcal{K}(1-s)\left(\frac{1}{(s-\varepsilon)p} - \frac{1}{sp}\right).
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, by Fatou Lemma, we have

$$\mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} \le \mathcal{K}(1-s)\liminf_{\varepsilon\to 0^+} |u|^p_{W^{s-\varepsilon,p}(\mathbb{R}^n)} \le \liminf_{j\to\infty} \lambda_1(s_j,p). \tag{4.12}$$

Since $\|u\|_{L^p(\Omega)} = 1$, by (4.4) and (4.12), we get

$$\lambda_1(s,p) \le \mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)} \le \liminf_{j\to\infty} \lambda_1(s_j,p).$$

Case $s = 1$. Let $\phi \in C_0^\infty(\Omega)$, $\phi \not\equiv 0$. Then, for any $j \in \mathbb{N}$

$$\lambda_1(s_j,p) \le \mathcal{K}(1-s_j)\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{n+s_j p}}\,dxdy}{\int_\Omega |\phi(x)|^p\,dx}.$$

Thus, by Remark 2.15 and the above inequality, we get

$$\limsup_{j\to\infty} \lambda_1(s_j,p) \le \frac{\int_\Omega |\nabla\phi(x)|^p\,dx}{\int_\Omega |\phi(x)|^p\,dx}.$$

As ϕ is arbitrary

$$\limsup_{j\to\infty} \lambda_1(s_j,p) \le \lambda_1(1,p).$$

As in the previous case, to prove (4.8), we need to show that

$$\liminf_{j\to\infty} \lambda_1(s_j,p) \ge \lambda_1(s,p).$$

Let $\{s_k\}_{k\in\mathbb{N}}$ be a subsequence of $\{s_j\}_{j\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \lambda_1(s_k,p) = \liminf_{j\to\infty} \lambda_1(s_j,p).$$

Let u_k be an eigenfunction of (3.1) with eigenvalue $\|u_k\|_{L^p(\Omega)} = 1$.Since

$$\lim_{k\to\infty} \mathcal{K}(1-s_k)|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)} = \lim_{k\to\infty} \lambda_1(s_k,p) \le \limsup_{j\to\infty} \lambda_1(s_j,p) \le \lambda_1(s,p).$$

Then $\{\mathcal{K}(1-s_k)|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)}\}_{k\in\mathbb{N}}$ is bounded, therefore, by Theorem 2.16, we can extract a subsequence (that we still denote by $\{u_k\}_{k\in\mathbb{N}}$) such that

$$u_k \rightharpoonup u \quad \text{weakly in } W^{1-\varepsilon,p}(\Omega), \tag{4.13}$$

$$u_k \to u \quad \text{strongly in } L^p(\Omega), \tag{4.14}$$

for all $\varepsilon > 0$, to some $u \in W^{1,p}_0(\Omega)$. Thus, by (4.14), we have $\|u\|_{L^p(\Omega)} = 1$ and then

$$\lambda_1(1,p) \le \int_\Omega |\nabla u|^p\,dx. \tag{4.15}$$

On the other hand, given $\varepsilon > 0$ there exists k_0 such that $1-\varepsilon < s_k$ for all $k \ge k_0$. Then, by Remark 2.17, we get

$$\mathcal{K}\varepsilon|u_k|^p_{W^{1-\varepsilon,p}(\Omega)} \le 2^{\varepsilon p}\mathrm{diam}(\Omega)^{s_k-1+\varepsilon}\mathcal{K}(1-s_k)|u_k|^p_{W^{s_k,p}(\mathbb{R}^n)} \quad \forall k \ge k_0.$$

Thus, by (4.13),

$$\mathcal{K}\varepsilon|u|^p_{W^{1-\varepsilon,p}(\Omega)} \le \liminf_{k\to\infty}\mathcal{K}\varepsilon|u_k|^p_{W^{1-\varepsilon,p}(\Omega)} \le 2^{\varepsilon p}\mathrm{diam}(\Omega)^{\varepsilon}\liminf_{j\to\infty}\lambda_1(s_j,p). \tag{4.16}$$

As $\varepsilon > 0$ is arbitrary, by (4.16) and Theorem 2.14, we have that

$$\int_\Omega |\nabla u(x)|^p dx = \lim_{\varepsilon\to 0^+}\mathcal{K}\varepsilon|u|^p_{W^{1-\varepsilon,p}(\Omega)} \le \liminf_{j\to\infty}\lambda_1(s_j,p). \tag{4.17}$$

Finally, by (4.15) and (4.17), we get

$$\lambda_1(1,p) \le \liminf_{j\to\infty}\lambda_1(s_j,p).$$

This completes the proof. □

For the proof of the following lemma we borrow ideas from [34, Lemma 2.3].

Lemma 4.13. *For every interval* $[a,b] \subset (0,1]$ *there exists* $\delta > 0$ *such that for all* $s \in [a,b]$ *there is no eigenvalue of* (4.1) *in* $(\lambda_1(s,p), \lambda_1(s,p)+\delta]$.

Proof. Suppose the lemma were false. Then we could find sequences $\{s_k\}_{k\in\mathbb{N}}$ in $(0,1]$, $\{\lambda_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}_+$ and $\{u_k\}_{k\in\mathbb{N}}$ in $\widetilde{W}^{s,p}(\Omega)\setminus\{0\}$ such that

$$\lim_{k\to\infty} s_k = s \in (0,1], \quad \lambda_k > \lambda_1(s_k,p)\ \forall k\in\mathbb{N}, \quad \lim_{k\to\infty}(\lambda_k - \lambda_1(s_k,p)) = 0,$$

and for all $k\in\mathbb{N}$ $\|u_k\|_{L^p(\Omega)} = 1$ and

$$u_k = \mathcal{R}_{s_k,p}(\lambda_k|u_k|^{p-2}u_k). \tag{4.18}$$

By Lemma 4.12,

$$\lim_{k\to\infty}\lambda_k = \lambda_1(s,p). \tag{4.19}$$

On the other hand $\{|u_k|^{p-2}u_k\}_{k\in\mathbb{N}}$ is bounded in $L^{p'}(\Omega)$ due to $\|u_k\|_{L^p(\Omega)} = 1$ for all $k \in \mathbb{N}$. Then, by Lemma 3.1, there exist $u \in L^p(\Omega)$ and a subsequence of $\{u_k\}_{k\in\mathbb{N}}$, still denoted $\{u_k\}_{k\in\mathbb{N}}$, such that $u_k \to u$ in $L^p(\Omega)$ as $k \to \infty$. Thus

$$|u_k|^{p-2}u_k \to |u|^{p-2}u \quad \text{strongly in } L^{p'}(\Omega). \tag{4.20}$$

Then, passing to the limit in (4.18), using (4.19), (4.20) and Lemma 3.1, we get

$$u = \mathcal{R}_{s,p}(\lambda_1(s,p)|u|^{p-2}u).$$

Therefore u is an eigenfunction associated to $\lambda_1(s,p)$. Then, by Corollary 4.5 and Theorem 4.8, we may assume without loss of generality, that $u > 0$.

On the other hand, given $s_0 \in (0, \min\{s, \frac{n}{p}\})$ there exists $k_0 \in N$ such that $s_k \geq s$ for all $k \geq k_0$ due to $s_k \to s$ as $k \to \infty$. Thus, by Lemma 4.10, we get

$$\left(C\frac{s_0(n-s_0p)^{p-1}\mathrm{diam}(\Omega)^{s_0p}}{\mathrm{diam}(\Omega)^{s_kp}\lambda_k\|h\|_{L^\infty(\Omega)}s_0p+\mathcal{K}}\right)^{\frac{p^\star_{s_0}}{p^\star_{s_0}-p}} \leq |\{x\in\Omega\colon u_k(x)<0\}| \quad \forall k \geq k_0.$$

Then, since $u_k \to u$ in $L^p(\Omega)$, u must change its sign in Ω, contrary to the fact that $u > 0$. $\square$

5. Bifurcation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1]$, and $p \in (1,\infty)$. In this section we consider the following non-linear problem:

$$\begin{cases} \mathcal{L}_{s,p}(u) = \lambda|u|^{p-2}u + f(x,u,\lambda) & \text{in } \Omega,\\ \qquad u = 0 & \text{in } \mathbb{R}^n\setminus\Omega, \end{cases} \tag{5.1}$$

where $f\colon \Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function such that

1. f satisfies a Carathéodory condition in the first two variables;
2. $f(x,t,\lambda) = o(|t|^{p-1})$ near $t = 0$, uniformly a.e. with respect to x and uniformly with respect to λ on bonded sets;
3. There exists $q \in (1, p^\star_s)$ such that

$$\lim_{|t|\to\infty}\frac{|f(x,t,\lambda)|}{|t|^{q-1}} = 0$$

uniformly a.e. with respect to x and uniformly with respect to λ on bounded sets.

A pair $(\lambda, u) \in \mathbb{R}\times\widetilde{W}^{s,p}(\Omega)$ is a weak solution of (5.1) if

$$\mathfrak{H}_{s,p}(u,v) = \int_\Omega \big(\lambda|u(x)|^{p-2}u(x) + f(x,u,\lambda)\big)\,v(x)\,dx,$$

for all $v \in \widetilde{W}^{s,p}(\Omega)$. Here

$$\mathfrak{H}_{s,p}(u,v) = \begin{cases} \mathcal{K}(1-s)\mathcal{H}_{s,p}(u,v) & \text{if } 0 < s < 1, \\ \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx & \text{if } s = 1. \end{cases}$$

Remark 5.1. The pair (λ, u) is weak solution of (5.1) iff (λ, u) satisfies

$$u = \mathscr{R}_\lambda(u)$$

where $\mathscr{R}_\lambda(u) = \mathcal{R}_{s,p}(\lambda|u|^{p-2}u + F(\lambda,u))$ and $F(\lambda,\cdot)$ is the Nemitsky operator associated with f.

We say that $(\lambda, 0) \in \mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ is a bifurcation point of (5.1) if in any neighbourhood of $(\lambda,0)$ in $\mathbb{R}\times\widetilde{W}^{s,p}(\Omega)$ there exists a nontrivial solution of (5.1).

The proof of the following result is analogous to that of [34, Proposition 2.1]

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1]$, and $p \in (1,\infty)$. If $(\lambda, 0)$ is a bifurcation point of (5.1) then λ is an eigenvalue of following eigenvalue problems*

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{5.2}$$

Let

$$\lambda_2(s,p) := \inf\left\{\lambda > \lambda_1(s,p) \colon \lambda \text{ is an eigenvalue of (5.2)}\right\}.$$

For $\lambda < \lambda_1(s,p)$ or $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ the function $u \equiv 0$ is the unique solution of

$$u = \mathcal{R}_{s,p}(\lambda|u|^{p-2}u).$$

Then for $\lambda < \lambda_1(s,p)$ or $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ we define the completely continuous operator $\mathcal{T}^\lambda_{s,p} : \widetilde{W}^{s,p}(\Omega) \to \widetilde{W}^{s,p}(\Omega)$

$$\mathcal{T}^\lambda_{s,p}(u) := \mathcal{R}_{s,p}(\lambda|u|^{p-2}u).$$

Thus $\deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathcal{T}^\lambda_{s,p}, B(0,r), 0)$ is well defined for any $\lambda < \lambda_1(s,p)$ or $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ and $r > 0$.

Theorem 5.3. *Let $s \in (0,1]$, $p \in (1,\infty)$, and $r > 0$. Then*

$$\deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathcal{T}^\lambda_{s,p}, B(0,r), 0) = \begin{cases} 1 & \text{if } \lambda < \lambda_1(s,p), \\ -1 & \text{if } \lambda_1(s,p) < \lambda < \lambda_2(s,p). \end{cases}$$

This result is a generalization of [34, Proposition 2.2], where the authors show that

$$\deg_{W_0^{1,p}(\Omega)}(I - \mathcal{T}_{1,p}^{\lambda}(u), B(0,r), 0) = \begin{cases} 1 & \text{if } \lambda < \lambda_1(1,p), \\ -1 & \text{if } \lambda_1(1,p) < \lambda < \lambda_2(1,p). \end{cases}$$

Proof of Theorem 5.3. Let $s_0 \in (0,1]$.

We begin by the case $\lambda_1(s_0,p) < \lambda < \lambda_2(s_0,p)$. By Lemma 4.13, there exists $\delta > 0$ such that there is no eigenvalues of (4.1) in $(\lambda_1(s,p), \lambda_1(s,p)+\delta]$ for any $s \in [s_0,1]$. Therefore, if $\lambda_1(s_0,p) < \lambda \leq \lambda_1(s_0,p)+\delta$, there exists a continuous function $\rho\colon [s_0,1] \to \mathbb{R}$ such that

$$\lambda_1(s,p) < \rho(s) < \lambda_2(s,p) \quad \forall s \in [s_0,1],$$

and $\rho(s_0) = \lambda$. Then it is sufficient to prove that the function $d\colon [s_0,1] \to \mathbb{R}$

$$d(s) \coloneqq \deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathcal{T}_{s,p}^{\rho(s)}, B(0,r), 0)$$

is constant due to $d(1) = -1$

Let $s \in [s_0,1]$. We define the operator $\mathcal{P}_s\colon L^p(\Omega) \to \widetilde{W}^{s,p}(\Omega)$ as

$$\mathcal{P}_s(u) \coloneqq \mathcal{R}_{s,p}(\rho(s)|u|^{p-2}u).$$

Then $\mathcal{P}_s$ is completely continuous and $\mathcal{T}_{s,p}^{\rho(s)} = \mathcal{P}_s \circ i$ where $i\colon \widetilde{W}^{s,p}(\Omega) \to L^p(\Omega)$ is the usual inclusion. Thus, by Lemma 2.21, we get

$$d(s) = \deg_{L^p(\Omega)}(I - i \circ \mathcal{P}_s, O, 0) \quad \forall s \in [s_0,1] \tag{5.3}$$

where O is any open bounded set in $L^p(\Omega)$ such that $0 \in O$.

On the other hand, since ρ is continuous and by Lemma 4.12, we get that the homotopy

$$\begin{aligned} &[s_0,1] \times L^p(\Omega) \to L^p(\Omega) \\ &\qquad (s,u) \to \mathcal{R}_{s,p}(\rho(s)|u|^{p-2}u) = (i \circ \mathcal{P}_s)(u) \end{aligned}$$

is completely continuous. Then $d(s)$ is constant in $[s_0,1]$ due to the invariance of the Leray-Schauder degree under compact homotopy and (5.3).

If $\lambda_1(s_0,p)+\delta < \lambda < \lambda_2(s_0,p)$, we take $\alpha\colon [s_0,1] \to \mathbb{R}$

$$\alpha(t) = \frac{1-t}{1-s_0}\lambda + \frac{t-s_0}{1-s_0}(\lambda_1(s_0,p)+\delta).$$

In this case, the degree

$$\deg_{\widetilde{W}^{s_0,p}(\Omega)}(I - \mathcal{R}_{s_0,p}(\alpha(t)\Psi_p(\cdot)), B(0,r), 0)$$

is well defined. Here $\Psi_p(u) = |u|^{p-2}u$. Then, from the invariance of the degree under homotopies, we get

$$\begin{aligned}&\deg_{\widetilde{W}^{s_0,p}(\Omega)}(I - \mathcal{R}_{s_0,p}(\alpha(t)\Psi_p(\cdot)), B(0,r), 0)\\&= \deg_{\widetilde{W}^{s_0,p}(\Omega)}(I - \mathcal{R}_{s_0,p}(\alpha(1)\Psi_p(\cdot)), B(0,r), 0)\\&= -1\end{aligned}$$

Finally, we consider the case $\lambda < \lambda_1(s_0, p)$. Given $a \in [0, 1]$, the degree

$$\deg_{\widetilde{W}^{s_0,p}(\Omega)}(I - \mathcal{R}_{s_0,p}(a\lambda\Psi_p(\cdot)), B(0,r), 0)$$

is well defined. Again, from the invariance of the degree under homotopies, we get

$$\deg_{\widetilde{W}^{s_0,p}(\Omega)}(I - \mathcal{R}_{s_0,p}(a\lambda\Psi_p(\cdot)), B(0,r), 0) = \deg_{\widetilde{W}^{s_0,p}(\Omega)}(I, B(0,r), 0) = 1$$

for all $a \in [0, 1]$. □

Finally, proceeding as in the proof of [34, Theorem 1.1], we prove Theorem 1.1.

Proof of Theorem 1.1. By contrary, suppose that $(\lambda_1(s,p), 0)$ is no bifurcation point of (5.1). Then there exist $\varepsilon, \delta_0 > 0$ such that for $|\lambda - \lambda_1(s,p)| \le \varepsilon$ and $\delta < \delta_0$ there is no non-trivial solution of

$$u - \mathscr{R}_\lambda(u) = 0$$

with $\|u\|_{W^{s,p}(\mathbb{R}^n)} = \delta$. Since the degree is invariance under compact homotopies

$$\deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathscr{R}_\lambda, B(0,\delta), 0) = \text{constant} \tag{5.4}$$

for all $\lambda \in [\lambda_1(s,p) - \varepsilon, \lambda_1(s,p) + \varepsilon]$.

Taking ε small enough, we can assume that there is no eigenvalue of (5.2) in $(\lambda_1(s,p), \lambda_1(s,p) + \varepsilon]$. Fix now $\lambda \in (\lambda_1(s,p), \lambda_1(s,p) + \varepsilon]$. We claim that if choose δ small enough then there is no solution of

$$u - \mathcal{R}_{s,p}(\lambda|u|^{p-2}u + tF(\lambda, u)) = 0$$

with $\|u\|_{W^{s,p}(\mathbb{R}^n)} = \delta$, for all $t \in [0, 1]$. Indeed, assuming the contrary and reasoning as in the proof of [34, Proposition 2.1] (see also Lemma 5.2), we would find that λ is an eigenvalue of (5.2), that is a contradiction.

Thus, since the degree is invariance under homotopies, by Theorem 5.3,

$$\deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathscr{R}_\lambda, B(0,\delta), 0) = \deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathcal{T}^{\lambda}_{s,p}, B(0,\delta), 0) = -1.$$

In similar manner, we can see that

$$\deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathscr{R}_\lambda, B(0,\delta), 0) = 1$$

for all $\lambda \in [\lambda_1(s,p) - \varepsilon, \lambda_1(s,p))$. Therefore $\deg_{\widetilde{W}^{s,p}(\Omega)}(I - \mathscr{R}_\lambda, B(0,\delta), 0)$ is no constant function. But this is a contradiction with (5.4) and so $(\lambda_1(s,p), 0)$ is a bifurcation point of (5.1).

The rest of the proof follows in the same manner as in [36]. $\square$

6. Existence of constant-sign solution

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, $p \in (1,\infty)$, and $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $g(0) = 0$. In this section, we will apply Theorem 1.1 to show that the following non-linear non-local problem

$$\begin{cases} (-\Delta)_p^s u = g(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{6.1}$$

has a non-trivial weak solution. Observe that $u \equiv 0$ is a solution of (6.1).

We will keep the following assumptions about g, throughout this section:

A1. $\frac{g(t)}{|t|^{p-2}t}$ is bounded;

A2. $\underline{\lambda} := \lim_{t\to 0} \frac{g(t)}{|t|^{p-2}t} < \lambda_1(s,p) < \liminf_{|t|\to\infty} \frac{g(t)}{|t|^{p-2}t}$.

Note that, if g satisfies A1 and A2 then

$$g(t) = \underline{\lambda}|t|^{p-2}t + f(t),$$

where $f(t) = o(|t|^{p-1})$ near $t = 0$. Then, our problem is related to the next bifurcation problem

$$\begin{cases} (-\Delta)_p^s u = \lambda|u|^{p-2}u + f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{6.2}$$

By Theorem 1.1 there exists a connected component $\mathscr{C}$ of the set of non-trivial solution of (6.2) in $\mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$ whose closure contains $(\lambda_1(s,p), 0)$ and it is either unbounded or contains a pair $(\lambda, 0)$ for some λ, eigenvalue of (4.1) with $\lambda > \lambda_1(s,p)$.

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $s \in (0,1)$, and $p \in (1,\infty)$. Then $\mathscr{C}$ is unbounded and*

$$\overline{\mathscr{C}} \subset \mathscr{H} := \{(\lambda_1(s,p), 0)\} \cup (\mathbb{R} \times \mathscr{P}),$$

where $\mathscr{P} := \{v \in \widetilde{W}^{s,p}(\Omega)\colon v \text{ has constant-sign in } \Omega\}$.

Proof. We split the proof in 3 steps.

Step 1. There exists a neighbourhood U of $(\lambda_1(s,p),0)$ in $\mathbb{R}\times\widetilde{W}^{s,p}(\Omega)$ such that $\mathscr{C}\cap U\setminus\{(\lambda_1(s,p),0)\}\subset\mathbb{R}\times\mathscr{P}$.

Let us assume by contradiction the existence of a sequence $\{(\lambda_k,u_k)\}_{k\in\mathbb{N}}$ of non-trivial solution of (6.2) such that u_k changes sign in Ω for all $k\in\mathbb{N}$ and $(\lambda_k,u_k)\to(\lambda_1(s,p),0)$ in $\mathbb{R}\times\widetilde{W}^{s,p}(\Omega)$ as $k\to\infty$.

For any $k\in\mathbb{N}$, since $h_k=\lambda_k+\frac{f(u_k)}{|u_k|^{p-2}u_k}$ is uniformly bounded in Ω and u_k changes sign in Ω, by Corollary 4.5 we have that 1 is an eigenvalue (4.2) with $h=h_k$ and $1>\lambda_1(s,p,h_k)$. Thus, by Lemma 4.10 and using that h_k is uniformly bounded in Ω, there exists a constant C independent of k such that

$$|\{x\in\Omega\colon u_k(x)>0\}|\geq C\quad\text{and}\quad|\{x\in\Omega\colon u_k(x)<0\}|\geq C\quad\forall k\in\mathbb{N}.\tag{6.3}$$

On the other hand, taking $\hat{u}_k:=\frac{u_k}{\|u_k\|_{\widetilde{W}^{s,p}(\Omega)}}$, it follows that the sequence $\{\hat{u}_k\}_{k\in\mathbb{N}}$ is bounded in $\widetilde{W}^{s,p}(\Omega)$ then, via a subsequence if necessary, we have that there exists $u\in\widetilde{W}^{s,p}(\Omega)$ such that

$$\begin{aligned}&\hat{u}_k\rightharpoonup u\text{ weakly in }\widetilde{W}^{s,p}(\Omega),\\&\hat{u}_k\to u\text{ strongly in }L^p(\Omega),\\&\hat{u}_k\to u\text{ a.e. in }\Omega.\end{aligned}$$

By (6.3),

$$u\not\equiv 0\text{ and }u\text{ changes sign.}\tag{6.4}$$

Moreover

$$\begin{aligned}\mathcal{K}(1-s)|u|^p_{W^{s,p}(\mathbb{R}^n)}&\leq\lim_{k\to\infty}\mathcal{K}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\hat{u}_k(x)-\hat{u}_k(y)|^p}{|x-y|^{n+sp}}dxdy\\&=\lim_{k\to\infty}\int_\Omega h_k(x)|\hat{u}_k|^p\,dx\\&=\lambda_1(s,p)\int_\Omega|u|^pdx,\end{aligned}$$

due to h_k is uniformly bounded in Ω, $h_k(x)\to\lambda_1(s,p)$ a.e. in Ω and $\hat{u}_k\to u$ strongly in $L^p(\Omega)$. Then

$$\mathcal{K}(1-s)\frac{\int_{\mathbb{R}^n}\int_{R^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy}{\int_\Omega|u|^pdx}\leq\lambda_1(s,p).$$

Thus, by definition of $\lambda_1(s,p)$, we have that u is an eigenfunction associated to $\lambda_1(s,p)$. Therefore, by Corollary 4.5, u has constant sign, this yield a contradiction with (6.4). Hence the claim follows.

Step 2. $\overline{\mathscr{C}} \subset \mathscr{H}$. Again we proceed by contradiction. Suppose that there exists $(\lambda_0, u_0) \in \overline{\mathscr{C}}$ such that can be approximated by elements of $\mathscr{C}$ from inside and from without $\mathscr{H}$, that is there exist $\{(\lambda_k, u_k)\}_{k\in} \subset \mathscr{C} \cap \mathscr{H}$ and $\{(\mu_k, v_k)\}_{k\in} \subset \mathscr{C} \cap \mathscr{H}^c$ such that $(\lambda_k, u_k) \to (\lambda_0, u_0)$ and $(\mu_k, v_k) \to (\lambda_0, u_0)$. By Step 1, $(\lambda_0, u_0) \neq (\lambda_1(s,p), 0)$.

Case $u_0 \equiv 0$. Thus $\lambda_0 \neq \lambda_1(s,p)$. Proceeding in a similar manner as in the previous step, we can see that λ_0 is an eigenvalue of (5.2) different to $\lambda_1(s,p)$ and arrive to a contradiction.

Case $u_0 \not\equiv 0$. We know that there exist $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}} \subset \mathscr{C} \cap \mathscr{H}$ such that $(\lambda_k, u_k) \to (\lambda_0, u_0)$ in $\mathbb{R} \times \widetilde{W}^{s,p}(\Omega)$. Therefore u_0 is either non-negative or non-positive and u_0 is a weak solution of

$$\begin{cases} (-\Delta)_p^s u = \left(\lambda_0 + \dfrac{f(u_0)}{|u_0|^{p-2}u_0}\right)|u|^{p-2}u & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Without loss of generality, we can assume that $u_0 \geq 0$ a.e. in Ω. Since $\lambda_0 + \frac{f(u_0)}{|u_0|^{p-2}u_0}$ is bounded, by Theorem 4.3, we have that $u_0 > 0$ a.e. in Ω. Thus $(\lambda_0, u_0) \in \mathscr{H}$. Argument in similar manner that in step 1, we can show that (λ_0, u_0) can not be approximated by elements of $\mathscr{C}$ from without $\mathscr{H}$, contradicting the fact that (λ_0, u_0) can be approximated by elements of $\mathscr{C}$ from without $\mathscr{H}$.

Step 3. $\mathscr{C}$ is unbounded. Since $\overline{\mathscr{C}} \subset \mathscr{H}$, $\mathscr{C}$ does not contain a pair $(\lambda, 0)$ for some λ, eigenvalue of (4.1) with $\lambda > \lambda_1(s,p)$. Then by Theorem 1.1, $\mathscr{C}$ is unbounded. $\square$

Our next aim is to show that $\mathscr{C} \cap \big([\underline{\lambda}, \infty) \times \widetilde{W}^{s,p}(\Omega)\big)$ is bounded. For this, we will need the following result. The proof is identical to the proof of [34, Lemma 3.2].

Lemma 6.2. *There exists a positive constant C such that if $(\lambda, u) \in \mathscr{C}$ then $\lambda \leq C$.*

Then for showing that $\mathscr{C} \cap \big([\underline{\lambda}, \infty) \times \widetilde{W}^{s,p}(\Omega)\big)$ is bounded, it is enough to prove the result given below.

Lemma 6.3. *There exists a positive constant M such that for any $(\lambda, u) \in \mathscr{C} \cap \big([\underline{\lambda}, C] \times \widetilde{W}^{s,p}(\Omega)\big)$ we have that $\|u\|_{\widetilde{W}^{s,p}(\Omega)} \leq M$. Here C is the constant of Lemma* 6.2.

Proof. Suppose by contradiction that there exists a sequence $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}}$ of elements of $\mathscr{C} \cap \big([\underline{\lambda}, C] \times \widetilde{W}^{s,p}(\Omega)\big)$ such that $\lambda_k \to \lambda_0$ and $\|u_k\|_{\widetilde{W}^{s,p}(\Omega)} \to \infty$ as $k \to \infty$. Without loss of generality we can assume that $u_k > 0$ for all $k \in \mathbb{N}$.

Taking $\hat{u}_k = \frac{u_k}{\|u_k\|_{\widetilde{W}^{s,p}(\Omega)}}$ and $h_k = \frac{f(u_k)}{|u_k|^{p-2}u_k}$, for any $k \in \mathbb{N}$ we have that

$$\hat{u}_k = \mathcal{R}_{s,p}\left(\lambda_k|\hat{u}_k|^{p-2}\hat{u}_k + \frac{f(u_k)}{|u_k|^{p-2}u_k}|\hat{u}_k|^{p-2}\hat{u}_k\right).$$

On the other hand, $\{\frac{f(u_k)}{|u_k|^{p-2}u_k}\}_{k\in\mathbb{N}}$ is uniformly bounded due to g satisfies A1, then there exists $h \in L^\infty(\Omega)$ such that

$$\frac{f(u_k)}{|u_k|^{p-2}u_k} \rightharpoonup h \text{ weakly in } L^q(\Omega) \quad \forall q > 1.$$

Since $\mathcal{R}_{s,p}$ to $L^{q'}(\Omega)$ with $q \in (1, p_s^\star)$ is a completely continuous operator, we have that there exists $u_0 \in \widetilde{W}^{s,p}(\Omega)$ such that $u_k \to u_0$ strongly in $\widetilde{W}^{s,p}(\Omega)$ and $u_0 = \mathcal{R}_{s,p}\left(\lambda_0|u_0|^{p-2}u_0 + h|u_0|^{p-2}u_0\right)$, that is u_0 is a weak solution of

$$\begin{cases} (-\Delta)_p^s u = (\lambda_0 + h(x))\,|u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

Observe that $u_0 \neq 0$ and $u \geq 0$ due to $\|\hat{u}_k\|_{\widetilde{W}^{s,p}(\Omega)} = 1$ and $\hat{u}_k > 0$ in Ω. Hence $\mu = 1$ is the first eigenvalue of

$$\begin{cases} (-\Delta)_p^s u = \mu\,(\lambda_0 + h(x))\,|u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{6.5}$$

and u_0 is an eigenfunction associated to 1. Then, by Corollary 4.5, we have that $u_0 > 0$ in Ω.

Claim. $h \geq \overline{\lambda} - \underline{\lambda}$ a.e. in Ω where $\lambda_1(s,p) < \overline{\lambda} < \liminf_{|s|\to\infty} \frac{g(s)}{|s|^{p-2}s}$.

Suppose the contrary, that is the set $A = \{x \in \Omega\colon h(x) < \overline{\lambda} - \underline{\lambda}\}$ has positive measure. Since $\hat{u}_k \to u_0 > 0$ a.e. in Ω, by the Egorov's theorem, there exists a set $U \subset \Omega$ such that $|\Omega \setminus U| < |A|$ and $u_k \to \infty$ uniformly in U. Then there exists $k_0 \in \mathbb{N}$ such that $\frac{f(u_k)}{|u_k|^{p-2}u_k} \geq \overline{\lambda} - \underline{\lambda}$ for all $k \geq k_0$ because

$$\lambda_1(s,p) < \overline{\lambda} < \liminf_{|s|\to\infty} \frac{g(s)}{|s|^{p-2}s} = \underline{\lambda} + \liminf_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s}$$

and therefore $h(x) \geq \overline{\lambda} - \underline{\lambda}$ a.e. in U. Thus $A \subset \Omega \setminus U$, then $|A| \leq |\Omega \setminus U| < |A|$, which is a contradiction. Hence, the claim follows.

Since $h(x) \geq \overline{\lambda} - \underline{\lambda}$ a.e. in Ω, $\lambda_0 - \underline{\lambda} \geq 0$ and $\overline{\lambda} > \lambda_1(s,p)$, we get $\lambda_0 + h(x) \geq \lambda_0 + \overline{\lambda} - \underline{\lambda} > \lambda_1(s,p)$.

On the other hand, since μ is the first eigenvalue of (6.5), we have that

$$1 \leq \mathcal{K}(1-s)\frac{|\phi|^p_{W^{s,p}(\mathbb{R}^n)}}{\int_\Omega (\lambda_0 + h(x))|\phi(x)|^p\,dx} \quad \forall \phi \in C_0^\infty(\Omega).$$

Then for any $\phi \in C_0^\infty(\Omega)$

$$(\lambda_0 + \overline{\lambda} - \underline{\lambda})\|\phi\|^p_{L^p(\Omega)} \leq \int_\Omega (\lambda_0 + h(x))|\phi(x)|^p \, dx \leq \mathcal{K}(1-s)|\phi|^p_{W^{s,p}(\mathbb{R}^n)}$$

due to our claim. Then $\lambda_0+\overline{\lambda}-\underline{\lambda} \leq \lambda_1(s,p) < \lambda_0+\overline{\lambda}-\underline{\lambda}$, getting a contradiction. Thus the lemma is true. □

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 6.2 and Lemma 6.3, $\mathscr{C} \cap \big([\underline{\lambda}, \infty) \times \widetilde{W}^{s,p}(\Omega)\big)$ is bounded. On other hand, by Lemma 6.1, $\mathscr{C}$ is unbounded. Then there exists $(\underline{\lambda}, \underline{u}) \in \mathscr{C}$, due to $\mathscr{C}$ is connected. By A2 $\underline{\lambda} < \lambda_1(s,p)$ and Lemma 6.1, $\underline{u}$ has constant-sign in Ω. Therefore u is a non-trivial weak solution of (6.1). □

Acknowledgement. This work was partially supported by the Mathamsud project 13MATH–03 – QUESP – Quasilinear Equations and Singular Problems. L. M. Del Pezzo was partially supported by PICT2012 0153 from ANPCyT (Argentina) and A. Quaas was partially supported by Fondecyt Grant No. 1151180 Programa Basal, CMM. U. de Chile and Millennium Nucleus Center for Analysis of PDE NC130017.

References

[1] Adams, R. A., *Sobolev Spaces*. Pure Appl. Math. 65. New York: Academic Press 1975.

[2] Amghibech, S., On the discrete version of Picone's identity. *Discrete Appl. Math.* 156 (2008)(1), 1 – 10.

[3] Anane, A., Simplicité et isolation de la première valeur propre du p-Laplacien avec poids (in French). *C. R. Acad. Sci. Paris Sér. I Math.* 305 (1987)(16), 725 – 728.

[4] Bandle, C. and Reichel, W., Solutions of quasilinear second-order elliptic boundary value problems via degree theory. In: *Stationary Partial Differential Equations. Vol. I* (eds.: M. Chipot et al.). Handb. Diff. Equ. Amsterdam: North-Holland 2004, pp. 1 – 70.

[5] Birindelli, I. and Demengel, F., Bifurcation for singular fully nonlinear equations. In: *On the Notions of Solution to Nonlinear Elliptic Problems: Results and Developments* (eds.: A. Alvino et al.). Quad. Mat. 23. Caserta: Dept. Math., Seconda Univ. Napoli 2008, pp. 117 – 144.

[6] Bjorland, C., Caffarelli, L. and Figalli, A., Non-local gradient dependent operators. *Adv. Math.* 230 (2012)(4-6), 1859 – 1894.

[7] Bjorland, C., Caffarelli, L. and Figalli, A., Nonlocal tug-of-war and the infinity fractional Laplacian. *Comm. Pure Appl. Math.* 65 (2012)(3), 337 – 380.

[8] Bourgain, J., Brezis, H. and Mironescu, P., Another look at Sobolev spaces. In: *Optimal Control and Partial Differential Equations* (Conference Paris 2000 in honour of Professor Alain Bensoussan's 60th Birthday; eds.: J. L. Menaldi et al.). Amsterdam: IOS Press 2001, pp. 439 – 455.

[9] Brasco, L. and Franzina, G., Convexity properties of Dirichlet integrals and Picone-type inequalities. *Kodai Math. J.* 37 (2014)(3), 769 – 799.

[10] Brasco, L. and Parini, E., The second eigenvalue of the fractional p-Laplacian. *Adv. Calc. Var.* (to appear).

[11] Busca, J., Esteban, M. J. and Quaas, A., Nonlinear eigenvalues and bifurcation problems for Pucci's operators. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 22 (2005)(2), 187 – 206.

[12] Chambolle, A., Lindgren, E. and Monneau, R., A Hölder infinity Laplacian. *ESAIM Control Optim. Calc. Var.* 18 (2012)(3), 799 – 835.

[13] Chang, K., *Methods in Nonlinear Analysis.* Springer Monogr. Math. Berlin: Springer 2005.

[14] Crandall, M. and Rabinowitz, P., Bifurcation from simple eigenvalues. *J. Funct. Anal.* 8 (1971), 321 – 340.

[15] Demengel, F. and Demengel, G., *Functional Spaces for the Theory of Elliptic Partial Differential Equations.* Transl. from the 2007 French original by R. Erné. Universitext. London: Springer 2012.

[16] Di Castro, A., Kuusi, T. and Palatucci, G., Local behavior of fractional p-minimizers, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33 (2016)(5), 1279 – 1299.

[17] Di Nezza, E., Palatucci, G. and Valdinoci, E., Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* 136 (2012)(5), 521 – 573.

[18] Drábek, P., *Solvability and Bifurcations of Nonlinear Equations.* Pitman Res. Notes Math. Ser. 264. Harlow: Longman 1992.

[19] Drábek, P., Girg, P., Takáč, P. and Ulm, M., The Fredholm alternative for the p-Laplacian: bifurcation from infinity, existence and multiplicity. *Indiana Univ. Math. J.* 53 (2004)(2), 433 – 482.

[20] Fiscella, A., Molica Bisci, G. and Servadei, R., Bifurcation and multiplicity results for critical nonlocal fractional Laplacian problems. *Bull. Sci. Math.* 140 (2016), 14 – 35.

[21] Franzina, G. and Palatucci, G., Fractional p-eigenvalues. *Riv. Mat. Univ. Parma (N.S.)* 5 (2014)(2), 373 – 386.

[22] García-Melián, J. and Sabina de Lis, J., A local bifurcation theorem for degenerate elliptic equations with radial symmetry, *J. Diff. Equ.* 179 (2002)(1), 27 – 43.

[23] Grisvard, P., *Elliptic Problems in Nonsmooth Domains.* Monogr. Stud. Math. 24. Boston (MA): Pitman 1985.

[24] Iannizzotto, A., Mosconi, S. and Squassina, M., Global Hölder regularity for the fractional p-Laplacian. *Rev. Mat. Iberoam.* (to appear).

[25] Ishii, H. and Nakamura, G., A class of integral equations and approximation of p-Laplace equations. *Calc. Var. Partial Diff. Equ.* 37 (2010)(3–4), 485 – 522.

[26] Kielhöfer, H., *Bifurcation Theory. An Introduction with Applications to PDEs.* Appl. Math. Sci. 156. New York: Springer 2004.

[27] Kuznetsov, Y., *Elements of Applied Bifurcation Theory.* Third edition. Appl. Math. Sci. 112. New York: Springer 2004.

[28] Lindgren, E. and Lindqvist, P., Fractional eigenvalues. *Calc. Var. Partial Diff. Equ.* 49 (2014)(1–2), 795 – 826.

[29] Maz′ya, V. and Shaposhnikova, T., On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.* 195 (2002)(2), 230 – 238.

[30] Molica Bisci, G. and Servadei, R., A bifurcation result for non-local fractional equations. *Anal. Appl. (Singap.)* 13 (2015)(4), 371 – 394.

[31] Papageorgiou, N. and Kyritsi-Yiallourou, S., *Handbook of Applied Analysis.* Adv. Mech. Math. 19, New York: Springer 2009.

[32] Perera, K., Squassina, M. and Yang, Y., Bifurcation and multiplicity results for critical fractional p-Laplacian problems. *Math. Nachr.* 289 (2015), 332 – 342.

[33] del Pino, M., Elgueta, M. and Manásevich, R., A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0,\ u(0) = u(T) = 0,\ p > 1$. *J. Diff. Equ.* 80 (1989)(1), 1 – 13.

[34] del Pino, M. and Manásevich, R., Global bifurcation from the eigenvalues of the p-Laplacian. *J. Diff. Equ.* 92 (1991)(2), 226 – 251.

[35] Rabinowitz, P., Some global results for nonlinear eigenvalue problems. *J. Funct. Anal.* 7 (1971), 487 – 513.

[36] Rabinowitz, P., Some aspects of nonlinear eigenvalue problems. *Rocky Mountain J. Math.* 3 (1973), 161 – 202.

[37] Rabinowitz, P., *Théorie du Degré Topologique et Applications à des Problèmes aux Limites Non Linéaires* (in French). Paris: Université Paris VI, Laboratoire Analyse Numérique 1975.

[38] Ros-Oton, X. and Serra, J., The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)* 101 (2014)(3), 275 – 302.

[39] Silvestre, L., Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.* 60 (2007)(1), 67 – 112.

[40] Stein, E., *Singular Integrals and Differentiability Properties of Functions.* Princeton Math. Ser. 30. Princeton (NJ): Princeton Univ. Press 1970.

[41] Struwe, M., *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems.* Fourth edition. Ergeb. Math. Grenzgeb. (3) 34. Berlin: Springer 2008.

Received December 15, 2014; revised March 16, 2016