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Which Functions are Fractionally Differentiable?

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Abstract. We examine the existence of fractional derivatives of a function in terms of the pointwise convergence or equiconvergence of certain improper integrals containing this function. The fractional differentiation operator is treated as the inverse to the Riemann-Liouville integral operator. Technically, we give a description of the range of the Riemann-Liouville operator. The results are reformulated also for Riemann-Liouville and Caputo fractional derivatives.

Keywords. Inversion of Riemann-Liouville operator, Riemann-Liouville fractional derivative, Caputo fractional derivative, description of fractionally differentiable functions

Mathematics Subject Classification (2010). 26A33

1. Introduction

Consider the Riemann-Liouville integral operator $J^\alpha : C[0,T] \to C[0,T]$ of order $\alpha > 0$, $\alpha \in \mathbb{R}$, defined by

$$(J^\alpha u)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds, \quad 0 \le t \le T,\ u \in C[0,T],$$

where Γ is the Euler Gamma-function. In particular, $(J^1 u)(t) = \int_0^t u(s)ds$. For $\alpha = m \in \mathbb{N} = \{1, 2, \ldots\}$, the range of the operator J^m is given by

$$J^m C[0,T] = \{v \in C^m[0,T] : v^{(k)}(0) = 0,\ k = 0, \ldots, m-1\} =: C_0^m[0,T],$$

and J^m is invertible on it, $(J^m)^{-1} v = D_0^m v$, where $D_0^m : C_0^m[0,T] \to C[0,T]$ is the restriction of the operator $D^m = \left(\frac{d}{dt}\right)^m : C^m[0,T] \to C[0,T]$. Due to the semigroup property (see e.g. [1,3])

$$J^\alpha J^\beta = J^\beta J^\alpha = J^{\alpha+\beta} \quad \text{for } \alpha > 0,\ \beta > 0, \tag{1}$$

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J^α is invertible on its range $J^\alpha C[0,T]$ also for fractional (noninteger) $\alpha > 0$. Indeed, if $J^\alpha u = 0$ for some $u \in C[0,T]$ then taking $m \in \mathbb{N}$, $m > \alpha$, we have $J^m u = J^{m-\alpha} J^\alpha u = 0$, $u = 0$.

The description of the range $J^\alpha C[0,T]$, $\alpha > 0$, is closely related to the description of the class of fractionally differentiable functions. Namely, one possible definition of the fractional differentiation operator of order $\alpha > 0$ is given by

$$D_0^\alpha v = (J^\alpha)^{-1} v, \quad v \in J^\alpha C[0,T]. \tag{2}$$

This most natural definition is used e.g. in the Mathematical Encyclopedia [8], however, the Riemann-Liouville modification $D^\alpha_{\text{R-L}}$ and the Caputo modification D^α_{Cap} of the definition are more suitable in applications and more popular in literature; in Sections 4 and 5 we examine the relations between D_0^α, $D^\alpha_{\text{R-L}}$ and D^α_{Cap} . In our considerations, the "pure" concept (2) is preferable, since due to (1), operator D_0^α has the property that

$$D_0^\alpha D_0^\beta = D_0^\beta D_0^\alpha = D_0^{\alpha+\beta} \quad \text{for } \alpha > 0,\ \beta > 0, \tag{3}$$

whereas for $D^\alpha_{\text{R-L}}$ and D^α_{Cap} this property is lost.

According to (2), a function $v \in C[0,T]$ is D_0^α-differentiable if and only if equation $J^\alpha u = v$ has a solution $u \in C[0,T]$. For $m < \alpha < m+1$, $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, this equation is equivalent to $J^{m+1-\alpha} J^\alpha u = J^{m+1-\alpha} v$, or to $J^{m+1} u = J^{m+1-\alpha} v$ which is solvable if and only if $J^{m+1-\alpha} v \in C_0^{m+1}[0,T]$; the solution is given by $u = D_0^\alpha v = D^{m+1} J^{m+1-\alpha} v$. Thus for $m < \alpha < m+1$, $m \in \mathbb{N}_0$, it holds that

$$D_0^\alpha v = D^{m+1} J^{m+1-\alpha} v \quad \text{for } v \in C[0,T] \text{ such that } \quad J^{m+1-\alpha} v \in C_0^{m+1}[0,T]. \tag{4}$$

This can be considered as a definition of D_0^α equivalent to (2).

Which functions are D_0^α-differentiable? Due to (4), an indirect answer is that a function $v \in C[0,T]$ is D_0^α-differentiable for an $\alpha \in (m, m+1)$ if and only if $J^{m+1-\alpha} v \in C_0^{m+1}[0,T]$. But here a new question arises: for which $v \in C[0,T]$ it holds $J^{m+1-\alpha} v \in C_0^{m+1}[0,T]$? We have not found in literature an exhaustive answer to these essential questions, only some simple sufficient conditions for the fractional differentiability are known, e.g. that $D^\alpha_{\text{Cap}} v$ exists for $v \in C^m[0,T]$, $m < \alpha < m+1$, if $v^{(m)} \in \mathcal{H}^\beta[0,T]$, $\alpha - m < \beta \le 1$. Our main result, Theorem 2.1, answers these questions to full extent in case $0 < \alpha < 1$ in terms of the pointwise convergence, majorized pointwise convergence and equiconvergence of the Riemann improper integrals

$$\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds := \lim_{\theta \uparrow 1} \int_0^{\theta t} (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds, \quad 0 < t \le T,$$

complemented by some further conditions on $v \in C[0,T]$. Theorem 2.2 extends the results to the case $m < \alpha < m+1$ with an arbitrary $m \in \mathbb{N}_0$.

2. Formulation of main results

By $\mathcal{H}^\alpha[0,T]$, $0 < \alpha \le 1$, we mean the standard Hölder space consisting of functions $v \in C[0,T]$ such that

$$\|v\|_{\mathcal{H}^\alpha} := \max_{0\le t\le T} |v(t)| + \sup_{0\le s<t\le T} \frac{|v(t)-v(s)|}{(t-s)^\alpha} < \infty,$$

and by $\mathcal{H}_0^\alpha[0,T]$, $0 < \alpha < 1$, we mean the closed [7] subspace of $\mathcal{H}^\alpha[0,T]$ consisting of functions $v \in \mathcal{H}^\alpha[0,T]$ such that

$$\sup_{0\le s<t\le T,\ t-s\le\varepsilon} \frac{|v(t)-v(s)|}{(t-s)^\alpha} \to 0 \quad \text{as } \varepsilon \to 0.$$

The central results of the article are formulated in the following theorem.

Theorem 2.1. *For an $\alpha \in (0,1)$ and a function $v \in C[0,T]$, the following conditions* (i), (ii), (ii′), (iii) *and* (iii′) *are equivalent:*

(i) *$v \in J^\alpha C[0,T]$, i.e. the fractional derivative $D_0^\alpha v := (J^\alpha)^{-1} v \in C[0,T]$ exists;*

(ii) *a finite limit $\gamma_0 := \lim_{t\to 0} t^{-\alpha} v(t)$ exists, and the Riemann improper integrals $\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds$, $0 < t \le T$, equiconverge in the sense that*

$$\lim_{\theta\uparrow 1} \sup_{0<t\le T} \left| \int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds - \int_0^{\theta t} (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds \right| = 0; \quad (5)$$

(ii′) *a finite limit $\gamma_0 := \lim_{t\to 0} t^{-\alpha} v(t)$ exists; the Riemann improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds =: w(t)$ converges for any $t \in (0,T]$ and defines a function $w \in C(0,T]$ which has a finite limit as $t \to 0$ (hence $w \in C[0,T]$); moreover, there is a majorant function $W \in L^1(0,T)$ such that*

$$\left| \int_0^{\theta t} (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds \right| \le W(t) \quad \textit{for } 0<t<T,\ 0<\theta<1; \quad (6)$$

(iii) *v has the structure $v = \gamma_0 t^\alpha + v_0$ where γ_0 is a constant, $v_0 \in \mathcal{H}_0^\alpha[0,T]$, $v_0(0) = 0$, and the improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds =: w(t)$ converges for any $t \in (0,T]$ and defines a function $w \in C(0,T]$ which has a finite limit $w(0) := \lim_{t\to 0} w(t)$ (so $w \in C[0,T]$);*

(iii′) *v has the structure $v = \gamma_0 t^\alpha + v_0$ where γ_0 is a constant, $v_0 \in \mathcal{H}_0^\alpha[0,T]$, $v_0(0) = 0$, and the improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds =: w_0(t)$ converges for any $t \in (0,T]$ and defines with $w_0(0) = 0$ a function $w_0 \in C[0,T]$.*

For $v \in J^\alpha C[0,T]$*, it holds for* $0 < t \le T$ *that*

$$\begin{aligned}(D_0^\alpha v)(t) &:= ((J^\alpha)^{-1}v)(t)\\ &= \frac{1}{\Gamma(1-\alpha)}\Big(t^{-\alpha}v(t) + \alpha\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds\Big); \qquad (7)\\ (D_0^\alpha v)(0) &:= ((J^\alpha)^{-1}v)(0) = \Gamma(\alpha+1)\gamma_0.\end{aligned}$$

The proof of Theorem 2.1 is presented in Section 6.

Perhaps, the equivalence of conditions (i), (ii) and (iii) is most interesting and useful. The equivalence of (i) and (ii) tells us that the existence of the fractional derivative $D_0^\alpha v$ can be always discovered with the help of equiconvergence (5), whereas the equivalence of (i) and (iii) tells us that, with the reservations formulated in condition (iii), the pointwise convergence of the improper integrals $\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds$, $0 < t \le T$, will also do.

Requiring in (ii), (ii′), (iii) or (iii′) the *absolute* convergence of the improper integrals we do not obtain the whole range $J^\alpha C[0,T]$. For instance, one can check that $v(t) = t(1-t)^\alpha \log(1-t)^{-1}\sin\log(1-t)$, $0 < t < 1$, $v(0) = v(1) = 0$, satisfies (iii) with $\gamma_0 = 0$, $T = 1$, hence $v \in J^\alpha C[0,1]$, but $\int_0^1 (1-s)^{-\alpha-1}\,|v(1)-v(s)|\,ds = \int_0^1 s(1-s)^{-1}\,|\log(1-s)|^{-1}\,|\sin\log(1-s)|\,ds = \infty$.

It is easy to extend Theorem 2.1 to the case of arbitrary fractional $\alpha > 0$; for the sake of brevity, we omit the claims corresponding to (ii') and (iii').

Theorem 2.2. *For* $m < \alpha < m+1$, $m \in \mathbb{N}_0 = \{0,1,2,\ldots\ldots\}$, *and* $v \in C[0,T]$, *the fractional derivative* $D_0^\alpha v \in C[0,T]$ *exists if and only if* $v \in C_0^m[0,T]$ *and the fractional derivative* $D_0^{\alpha-m}v^{(m)} \in C[0,T]$ *exists; in other words, the following conditions* ($\mathrm{i_m}$), ($\mathrm{ii_m}$) *and* ($\mathrm{iii_m}$) *are equivalent:*

($\mathrm{i_m}$) $v \in J^\alpha C[0,T]$*;*

($\mathrm{ii_m}$) $v \in C_0^m[0,T]$*, a finite limit* $\lim_{t\to0} t^{m-\alpha}v^{(m)}(t) =: \gamma_m$ *exists, and*

$$\sup_{0<t\le T}\left|\int_{\theta t}^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds\right| \to 0 \quad \text{as } \theta \to 1;$$

($\mathrm{iii_m}$) $v \in C_0^m[0,T]$*, function* $v^{(m)}$ *has the structure* $v^{(m)} = \gamma_m t^{\alpha-m} + v_m$ *where* γ_m *is a constant,* $v_m \in \mathcal{H}_0^{\alpha-m}[0,T]$*,* $v_m(0) = 0$*, and the Riemann improper integral* $\int_0^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds =: w_m(t)$ *converges for any* $t \in (0,T]$ *and defines a function* $w_m \in C(0,T]$ *which has a finite limit* $w_m(0) := \lim_{t\to0} w_m(t)$*.*

For $v \in J^\alpha C[0,T]$*, it holds* $D_0^\alpha v = D_0^{\alpha-m}v^{(m)}$*, i.e.* $(D_0^\alpha v)(0) = \Gamma(\alpha+1-m)\gamma_m$*,*

$$\begin{aligned}(D_0^\alpha v)(t) = \frac{1}{\Gamma(m+1-\alpha)}&\Big(t^{m-\alpha}v^{(m)}(t)\\ &+ (\alpha-m)\int_0^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds\Big), \quad 0 < t \le T.\end{aligned}$$

Proof. For $m = 0$, the claims of Theorem 2.2 are contained in Theorem 2.1. For $m \geq 1$, $m < \alpha < m+1$, due to (3), it holds that $D_0^\alpha = D_0^{\alpha-m} D_0^m$, and Theorem 2.2 follows from Theorem 2.1 with $D_0^m v$ in the role of v and $\alpha - m \in (0,1)$ in the role of α. □

For $\alpha = m \in \mathbb{N}$ a formal counterpart of Theorem 2.2 does not hold; actually we do not need it since $J^m C[0,T] = C_0^m[0,T]$.

3. Examples

3.1. Functions $v \in \mathcal{H}^\beta[0,T]$, $0 < \alpha < \beta \leq 1$, $v(0) = 0$. Such a function satisfies condition (ii) of Theorem 2.1, hence $D_0^\alpha v \in C[0,T]$ is well-defined and formula (7) holds for it.

This simple observation is useful examining the D_0^α-differentiability of functions from the Sobolev space

$$W_0^{m+1,p}(0,T) = \left\{ v \in C^m[0,T] \colon v^{(m+1)} \in L^p(0,T),\ v^{(k)}(0) = 0,\ k = 0, \dots, m \right\}.$$

Proposition 3.1. *For $m < \alpha < m+1$, $m \in \mathbb{N}_0$, $p > \frac{1}{m+1-\alpha}$, it holds*

$$W_0^{m+1,p}(0,T) \subset J^\alpha C[0,T], \quad D_0^\alpha v = J^{m+1-\alpha} v^{(m+1)} \quad \text{for } v \in W_0^{m+1,p}(0,T). \quad (8)$$

Proof. Consider first the case $m = 0$. Inequality $p > \frac{1}{1-\alpha}$ implies that $\frac{p-1}{p} > \alpha$. For $v \in W_0^{1,p}(0,T)$, $0 \leq s < t \leq T$, $t - s \leq 1$, we get with the help of Hölder's inequality that

$$| v(t) - v(s) | = \left| \int_s^t v'(\tau) d\tau \right| \leq \|v'\|_{L^p} (t-s)^{\frac{p-1}{p}},$$

thus $v \in \mathcal{H}^\beta[0,T]$, $\beta = \frac{p-1}{p} > \alpha$. Hence $v \in J^\alpha C[0,T]$ and (7) holds. Integration by parts brings (7) to the form $D_0^\alpha v = J^{1-\alpha} v'$, i.e. to (8) for $m = 0$, completing so the proof of the Proposition for $m = 0$. For $m < \alpha < m+1$, $m \geq 1$, the Proposition follows from case $m = 0$ due to the equality $D_0^\alpha = D_0^{\alpha-m} D_0^m$. □

For $v \in C^{m+1}[0,T]$, more generally for $v \in W^{m+1,p}(0,T)$), the Caputo fractional derivative $D_{\mathrm{Cap}}^\alpha v$ of an order $\alpha \in (m, m+1)$ is often defined by $D_{\mathrm{Cap}}^\alpha v = J^{m+1-\alpha} v^{(m+1)}$ (see e.g. [5,6]). We see from Proposition 3.1 that $D_0^\alpha v = D_{\mathrm{Cap}}^\alpha v$ for $v \in W_0^{m+1,p}(0,T)$ with $p > \frac{1}{m+1-\alpha}$, in particular, for $v \in C_0^{m+1}[0,T]$. In Section 5 we use somewhat more general concept of Caputo fractional derivative [1,3] applicable to a more wide class of functions $v \in C^m[0,T]$ that need not to be $m+1$ times differentiable; for $v \in C^{m+1}[0,T]$ the two definitions are equivalent.

3.2. Function t^α, $\alpha > 0$. Since $(J^\alpha 1)(t) = \frac{1}{\Gamma(\alpha+1)} t^\alpha$, function t^α is D_0^α-differentiable. By Theorem 2.1 t^α satisfies for $0 < \alpha < 1$ condition (ii). But we need an independent establishment of this fact since the proof of Theorem 2.1 itself uses (ii) for t^α, $0 < \alpha < 1$. A direct proof is relatively simple: $\gamma_0 = \lim_{t\to 0} t^{-\alpha} t^\alpha = 1$, and the integration by parts and the change of variables $s = tx$ yield

$$\int_{\theta t}^{t} (t-s)^{-\alpha-1}\big(t^\alpha - s^\alpha\big)ds = \frac{t^\alpha - s^\alpha}{\alpha(t-s)^\alpha}\Big|_{s=\theta t}^{t} + \int_{\theta t}^{t} (t-s)^{-\alpha} s^{\alpha-1} ds$$
$$= \underbrace{-\frac{1-\theta^\alpha}{\alpha(1-\theta)^\alpha} + \int_\theta^1 (1-x)^{-\alpha} x^{\alpha-1} dx}_{\to 0 \text{ as } \theta\uparrow 1} \qquad (9)$$

(we see that the integral is actually independent of t); the L'Hospital's rule helps to observe that $\lim_{s\uparrow t} \frac{t^\alpha - s^\alpha}{(t-s)^\alpha} = 0$ and $\lim_{\theta\uparrow 1} \frac{1-\theta^\alpha}{(1-\theta)^\alpha} = 0$.

Although function $v(t) = (T-t)^\alpha - T^\alpha$, $0 \le t \le T$, is for $0 < \alpha < 1$ of the same smoothness class as t^α (still $v \in \mathcal{H}^\alpha[0,T]$ and $v(0) = 0$), nevertheless, $v \notin J^\alpha C[0,T]$, since $\int_0^t (t-s)^{-\alpha-1}\big(v(t) - v(s)\big)ds$ diverges for $t = T$ and hence (iii) is violated:

$$\int_0^T (T-s)^{-\alpha-1}\big(v(T) - v(s)\big)ds = -\int_0^T (T-s)^{-1} ds = -\infty.$$

3.3. Functions of type $t^\alpha v(t)$. Such functions appear treating singular fractional differential equations [4].

Proposition 3.2. *For $0 < \alpha < 1$, $v \in C[0,T]$, the function $t^\alpha v(t)$ is D_0^α-differentiable (i.e. $t^\alpha v(t)$ belongs to $J^\alpha C[0,T]$) if and only if the improper integrals $\int_0^t t^\alpha (t-s)^{-\alpha-1}(v(t) - v(s))ds$, $0 < t \le T$, equiconverge.*

Proof. We show that $t^\alpha v(t)$ satisfies (ii) iff $\int_0^t t^\alpha (t-s)^{-\alpha-1}(v(t)-v(s))ds$, $0<t\le T$, equiconverge. First, the limit $\lim_{t\to 0} t^{-\alpha}(t^\alpha v(t)) = v(0)$ exists. Second,

$$\int_0^t (t-s)^{-\alpha-1}(t^\alpha v(t) - s^\alpha v(s))ds$$
$$= \int_0^t (t-s)^{-\alpha-1}\big(t^\alpha - s^\alpha\big)v(s)ds + \int_0^t t^\alpha (t-s)^{-\alpha-1}(v(t) - v(s))ds.$$

Due to (9), $\int_0^t (t-s)^{-\alpha-1}\big(t^\alpha - s^\alpha\big)v(s)ds$, $0 < t \le T$, equiconverge (even absolutely). Thus $\int_0^t (t-s)^{-\alpha-1}(t^\alpha v(t) - s^\alpha v(s))ds$, $0 < t \le T$, equiconverge iff $\int_0^t t^\alpha (t-s)^{-\alpha-1}(v(t) - v(s))ds$, $0 < t \le T$, equiconverge. □

Proposition 3.3. *Assume that, for a* $v \in C[0,T]$ *and an* $\alpha \in (0,1)$*, function* $t^\alpha v(t)$ *is* D_0^α*-differentiable (i.e.* $t^\alpha v(t)$ *belongs to* $J^\alpha C[0,T]$*). Then function* $t^{\alpha'} v(t)$ *is* $D_0^{\alpha'}$*-differentiable for* $0 < \alpha' < \alpha$ *(i.e.* $t^{\alpha'} v(t)$ *belongs to* $J^{\alpha'} C[0,T]$*).*

Proof. Assume the conditions of the proposition. According to Proposition 3.2 we have to show that $\int_0^t t^{\alpha'}(t-s)^{-\alpha'-1}(v(t)-v(s))ds$, $0 < t \le T$, equiconverge. Representing $v(t)-v(s) = t^{-\alpha}(t^\alpha v(t) - s^\alpha v(s)) + (t^{-\alpha} - s^{-\alpha})s^\alpha v(s)$, it is sufficient to show that the improper integrals $I_1 = \int_0^t t^{\alpha'-\alpha}(t-s)^{-\alpha'-1}(t^\alpha v(t) - s^\alpha v(s))ds$ and $I_2 = \int_0^t t^{\alpha'}(t-s)^{-\alpha'-1}(t^{-\alpha} - s^{-\alpha})s^\alpha v(s)ds$, $0 < t \le T$, equiconverge. Due to equivalence of conditions (i) and (iii) for $t^\alpha v(t)$ this function belongs to $\mathcal{H}^\alpha[0,T]$, resulting to the equiconverge of I_1:

$$\begin{aligned}\int_{\theta t}^t t^{\alpha'-\alpha}(t-s)^{-\alpha'-1}(t^\alpha v(t) - s^\alpha v(s))ds &\le c\int_{\theta t}^t t^{\alpha'-\alpha}(t-s)^{\alpha-\alpha'-1}ds \\ &= c\int_{\theta t}^t t^{-1}\Big(1-\frac{s}{t}\Big)^{\alpha-\alpha'-1}ds \\ &= c\int_\theta^1 (1-x)^{\alpha-\alpha'-1}dx \to 0 \quad \text{as } \theta \uparrow 1.\end{aligned}$$

Also I_2 equiconverge:

$$\int_{\theta t}^t t^{\alpha'}(t-s)^{-\alpha'-1}(t^{-\alpha} - s^{-\alpha})s^\alpha ds = \int_\theta^1 (1-x)^{-\alpha'-1}(x^\alpha - 1)dx \to 0 \quad \text{as } \theta \uparrow 1,$$

since $(1-x)^{-\alpha'-1}(x^\alpha - 1) = \frac{x^\alpha - 1}{(1-x)^\alpha}(1-x)^{\alpha-\alpha'-1}$ where $\frac{x^\alpha-1}{(1-x)^\alpha}$ is bounded in $(0,1)$ (note that $\frac{x^\alpha-1}{(1-x)^\alpha} \to 0$ as $x \to 1$) and $(1-x)^{\alpha-\alpha'-1}$ belongs to $L^1(0,1)$. □

Remark 3.4. Proposition 3.3 admits an extension: *If* $t^r v(t)$ *is* D_0^r*-differentiable for a* $v \in C[0,T]$ *and an* $r \in \mathbb{R}$*,* $r > 0$*, then* $t^\varrho v(t)$ *is* D_0^ϱ*-differentiable for* $0 < \varrho \le r$*.*

The proof can be constructed establishing for $v \in C[0,T]$ the following claims:

1. if $t^{m+\alpha}v(t)$ belongs to $J^{m+\alpha}C[0,T]$, $m \in \mathbb{N}$, $0 < \alpha < 1$, then $t^{m+\alpha'}v(t)$ belongs for $0 < \alpha' < \alpha$ to $J^{m+\alpha'}C[0,T]$ (here Proposition 3.3 can be used) as well as $t^m v(t)$ belongs to $J^m C[0,T]$;
2. if $t^m v(t)$ belongs to $J^m C[0,T] = C_0^m[0,T]$, $m \in \mathbb{N}$, then $t^k v(t)$ belongs to $C_0^k[0,T] = J^k C[0,T]$ for $k \in \mathbb{N}$, $k \le m$;
3. if $t^k v(t)$ belongs to $J^k C[0,T]$, $k \in \mathbb{N}$, then $t^\varrho v(t)$ belongs to $J^\varrho C[0,T]$ for $\varrho \in (k-1, k]$.

We omit the details. Claim 2 concerns usual (integer order) differentiation.

4. Reformulation for Riemann-Liouville fractional derivative

For $m < \alpha < m+1$, $m \in \mathbb{N}_0$, and $v \in C[0,T]$ such that $J^{m+1-\alpha}v \in C^{m+1}[0,T]$, the Riemann-Liouville fractional derivative of order α is defined [1,3,6] by

$$D^\alpha_{\text{R-L}} v = D^{m+1} J^{m+1-\alpha} v. \tag{10}$$

The difference between (10) and (4) is that now we assume $J^{m+1-\alpha}v \in C^{m+1}[0,T]$ instead of $J^{m+1-\alpha}v \in C_0^{m+1}[0,T]$. Hence, for $m < \alpha < m+1$, a D_0^α-differentiable function v is also $D^\alpha_{\text{R-L}}$-differentiable and $D_0^\alpha v = D^\alpha_{\text{R-L}} v$. The inverse statement is true for functions $v \in C^m[0,T]$ (Proposition 4.1), whereas for lesser smooth v the relation between $D^\alpha_{\text{R-L}} v$ and $D_0^\alpha v$ is somewhat more complicated (Propositions 4.2 and 4.3).

Proposition 4.1. *For $m < \alpha < m+1$, $m \in \mathbb{N}_0$, a function $v \in C^m[0,T]$ is $D^\alpha_{R\text{-}L}$-differentiable if and only if v is D_0^α-differentiable. Besides $D^\alpha_{R\text{-}L} v = D_0^\alpha v$.*

Proof. Let $v \in C^m[0,T]$ be $D^\alpha_{\text{R}-\text{L}}$-differentiable, i.e. $J^{m+1-\alpha}v \in C^{m+1}[0,T]$. We have to establish that $J^{m+1-\alpha}v \in C_0^{m+1}[0,T]$, i.e. $(J^{m+1-\alpha}v)^{(k)}(0) = 0$ for $k = 0, 1, \ldots, m$. For $m = 0$, $0 < \alpha < 1$, $v \in C[0,T]$, this really holds. For $m \geq 1$, $v \in C^1[0,T]$, we transform

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}v)(t) = \int_0^t (t-s)^{m-\alpha} v(s)ds = \int_0^t s^{m-\alpha} v(t-s)ds;$$

the last integral admits a differentiation, and we obtain

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}v)'(t) = v(0)t^{m-\alpha} + \int_0^t s^{m-\alpha} v'(t-s)ds.$$

The function $t^{m-\alpha}$ has a singularity at $t = 0$, whereas the integral term belongs to $C[0,T]$. Since $(J^{m+1-\alpha}v)' \in C[0,T]$ by condition, we conclude that $v(0) = 0$ and

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}v)'(t) = \int_0^t s^{m-\alpha} v'(t-s)ds.$$

If $m \geq 2$, we can repeat the argument with differentiation. We obtain recursively that $v^{(k-1)}(0) = 0$, $k = 1, \ldots, m$, and

$$\Gamma(m+1-\alpha)(J^{m+1-\alpha}v)^{(k)}(t) = \int_0^t s^{m-\alpha} v^{(k)}(t-s)ds = \int_0^t (t-s)^{m-\alpha} v^{(k)}(s)ds$$

for $k = 0, \ldots, m$. We conclude that $(J^{m+1-\alpha}v)^{(k)}(0) = 0$, $k = 0, 1, \ldots, m$. $\square$

It is known (see e.g. [1]) and it is easy to check that

$$J^\alpha t^r = \frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)} t^{r+\alpha} \quad \text{for } \alpha > 0,\ r > -1. \tag{11}$$

In particular, for $m < \alpha < m+1$ we have

$$J^{m+1-\alpha} t^{\alpha-m-1+k} = \frac{\Gamma(\alpha - m + k)}{k!} t^k, \quad k = 0, 1, 2, \ldots, \tag{12}$$

$$D^\alpha_{\text{R-L}} t^{\alpha-m-1+k} = D^{m+1} J^{m+1-\alpha} t^{\alpha-m-1+k} = 0, \quad k = 0, \ldots, m, \tag{13}$$

thus $D^\alpha_{\text{R-L}}$ has an m-dimensional null-space span$\{t^{\alpha-m-1+k} : k = 1, \ldots, m\}$ in $C[0,T]$; in case $0 < \alpha < 1$, i.e. $m = 0$, the null-space of $D^\alpha_{\text{R-L}}$ in $C[0,T]$ is trivial. (In $L^1(0,T)$ operator $D^\alpha_{\text{R-L}}$ has an $(m+1)$-dimensional null-space span$\{t^{\alpha-m-1+k} : k = 0, \ldots, m\}$.)

Proposition 4.2. *For $m < \alpha < m+1$, $m \geq 1$, a function $v \in C[0,T]$ has the fractional derivative $D^\alpha_{\text{R}-\text{L}} v \in C[0,T]$ if and only if v has the structure*

$$v = v_0 + \sum_{k=1}^{m} \gamma_k t^{\alpha-m-1+k}, \quad \gamma_k = \text{const}, \tag{14}$$

where $v_0 \in C[0,T]$ is D^α_0-differentiable, i.e. $v_0 \in J^\alpha C[0,T]$, implying $J^{m+1-\alpha} v_0 \in C^{m+1}_0[0,T]$. Besides $D^\alpha_{\text{R}-\text{L}} v = D^\alpha_0 v_0$.

Proof. Assume the representation (14) where $v_0 \in C[0,T]$ is D^α_0-differentiable. From (13) and (14) it follows immediately that $D^\alpha_{\text{R-L}} v = D^\alpha_{\text{R-L}} v_0 = D^\alpha_0 v_0$.

Conversely, if $v \in C[0,T]$ is $D^\alpha_{\text{R-L}}$-differentiable then $J^{m+1-\alpha} v \in C^{m+1}[0,T]$ according to (10). Put $v_0 = v - \sum_{k=1}^m \gamma_k t^{\alpha-m+k} \in C[0,T]$, $\gamma_k = \frac{(J^{m+1-\alpha}v)^{(k)}(0)}{\Gamma(\alpha-m+k)}$, $k = 0, \ldots, m$; observe that $\gamma_0 = 0$ since $(J^{m+1-\alpha} v)(0) = 0$. Due to (12)

$$J^{m+1-\alpha} v_0 = J^{m+1-\alpha} v - \sum_{k=1}^{m} \gamma_k \frac{\Gamma(\alpha-m+k)}{k!} t^k = J^{m+1-\alpha} v - \sum_{k=1}^{m} \frac{(J^{m+1-\alpha}v)^{(k)}(0)}{k!} t^k,$$

$$(J^{m+1-\alpha} v_0)^{(k)}(0) = 0, \quad k = 0, \ldots, m.$$

Thus we have for v representation (14) such that $J^{m+1-\alpha} v_0 \in C^{m+1}_0[0,T]$, hence $D^\alpha_0 v_0$ is well defined (see (4)) and $D^\alpha_0 v_0 = D^\alpha_{\text{R-L}} v_0 = D^\alpha_{\text{R-L}} v$ due to (13. □

For a $D^\alpha_{\text{R-L}}$-differentiable function $v \in C$ the representation (14) is unique.

Proposition 4.3. *For $m < \alpha < m+1$, $m \geq 2$, $1 \leq \ell \leq m-1$, a function $v \in C^\ell[0,T]$ has the fractional derivative $D^\alpha_{\text{R-L}} v \in C[0,T]$ if and only if v has the structure*

$$v = v_0 + \sum_{k=\ell+1}^{m} \gamma_k t^{\alpha-m-1+k}, \quad \gamma_k = \text{const}, \tag{15}$$

where $v_0 \in C^\ell[0,T]$ is D^α_0-differentiable, i.e. $v_0 \in J^\alpha C[0,T]$, implying $J^{m+1-\alpha} v_0 \in C^{m+1}_0[0,T]$. Besides $D^\alpha_{\text{R}-\text{L}} v = D^\alpha_0 v_0$.

(The case $\ell = m$ is covered by Proposition 4.1, and the case $\ell = 0$ is covered by Proposition 4.2). The proof of Proposition 4.3 can be easily constructed combining the proof arguments of Propositions 4.1 and 4.2. For a $D^\alpha_{\mathrm{R-L}}$-differentiable function $v \in C^\ell[0,T]$ the representation (15) is unique.

Due to Proposition 4.1, the reformulation of Theorem 2.2 for $D^\alpha_{\text{R-L}}v$, $m < \alpha < m+1$, $v \in C^m[0,T]$, $m \geq 0$, contains no essential changes, except that condition (i_m) now reads as follows: *The fractional derivative* $D^\alpha_{R\text{-}L}v \in C[0,T]$ *exists.*

Using Propositions 4.2 and 4.3 it is easy to reformulate Theorem 2.2 also for $D^\alpha_{\text{R-L}}v$, $v \in C^\ell[0,T]$, $0 \leq \ell \leq m-1$.

Remark 4.4. For $m < \alpha < m+1$, $m \geq 1$, $f \in L^1(0,T)$ (then $J^\alpha f \in C^{m-1}[0,T]$, $(J^\alpha f)^{(i)}(0) = 0$, $i = 0, \ldots, m-1$), formula (14) with $v_0 = J^\alpha f$ presents the general solution of equation $D^\alpha_{\text{R-L}}v = f$ in $C[0,T]$; the general solution in $L^1(0,T)$ contains also the term $\gamma_0 t^{\alpha-m-1}$, corresponding to $k = 0$. Similarly, formula (15) with $v_0 = J^\alpha f$ presents the general solution of $D^\alpha_{\text{R-L}}v = f$ in $C^\ell[0,T]$.

5. Reformulation for Caputo fractional derivative

For $v \in C^m[0,T]$, denote $(\Pi_m v)(t) = \sum_{k=0}^m \frac{v^{(k)}(0)}{k!}t^k$. For $m < \alpha < m+1$, $m \in \mathbb{N}_0$, and $v \in C^m[0,T]$ such that $J^{m+1-\alpha}(v - \Pi_m v) \in C^{m+1}[0,T]$, the Caputo fractional derivative of order α is defined [1,3] by

$$D^\alpha_{\text{Cap}}v = D^{m+1}J^{m+1-\alpha}(v - \Pi_m v) = D^\alpha_{\mathrm{R-L}}(v - \Pi_m v). \qquad (16)$$

In particular, for $v \in C^{m+1}[0,T]$ the condition $J^{m+1-\alpha}(v - \Pi_m v) \in C^{m+1}[0,T]$ is fulfilled, and an equivalent formulation of (16) can be given by (cf. [5,6]) $D^\alpha_{\text{Cap}}v = J^{m+1-\alpha}D^{m+1}v$ commented in the end of Section 3.1.

Proposition 5.1. *A function* $v \in C^m[0,T]$ *has the Caputo fractional derivative* $D^\alpha_{\text{Cap}}v \in C[0,T]$, $m < \alpha < m+1$, $m \in \mathbb{N}_0$, *if and only if* $v - \Pi_m v$ *has the fractional derivative* $D^\alpha_0(v - \Pi_m v) \in C[0,T]$. *Besides* $D^\alpha_{Cap}v = D^\alpha_0(v - \Pi_m v)$.

Proof. For $m < \alpha < m+1$, conditions $u \in C^m[0,T]$, $u^{(k)}(0) = 0$, $k = 0, \ldots, m$, imply that $(J^{m+1-\alpha}u)^{(k)}(0) = 0$, $k = 0, \ldots, m$. Therefore the preassumption $J^{m+1-\alpha}(v - \Pi_m v) \in C^{m+1}[0,T]$ in (16) is equivalent to $J^{m+1-\alpha}(v - \Pi_m v) \in C_0^{m+1}[0,T]$, thus $D^\alpha_0(v - \Pi_m v)$ is well defined by (4), and (16) can be continued as follows: $D^\alpha_{\text{Cap}}v = D^{m+1}J^{m+1-\alpha}(v - \Pi_m v) = D^\alpha_0(v - \Pi_m v)$. $\square$

Taking into account that $(v - \Pi_m v)^{(m)} = v^{(m)} - v^{(m)}(0)$ we obtain the following reformulation of Theorem 2.2 for $D^\alpha_{\text{Cap}}v$, $v \in C^m[0,T]$.

Theorem 5.2. *For $m < \alpha < m+1$, $m \in \mathbb{N}_0$, and $v \in C^m[0,T]$, the following conditions* $(\mathrm{i_m})$, $(\mathrm{ii_m})$ *and* $(\mathrm{iii_m})$ *are equivalent:*

$(\mathrm{i_m})$ *the fractional derivative $D^\alpha_{Cap} v \in C[0,T]$ exists;*

$(\mathrm{ii_m})$ *a finite limit $\lim_{t\to 0} t^{m-\alpha}\big(v^{(m)}(t) - v^{(m)}(0)\big) =: \gamma_m$ exists, and*

$$\sup_{0<t\le T}\left|\int_{\theta t}^{t}(t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds\right| \to 0 \quad \text{as } \theta \uparrow 1;$$

$(\mathrm{iii_m})$ *$v^{(m)}$ has the structure $v^{(m)} - v^{(m)}(0) = \gamma_m t^{\alpha-m} + v_m$ where γ_m is a constant, $v_m \in \mathcal{H}_0^{\alpha-m}[0,T]$, and $\int_0^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds =: w_m(t)$ converges for every $t \in (0,T]$ defining a function $w_m \in C(0,T]$ which has a finite limit $\lim_{t\to 0} w_m(t) =: w_m(0)$.*

For $v \in C^m[0,T]$ with $D^\alpha_{Cap} v \in C[0,T]$, it holds $(D^\alpha_{\mathrm{Cap}} v)(0) = \Gamma(\alpha+1-m)\gamma_m$,

$$(D^\alpha_{\mathrm{Cap}} v)(t) = \frac{1}{\Gamma(m+1-\alpha)}\Big(t^{m-\alpha}\big(v^{(m)}(t)-v^{(m)}(0)\big) + (\alpha-m)\int_0^t (t-s)^{m-\alpha-1}\big(v^{(m)}(t)-v^{(m)}(s)\big)ds\Big), \quad 0<t\le T.$$

6. Proof of Theorem 2.1

6.1. Differentiation of $J^{1-\alpha}v_0$. The proof of implication (iii')→(i) is based on Proposition 6.3 about the differentiation of the fractional integral $J^{1-\alpha}v_0$. First we establish some auxiliary results (Lemmas 6.1 and 6.2).

Lemma 6.1. *For $0<\alpha<1$, $t>0$, $0<h<t$ it holds that*

$$0 \le \int_0^{t-h}\big((t-s)^{-\alpha-1} - (t+h-s)^{-\alpha-1}\big)(t-s)^\alpha ds \le c_\alpha^+ := \frac{1}{\alpha}\big(1-2^{-\alpha}\big) + \log 2. \quad (17)$$

For $0<\alpha<1$, $t>0$, $0<2h<t$ it holds that

$$0 \le \int_0^{t-2h}\big((t-h-s)^{-\alpha-1} - (t-s)^{-\alpha-1}\big)(t-s)^\alpha ds \le c_\alpha^- := \frac{1}{\alpha}2^\alpha + \log 2. \quad (18)$$

Proof. To establish (17), integrate the term $\int_0^{t-h}(t+h-s)^{-\alpha-1}(t-s)^\alpha ds$ by parts and estimate the result from below:

$$\begin{aligned}\int_0^{t-h}(t+h-s)^{-\alpha-1}(t-s)^\alpha ds &= \frac{1}{\alpha}\left(2^{-\alpha} - \left(\frac{t}{t+h}\right)^\alpha\right) + \int_0^{t-h}(t+h-s)^{-\alpha}(t-s)^{\alpha-1}ds \\ &\ge \frac{1}{\alpha}\left(2^{-\alpha} - \left(\frac{t}{t+h}\right)^\alpha\right) + \int_0^{t-h}(t+h-s)^{-1}ds.\end{aligned}$$

Now (17) follows:

$$\begin{aligned}0 &\le \int_0^{t-h} \big((t-s)^{-\alpha-1} - (t+h-s)^{-\alpha-1}\big)(t-s)^\alpha ds\\ &\le -\frac{1}{\alpha}\left(2^{-\alpha} - \left(\frac{t}{t+h}\right)^\alpha\right) + \int_0^{t-h} \big((t-s)^{-1} - (t+h-s)^{-1}\big) ds\\ &= \frac{1}{\alpha}\left(\left(\frac{t}{t+h}\right)^\alpha - 2^{-\alpha}\right) + \log 2 - \log\frac{t+h}{t}\\ &\le \frac{1}{\alpha}\big(1 - 2^{-\alpha}\big) + \log 2.\end{aligned}$$

To establish (18), integrate the term $\int_0^{t-2h}\big((t-h-s)^{-\alpha-1}(t-s)^\alpha ds$ by parts and estimate the result from above:

$$\begin{aligned}\int_0^{t-2h}(t-h-s)^{-\alpha-1}(t-s)^\alpha ds &= \frac{1}{\alpha}\left(2^\alpha - \left(\frac{t}{t-h}\right)^\alpha\right) + \int_0^{t-2h}(t-h-s)^{-\alpha}(t-s)^{\alpha-1}ds\\ &\le \frac{1}{\alpha}\left(2^\alpha - \left(\frac{t}{t-h}\right)^\alpha\right) + \int_0^{t-2h}(t-h-s)^{-1}ds.\end{aligned}$$

Now (18) follows:

$$\begin{aligned}0 &\le \int_0^{t-2h} \big((t-h-s)^{-\alpha-1} - (t-s)^{-\alpha-1}\big)(t-s)^\alpha ds\\ &\le \frac{1}{\alpha}\left(2^\alpha - \left(\frac{t}{t-h}\right)^\alpha\right) + \int_0^{t-2h} \big((t-h-s)^{-1} - (t-s)^{-1}\big) ds\\ &= \frac{1}{\alpha}\left(2^\alpha - \left(\frac{t}{t-h}\right)^\alpha\right) + \log 2 - \log\frac{t}{t-h}\\ &\le \frac{1}{\alpha}2^\alpha + \log 2.\end{aligned}$$

□

Lemma 6.2. *Let $v_0 \in \mathcal{H}_0^\alpha[0,T]$, $0 < \alpha < 1$. Assume that the improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v_0(t) - v_0(s)\big)ds$ converges for a $t \in (0,T]$. Then for this t and for*

$$V_+(t,h) := \int_0^t \frac{(t+h-s)^{-\alpha} - (t-s)^{-\alpha}}{h}\big(v_0(t) - v_0(s)\big)ds, \qquad 0 < h < T - t,$$

$$V_-(t,h) := \int_0^{t-h} \frac{(t-s)^{-\alpha} - (t-h-s)^{-\alpha}}{h}\big(v_0(t) - v_0(s)\big)ds, \quad 0 < h < t,$$

it holds that

$$\lim_{h\to 0} V_+(t,h) = \lim_{h\to 0} V_-(t,h) = -\alpha\int_0^t (t-s)^{-\alpha-1}\big(v_0(t) - v_0(s)\big)ds \tag{19}$$

(for $t = T$ the claim concerns only $\lim_{h\to 0} V_-(t,h)$).

Proof. Note that $\int_{t-h}^{t}(t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds \to 0$ as $h \to 0$ since, by assumption, the improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds$ converges. To prove (19) for $V_+(t,h)$, it is sufficient to establish the relation

$$V_+(t,h) + \alpha \int_0^{t-h} (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds \to 0 \quad \text{as } 0<h\to 0.$$

To this end, represent

$$\begin{aligned} &V_+(t,h) + \alpha \int_0^{t-h} (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds \\ &= \int_{t-h}^{t} \frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h}\big(v_0(t)-v_0(s)\big)ds \\ &\quad + \int_0^{t-h}\left(\frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h} + \alpha(t-s)^{-\alpha-1}\right)\big(v_0(t)-v_0(s)\big)ds \end{aligned}$$

and show that

$$\int_{t-h}^{t}\left|\frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h}\right|\big|v_0(t)-v_0(s)ds\big| \to 0, \tag{20}$$

$$\int_0^{t-h}\left|\frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h} + \alpha(t-s)^{-\alpha-1}\right| |v_0(t)-v_0(s)|\,ds \to 0 \tag{21}$$

as $0<h\to 0$. Indeed, fix an arbitrary small $\varepsilon>0$. Since $v_0 \in \mathcal{H}_0^\alpha(0,T]$, there is a $\delta>0$ such that

$$|v_0(t)-v_0(s)| \le \varepsilon(t-s)^\alpha \quad \text{for } 0 \le t-s \le \delta. \tag{22}$$

Hence, for $0<h\le\delta$ we obtain $\int_{t-h}^{t}\left|\frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h}\right| |v_0(t)-v_0(s)|\,ds \le \varepsilon\int_{t-h}^{t}\frac{(t-s)^{-\alpha}-(t+h-s)^{-\alpha}}{h}(t-s)^\alpha ds = \varepsilon\frac{1}{h}\int_{t-h}^{t}\left(1-\left(\frac{t-s}{t+h-s}\right)^\alpha\right)ds \le \varepsilon\frac{1}{h}\int_{t-h}^{t}ds = \varepsilon$ proving (20). To prove (21), represent

$$\frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h} = -\alpha\,(t+h'-s)^{-\alpha-1}, \quad \text{with some } h' \in (0,h),$$

and using (22) and (17) estimate

$$\begin{aligned} &\int_0^{t-h}\left|\frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h} + \alpha(t-s)^{-\alpha-1}\right| |v_0(t)-v_0(s)|\,ds \\ &\le \varepsilon\alpha\int_0^{t-h}\big((t-s)^{-\alpha-1}-(t+h'-s)^{-\alpha-1}\big)(t-s)^\alpha ds \\ &\le \varepsilon\alpha\int_0^{t-h}\big((t-s)^{-\alpha-1}-(t+h-s)^{-\alpha-1}\big)(t-s)^\alpha ds \\ &\le c_\alpha^+\alpha\varepsilon \quad \text{for } 0<h<\delta. \end{aligned}$$

This completes the proof of (19) for $V_+(t,h)$.

Similarly, to prove (19) for $V_-(t,h)$, it is sufficient to establish the relation

$$V_-(t,h) + \alpha \int_0^{t-2h} (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds \to 0 \quad \text{as } 0<h\to 0.$$

Representing

$$\begin{aligned} &V_-(t,h) + \alpha \int_0^{t-2h} (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds \\ &= \int_{t-2h}^{t-h} \frac{(t-s)^{-\alpha}-(t-h-s)^{-\alpha}}{h}\big(v_0(t)-v_0(s)\big)ds \\ &\quad + \int_0^{t-2h}\left(\frac{(t-s)^{-\alpha}-(t-h-s)^{-\alpha}}{h} + \alpha\,(t-s)^{-\alpha-1}\right)\big(v_0(t)-v_0(s)\big)ds \end{aligned}$$

it is easy to verify that

$$\int_{t-2h}^{t-h}\left|\frac{(t-s)^{-\alpha}-(t-h-s)^{-\alpha}}{h}\right|\big|v_0(t)-v_0(s)\big|ds \to 0,$$

$$\int_0^{t-2h}\left|\frac{(t-s)^{-\alpha}-(t-h-s)^{-\alpha}}{h} + \alpha\,(t-s)^{-\alpha-1}\right|\big|v_0(t)-v_0(s)\big|ds \to 0$$

as $0<h\to 0$. The proof of the latter relation exploits (18) instead of (17). □

Proposition 6.3. *Assume that $v_0 \in \mathcal{H}_0^\alpha[0,T]$, $0<\alpha<1$, and that the improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds$ is convergent for a $t\in(0,T]$. Then for this t, the derivative $(J^{1-\alpha}v_0)'(t)$ (the left-hand derivative if $t=T$) exists, and*

$$(J^{1-\alpha}v_0)'(t) = \frac{1}{\Gamma(1-\alpha)}\Big(v_0(t)t^{-\alpha} + \alpha\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds\Big). \tag{23}$$

Proof. Let us analyse the existence of the right-hand derivative of $(J^{1-\alpha}v_0)(t)$. For $0<h\le T-t$ we have

$$\begin{aligned} &\frac{(J^{1-\alpha}v_0)(t+h)-(J^{1-\alpha}v_0)(t)}{h} \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{1}{h}\left\{\int_0^{t+h}(t+h-s)^{-\alpha}v_0(s)ds - \int_0^t (t-s)^{-\alpha}v_0(s)ds\right\} \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{1}{h}\left\{\int_0^{t+h}(t+h-s)^{-\alpha}\big(v_0(s)-v_0(t)\big)ds\right. \\ &\qquad \left. - \int_0^t (t-s)^{-\alpha}\big(v_0(s)-v_0(t)\big)ds + \frac{v_0(t)}{1-\alpha}\big((t+h)^{1-\alpha}-t^{1-\alpha}\big)\right\} \\ &= \frac{1}{\Gamma(1-\alpha)}\left\{\int_0^t \frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h}\big(v_0(s)-v_0(t)\big)ds\right. \\ &\qquad \left. + \frac{1}{h}\int_t^{t+h}(t+h-s)^{-\alpha}\big(v_0(s)-v_0(t)\big)ds + v_0(t)\frac{(t+h)^{1-\alpha}-t^{1-\alpha}}{(1-\alpha)h}\right\}. \end{aligned}$$

Here $\frac{(t+h)^{1-\alpha}-t^{1-\alpha}}{(1-\alpha)h} \to \frac{1}{1-\alpha}\frac{d}{dt}t^{1-\alpha} = t^{-\alpha}$ as $h \to 0$. Further, due to (22),

$$\left|\frac{1}{h}\int_t^{t+h}(t+h-s)^{-\alpha}\big(v_0(s)-v_0(t)\big)ds\right| \le \frac{\varepsilon}{h}\int_t^{t+h}\Big(\frac{t-s}{t+h-s}\Big)^{\alpha}ds \le \varepsilon.$$

Hence $\frac{1}{h}\int_t^{t+h}(t+h-s)^{-\alpha}\big(v_0(t)-v_0(s)\big)ds \to 0$ as $h \to 0$. Finally, by Lemma 6.2

$$\int_0^t \frac{(t+h-s)^{-\alpha}-(t-s)^{-\alpha}}{h}\big(v_0(s)-v_0(t)\big)ds$$
$$= -V_+(t,h) \to \alpha\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds \quad \text{as } 0<h\to 0.$$

Thus the right-hand derivative of $(J^{1-\alpha}v_0)(t)$ exists and formula (23) holds for it.

The treatment of the left-hand derivative of $J^{1-\alpha}v$ at $t \in (0,T]$ is similar:

$$\frac{(J^{1-\alpha}v_0)(t)-(J^{1-\alpha}v_0)(t-h)}{h}$$
$$= \frac{1}{\Gamma(1-\alpha)}\Bigg\{\int_0^{t-h}\frac{(t-s)^{-\alpha}-(t-h-s)^{-\alpha}}{h}\big(v_0(s)-v_0(t)\big)ds$$
$$+\frac{1}{h}\int_{t-h}^{t}(t-s)^{-\alpha}\big(v_0(s)-v_0(t)\big)ds + v_0(t)\frac{t^{1-\alpha}-(t-h)^{1-\alpha}}{(1-\alpha)h}\Bigg\}.$$

Again $\frac{t^{1-\alpha}-(t-h)^{1-\alpha}}{(1-\alpha)h} \to t^{-\alpha}$, $\frac{1}{h}\int_{t-h}^t(t-s)^{-\alpha}\big(v_0(t)-v_0(s)\big)ds \to 0$ as $0<h\to 0$, and by Lemma 6.2 the term $\int_0^{t-h}\frac{(t-h-s)^{-\alpha}-(t-s)^{-\alpha}}{h}\big(v_0(s)-v_0(t)\big)ds = -V_-(t,h)$ is convergent as $0<h\to 0$ and its limit is $\alpha\int_0^t(t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds$.

For both one-side derivatives we get expression (23), thus the one-side derivatives coincide, the derivative $(J^{1-\alpha}v_0)'(t)$ exists, and formula (23) holds. This finishes the proof. □

6.2. Some properties of fractional integral $J^\alpha u$. The following proposition collects the properties of $J^\alpha u$, $u \in C[0,T]$, needed in the proof of implications (i) ⇒ (ii) and (i) ⇒ (iii′).

Proposition 6.4. *For $u \in C[0,T]$, $0<\alpha<1$, the function $v = J^\alpha u$ has the structure*

$$v = \gamma_0 t^\alpha + v_0, \quad \textit{where } \gamma_0 = \frac{u(0)}{\Gamma(\alpha+1)},\ v_0 = J^\alpha(u-u(0)) \in \mathcal{H}_0^\alpha[0,T], \tag{24}$$

implying that $\lim_{t\to 0} t^{-\alpha}v(t) = \gamma_0$. *Moreover,* $\int_0^t(t-s)^{-\alpha-1}\big(v(t)-(v(s)\big)ds =: w(t)$ *equiconverge for $t \in (0,T]$, i.e. (5) holds true, and $w \in C[0,T]$ with*

$$w(0) := \lim_{t\to 0} w(t) = \frac{1}{\alpha}\Big(\Gamma(1-\alpha)-\frac{1}{\Gamma(\alpha+1)}\Big)u(0) = \frac{\gamma_0}{\alpha}\big(\Gamma(1-\alpha)\Gamma(\alpha+1)-1\big).$$

Finally, $u \in C[0,T]$ *can be recovered from* $v = J^\alpha u$ *by inversion formula* (7)*:* $u(0) = \Gamma(\alpha+1)\gamma_0$,

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\Big(t^{-\alpha}v(t) + \alpha\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds\Big), \quad 0 < t \le T. \quad (25)$$

Proof. Representing $u = u(0) + (u - u(0))$ and exploiting (11) we immediately get the representation $J^\alpha u = \frac{u(0)}{\Gamma(\alpha+1)}t^\alpha + J^\alpha(u - u(0))$. To obtain (24), it remains to show that $J^\alpha(u - u(0)) \in \mathcal{H}_0^\alpha[0,T]$. Introducing the operator

$$K^\alpha u = J^\alpha(u - u(0)), \quad u \in C[0,T],$$

it is sufficient to observe that $K^\alpha \in \mathcal{L}(C[0,T], \mathcal{H}_0^\alpha[0,T])$, i.e. that K^α is a linear bounded operator from $C[0,T]$ into $\mathcal{H}_0^\alpha[0,T]$. An elementary well-known fact is that $J^\alpha \in \mathcal{L}(C[0,T], \mathcal{H}^\alpha[0,T])$ (see e.g. [1] or, in a more suitable form, [2, Satz 3.4.9]), hence also $K^\alpha \in \mathcal{L}(C[0,T], \mathcal{H}^\alpha[0,T])$. To see that actually it holds $K^\alpha \in \mathcal{L}(C[0,T], \mathcal{H}_0^\alpha[0,T])$, approximate a given $u \in C[0,T]$ by $u_n \in C^1[0,T]$ so that $u_n(0) = u(0)$ and $\|u_n - u\|_C \to 0$ as $n \to \infty$. Integrating by parts we get

$$\big(K^\alpha u_n\big)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(u_n(s) - u_n(0)\big)ds = \frac{1}{\Gamma(\alpha+1)}\int_0^t (t-s)^\alpha u_n'(s)ds,$$

and taking into account that $u \longmapsto \int_0^t (t-s)^\alpha u'(s)ds$ is a bounded linear operator in $C^1[0,T]$ (observe that $\frac{d}{dt}\int_0^t (t-s)^\alpha u'(s)ds = \alpha\int_0^t (t-s)^{\alpha-1}u'(s)ds$), we obtain that $K^\alpha u_n = J^\alpha(u_n - u_n(0)) \in C^1[0,T] \subset \mathcal{H}_0^\alpha[0,T]$. Due to equality $u_n(0) - u(0) = 0$, it holds that $K^\alpha u_n - K^\alpha u = J^\alpha(u_n - u)$, and

$$\|K^\alpha u_n - K^\alpha u\|_{\mathcal{H}^\alpha} = \|J^\alpha(u_n - u)\|_{\mathcal{H}^\alpha} \le \big\|J^\alpha\big\|_{\mathcal{L}(C,\mathcal{H}^\alpha)} \|u_n - u\|_C \to 0.$$

Since $\mathcal{H}_0^\alpha[0,T]$ is a closed subspace of $\mathcal{H}^\alpha[0,T]$, we can conclude that together with $K^\alpha u_n$ also $K^\alpha u$ belongs to $\mathcal{H}_0^\alpha[0,T]$. Hence $K^\alpha \in \mathcal{L}(C[0,T], \mathcal{H}_0^\alpha[0,T])$.

Observe that (24) really implies that $\lim_{t\to 0} t^{-\alpha}v(t) = \gamma_0$ since, due to inclusion $v_0 \in \mathcal{H}_0^\alpha[0,T]$, it holds $t^{-\alpha}v_0(t) = t^{-\alpha}\big(v_0(t) - v_0(0)\big) \to 0$ as $t \to 0$, and $\lim_{t\to 0} t^{-\alpha}v(t) = \lim_{t\to 0} t^{-\alpha}(\gamma_0 t^\alpha) = \gamma_0$.

The proof of the claims about the integral $\int_0^t (t-s)^{-\alpha-1}\big(v(t) - (v(s)\big)ds$, $v = J^\alpha u$, is more labour-consuming. Introduce the integral operators A and A_θ by

$$(A_\theta u)(t) := \int_0^{\theta t} (t-s)^{-\alpha-1}\big((J^\alpha u)(t) - (J^\alpha u)(s)\big)ds, \quad 0 < t \le T,\ 0 < \theta < 1,$$

$$(Au)(t) := \int_0^t (t-s)^{-\alpha-1}\big((J^\alpha u)(t) - (J^\alpha u)(s)\big)ds, \quad 0 < t \le T.$$

For $u \in C[0,T]$, it holds $A_\theta u \in C(0,T]$ (observe that $t - s \geq (1-\theta)t$ under the integral). The domain of operator A consists of functions $u \in C[0,T]$ for which the improper integrals $\int_0^t (t-s)^{-\alpha-1}\big((J^\alpha u)(t) - (J^\alpha u)(s)\big)ds$ converge for $0 < t \leq T$. Below we prove that $A_\theta, A \in \mathcal{L}(C[0,T])$ and $\big\|A_\theta u - Au\big\|_C \to 0$ for any $u \in C[0,T]$ as $\theta \uparrow 1$ establishing so (5) for $v = J^\alpha u$. Let us transform

$$\begin{aligned}(A_\theta u)(t) &= \int_0^{\theta t} (t-s)^{-\alpha-1} ds\, (J^\alpha u)(t) - \int_0^{\theta t} (t-s)^{-\alpha-1} (J^\alpha u)(s) ds \\ &= \frac{1}{\alpha\Gamma(\alpha)} \bigg(\big((1-\theta)^{-\alpha} - 1\big) t^{-\alpha} \int_0^t (t-\sigma)^{\alpha-1} u(\sigma) d\sigma \\ &\quad - \alpha \int_0^{\theta t} (t-s)^{-\alpha-1} \int_0^s (s-\sigma)^{\alpha-1} u(\sigma) d\sigma ds \bigg).\end{aligned}$$

In the last integral we change the integration order; this is legitimate since the function $(t-s)^{-\alpha-1}(s-\sigma)^{\alpha-1}$ has for fixed $t \in (0,T]$ and $\theta \in (0,1)$ in the closure of the region $0 < \sigma < s < \theta t$ only a weak singularity on the diagonal $s = \sigma$, whereas $t - s \geq (1-\theta)t$. We obtain

$$\int_0^{\theta t} (t-s)^{-\alpha-1} \int_0^s (s-\sigma)^{\alpha-1} u(\sigma) d\sigma ds = \int_0^{\theta t} \Big(\int_\sigma^{\theta t} (t-s)^{-\alpha-1} (s-\sigma)^{\alpha-1} ds \Big) u(\sigma) d\sigma.$$

The change of variables $s = (t-\sigma)x + \sigma$ (then $x = \frac{s-\sigma}{t-\sigma}$, $s - \sigma = (t-\sigma)x$, $t - s = (t-\sigma)(1-x)$, $ds = (t-\sigma)dx$) yields

$$\int_\sigma^{\theta t} (t-s)^{-\alpha-1} (s-\sigma)^{\alpha-1} ds = \frac{1}{t-\sigma} \int_0^{\frac{\theta t - \sigma}{t-\sigma}} x^{\alpha-1} (1-x)^{-\alpha-1} dx.$$

The last integral can be computed by the formula $\alpha \int_0^\xi x^{\alpha-1}(1-x)^{-\alpha-1} dx = \xi^\alpha (1-\xi)^{-\alpha}$, $0 < \xi < 1$, which can be checked by differentiation with respect to ξ. We get

$$\alpha \int_0^{\frac{\theta t - x}{t-\sigma}} x^{\alpha-1} (1-x)^{-\alpha-1} dx = \left(\frac{\theta t - \sigma}{t-\sigma}\right)^\alpha \left(1 - \frac{\theta t - \sigma}{t - \sigma}\right)^{-\alpha} = (1-\theta)^{-\alpha} t^{-\alpha} (\theta t - \sigma)^\alpha.$$

Thus $\alpha \int_0^{\theta t} (t-s)^{-\alpha-1} \int_0^s (s-\sigma)^{\alpha-1} u(\sigma) d\sigma = (1-\theta)^{-\alpha} t^{-\alpha} \int_0^{\theta t} \frac{(\theta t - \sigma)^\alpha}{t-\sigma} u(\sigma) d\sigma$, and

$$\begin{aligned}(A_\theta u)(t) &= \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} t^{-\alpha} \int_0^{\theta t} (t-\sigma)^{\alpha-1} \left(1 - \frac{(\theta t - \sigma)^\alpha}{(t-\sigma)^\alpha}\right) u(\sigma) d\sigma \\ &\quad - \frac{1}{\Gamma(\alpha+1)} t^{-\alpha} \int_0^t (t-\sigma)^{\alpha-1} u(\sigma) d\sigma + \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} t^{-\alpha} \int_{\theta t}^t (t-\sigma)^{\alpha-1} u(\sigma) d\sigma,\end{aligned}$$

or after the change of variables $\sigma = tx$,

$$\begin{aligned}(A_\theta u)(t) = &\frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\theta (1-x)^{\alpha-1} \left(1 - \left(\frac{\theta - x}{1-x}\right)^\alpha\right) u(tx) dx \\ &- \frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-x)^{\alpha-1} u(tx) dx + \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} \int_\theta^1 (1-x)^{\alpha-1} u(tx) dx,\end{aligned} \tag{26}$$

where $0 < t \le T$. Herefrom we see that for a fixed $\theta \in (0,1)$ it holds

$$\begin{aligned} (A_\theta u)(0) &:= \lim_{t\to 0}(A_\theta u)(t) \\ &= \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\theta (1-x)^{\alpha-1}\left(1-\left(\frac{\theta-x}{1-x}\right)^\alpha\right) dx\, u(0); \end{aligned} \tag{27}$$

the other two integrals in (26) reduce as $t \to 0$ since

$$\begin{aligned} \frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-x)^{\alpha-1} dx &= \frac{1}{\Gamma(\alpha+1)\alpha}, \\ \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} \int_\theta^1 (1-x)^{\alpha-1} dx &= \frac{1}{\Gamma(\alpha+1)\alpha}. \end{aligned}$$

So for $u \in C[0,T]$, $\theta \in (0,1)$, we have $A_\theta u \in C[0,T]$. The boundedness and even the uniform boundedness of $A_\theta \in \mathcal{L}(C[0,T])$, $0 < \theta < 1$, follows from (26) since, as we soon show,

$$0 \le \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\theta (1-x)^{\alpha-1}\left(1-\left(\frac{\theta-x}{1-x}\right)^\alpha\right) dx \le \frac{\Gamma(1-\alpha)}{\alpha} \tag{28}$$

(observe that $1-\left(\frac{\theta-x}{1-x}\right)^\alpha \ge 0$ for $0 \le x \le \theta$); this results to

$$\|A_\theta\|_{\mathcal{L}(C)} \le \frac{\Gamma(1-\alpha)}{\alpha} + \frac{2}{\Gamma(\alpha+1)\alpha}, \quad 0 < \theta < 1. \tag{29}$$

Claim (5) for $v = J^\alpha u$ means that $\|A_\theta u - Au\|_C \to 0$ as $\theta \to 1$. By the Banach-Steinhaus theorem, in view of (29), we obtain the convergence $\|A_\theta u - Au\|_C \to 0$ as $\theta \to 1$ for any $u \in C[0,T]$ by checking that this convergence takes place on a dense set of $C[0,T]$, concretely, for any $u \in C^1[0,T]$: then also $v = J^\alpha u \in C^1[0,T]$, and the integration by parts yields

$$\begin{aligned} (A_\theta u)(t) - (Au)(t) &= \int_{\theta t}^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big) ds \\ &= \frac{1}{\alpha}\left(\frac{v(t)-v(\theta t)}{(t-\theta t)^\alpha} + \int_{\theta t}^t (t-s)^{-\alpha} v'(s) ds\right), \\ |\, (A_\theta u)(t) - (Au)(t)\, | &\le \frac{1}{\alpha}\left(1+\frac{1}{1-\alpha}\right)(t-\theta t)^{1-\alpha} \,\|\, v' \,\|_C, \quad 0 < t \le T, \end{aligned}$$

$$\max_{0\le t\le T} |\, (A_\theta u)(t) - (Au)(t)\, | \to 0 \quad \text{as } \theta \to 1. \tag{30}$$

It remains to establish (28). For $u_1(t) \equiv 1$, the latter two integrals in (26) reduce, and $(A_\theta u_1)(t) = \frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)} \int_0^\theta (1-x)^{\alpha-1}\left(1-\left(\frac{\theta-x}{1-x}\right)^\alpha\right) dx$. We can compute $(A_\theta u_1)(t)$ also directly from the definition of A_θ: now

$$(J^\alpha u_1)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds = \frac{1}{\Gamma(\alpha+1)} t^\alpha,$$

and integrating by parts we obtain

$$\begin{aligned}(A_\theta u_1)(t) &= \frac{1}{\Gamma(\alpha+1)}\int_0^{\theta t}(t-s)^{-\alpha-1}(t^\alpha - s^\alpha)ds\\ &= \frac{1}{\Gamma(\alpha+1)}\Big(\frac{1}{\alpha}(t-s)^{-\alpha}(t^\alpha-s^\alpha)\Big|_{s=0}^{\theta t} + \int_0^{\theta t}(t-s)^{-\alpha}s^{\alpha-1}ds\Big)\\ &= \frac{1}{\Gamma(\alpha+1)\alpha}\Big(\frac{1-\theta^\alpha}{(1-\theta)^\alpha}-1\Big) + \frac{1}{\Gamma(\alpha+1)}\int_0^\theta (1-x)^{-\alpha}x^{\alpha-1}dx.\end{aligned}$$

The two representations of $A_\theta u_1$ imply that

$$\begin{aligned}&\frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)}\int_0^\theta (1-x)^{\alpha-1}\Big(1-\Big(\frac{\theta-x}{1-x}\Big)^\alpha\Big)dx\\ &= \frac{1}{\Gamma(\alpha+1)\alpha}\Big(\frac{1-\theta^\alpha}{(1-\theta)^\alpha}-1\Big) + \frac{1}{\Gamma(\alpha+1)}\int_0^\theta (1-x)^{-\alpha}x^{\alpha-1}dx.\end{aligned} \tag{31}$$

Since $\frac{1-\theta^\alpha}{(1-\theta)^\alpha} \le 1$ for $0 < \theta < 1$ and $\int_0^1 (1-x)^{-\alpha}x^{\alpha-1}dx = \Gamma(\alpha)\Gamma(1-\alpha)$, inequality (28) follows.

The continuity of $w = Au$ in $[0,T]$ follows from the continuity of $w_\theta = A_\theta u$ and the uniform convergence (30). In particular, due to (27) and (31)

$$\begin{aligned}w(0) &= \lim_{\theta\to1}(A_\theta u)(0)\\ &= \lim_{\theta\to1}\frac{(1-\theta)^{-\alpha}}{\Gamma(\alpha+1)}\int_0^\theta (1-x)^{\alpha-1}\left(1-\left(\frac{\theta-x}{1-x}\right)^\alpha\right)dx\, u(0)\\ &= \lim_{\theta\to1}\left(\frac{1}{\Gamma(\alpha+1)\alpha}\left(\frac{1-\theta^\alpha}{(1-\theta)^\alpha}-1\right)+\frac{1}{\Gamma(\alpha+1)}\int_0^\theta(1-x)^{-\alpha}x^{\alpha-1}dx\right)u(0)\\ &= \frac{1}{\Gamma(\alpha+1)}\left(-\frac{1}{\alpha}+\Gamma(\alpha)\Gamma(1-\alpha)\right)u(0)\\ &= \frac{1}{\alpha}\left(\Gamma(1-\alpha)-\frac{1}{\Gamma(\alpha+1)}\right)u(0).\end{aligned}$$

This completes the proof of the claims concerning $\int_0^t (t-s)^{-\alpha-1}\big(v(t)-(v(s)\big)ds$ for $v = J^\alpha u$.

Finally, let us establish formula (25). According to (24) and (4), it holds $u = (J^\alpha)^{-1}v = \gamma_0 D_0^\alpha t^\alpha + D_0^\alpha v_0 = \gamma_0\Gamma(\alpha+1) + (J^{1-\alpha}v_0)'$. Since $v_0 \in \mathcal{H}_0^\alpha$ and improper integrals $\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-(v_0(s)\big)ds$, $0 < t \le T$, converge (even equiconverge), Proposition 6.3 is applicable to confirm the differentiability of

$J^{1-\alpha}v_0$, and according to (23) we have

$$u(t) = \gamma_0\Gamma(\alpha+1) + \frac{1}{\Gamma(1-\alpha)}\Big(v_0(t)t^{-\alpha} + \alpha\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds\Big)$$
$$= \gamma_0\Gamma(\alpha+1) + \frac{1}{\Gamma(1-\alpha)}\Big(v(t)t^{-\alpha} + \alpha\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds\Big)$$
$$- \frac{\gamma_0}{\Gamma(1-\alpha)}\Big(1 + \alpha\int_0^t (t-s)^{-\alpha-1}\big(t^\alpha - s^\alpha\big)ds\Big).$$

Herefrom (25) follows, since (apply the equality part of (9) with $\theta = 0$)

$$\alpha\int_0^t (t-s)^{-\alpha-1}\big(t^\alpha - s^\alpha\big)ds = \Gamma(1-\alpha)\Gamma(\alpha+1) - 1.$$

The proof of Proposition 6.4 is complete. □

The following proposition will be used in the proof of implication (ii′) ⇒ (i).

Proposition 6.5. *If* $v \in C[0,T]$ *satisfies* (ii′), *and*

$$t^\alpha v(t) + \alpha\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds = 0 \quad \text{for } 0 < t \le T, \tag{32}$$

then $v(t) = 0$ *for* $0 \le t \le T$.

Proof. Since the improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds$ converges, condition (32) implies that, for any (fixed) $t \in (0,T]$,

$$t^{-\alpha}v(t) + \alpha\int_0^{\theta t} (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds \to 0 \quad \text{as } \theta \to 1.$$

Together with (6) this implies by Lebesgue's theorem that, for any $t' \in (0,T]$,

$$\int_0^{t'}\Big(t^{-\alpha}v(t) + \alpha\int_0^{\theta t}(t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds\Big)dt \to 0 \quad \text{as } \theta \to 1,$$

$$\text{or} \quad (1-\theta)^{-\alpha}\int_0^{t'} t^{-\alpha}v(t)dt - \alpha\int_0^{t'}\int_0^{\theta t}(t-s)^{-\alpha-1}v(s)dsdt \to 0 \quad \text{as } \theta \to 1.$$

In the last integral, the change of the order of integrations is legitimate, and we get

$$\alpha\int_0^{t'}\int_0^{\theta t}(t-s)^{-\alpha-1}v(s)dsdt = \int_0^{\theta t'}\alpha\int_{\frac{s}{\theta}}^{t'}(t-s)^{-\alpha-1}dt\, v(s)ds$$
$$= \theta^\alpha(1-\theta)^{-\alpha}\int_0^{t'} s^{-\alpha}v(s)ds - \int_0^{\theta t'}(t'-s)^{-\alpha}v(s)ds.$$

Thus $\frac{1-\theta^\alpha}{(1-\theta)^\alpha}\int_0^{t'} s^{-\alpha}v(s)ds + \int_0^{\theta t'}(t'-s)^{-\alpha}v(s)ds \to 0$, or, since $\lim_{\theta\to1}\frac{1-\theta^\alpha}{(1-\theta)^\alpha} = 0$, also $\int_0^{\theta t'}(t'-s)^{-\alpha}v(s)ds \to 0$ as $\theta \to 1$. This means that $\int_0^{t'}(t'-s)^{-\alpha}v(s)ds = 0$ for $0 < t' \le T$, implying $v(s) \equiv 0$. □

6.3. Equivalence of conditions (i), (ii), (ii′), (iii) and (iii′). We establish the implications (i) ⇒ (ii) ⇒ (ii′) ⇒ (i), (i) ⇒ (iii′) ⇒ (i) and (iii) ⇔ (iii′).

(i) ⇒ (ii). This implication holds due to Proposition 6.4.

(ii) ⇒ (ii′). This implication is clear.

(ii′) ⇒ (i). Let $v \in C[0,T]$ satisfy (ii′), i.e. $v_\alpha(t) = t^{-\alpha}v(t)$ has a continuous extension to $t = 0$, $w \in C[0,T]$ for $w(t) := \int_0^t (t-s)^{-\alpha-1}\big(v(t) - v(s)\big)ds$, and $|w_\theta(t)| \le W(t)$, $0 < t < T$, for $w_\theta(t) := \int_0^{\theta t}(t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds$, where $W \in L^1(0,T)$. We have to show that $v \in J^\alpha C[0,T]$. The function

$$u(t) := \frac{1}{\Gamma(1-\alpha)}\left(t^{-\alpha}v(t) + \alpha\int_0^t (t-s)^{-\alpha-1}\big(v(t)-v(s)\big)ds\right)$$

is continuous in $[0,T]$ as the sum of two continuous functions. Denoting $\tilde v := J^\alpha u$ we have by Proposition 6.4 that a finite limit $\tilde\gamma_0 := \lim_{t\to 0} t^{-\alpha}\tilde v(t)$ exists and condition (5) is fulfilled for $\tilde v$, hence also $\tilde w_\theta(t) := \int_0^{\theta t}(t-s)^{-\alpha-1}\big(\tilde v(t) - \tilde v(s)\big)ds$ has an integrable (even bounded) majorant. Inversion formula (25) yields

$$u(t) = \frac{1}{\Gamma(1-\alpha)}\left(t^{-\alpha}\tilde v(t) + \alpha\int_0^t (t-s)^{-\alpha-1}\big(\tilde v(t)-\tilde v(s)\big)ds\right).$$

From the two formulas for $u(t)$ we conclude that

$$t^{-\alpha}\left(v(t)-\tilde v(t)\right) + \alpha\int_0^t (t-s)^{-\alpha-1}\left[\big(v(t)-\tilde v(t)\big) - \big(v(s)-\tilde v(s)\big)\right]ds \equiv 0,$$

and since also $\int_0^{\theta t}(t-s)^{-\alpha-1}\left[\big(v(t)-\tilde v(t)\big) - \big(v(s)-\tilde v(ts)\big)\right]ds$ has an integrable majorant, we obtain $v(t) - \tilde v(t) \equiv 0$ by Proposition 6.5. Thus $v = \tilde v = J^\alpha u \in J^\alpha C[0,T]$.

(i) ⇒ (iii′). Also this implication holds due to Proposition 6.4.

(iii′) ⇒ (i). Assume (iii′): $v \in C[0,T]$ has the representation $v = \gamma_0 t^\alpha + v_0$, $0 < \alpha < 1$, where $\gamma_0 = \text{const}$, $v_0 \in \mathcal{H}_0^\alpha[0,T]$, $v_0(0) = 0$, and the improper integral $\int_0^t (t-s)^{-\alpha-1}\big(v_0(t)-v_0(s)\big)ds =: w_0(t)$ converges for any $t \in (0,T]$ defining a function $w_0 \in C[0,T]$ with $w_0(0) = 0$. We have to show that $v \in J^\alpha C[0,T]$. For t^α this is clear, so we have to show that $v_0 \in J^\alpha C[0,T]$. By (4) for $m = 0$, $\alpha \in (0,1)$, this means that $J^{1-\alpha}v_0 \in C_0^1[0,T]$. This inclusion really holds. Indeed, by Proposition 6.3, $J^{1-\alpha}v_0$ is differentiable and formula (23) holds for $(J^{1-\alpha}v_0)'(t)$, $t \in (0,T]$. Assumption (iii’) yields that both terms in the r.h.s. of (23) belong to $C[0,T]$, since also $\lim_{t\to 0} t^{-\alpha}v_0(t) = 0$ for $v_0 \in \mathcal{H}_0^\alpha[0,T]$ with $v_0(0) = 0$. So $J^{1-\alpha}v_0 \in C^1[0,T]$. Clearly $(J^{1-\alpha}v_0)(0) = 0$, thus $J^{1-\alpha}v_0 \in C_0^1[0,T]$.

(iii) ⇔ (iii′). This equivalence relation is a consequence of (9).

The fractional differentiation formula (7) for $v \in J^\alpha C[0,T]$ is established in Proposition 6.4, see formula (25).

The proof of Theorem 2.1 is complete.

7. Future work: fractional differentiation in $L^p(0,T)$

It is of interest to describe also the set $J^\alpha L^p(0,T)$, $0 < \alpha < 1$, $1 \le p \le \infty$, modifying so Theorem 2.1 and other results for the fractional differentiation in spaces $L^p(0,T)$. In particular, the following conjecture seems to be true.

Conjecture 7.1. A function $v \in L^p(0,T)$, $1 < p \le \infty$, has the fractional derivative $D_0^\alpha v \in L^p(0,T)$, $0 < \alpha < 1$, if and only if the function $t^{-\alpha}v(t)$ belongs to $L^p(0,T)$, the Lebesgue integral $\int_0^t (t-s)^{-\alpha-1}(v(t)-v(s))ds =: w(t)$ is for almost every $t \in (0,T)$ well defined, and $w \in L^p(0,T)$. For v satisfying these conditions, formula (7) holds almost everywhere.

For $p = 1$ the formulation is more complicated, this is connected with the unboundedness of the operator $t^{-\alpha}J^\alpha$ in $L^1(0,T)$, a consequence of (11). In $L^p(0,T)$, $1 < p \le \infty$, this operator occurs to be bounded.

We hope to continue elsewhere.

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