Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 36 (2017), 1–16
DOI: 10.4171/ZAA/1576

Approximation by Riesz Means of Hexagonal Fourier Series

Ali Guven

Dedicated to Professor Daniyal M. Israfilov for his 60th birthday

Abstract. Let f be an H-periodic (periodic with respect to the hexagon lattice) Hölder continuous function of two real variables. The error $\|f - R_n(p_k; f)\|$ is estimated in the uniform norm and in the Hölder norm, where (p_k) is a sequence of numbers such that $0 < p_0 \leq p_1 \leq \cdots$ and $R_n(p_k; f)$ is the nth Riesz mean of hexagonal Fourier series of f with respect to (p_k).

Keywords. Hexagonal Fourier series, Hölder class, Riesz mean

Mathematics Subject Classification (2010). 42B08, 41A25, 41A63

1. Introduction

In general, approximation problems of functions of several real variables defined on cubes of the Euclidean space are studied by assuming that the functions are periodic in each of their variables (see, for example [9, Sections 5.3 and 6.3] and [11, Vol. II, Chapter XVII]). But in the case of non-tensor product domains, for example in hexagonal domains of $\mathbb{R}^2$, another definition of periodicity is needed. For such domains most useful periodicity is the periodicity defined by lattices. We refer to [6] for general information about lattices.

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2}, \frac{1}{2}\right)^2$, the simplest lattice is the hexagon lattice and the simplest spectral set is the regular hexagon.

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H = \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}$$

and

$$\Omega_H = \left\{ (x_1, x_2) \in \mathbb{R}^2 : -1 \leq x_2, \frac{\sqrt{3}}{2} x_1 \pm \frac{1}{2} x_2 < 1 \right\}.$$

A. Guven: Department of Mathematics, Faculty of Arts and Sciences, Balikesir University, 10145 Balikesir, Turkey; guvennali@gmail.com

It is more convenient to use the homogeneous coordinates (t_1, t_2, t_3) that satisfies $t_1 + t_2 + t_3 = 0$. If we define

$$t_1 := -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2},\ t_2 := x_2,\ t_3 := -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2},$$

the hexagon Ω_H becomes

$$\Omega = \left\{(t_1, t_2, t_3) \in \mathbb{R}^3 : -1 \leq t_1, t_2, -t_3 < 1, \quad t_1 + t_2 + t_3 = 0\right\},$$

which is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1 + t_2 + t_3 = 0$, that is

$$\mathbb{R}^3_H = \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\right\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is $\mathbb{Z}^3_H = \mathbb{Z}^3 \cap \mathbb{R}^3_H$.

A function $f : \mathbb{R}^2 \to \mathbb{C}$ is called H-periodic if

$$f(x + Hk) = f(x)$$

for all $k \in \mathbb{Z}^2$ and $x \in \mathbb{R}^2$. If we define $\mathbf{t} \equiv \mathbf{s} \pmod 3$ as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \pmod 3$$

for $\mathbf{t} = (t_1, t_2, t_3)$, $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3_H$, it follows that the function f is H-periodic if and only if $f(\mathbf{t}) = f(\mathbf{t} + \mathbf{s})$ whenever $\mathbf{s} \equiv \mathbf{0} \pmod 3$. It is clear that

$$\int_\Omega f(\mathbf{t} + \mathbf{s})\, d\mathbf{t} = \int_\Omega f(\mathbf{t})\, d\mathbf{t}, \quad \left(\mathbf{s} \in \mathbb{R}^3_H\right)$$

holds for H-periodic integrable function f (see [10]).

$L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f, g\rangle_H := \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t})\, \overline{g(\mathbf{t})} d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω. The functions

$$\phi_{\mathbf{j}}(\mathbf{t}) := e^{\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t}\rangle}, \quad \left(\mathbf{t} \in \mathbb{R}^3_H\right)$$

are H-periodic and by a theorem of B. Fuglede ([3]) the set

$$\left\{\phi_{\mathbf{j}}(\mathbf{t}) : \mathbf{j} \in \mathbb{Z}^3_H\right\}$$

becomes an orthonormal basis of $L^2(\Omega)$ (see also [6]).

For every natural number n, we define a subset of $\mathbb{Z}_H^3$ by

$$\mathbb{H}_n := \left\{ \mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \le j_1, j_2, j_3 \le n \right\}.$$

Note that, $\mathbb{H}_n$ consists of all integer points inside the hexagon $n\overline{\Omega}$. Members of the set

$$\mathcal{H}_n := \operatorname{span}\left\{ \phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n \right\}, \quad (n \in \mathbb{N})$$

are called hexagonal trigonometric polynomials. It is clear that the dimension of $\mathcal{H}_n$ is $\#\mathbb{H}_n = 3n^2 + 3n + 1$.

The hexagonal Fourier series of an H-periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{1}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}) \overline{\phi_{\mathbf{j}}(\mathbf{t})} d\mathbf{t}, \quad \left(\mathbf{j} \in \mathbb{Z}_H^3\right).$$

The nth partial sum of the series (1) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \quad (n \in \mathbb{N}).$$

It is clear that

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t} - \mathbf{s}) D_n(\mathbf{s}) d\mathbf{s}, \tag{2}$$

where D_n is the Dirichlet kernel, defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \phi_{\mathbf{j}}(\mathbf{t}).$$

It is known that ([6, 8]) the Dirichlet kernel can be expressed as

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}), \quad (n \in \mathbb{N}), \tag{3}$$

where

$$\Theta_n(\mathbf{t}) := \frac{\sin \frac{(n+1)(t_1 - t_2)\pi}{3} \sin \frac{(n+1)(t_2 - t_3)\pi}{3} \sin \frac{(n+1)(t_3 - t_1)\pi}{3}}{\sin \frac{(t_1 - t_2)\pi}{3} \sin \frac{(t_2 - t_3)\pi}{3} \sin \frac{(t_3 - t_1)\pi}{3}} \tag{4}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}_H^3$.

More detailed information on hexagonal Fourier series can be found in [6,10].

2. Main results

Let $C_H\left(\overline{\Omega}\right)$ be the Banach space of complex valued H-periodic continuous functions defined on $\mathbb{R}^3_H$, whose norm is the uniform norm:

$$\|f\|_{C_H\left(\overline{\Omega}\right)} = \sup\left\{|f(\mathbf{t})| : \mathbf{t} \in \overline{\Omega}\right\}.$$

A function $f \in C_H\left(\overline{\Omega}\right)$ is said to belong to the Hölder space $H^\alpha\left(\overline{\Omega}\right)$ $(0 < \alpha \leq 1)$ if

$$A^\alpha(f) := \sup_{\mathbf{t}\neq\mathbf{s}} \frac{|f(\mathbf{t}) - f(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\alpha} < \infty,$$

where $\|\mathbf{t}\| = \max\{|t_1|, |t_2|, |t_3|\}$. $H^\alpha\left(\overline{\Omega}\right)$ becomes a Banach space with respect to the Hölder norm

$$\|f\|_{H^\alpha\left(\overline{\Omega}\right)} := \|f\|_{C_H\left(\overline{\Omega}\right)} + A^\alpha(f).$$

Fejér and Abel summability of hexagonal Fourier series of functions belong to $C_H\left(\overline{\Omega}\right)$ was studied by Y. Xu in [10]. In [10], it was proved that Fejér and Abel-Poisson means of hexagonal Fourier series of a function $f \in C_H\left(\overline{\Omega}\right)$ converges uniformly to this function on $\overline{\Omega}$. Later, in [4,5], the order of convergence of Fejér and Abel-Poisson means of hexagonal Fourier series of functions belong to $H^\alpha\left(\overline{\Omega}\right)$ was estimated in uniform and Hölder norms, respectively. In this work we give estimates for the order of approximation of Riesz means of hexagonal Fourier series in uniform and Hölder norms, and by this way we obtain analogues of theorems given in [1,2].

Let (p_k) be a sequence of numbers such that $0 < p_0 \leq p_1 \leq \cdots$. The nth Riesz mean of the series (1) with respect to the sequence (p_k) is defined by

$$R_n(p_k; f)(\mathbf{t}) := \frac{1}{P_n} \sum_{k=0}^{n} p_k S_k(f)(\mathbf{t}), \quad (n \in \mathbb{N}),$$

where $P_n := \sum_{k=0}^{n} p_k$. By considering (2), it can be easily shown that

$$R_n(p_k; f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_\Omega f(\mathbf{t}-\mathbf{s}) L_n(p_k; \mathbf{s})\, d\mathbf{s},$$

where

$$L_n(p_k; \mathbf{s}) = \frac{1}{P_n} \sum_{k=0}^{n} p_k D_k(\mathbf{s}). \tag{5}$$

In the case $p_k = 1$ $(k = 0, 1, \ldots)$, the Riesz mean $R_n(p_k; f)(\mathbf{t})$ coincide with the Fejér means

$$S_n^{(1)}(f)(\mathbf{t}) := \frac{1}{n+1} \sum_{k=0}^{n} S_k(f)(\mathbf{t}), \quad (n \in \mathbb{N}).$$

We shall write $A \lesssim B$ for the quantities A and B, if there exists a constant $K > 0$ (K is an absolute constant, or a constant depending only on parameters which are not important for the questions involve in the paper) such that $A \leq KB$ holds.

We estimate the rate of approximation of Riesz means of hexagonal Fourier series of H-periodic Hölder continuous functions as follows:

Theorem 2.1. *Let (p_k) be a sequence of numbers such that $0 < p_0 \leq p_1 \leq \cdots$, and let $f \in H^{\alpha}\left(\overline{\Omega}\right)$ $(0 < \alpha \leq 1)$. Then,*

$$\left\| f - R_n\left(p_k; f\right)\right\|_{C_H\left(\overline{\Omega}\right)} \lesssim \begin{cases} \left(\frac{p_n}{P_n}\right)^{\alpha} \log n, & \alpha < 1 \\ \frac{p_n}{P_n}\left(1 + \log \frac{P_n}{p_n}\right) \log n, & \alpha = 1 \end{cases} \tag{6}$$

for $n \geq 2$.

Proof. By considering (3) and (5), we get

$$\begin{aligned} \left| f\left(\mathbf{t}\right) - R_n\left(p_k; f\right)\left(\mathbf{t}\right)\right| &\leq \frac{1}{|\Omega|}\int_{\Omega} \left| f\left(\mathbf{t}\right) - f\left(\mathbf{t}-\mathbf{s}\right)\right| \left| L_n\left(p_k; \mathbf{s}\right)\right| d\mathbf{s} \\ &\lesssim \int_{\Omega} \|\mathbf{s}\|^{\alpha} \left| L_n\left(p_k; \mathbf{s}\right)\right| d\mathbf{s} \\ &= \frac{1}{P_n}\int_{\Omega} \|\mathbf{s}\|^{\alpha} \left|\sum_{k=1}^{n} p_k D_k\left(\mathbf{s}\right)\right| d\mathbf{s} \\ &= \frac{1}{P_n}\int_{\Omega} \|\mathbf{s}\|^{\alpha} \left| p_0 + \sum_{k=1}^{n} p_k\left(\Theta_k\left(\mathbf{s}\right) - \Theta_{k-1}\left(\mathbf{s}\right)\right)\right| d\mathbf{s}, \end{aligned} \tag{7}$$

because of $f \in H^{\alpha}(\overline{\Omega})$. Since the function $\|\mathbf{t}\|^{\alpha}|p_0 + \sum_{k=1}^{n} p_k(\Theta_k(\mathbf{t}) - \Theta_{k-1}(\mathbf{t}))|$ is symmetric with respect to t_1, t_2 and t_3, where $\mathbf{t} = (t_1, t_2, t_3) \in \Omega$, it is sufficient to estimate the integral

$$I_n := \int_{\Delta} \|\mathbf{t}\|^{\alpha} \left| p_0 + \sum_{k=1}^{n} p_k\left(\Theta_k\left(\mathbf{t}\right) - \Theta_{k-1}\left(\mathbf{t}\right)\right)\right| d\mathbf{t},$$

where

$$\Delta := \left\{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_H \colon 0 \leq t_1, t_2, -t_3 \leq 1\right\} = \{(t_1, t_2) \colon t_1 \geq 0,\ t_2 \geq 0,\ t_1 + t_2 \leq 1\},$$

which is one of the six equilateral triangles in Ω. If we use the formula (4), we obtain

$$\int_{\Delta}\|\mathbf{t}\|^{\alpha}\left|p_0+\sum_{k=1}^{n}p_k\left(\Theta_k\left(\mathbf{t}\right)-\Theta_{k-1}\left(\mathbf{t}\right)\right)\right|d\mathbf{t}$$

$$=\int_{\Delta}(t_1+t_2)^{\alpha}\left|p_0+\sum_{k=1}^{n}p_k\left(\begin{array}{c}\frac{\sin\frac{(k+1)(t_1-t_2)\pi}{3}\sin\frac{(k+1)(t_2-t_3)\pi}{3}\sin\frac{(k+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\\-\frac{\sin\frac{k(t_1-t_2)\pi}{3}\sin\frac{k(t_2-t_3)\pi}{3}\sin\frac{k(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}\end{array}\right)\right|d\mathbf{t}.$$

If we use the change of variables

$$s_1:=\frac{t_1-t_3}{3}=\frac{2t_1+t_2}{3},\quad s_2:=\frac{t_2-t_3}{3}=\frac{t_1+2t_2}{3} \tag{8}$$

as in [10], we get

$$I_n=3\int_{\widetilde{\Delta}}(s_1+s_2)^{\alpha}\left|p_0+\sum_{k=1}^{n}p_k\left(\begin{array}{c}\frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\\-\frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\end{array}\right)\right|ds_1ds_2,$$

where $\widetilde{\Delta}$ is the image of Δ in the plane, that is

$$\widetilde{\Delta}:=\{(s_1,s_2):0\leq s_1\leq 2s_2,\ 0\leq s_2\leq 2s_1,\ s_1+s_2\leq 1\}.$$

Since the integrated function is symmetric with respect to s_1 and s_2, we have

$$I_n=6\int_{\Delta^*}(s_1+s_2)^{\alpha}\left|p_0+\sum_{k=1}^{n}p_k\left(\begin{array}{c}\frac{\sin((k+1)(s_1-s_2)\pi)\sin((k+1)s_2\pi)\sin((k+1)(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\\-\frac{\sin(k(s_1-s_2)\pi)\sin(ks_2\pi)\sin(k(-s_1\pi))}{\sin((s_1-s_2)\pi)\sin(s_2\pi)\sin(-s_1\pi)}\end{array}\right)\right|ds_1ds_2,$$

where Δ^* is the half of $\widetilde{\Delta}$:

$$\Delta^*:=\left\{(s_1,s_2)\in\widetilde{\Delta}:s_1\leq s_2\right\}=\{(s_1,s_2):s_1\leq s_2\leq 2s_1,\ s_1+s_2\leq 1\}.$$

The change of variables

$$s_1:=\frac{u_1-u_2}{2},\quad s_2:=\frac{u_1+u_2}{2} \tag{9}$$

transforms the triangle Δ^* to the triangle

$$\Gamma:=\left\{(u_1,u_2):0\leq u_2\leq\frac{u_1}{3},\quad 0\leq u_1\leq 1\right\},$$

and the integral becomes

$$I_n=3\int_{\Gamma}u_1^{\alpha}\left|p_0+\sum_{k=1}^{n}p_kD_k^*\left(u_1,u_2\right)\right|du_1du_2,$$

where

$$D_k^*(u_1,u_2)=\frac{\sin\left((k+1)(-u_2)\pi\right)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\left(-\frac{u_1-u_2}{2}\pi\right)\right)}{\sin\left((-u_2)\pi\right)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(-\frac{u_1-u_2}{2}\pi\right)}$$
$$-\frac{\sin\left(k(-u_2)\pi\right)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(k\left(-\frac{u_1-u_2}{2}\pi\right)\right)}{\sin\left((-u_2)\pi\right)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(-\frac{u_1-u_2}{2}\pi\right)}.$$

By elementary trigonometric identities, we have

$$D_k^*(u_1,u_2)=D_{k,1}^*(u_1,u_2)+D_{k,2}^*(u_1,u_2)+D_{k,3}^*(u_1,u_2),$$

where

$$D_{k,1}^*(u_1,u_2):=2\cos\left(\left(k+\frac{1}{2}\right)u_2\pi\right)\times\frac{\sin\left(\frac{1}{2}u_2\pi\right)\sin\left((k+1)\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$
$$D_{k,2}^*(u_1,u_2):=2\cos\left(\left(k+\frac{1}{2}\right)\frac{u_1+u_2}{2}\pi\right)\times\frac{\sin(ku_2\pi)\sin\left(\frac{1}{2}\frac{u_1+u_2}{2}\pi\right)\sin\left((k+1)\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}$$
$$D_{k,3}^*(u_1,u_2):=2\cos\left(\left(k+\frac{1}{2}\right)\frac{u_1-u_2}{2}\pi\right)\times\frac{\sin(ku_2\pi)\sin\left(k\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{1}{2}\frac{u_1-u_2}{2}\pi\right)}{\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}.$$

Also, we will need another formula for $D_k^*(u_1,u_2)$:

$$D_k^*(u_1,u_2)=\frac{\cos\left((2k+1)u_2\pi\right)}{2\sin\left(\frac{u_1+u_2}{2}\pi\right)\sin\left(\frac{u_1-u_2}{2}\pi\right)}-\frac{\cos\left((2k+1)\frac{u_1+u_2}{2}\pi\right)}{2\sin(u_2\pi)\sin\left(\frac{u_1-u_2}{2}\pi\right)}+\frac{\cos\left((2k+1)\frac{u_1-u_2}{2}\pi\right)}{2\sin(u_2\pi)\sin\left(\frac{u_1+u_2}{2}\pi\right)}, \tag{10}$$

which easily follows from the fact that $\sin 2x+\sin 2y+\sin 2z=-4\sin x\sin y\sin z$ if $x+y+z=0$.

We partition the triangle Γ as

$$\Gamma_1:=\left\{(u_1,u_2)\in\Gamma:u_1\le\frac{p_n}{P_n}\right\},$$
$$\Gamma_2:=\left\{(u_1,u_2)\in\Gamma:u_1\ge\frac{p_n}{P_n},\ u_2\le\frac{p_n}{3P_n}\right\},$$

$$\Gamma_3 := \left\{ (u_1, u_2) \in \Gamma : u_1 \geq \frac{p_n}{P_n},\ u_2 \geq \frac{p_n}{3P_n} \right\}.$$

To estimate the integrals, we will use the well known inequalities

$$\left| \frac{\sin nt}{\sin t} \right| \leq n, \quad (n \in \mathbb{N}), \tag{11}$$

and

$$\sin t \geq \frac{2}{\pi} t, \quad \left(0 \leq t \leq \frac{\pi}{2} \right). \tag{12}$$

It is clear that,

$$\int_\Gamma u_1^\alpha p_0 du_1 du_2 = p_0 \int_\Gamma u_1^\alpha du_1 du_2 \lesssim p_0 \leq p_n \quad (0 < \alpha \leq 1). \tag{13}$$

We have

$$\begin{aligned} &\int_{\Gamma_1} u_1^\alpha \left| \sum_{k=1}^{n} p_k D_k^* (u_1, u_2) \right| du_1 du_2 \\ &= \left(\int_{\Gamma_1'} + \int_{\Gamma_1''} + \int_{\Gamma_1'''} \right) u_1^\alpha \left| \sum_{k=1}^{n} p_k D_k^* (u_1, u_2) \right| du_1 du_2, \end{aligned}$$

where

$$\begin{aligned} \Gamma_1' &:= \left\{ (u_1, u_2) \in \Gamma_1 : u_1 \leq \frac{1}{n+1} \right\}, \\ \Gamma_1'' &:= \left\{ (u_1, u_2) \in \Gamma_1 : u_1 \geq \frac{1}{n+1}, \quad u_2 \leq \frac{1}{3(n+1)} \right\}, \\ \Gamma_1''' &:= \left\{ (u_1, u_2) \in \Gamma_1 : u_1 \geq \frac{1}{n+1}, \quad u_2 \geq \frac{1}{3(n+1)} \right\}. \end{aligned}$$

For $j = 1, 2, 3$ and for $0 < \alpha \leq 1$, we obtain

$$\begin{aligned} \int_{\Gamma_1'} u_1^\alpha \left| \sum_{k=1}^{n} p_k D_{k,j}^* (u_1, u_2) \right| du_1 du_2 &\lesssim \int_{\Gamma_1'} u_1^\alpha \left(\sum_{k=1}^{n} (k+1)^2 p_k \right) du_1 du_2 \\ &\lesssim (n+1)^2 P_n \int_0^{\frac{1}{3(n+1)}} \int_{3u_2}^{\frac{1}{n+1}} u_1^\alpha du_1 du_2 \\ &\lesssim P_n \left(\frac{1}{n+1} \right)^\alpha \\ &\leq P_n \left(\frac{p_n}{P_n} \right)^\alpha, \end{aligned}$$

by considering (11). By (11) and (12), we get

$$\begin{aligned}\int_{\Gamma_1''} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_{k,j}^{*}(u_1,u_2)\right| du_1du_2 &\lesssim \int_{\Gamma_1''}\left(\sum_{k=1}^{n}(k+1)\,p_k\right) u_1^{\alpha-1}du_1du_2\\ &\lesssim (n+1)\,P_n\int_0^{\frac{1}{3(n+1)}}\int_{\frac{1}{n+1}}^{\frac{p_n}{P_n}} u_1^{\alpha-1}du_1du_2\\ &\leq P_n\left(\frac{p_n}{P_n}\right)^{\alpha},\end{aligned}$$

for $j=1,2,3$ and for $0<\alpha\leq 1$. The inequality (12) yields

$$\begin{aligned}\int_{\Gamma_1'''} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_{k,1}^{*}(u_1,u_2)\right| du_1du_2 &\lesssim P_n\int_{\frac{1}{3(n+1)}}^{\frac{p_n}{3P_n}}\int_{3u_2}^{\frac{p_n}{P_n}} u_1^{\alpha-2}du_1du_2\\ &\lesssim P_n\begin{cases}\left(\frac{p_n}{P_n}\right)^{\alpha}, & \alpha<1\\ \frac{p_n}{P_n}\log\left((n+1)\frac{p_n}{P_n}\right), & \alpha=1.\end{cases}\end{aligned}$$

By considering (12) again we get

$$\begin{aligned}\int_{\Gamma_1'''} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_{k,j}^{*}(u_1,u_2)\right| du_1du_2 &\lesssim P_n\int_{\frac{1}{3(n+1)}}^{\frac{p_n}{3P_n}}\int_{3u_2}^{\frac{p_n}{P_n}} u_1^{\alpha-1}\frac{1}{u_2}du_1du_2\\ &\lesssim P_n\left(\frac{p_n}{P_n}\right)^{\alpha}\log\left((n+1)\frac{p_n}{P_n}\right)\end{aligned}$$

for $j=2,3$ and for $0<\alpha\leq 1$.

If we combine above estimates we obtain

$$\int_{\Gamma_1} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_{k}^{*}(u_1,u_2)\right| du_1du_2 \lesssim P_n\left(\frac{p_n}{P_n}\right)^{\alpha}\left(1+\log\left((n+1)\frac{p_n}{P_n}\right)\right) \tag{14}$$

for $0<\alpha\leq 1$.

We partition the rectangle Γ_2 as

$$\Gamma_2' := \left\{(u_1,u_2)\in\Gamma_2 : u_2\leq\frac{1}{3\,(n+1)^2}\right\}$$

and

$$\Gamma_2'' := \left\{(u_1,u_2)\in\Gamma_2 : u_2\geq\frac{1}{3\,(n+1)^2}\right\}$$

in order to estimate the integral

$$\int_{\Gamma_2} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_{k}^{*}(u_1,u_2)\right| du_1du_2.$$

By considering (11) we get

$$\int_{\Gamma_2'} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_{k,1}^{*}(u_1,u_2)\right| du_1du_2 \lesssim P_n \int_0^{\frac{1}{3(n+1)^2}} \int_{\frac{p_n}{P_n}}^{1} u_1^{\alpha-2} du_1 du_2$$

$$\lesssim P_n \begin{cases} \left(\frac{p_n}{P_n}\right)^{\alpha}, & \alpha<1 \\ \frac{p_n}{P_n}\log\left(\frac{P_n}{p_n}\right), & \alpha=1. \end{cases}$$

By (11) and (12) we obtain

$$\int_{\Gamma_2'} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_{k,j}^{*}(u_1,u_2)\right| du_1du_2 \lesssim (n+1)\, P_n \int_0^{\frac{1}{3(n+1)^2}} \int_{\frac{p_n}{P_n}}^{1} u_1^{\alpha-1} du_1 du_2 \lesssim P_n \left(\frac{p_n}{P_n}\right)^{\alpha}$$

for $j=2,3$ and for $0<\alpha\leq 1$.

We use the expression (10) of the function $D_k^{*}(u_1,u_2)$ to estimate the integral over Γ_2'' and Γ_3:

Since $\sum_{k=1}^{n}\cos((2k+1)x) = \sin x \sum_{k=1}^{n}\cos(2kx) - \cos x \sum_{k=1}^{n}\sin(2kx)$ and $\left|\sum_{k=1}^{n}\cos(2kx)\right| \leq \frac{1}{|\sin x|}$, $\left|\sum_{k=1}^{n}\sin(2kx)\right| \leq \frac{1}{|\sin x|}$ for $x\neq m\pi$ $(m\in\mathbb{Z})$, we get

$$\left|\sum_{k=1}^{n}\cos((2k+1)x)\right| \lesssim \frac{1}{|\sin x|},$$

and hence Abel's transform yields

$$\left|\sum_{k=1}^{n} p_k \cos((2k+1)x)\right| \lesssim \frac{p_n}{|\sin x|}. \tag{15}$$

Therefore, by taking into account the inequality

$$\sin\left(\frac{u_1\pi}{2}\right) \lesssim \sin\left(\frac{(u_1+u_2)\pi}{2}\right), \quad (u_1,u_2)\in\Gamma$$

and (15), we obtain

$$\left|\sum_{k=1}^{n} p_k D_k^{*}(u_1,u_2)\right| \lesssim \frac{p_n}{u_1^2 u_2} \tag{16}$$

for $(u_1,u_2)\in\Gamma_2''\cup\Gamma_3$. Hence,

$$\int_{\Gamma_2''} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_k^{*}(u_1,u_2)\right| du_1du_2 \lesssim p_n \int_{\frac{1}{3(n+1)^2}}^{\frac{p_n}{3P_n}} \int_{\frac{p_n}{P_n}}^{1} u_1^{\alpha-2}\frac{1}{u_2} du_1 du_2$$

$$\lesssim p_n \begin{cases} \left(\frac{p_n}{P_n}\right)^{\alpha-1}\log\left((n+1)^2\frac{p_n}{P_n}\right), & \alpha<1 \\ \log\left(\frac{P_n}{p_n}\right)\log\left((n+1)^2\frac{p_n}{P_n}\right), & \alpha=1. \end{cases}$$

Therefore, the estimate

$$\int_{\Gamma_2} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_k^{*}(u_1,u_2)\right| du_1du_2 \tag{17}$$
$$\lesssim \begin{cases} P_n\left(\frac{p_n}{P_n}\right)^{\alpha}+p_n\left(\frac{p_n}{P_n}\right)^{\alpha-1}\log\left((n+1)^2\frac{p_n}{P_n}\right), & \alpha<1 \\ p_n+p_n\log\left(\frac{P_n}{p_n}\right)+p_n\log\left(\frac{P_n}{p_n}\right)\log\left((n+1)^2\frac{p_n}{P_n}\right), & \alpha=1 \end{cases}$$

follows.

If we use (16), we get

$$\int_{\Gamma_3} u_1^{\alpha}\left|\sum_{k=1}^{n} p_k D_k^{*}(u_1,u_2)\right| du_1du_2 \lesssim p_n\int_{\frac{p_n}{3P_n}}^{\frac{1}{3}}\int_{3u_2}^{1} u_1^{\alpha-2}\frac{1}{u_2}du_1du_2 \tag{18}$$
$$\lesssim p_n\begin{cases} \left(\frac{p_n}{P_n}\right)^{\alpha-1}, & \alpha<1 \\ \left(\log\left(\frac{P_n}{p_n}\right)\right)^2, & \alpha=1. \end{cases}$$

Hence, combining (13), (14), (17), (18) and considering (7) completes the proof of Theorem 2.1. □

Remark 2.2. Analogue of Theorem 2.1 for classical Fourier series was proved by P. Chandra in [1].

Remark 2.3. If we take $p_k = 1$ $(k = 0, 1, \ldots)$, (6) gives the estimate

$$\left\|f - S_n^{(1)}(f)\right\|_{C_H(\overline{\Omega})} \lesssim \begin{cases} \frac{\log n}{n^{\alpha}}, & \alpha<1 \\ \frac{(\log n)^2}{n}, & \alpha=1 \end{cases}$$

for Fejér means.

In the following theorem, we give an estimate for the order of approximation of $R_n(p_k; f)$ in the Hölder norm.

Theorem 2.4. *Let (p_k) be a sequence of numbers such that $0 < p_0 \le p_1 \le \cdots$, $0 \le \beta < \alpha \le 1$ and $f \in H^{\alpha}\left(\overline{\Omega}\right)$. Then,*

$$\|f - R_n(p_k; f)\|_{H^{\beta}(\overline{\Omega})} \lesssim \begin{cases} \left(\dfrac{p_n}{P_n}\right)^{\alpha-\beta}(\log n)^{1+\frac{\beta}{\alpha}}, & \alpha<1 \\ \left(\dfrac{p_n}{P_n}\right)^{1-\beta}\left(1+\log\dfrac{P_n}{p_n}\right)^{1-\beta}(\log n)^{1+\beta}, & \alpha=1 \end{cases} \tag{19}$$

for $n \ge 2$.

Proof. Set $e_n(\mathbf{t}) := f(\mathbf{t}) - R_n(p_k; f)(\mathbf{t})$. Hence,

$$\|f - R_n(p_k; f)\|_{H^\beta(\overline{\Omega})} = \|e_n\|_{C_H(\overline{\Omega})} + A^\beta(e_n).$$

Since $e_n(\mathbf{t}) - e_n(\mathbf{s}) = \frac{1}{|\Omega|}\int_\Omega (f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})) L_n(p_k; \mathbf{u})\, d\mathbf{u}$, we have

$$\begin{aligned} |e_n(\mathbf{t}) - e_n(\mathbf{s})| &\lesssim \int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \, |L_n(p_k; \mathbf{u})|\, d\mathbf{u} \\ &= \frac{1}{P_n}\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left| p_0 + \sum_{k=1}^n p_k D_k(\mathbf{u}) \right| d\mathbf{u}, \end{aligned}$$

thus

$$|e_n(\mathbf{t}) - e_n(\mathbf{s})| \lesssim \frac{1}{P_n} J_n,$$

where

$$J_n := \int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left| p_0 + \sum_{k=1}^n p_k D_k(\mathbf{u}) \right| d\mathbf{u}.$$

Since $f \in H^\alpha\left(\overline{\Omega}\right)$, we have

$$|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \lesssim \|\mathbf{t} - \mathbf{s}\|^\alpha$$

and

$$|f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \lesssim \|\mathbf{u}\|^\alpha.$$

Hence,

$$\begin{aligned} (J_n)^{\frac{\beta}{\alpha}} &= \left(\int_\Omega |f(\mathbf{t}) - f(\mathbf{t}-\mathbf{u}) - f(\mathbf{s}) + f(\mathbf{s}-\mathbf{u})| \left| p_0 + \sum_{k=1}^n p_k D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{\frac{\beta}{\alpha}} \\ &\lesssim \|\mathbf{t} - \mathbf{s}\|^\beta \left(\int_\Omega \left| p_0 + \sum_{k=1}^n p_k D_k(\mathbf{u}) \right| d\mathbf{u} \right)^{\frac{\beta}{\alpha}}. \end{aligned}$$

As in proof of Theorem 2.1, it is sufficient to estimate the integral

$$\int_\Omega \left| p_0 + \sum_{k=1}^n p_k D_k(\mathbf{u}) \right| d\mathbf{u}.$$

By transforms (8) and (9) we obtain

$$\int_\Omega \left| p_0 + \sum_{k=1}^n p_k D_k(\mathbf{u}) \right| d\mathbf{u} = 3\int_\Gamma \left| p_0 + \sum_{k=1}^n p_k D_k^*(u_1, u_2) \right| du_1 du_2.$$

It is clear that $\int_{\Gamma} p_0 du_1 du_2 \le p_0 \le P_n$. By (11), it follows that

$$\begin{aligned}\int_{\Gamma_1'} \left| \sum_{k=1}^{n} p_k D_{k,j}^* (u_1, u_2) \right| du_1 du_2 &\lesssim \int_{\Gamma_1'} \left(\sum_{k=1}^{n} (k+1)^2 p_k \right) du_1 du_2 \\ &\lesssim (n+1)^2 P_n \int_0^{\frac{1}{3(n+1)}} \int_{3u_2}^{\frac{1}{n}+1} du_1 du_2 \\ &\lesssim P_n\end{aligned}$$

for $j = 1, 2, 3$. By (12),

$$\int_{\Gamma_1''} \left| \sum_{k=1}^{n} p_k D_{k,1}^* (u_1, u_2) \right| du_1 du_2 \lesssim P_n \int_0^{\frac{1}{3(n+1)}} \int_{\frac{1}{n+1}}^{\frac{p_n}{P_n}} u_1^{-2} du_1 du_2 \lesssim P_n,$$

and using (11) and (12) yields,

$$\begin{aligned}\int_{\Gamma_1''} \left| \sum_{k=1}^{n} p_k D_{k,j}^* (u_1, u_2) \right| du_1 du_2 &\lesssim (n+1) P_n \int_0^{\frac{1}{3(n+1)}} \int_{\frac{1}{n+1}}^{\frac{p_n}{P_n}} u_1^{-1} du_1 du_2 \\ &\lesssim P_n \log \left((n+1) \frac{p_n}{P_n} \right)\end{aligned}$$

for $j = 2, 3$.

By considering (12) again, we obtain

$$\begin{aligned}\int_{\Gamma_1'''} \left| \sum_{k=1}^{n} p_k D_{k,1}^* (u_1, u_2) \right| du_1 du_2 &\lesssim P_n \int_{\frac{1}{3(n+1)}}^{\frac{p_n}{3P_n}} \int_{3u_2}^{\frac{p_n}{P_n}} u_1^{-2} du_1 du_2 \\ &\lesssim P_n \log \left((n+1) \frac{p_n}{P_n} \right),\end{aligned}$$

and for $j = 2, 3$

$$\begin{aligned}\int_{\Gamma_1'''} \left| \sum_{k=1}^{n} p_k D_{k,j}^* (u_1, u_2) \right| du_1 du_2 &\lesssim P_n \int_{\frac{1}{3(n+1)}}^{\frac{p_n}{3P_n}} \int_{3u_2}^{\frac{p_n}{P_n}} \frac{1}{u_1 u_2} du_1 du_2 \\ &\lesssim P_n \left[\log \left((n+1) \frac{p_n}{P_n} \right) \right]^2.\end{aligned}$$

By (12),

$$\int_{\Gamma_2'} \left| \sum_{k=1}^{n} p_k D_{k,1}^* (u_1, u_2) \right| du_1 du_2 \lesssim P_n \int_0^{\frac{1}{3(n+1)}} \int_{\frac{p_n}{3P_n}}^{1} u_1^{-2} du_1 du_2 \lesssim p_n \le P_n,$$

and by (11) and (12),

$$\int_{\Gamma_2'}\left|\sum_{k=1}^{n}p_kD_{k,j}^{*}(u_1,u_2)\right|du_1du_2 \lesssim (n+1)P_n^2\int_0^{\frac{1}{3(n+1)}}\int_{\frac{p_n}{3P_n}}^{1}u_1^{-1}du_1du_2$$
$$\lesssim P_n\frac{p_n}{P_n}\log\left(\frac{P_n}{p_n}\right)$$
$$\leq P_n$$

for $j=2,3$.

If we use (16), we obtain

$$\int_{\Gamma_2''}\left|\sum_{k=1}^{n}p_kD_k^{*}(u_1,u_2)\right|du_1du_2 \lesssim p_n\int_{\frac{1}{3(n+1)^2}}^{\frac{p_n}{3P_n}}\int_{\frac{p_n}{P_n}}^{1}\frac{1}{u_1^2u_2}du_1du_2$$
$$\lesssim P_n\log\left((n+1)^2\frac{p_n}{P_n}\right),$$

and

$$\int_{\Gamma_3}\left|\sum_{k=1}^{n}p_kD_k^{*}(u_1,u_2)\right|du_1du_2 \lesssim p_n\int_{\frac{p_n}{3P_n}}^{\frac{1}{3}}\int_{3u_2}^{1}\frac{1}{u_1^2u_2}du_1du_2 \lesssim P_n.$$

By combining these estimates, we get

$$\int_{\Omega}\left|p_0+\sum_{k=1}^{n}p_kD_k(\mathbf{u})\right|d\mathbf{u}$$
$$\lesssim P_n\left[1+\log\left((n+1)\frac{p_n}{P_n}\right)+\log\left((n+1)^2\frac{p_n}{P_n}\right)+\left(\log\left((n+1)\frac{p_n}{P_n}\right)\right)^2\right]$$
$$\lesssim P_n(\log n)^2,$$

which yields

$$(J_n)^{\frac{\beta}{\alpha}} \lesssim \|\mathbf{t}-\mathbf{s}\|^{\beta}\left(P_n(\log n)^2\right)^{\frac{\beta}{\alpha}}$$

for $0<\alpha\leq 1$. On the other hand,

$$J_n^{1-\frac{\beta}{\alpha}} \lesssim \left(\int_{\Omega}\|\mathbf{u}\|^{\alpha}\left|p_0+\sum_{k=1}^{n}p_kD_k(\mathbf{u})\right|d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}}$$
$$\lesssim \begin{cases}(P_n\log n)^{1-\frac{\beta}{\alpha}}\left(\frac{p_n}{P_n}\right)^{\alpha-\beta}, & \alpha<1\\ (P_n\log n)^{1-\beta}\left(\frac{p_n}{P_n}\right)^{1-\beta}\left(1+\log\left(\frac{P_n}{p_n}\right)\right)^{1-\beta}, & \alpha=1\end{cases},$$

which follows from Theorem 2.1.

Therefore, for $\alpha < 1$,

$$J_n = J_n^{\frac{\beta}{\alpha}} J_n^{1-\frac{\beta}{\alpha}} \lesssim \|\mathbf{t}-\mathbf{s}\|^{\beta} P_n (\log n)^{1+\frac{\beta}{\alpha}} \left(\frac{p_n}{P_n}\right)^{\alpha-\beta},$$

and hence

$$\frac{|e_n(\mathbf{t}) - e_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^{\beta}} \lesssim \left(\frac{p_n}{P_n}\right)^{\alpha-\beta} (\log n)^{1+\frac{\beta}{\alpha}}. \tag{20}$$

For $\alpha = 1$,

$$J_n = J_n^{\beta} J_n^{1-\beta} \lesssim \|\mathbf{t}-\mathbf{s}\|^{\beta} P_n (\log n)^{1+\beta} \left(\frac{p_n}{P_n}\right)^{1-\beta} \left(1 + \log\left(\frac{P_n}{p_n}\right)\right)^{1-\beta},$$

thus

$$\frac{|e_n(\mathbf{t}) - e_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^{\beta}} \lesssim (\log n)^{1+\beta} \left(\frac{p_n}{P_n}\right)^{1-\beta} \left(1 + \log\left(\frac{P_n}{p_n}\right)\right)^{1-\beta}. \tag{21}$$

Considering (20), (21) and corresponding parts of Theorem 2.1 for $\alpha < 1$ and $\alpha = 1$ finishes the proof of Theorem 2.4. □

Remark 2.5. Analogue of Theorem 2.4 for classical Fourier series was proved by P. Chandra in [2].

Remark 2.6. In the case $p_k = 1$ $(k = 0, 1, \ldots)$, the estimate (19) reduces to

$$\left\|f - S_n^{(1)}(f)\right\|_{H^{\beta}(\overline{\Omega})} \lesssim \begin{cases} \frac{(\log n)^{1+\frac{\beta}{\alpha}}}{n^{\alpha-\beta}}, & \alpha < 1 \\ \frac{(\log n)^2}{n^{1-\beta}}, & \alpha = 1. \end{cases} \tag{22}$$

Analogue of (22) was obtained by S. Prössdorf in [7] for classical Fourier series.

Acknowledgement. This paper was supported by Balikesir University, Grant Number: 2014/49. The author wishes to thank the referee for his/her valuable suggestions, which improved the paper.

References

[1] Chandra, P., On the degree of approximation of functions belonging to the Lipschitz class. *Nanta Math.* 8 (1975), 88 – 91.

[2] Chandra, P., On the generalized Fejér means in the metric of Hölder space. *Math. Nachr.* 109 (1982), 39 – 45.

[3] Fuglede, B., Commuting self-adjoint partial differential operators and a group theoretic problem. *J. Funct. Anal.* 16 (1974), 101 – 121.

[4] Guven, A., Approximation by means of hexagonal Fourier series in Hölder norms. *J. Class. Anal.* 1 (2012), 43 – 52.

[5] Guven, A., Approximation by $(C, 1)$ and Abel-Poisson means of Fourier series on hexagonal domains. *Math. Inequal. Appl.* 16 (2013), 175 – 191.

[6] Li, H., Sun, J. and Xu, Y., Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle. *SIAM J. Numer. Anal.* 46 (2008), 1653 – 1681.

[7] Prössdorf, S., Zur Konvergenz der Fourierreihen hölderstetiger Funktionen, *Math. Nachr.* 69 (1975), 7 – 14.

[8] Sun, J. , Multivariate Fourier series over a class of non tensor-product partition domains. *J. Comput. Math.* 21 (2003), 53 – 62.

[9] Timan, A., F., *Theory of Approximation of Functions of a Real Variable.* New York: Pergamon Press 1963.

[10] Xu, Y., Fourier series and approximation on hexagonal and triangular domains. *Constr. Approx.* 31 (2010), 115 – 138.

[11] Zygmund, A., *Trigonometric Series. Vols. I, II.* Second edition. New York: Cambridge Univ. Press 1963.

Received July 23, 2014; revised June 16, 2016