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Generalized Morrey Spaces – Revisited

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Abstract. The generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ was defined by Mizuhara 1991 and Nakai in 1994. It is equipped with a parameter $0 < p < \infty$ and a function $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$. Our experience shows that $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is easy to handle when $1 < p < \infty$. However, when $0 < p \le 1$, the function space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is difficult to handle as many examples show. We propose a way to deal with $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ for $0 < p \le 1$, in particular, to obtain some estimates of the Hardy-Littlewood maximal operator on these spaces. Especially, the vector-valued estimates obtained in the earlier papers are refined. The key tool is the weighted dual Hardy operator. Much is known on the weighted dual Hardy operator.

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1. Introduction

We are concerned with generalized Morrey spaces in the present paper. The generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is equipped with a function ϕ and a positive parameter $0 < p < \infty$. The generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ was defined independently by Mizuhara in 1991 [21] and Nakai in 1994 [23].

Let $0 < p < \infty$. Denote by $\mathcal{G}_p$ the set of all the functions $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ decreasing in the second variable such that $t \in (0,\infty) \mapsto t^{\frac{n}{p}} \phi(x,t) \in (0,\infty)$ is almost increasing uniformly over the first variable x, so that there exists a

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constant $C > 0$ such that

$$\phi(x,r) \le \phi(x,s), \quad C\phi(x,r)r^{\frac{n}{p}} \ge \phi(x,s)s^{\frac{n}{p}}$$

for all $x \in \mathbb{R}^n$ and $0 < s \le r < \infty$. In this paper we often assume $0 < p \le 1$.

All "cubes" in $\mathbb{R}^n$ are assumed to have their sides parallel to the coordinate axes. Denote by $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^n)$ the set of all cubes. For a cube $Q \in \mathcal{Q}$, the symbol $\ell(Q)$ stands for the side-length of the cube Q; $\ell(Q) \equiv |Q|^{\frac{1}{n}}$, where $|E|$ denotes the Lebesgue measure of a measurable set E. When we are given a cube Q, we use the following abuse of notation: $\phi(Q) \equiv \phi(c(Q), \ell(Q))$, where $c(Q)$ denotes the center of Q.

Let $0 < p < \infty$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a function which is not necessarily in $\mathcal{G}_p$. The generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is defined as the set of all measurable functions f for which the quasi-norm

$$\|f\|_{\mathcal{M}_{p,\phi}} \equiv \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(Q)} \left(\frac{1}{|Q|} \int_Q |f(y)|^p \, dy \right)^{\frac{1}{p}}$$

is finite. Seemingly the requirement $\phi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, in the case when $1 \le p < \infty$, Nakai established that there exists a function ρ such that ρ itself is decreasing, that $\rho(t)t^{\frac{n}{p}} \le \rho(T)T^{\frac{n}{p}}$ for all $0 < t \le T < \infty$ and that $\mathcal{M}_{p,\phi}(\mathbb{R}^n) = \mathcal{M}_{p,\rho}(\mathbb{R}^n)$ [24, p. 446]. See [35, (1.2)] for the case when $0 < p \le 1$. This assumption will turn out to be natural even when ϕ depends on x. See Section 2 of this paper in the case when ϕ depends on x.

Observe that, if $\phi(x,r) = r^{-\frac{n}{p}}$, then $\mathcal{M}_{p,\phi}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. In the special case when $\phi(x,r) \equiv r^{\frac{\lambda}{p}-\frac{n}{p}}$, we write $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ instead of $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$.

We adopt the following notation:

1. We define $\mathbb{N}_0 \equiv \{0, 1, \ldots\}$.
2. We let
$$\|x\|_\infty = \|(x_1, x_2, \ldots, x_n)\|_\infty \equiv \max_{j=1,2,\ldots,n} |x_j|$$
when we have $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.
3. Let $A, B \ge 0$. Then $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \le CB$ and $A \sim B$ stands for $A \lesssim B \lesssim A$, where C depends only on the parameters of importance.
4. By a "cube" we mean a compact cube whose edges are parallel to the coordinate *axes*. The metric closed ball defined by ℓ^∞ is called a *cube*. If a cube has center x and radius r, we denote it by $Q(x,r)$. Namely, we write
$$Q(x,r) \equiv \left\{ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \max_{j=1,2,\ldots,n} |x_j - y_j| \le r \right\}$$

when $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$. From the definition of $Q(x, r)$, its volume is $(2r)^n$. We write $Q(r)$ instead of $Q(0, r)$, where 0 denotes the origin. Given a cube Q, we denote by $c(Q)$ *the center of* Q and by $\ell(Q)$ the *sidelength of* Q: $\ell(Q) = |Q|^{1/n}$, where $|Q|$ denotes the volume of the cube Q.

5. Given a cube Q and $k > 0$, $k\,Q$ means *the cube concentric to* Q *with sidelength* $k\,\ell(Q)$.
6. By a *dyadic cube*, we mean a set of the form $2^{-j}m + [0, 2^{-j}]^n$ for some $m \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$. The set of all dyadic cubes will be denoted by $\mathcal{D}$.
7. Let $\mathcal{Q}_x(\mathbb{R}^n)$ be a collection of all cubes that contain $x \in \mathbb{R}^n$.
8. The symbol $B(x, r)$ stands for the open ball centered at $x \in \mathbb{R}$ and of radius $r > 0$. Abbreviate $B(0, r)$ to $B(r)$.
9. We adopt the following definition of the Hardy-Littlewood maximal operator to estimate some integrals: The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}_x(\mathbb{R}^n)} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

for a locally integrable function f on $\mathbb{R}^n$.

10. Let $0 < p < \infty$ and $0 < q \le \infty$. If $\{f_j\}_{j=1}^\infty$ is a sequence of complex-valued Lebesgue measurable functions, then define

$$\left\| \{f_j\}_{j=1}^\infty \right\|_{\ell_q(L_p)} \equiv \left\| \left\{ \|f_j\|_{L_p} \right\}_{j=1}^\infty \right\|_{\ell_q}$$

and

$$\left\| \{f_j\}_{j=1}^\infty \right\|_{L_p(\ell_q)} \equiv \left\| \| \{f_j(\cdot)\}_{j=1}^\infty \|_{\ell_q} \right\|_{L_p}.$$

The weak L_p space is the set of all functions f for which the quasi-norm

$$\|f\|_{WL_p} \equiv \sup_{\lambda > 0} \lambda \|\chi_{\{|f| > \lambda\}}\|_{L_p}$$

is finite. If $\{f_j\}_{j=1}^\infty$ is a sequence of complex-valued Lebesgue measurable functions, then define

$$\left\| \{f_j\}_{j=1}^\infty \right\|_{\ell_q(WL_p)} \equiv \left\| \left\{ \|f_j\|_{WL_p} \right\}_{j=1}^\infty \right\|_{\ell_q}$$

and

$$\left\| \{f_j\}_{j=1}^\infty \right\|_{WL_p(\ell_q)} \equiv \left\| \left\| \{f_j\}_{j=1}^\infty \right\|_{\ell_q} \right\|_{WL_p}.$$

11. Let $0 < p < \infty$ and $0 < q \le \infty$ and $\phi \in \mathcal{G}_p$. The space $\mathcal{M}_{p,\phi}(l_q, \mathbb{R}^n)$ stands for the set of all sequences $\{f_j\}_{j=1}^{\infty}$ of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$ for which

$$\left\|\{f_j\}_{j=1}^{\infty}\right\|_{\mathcal{M}_{p,\phi}(l_q)} \equiv \left\|\left\|\{f_j\}_{j=1}^{\infty}\right\|_{l_q}\right\|_{\mathcal{M}_{p,\phi}} < \infty.$$

Denote by $W\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ the set of all measurable functions f for which the quasi-norm $\|f\|_{W\mathcal{M}_{p,\phi}} \equiv \sup_{\lambda>0} \lambda \|\chi_{\{|f|>\lambda\}}\|_{\mathcal{M}_{p,\phi}}$ is finite. Likewise denote by $W\mathcal{M}_{p,\phi}(l_q, \mathbb{R}^n)$ the set of all sequences $\{f_j\}_{j=1}^{\infty}$ for which the quasi-norm $\left\|\{f_j\}_{j=1}^{\infty}\right\|_{W\mathcal{M}_{p,\phi}(l_q)} \equiv \left\|\left\|\{f_j\}_{j=1}^{\infty}\right\|_{l_q}\right\|_{W\mathcal{M}_{p,\phi}}$ is finite. The vector-valued spaces $l_q\left(\mathcal{M}_{p,\phi}(\mathbb{R}^n)\right)$ and $l_q\left(W\mathcal{M}_{p,\phi}(\mathbb{R}^n)\right)$ can be also defined similarly by the norms

$$\left\|\{f_j\}_{j=1}^{\infty}\right\|_{l_q\left(\mathcal{M}_{p,\phi}\right)} \equiv \left\|\left\{\|f_j\|_{\mathcal{M}_{p,\phi}}\right\}_{j=1}^{\infty}\right\|_{l_q} < \infty$$

and

$$\left\|\{f_j\}_{j=1}^{\infty}\right\|_{l_q\left(W\mathcal{M}_{p,\phi}\right)} \equiv \left\|\left\{\|f_j\|_{W\mathcal{M}_{p,\phi}}\right\}_{j=1}^{\infty}\right\|_{l_q} < \infty,$$

respectively.

12. For a measurable function h and a sequence of measurable functions $\{f_j\}_{j=1}^{\infty}$, we write $h\{f_j\}_{j=1}^{\infty} \equiv \{h \cdot f_j\}_{j=1}^{\infty}$.

2. Structure of generalized Morrey spaces

We shall show that the definition of $\mathcal{G}_1$ is suitable when we consider $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$.

Lemma 2.1. *Let $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a function and let $0 < p < \infty$. Then there exists a function $\psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfying*

$$\psi(y,s)s^{\frac{n}{p}} \le \psi(x,r)r^{\frac{n}{p}} \tag{1}$$

for all $x, y \in \mathbb{R}^n$ and $r, s > 0$ with $\|x-y\|_\infty \le r-s$ such that $\mathcal{M}_{p,\phi}(\mathbb{R}^n) = \mathcal{M}_{p,\psi}(\mathbb{R}^n)$ with norm coincidence.

Proof. Let us set

$$\psi(x,r) \equiv \inf_{y\in\mathbb{R}^n}\left(\inf_{v\in[r+\|x-y\|_\infty,\infty)} \phi(y,v)\left(\frac{v}{r}\right)^{\frac{n}{p}}\right) \quad (x \in \mathbb{R}^n, r > 0). \tag{2}$$

Then $\phi \ge \psi$ trivially and hence $\|f\|_{\mathcal{M}_{p,\phi}} \le \|f\|_{\mathcal{M}_{p,\psi}}$ for any measurable function f. Meanwhile for any measurable function f,

$$\begin{aligned}\|f\|_{\mathcal{M}_{p,\psi}} &= \sup_{Q\in\mathcal{Q}} \frac{1}{\psi(c(Q),\ell(Q))}\left(\frac{1}{|Q|}\int_Q |f(y)|^p\,dy\right)^{\frac{1}{p}} \\ &= \sup_{Q\in\mathcal{Q}} \sup_{y\in\mathbb{R}^n}\left(\sup_{v\in[\ell(Q)+\|c(Q)-y\|_\infty,\infty)} \frac{1}{\phi(y,v)}\left(\frac{1}{v^n}\int_Q |f(y)|^p\,dy\right)^{\frac{1}{p}}\right) \\ &\le \sup_{Q\in\mathcal{Q}} \sup_{y\in\mathbb{R}^n}\left(\sup_{v\in[\ell(Q)+\|c(Q)-y\|_\infty,\infty)} \frac{1}{\phi(y,v)}\left(\frac{1}{v^n}\int_{Q(y,v)} |f(y)|^p\,dy\right)^{\frac{1}{p}}\right) \\ &= \|f\|_{\mathcal{M}_{p,\phi}}.\end{aligned}$$

Thus we have $\mathcal{M}_{p,\phi}(\mathbb{R}^n) = \mathcal{M}_{p,\psi}(\mathbb{R}^n)$ with norm coincidence.

From the definition of ψ, it is easy to check (1). □

Lemma 2.2. *Let $0 < p < \infty$ and let $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a function satisfying (1). Then there exists a function $\psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ satisfying*

$$\psi(x,r) \le \psi(x,s) \tag{3}$$

for all $x \in \mathbb{R}^n$ and $0 < s \le r < \infty$ such that $\mathcal{M}_{p,\phi}(\mathbb{R}^n) = \mathcal{M}_{p,\psi}(\mathbb{R}^n)$ with norm equivalence.

Proof. Let us set

$$\psi(x,r) \equiv \inf_{0<s\le r}\left(\sup_{y\in Q(x,r)} \phi(y,s)\right) \quad (x \in \mathbb{R}^n, r > 0).$$

It is easy to verify (3). Furthermore,

$$\psi(x,r) \le \sup_{y\in Q(x,r)} \phi(y,r) \le 3^{\frac{n}{p}}\phi(x,3r) \quad (x \in \mathbb{R}^n, r > 0)$$

from (1) and hence $\|f\|_{\mathcal{M}_{p,\phi}} \le 3^{\frac{n}{p}}\|f\|_{\mathcal{M}_{p,\psi}}$. Meanwhile, for all $Q \in \mathcal{Q}$ and $0 < s \le \ell(Q)$ such that $\log_2(s\ell(Q)^{-1}) \in \mathbb{Z}$, we can find a cube $R = R_Q(s)$ contained in Q such that $\ell(R) = s$ and that

$$\left(\frac{1}{|R|}\int_R |f(y)|^p\,dy\right)^{\frac{1}{p}} \ge 2^{-\frac{n}{p}}\left(\frac{1}{|Q|}\int_Q |f(y)|^p\,dy\right)^{\frac{1}{p}}$$

by the pegion hole principle. Thus for all $Q \in \mathcal{Q}$ and $0 < s \le \ell(Q)$, we can find a cube $R = R_Q(s)$ contained in Q such that $\ell(R) = s$ and that

$$\left(\frac{1}{|R|}\int_R |f(y)|^p\,dy\right)^{\frac{1}{p}} \ge 4^{-\frac{n}{p}}\left(\frac{1}{|Q|}\int_Q |f(y)|^p\,dy\right)^{\frac{1}{p}}.$$

Therefore, it follows that

$$\begin{aligned}
\|f\|_{\mathcal{M}_{p,\psi}} &= \sup_{Q\in\mathcal{Q}} \frac{1}{\psi(c(Q),\ell(Q))}\left(\frac{1}{|Q|}\int_Q |f(y)|^p\,dy\right)^{\frac{1}{p}}\\
&= \sup_{Q\in\mathcal{Q}}\sup_{0<s<r}\left(\inf_{y\in Q}\frac{1}{\phi(y,s)}\left(\frac{1}{|Q|}\int_Q |f(y)|^p\,dy\right)^{\frac{1}{p}}\right)\\
&\le 4^{\frac{n}{p}}\sup_{Q\in\mathcal{Q}}\sup_{0<s<r}\left(\inf_{y\in Q}\frac{1}{\phi(y,s)}\left(\frac{1}{|R_Q(s)|}\int_{R_Q(s)} |f(y)|^p\,dy\right)^{\frac{1}{p}}\right)\\
&\le 4^{\frac{n}{p}}\sup_{Q\in\mathcal{Q}}\sup_{0<s<r}\left(\frac{1}{\phi(R_Q(s))}\left(\frac{1}{|R_Q(s)|}\int_{R_Q(s)} |f(y)|^p\,dy\right)^{\frac{1}{p}}\right)\\
&\le 4^{\frac{n}{p}}\|f\|_{\mathcal{M}_{p,\phi}},
\end{aligned}$$

as was to be shown. □

Lemma 2.3. *Let $0<p<\infty$ and let $\phi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ be a function satisfying*

$$\phi(x,r)\le\phi(x,s) \tag{4}$$

for all $0<s\le r<\infty$ and $x\in\mathbb{R}^n$. Then there exists a function $\psi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ satisfying (1) and (3) such that $\mathcal{M}_{p,\phi}(\mathbb{R}^n)=\mathcal{M}_{p,\psi}(\mathbb{R}^n)$ with norm coincidence.

Proof. Let us define ψ by (2). Then as we have seen in Lemma 2.1, $\psi:\mathbb{R}^n\times(0,\infty)\to(0,\infty)$ satisfies (1) and $\mathcal{M}_{p,\phi}(\mathbb{R}^n)=\mathcal{M}_{p,\psi}(\mathbb{R}^n)$ with norm coincidence. It remains to check (3). Let $R<R'$. Then, from (4), we obtain

$$\begin{aligned}
\psi(x,R') &= \inf_{y\in\mathbb{R}^n}\left(\inf_{v\in[R'+\|x-y\|_\infty,\infty)}\phi(y,v)\left(\frac{v}{R'}\right)^{\frac{n}{p}}\right)\\
&\le \inf_{y\in\mathbb{R}^n}\left(\inf_{v\in[R'+R'\|x-y\|_\infty/R,\infty)}\phi(y,v)\left(\frac{v}{R'}\right)^{\frac{n}{p}}\right)\\
&= \inf_{y\in\mathbb{R}^n}\left(\inf_{v\in[R+\|x-y\|_\infty,\infty)}\phi\left(y,\frac{R'v}{R}\right)\left(\frac{v}{R}\right)^{\frac{n}{p}}\right)\\
&\le \inf_{y\in\mathbb{R}^n}\left(\inf_{v\in[R+\|x-y\|_\infty,\infty)}\phi(y,v)\left(\frac{v}{R}\right)^{\frac{n}{p}}\right)=\psi(x,R).
\end{aligned}$$

This proves (3). □

The following compatibility condition:

$$\psi(x,r)\sim\psi(y,r)\quad(|x-y|\le r) \tag{5}$$

can be naturally postulated.

Proposition 2.4. *Let* $0 < p < \infty$ *and let* $\psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *be a function satisfying* (1) *and* (3)*. Then* ψ *satisfies* (5)*.*

Proof. By (3), we have $\psi(x,r) \geq \psi(x,3r)$ and by (1) $\psi(x,3r) \gtrsim \psi(y,r)$. □

With Lemma 2.3 in mind, we always assume that $\phi \in \mathcal{G}_p$ satisfies (1). We summarize our observations.

Theorem 2.5. *Let* $0 < p < \infty$ *and let* $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *be a function. Then there exists a function* $\psi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ *satisfying* (1)*,* (3) *and* (5) *such that* $\mathcal{M}_{p,\phi}(\mathbb{R}^n) = \mathcal{M}_{p,\psi}(\mathbb{R}^n)$ *with norm coincidence.*

The main structure of this generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is as follows:

Proposition 2.6. *Let* $0 < p < \infty$ *and* $\phi \in \mathcal{G}_p$*.*

1. *If* $r \in (0,p)$*, then*

$$\|f\|_{\mathcal{M}_{r,\phi}} \leq \|f\|_{\mathcal{M}_{p,\phi}} \quad \text{for all } f \in \mathcal{M}_{p,\phi}(\mathbb{R}^n). \tag{6}$$

2. *If* $u \in (0,\infty)$*, then*

$$\||f|^u\|_{\mathcal{M}_{\frac{p}{u},\phi^u}} = (\|f\|_{\mathcal{M}_{p,\phi}})^u \quad \text{for all } f \in \mathcal{M}_{p,\phi}(\mathbb{R}^n). \tag{7}$$

3. *Assume* (1)*. Then*

$$\frac{1}{\phi(Q)} \leq \|\chi_Q\|_{\mathcal{M}_{p,\phi}} \lesssim \frac{1}{\phi(Q)}. \tag{8}$$

See [34, Proposition 2.1] for the case when $p \geq 1$ and ϕ is independent of x. The same proof works for this case but for the sake of convenience for readers we supply the whole proof.

Proof. It is easy to check (6) by using the Hölder inequality and (7) by a direct calculation. Let us check (8).

By the definition,

$$\|\chi_Q\|_{\mathcal{M}_{p,\phi}} = \sup_{R \in \mathcal{Q}} \frac{1}{\phi(R)} \left(\frac{|Q \cap R|}{|R|} \right)^{\frac{1}{p}}.$$

Thus, the left inequality is clear.

Let us check the right inequality. If we write the norm fully, then

$$\|\chi_Q\|_{\mathcal{M}_{p,\phi}} = \sup_{R \in \mathcal{Q}, Q \cap R \neq \emptyset} \frac{1}{\phi(R)} \left(\frac{|Q \cap R|}{|R|} \right)^{\frac{1}{p}}.$$

We next claim

$$\|\chi_Q\|_{\mathcal{M}_{p,\phi}} \le \sup\left\{\frac{1}{\phi(R)}\left(\frac{|Q\cap R|}{|R|}\right)^{\frac{1}{p}} : R\in\mathcal{Q}, Q\cap R\neq\emptyset, Q\subset 3R\right\}. \quad (9)$$

In fact, if R is a cube that intersects both Q and $\mathbb{R}^n\setminus 3Q$, then choose a cube S such that $Q\cap R\subset S\subset R$ and that $\ell(S)=\ell(Q)$. Then $S\subset 3Q$ and

$$\frac{1}{\phi(R)}\left(\frac{|Q\cap R|}{|R|}\right)^{\frac{1}{p}} \le \frac{1}{\phi(S)}\left(\frac{|Q\cap R|}{|S|}\right)^{\frac{1}{p}} \le \frac{1}{\phi(S)}\left(\frac{|Q\cap S|}{|S|}\right)^{\frac{1}{p}}$$

from (1), which yields (9).

Let R be a cube intersecting Q. We let S be a cube concentric to R having sidelength $10\ell(Q)$. Then since $\phi\in\mathcal{G}_p$,

$$\frac{1}{\phi(R)}\left(\frac{|Q\cap R|}{|R|}\right)^{\frac{1}{p}} \le \frac{1}{\phi(R)} \lesssim \frac{1}{\phi(S)} \lesssim \frac{1}{\phi(Q)}.$$

Thus, we obtain the right inequality. □

A direct consequence of the assumption $\phi\in\mathcal{G}_p$ is the following equivalent expression:

$$\|f\|_{\mathcal{M}_{p,\phi}} \sim \sup_{Q\in\mathcal{D}}\frac{1}{\phi(Q)}\left(\frac{1}{|Q|}\int_Q |f(y)|^p\,dy\right)^{\frac{1}{p}}.$$

3. Boundedness of the maximal operator

The following result is standard and we aim to extend it to generalized Morrey spaces:

Theorem 3.1 ([4]). *Let $0<\lambda<n$. Then*

(1) *M is bounded on $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ if $1<p<\infty$;*
(2) *M is bounded from $\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\lambda}(\mathbb{R}^n)$.*

We denote by $L_{\infty,v}(0,\infty)$ the space of all non-negative functions $g(t)$, $t>0$ such that

$$\|g\|_{L_{\infty,v}(0,\infty)} \equiv \sup_{t>0} v(t)g(t)$$

is finite. The space $\mathfrak{M}(0,\infty)$ is defined to be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ is defined to be its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} \equiv \left\{\phi\in\mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+}\phi(t)=0\right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) \equiv \|u\, g\|_{L_\infty(t,\infty)}, \quad t \in (0,\infty).$$

We invoke the following theorem:

Theorem 3.2 ([2, Lemma 5.2]). *Let v_1, $v_2 \in \mathfrak{M}^+(0,\infty)$ satisfy*

$$0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$$

for any $t > 0$ and let $u \in \mathfrak{M}^+(0,\infty)$ be continuous.

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if $\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty$.

We recall that the dual weighted Hardy operator is given by

$$H_w^* g(t) \equiv \int_t^\infty g(s) w(s) ds, \quad 0 < t < \infty,$$

where w is a weight. Remark that the following theorem in the special case $w = 1$ was proved in [2, Theorem 5.1]:

Theorem 3.3 ([9, Theorem 3.1]). *Let v_1, v_2 and w be weights on $(0,\infty)$ and assume that v_1 is bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \leq C \sup_{t>0} v_1(t) g(t) \tag{10}$$

holds for some $C > 0$ for $g \in \mathfrak{M}^+(0,\infty)$ if and only if

$$B \equiv \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{11}$$

Moreover, the value $C = B$ is the best constant for (10).

Remark 3.4. In (10) and (11) it will be understood that $\frac{1}{\infty} \equiv 0$ and $0 \cdot \infty \equiv 0$. See [10, Theorem 1] as well for some applications.

The following statement, extending the results in Mizuhara and Nakai [21, 23], was proved in [6–8]:

Proposition 3.5. *Let $1 \leq p < \infty$. Moreover, let ϕ_1, $\phi_2 \in \mathcal{G}_p$ satisfy*

$$\int_r^\infty \phi_1(x,t) \frac{dt}{t} \lesssim \phi_2(x,r) \tag{12}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Then, for $p > 1$, M is bounded from $\mathcal{M}_{p,\phi_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\phi_2}(\mathbb{R}^n)$ and, for $p = 1$, M is bounded from $\mathcal{M}_{1,\phi_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\phi_2}(\mathbb{R}^n)$.

As is seen from the paper [30, Theorem 2.3], (12) is too strong. The following statements, containing Proposition 3.5, was proved by Akbulut, Guliyev and Mustafayev [1, Theorem 3.4]; note that (12) is stronger than (13):

Proposition 3.6. *Let $1 \le p < \infty$ and suppose the couple (ϕ_1, ϕ_2) of the functions in $\mathcal{G}_p$ satisfies the following condition:*

$$\phi_1(x,t) \lesssim \phi_2(x,t) \tag{13}$$

where the impicit constant does not depend on x and t.

1. *If $p > 1$, then M is bounded from $\mathcal{M}_{p,\phi_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\phi_2}(\mathbb{R}^n)$. Namely, if $p > 1$, $\|Mf\|_{\mathcal{M}_{p,\phi_2}} \lesssim \|f\|_{\mathcal{M}_{p,\phi_1}}$ for all $f \in \mathcal{M}_{p,\phi_1}(\mathbb{R}^n)$*
2. *If $p \ge 1$, then M is bounded from $\mathcal{M}_{1,\phi_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\phi_2}(\mathbb{R}^n)$. Namely, if $p \ge 1$, $\|Mf\|_{W\mathcal{M}_{p,\phi_2}} \lesssim \|f\|_{\mathcal{M}_{p,\phi_1}}$ for all $f \in \mathcal{M}_{p,\phi_1}(\mathbb{R}^n)$.*

From this proposition, when $\phi_1 = \phi_2 = \phi$, we have the following boundedness:

Corollary 3.7 ([23, Theorem 1], [30, Theorem 2.3]). *Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$.*

1. *Let $1 < p < \infty$. Then $\|Mf\|_{\mathcal{M}_{p,\phi}} \lesssim \|f\|_{\mathcal{M}_{p,\phi}}$ for all $f \in \mathcal{M}_{p,\phi}(\mathbb{R}^n)$.*
2. *Let $1 \le p < \infty$. Then $\|Mf\|_{W\mathcal{M}_{p,\phi}} \lesssim \|f\|_{\mathcal{M}_{p,\phi}}$ for all $f \in \mathcal{M}_{p,\phi}(\mathbb{R}^n)$.*

We know that there is no requirement when we consider the boundedness of the maximal operator [30, Theorem 2.3] when ϕ is independent of x. Proposition 3.6 naturally extends the assertion above.

4. Vector-valued boundedness of the maximal operator

Our aim here is to extend the Fefferman-Stein vector-valued inequality to our function spaces for M in addition to Corollary 5.3;

$$\|\{Mf_j\}_{j=1}^{\infty}\|_{L_p(\ell_u)} \lesssim \|\{f_j\}_{j=1}^{\infty}\|_{L_p(\ell_u)}, \tag{14}$$

and

$$\|\{Mf_j\}_{j=1}^{\infty}\|_{WL_1(\ell_u)} \lesssim \|\{f_j\}_{j=1}^{\infty}\|_{L_1(\ell_u)}, \tag{15}$$

where $1 < p < \infty$ and $1 < u \le \infty$; see [5, Theorem 1(1)] and [5, Theorem 1(2)] for the proof of (14) and (15), respectively. When $q = \infty$, it is understood that (14) reads

$$\left\| \sup_{j\in\mathbb{N}} Mf_j \right\|_{L_p} \lesssim \left\| \sup_{j\in\mathbb{N}} |f_j| \right\|_{L_p}.$$

Write $MF \equiv \{Mf_j\}_{j=1}^{\infty}$, when we are given a sequence $F = \{f_j\}_{j=1}^{\infty}$. Thus, (14) reads

$$\|MF\|_{L_p(\ell_u)} \lesssim \|F\|_{L_p(\ell_u)}.$$

Our main result here is as follows:

Theorem 4.1. *Let $1 \le p < \infty$, $1 < q \le \infty$ and suppose that the couple $(\phi_1, \phi_2) \in \mathcal{G}_p \times \mathcal{G}_p$ satisfies the condition*

$$\int_r^\infty \phi_1(x,t)\frac{dt}{t} \lesssim \phi_2(x,r) \quad (x \in \mathbb{R}^n, r > 0), \tag{16}$$

where the implicit constant does not depend on x and t.

(1) *For $1 < p < \infty$, M is bounded from $\mathcal{M}_{p,\phi_1}(l_q, \mathbb{R}^n)$ to $\mathcal{M}_{p,\phi_2}(l_q, \mathbb{R}^n)$, i.e., $\|MF\|_{\mathcal{M}_{p,\phi_2}(l_q)} \lesssim \|F\|_{\mathcal{M}_{p,\phi_1}(l_q)}$ holds for all $F \in \mathcal{M}_{p,\phi_1}(l_q, \mathbb{R}^n)$.*

(2) *For $1 \le p < \infty$, M is bounded from $\mathcal{M}_{p,\phi_1}(l_q, \mathbb{R}^n)$ to $W\mathcal{M}_{p,\phi_2}(l_q, \mathbb{R}^n)$, i.e., $\|MF\|_{W\mathcal{M}_{1,\phi_2}(l_q)} \lesssim \|F\|_{\mathcal{M}_{1,\phi_1}(l_q)}$ holds for all $F \in \mathcal{M}_{1,\phi_1}(l_q, \mathbb{R}^n)$.*

Remark that (16) is natural in view of [11, Theorem 4].

As a corollary, by letting $\phi_1 = \phi_2$ we can recover the vector-valued inequality obtained in [34, Theorem 5.3].

Corollary 4.2 ([34, Theorem 5.3])**.** *Let $1 < p < \infty$ and $1 < u < \infty$. Assume that ϕ is independent of x.*

1. *Assume in addition that $\phi \in \mathcal{G}_p$ satisfies (12). Then $\|MF\|_{\mathcal{M}_{p,\phi}(\ell_u)} \lesssim \|F\|_{\mathcal{M}_{p,\phi}(\ell_u)}$ for any sequence of measurable functions $F = \{f_j\}_{j=1}^\infty \in \mathcal{M}_{p,\phi}(\ell_u)$.*
2. *We have $\|MF\|_{\mathcal{M}_{p,\phi}(\ell_\infty)} \lesssim \|F\|_{\mathcal{M}_{p,\phi}(\ell_\infty)}$ for any sequence of measurable functions $F = \{f_j\}_{j=1}^\infty \in \mathcal{M}_{p,\phi}(\ell_\infty)$.*

Now we are oriented to the proof of Theorem 4.1. We first prove the following auxiliary estimate:

Lemma 4.3. *Let $1 < p < \infty$ and $1 < q \le \infty$.*

1. *The inequality*

$$\begin{aligned}\|\chi_{B(x,r)}MF\|_{L_p(\ell_q)} &\lesssim \|\chi_{B(x,2r)}F\|_{L_p(\ell_q)} + r^{\frac{n}{p}}\int_r^\infty \frac{\|\chi_{B(x,t)}F\|_{L_1(\ell_q)}}{t^{n+1}}\,dt \\ &\sim \|\chi_{B(x,2r)}F\|_{L_p(\ell_q)} + r^{\frac{n}{p}}\int_{2r}^\infty \frac{\|\chi_{B(x,t)}F\|_{L_1(\ell_q)}}{t^{n+1}}\,dt\end{aligned} \tag{17}$$

holds for all $F = \{f_j\}_{j=0}^\infty \subset L_{p,\mathrm{loc}}(\mathbb{R}^n)$ and for any ball $B = B(x,r)$.

2. *The inequality*

$$\begin{aligned}\|\chi_{B(x,r)}MF\|_{WL_1(\ell_q)} &\lesssim \|\chi_{B(x,2r)}F\|_{L_1(\ell_q)} + r^n\int_r^\infty \frac{\|\chi_{B(x,t)}F\|_{L_1(\ell_q)}}{t^{n+1}}\,dt \\ &\sim \|\chi_{B(x,2r)}F\|_{L_1(\ell_q)} + r^n\int_{2r}^\infty \frac{\|\chi_{B(x,t)}F\|_{L_1(\ell_q)}}{t^{n+1}}\,dt\end{aligned} \tag{18}$$

holds for all $F = \{f_j\}_{j=0}^\infty \subset L_{1,\mathrm{loc}}(\mathbb{R}^n)$ and for any ball $B = B(x,r)$.

Proof. We split $F = \{f_j\}_{j=1}^{\infty}$ with $F_1 = \chi_{B(x,2r)}F$ and $F_2 = F - F_1$. We estimate each term.

By the triangle inequality

$$\|\chi_{B(x,r)}MF\|_{L_p(\ell_q)} \le \|\chi_{B(x,r)}MF_1\|_{L_p(\ell_q)} + \|\chi_{B(x,r)}MF_2\|_{L_p(\ell_q)}.$$

First, we shall estimate $\|\chi_{B(x,r)}MF_1\|_{L_p(\ell_q)}$ for $1 < p < \infty$ and $1 < q \le \infty$. Thanks to (14), we have

$$\|\chi_{B(x,r)}MF_1\|_{L_p(\ell_q)} \le \|MF_1\|_{L_p(\ell_q)} \lesssim \|F_1\|_{L_p(\ell_q)} = \|\chi_{B(x,2r)}F\|_{L_p(\ell_q)}, \tag{19}$$

where the implicit constant is independent of the vector-valued function F. Thus, the estimate for MF_1 is valid.

Now we handle MF_2. Freeze a point y in $B(x,r)$. For all $j \in \mathbb{N}_0$ and $y \in B(x,r)$,

$$\|MF_2(y)\|_{\ell_q} \lesssim \sum_{k=1}^{\infty} \frac{1}{|2^k Q(x,r)|} \int_{2^k Q(x,r)} \|F(y)\|_{\ell_q}\, dy$$

according to [34, Lemma 4.2]. As a result, we obtain

$$r^{-\frac{n}{p}} \|\chi_{B(x,r)}MF_2\|_{L_p(\ell_q)} \lesssim \int_{2r}^{\infty} s^{-n-1} \|\chi_{B(x,s)}F\|_{L_1(\ell_q)}\, ds \tag{20}$$

for all $y \in B(x,s)$. Thus, the estimate for MF_2 is valid. Then we obtain (17) from (19) and (20).

Let $p = 1$. By the quasi-triangle inequality for any ball $B = B(x,r)$

$$\|\chi_{B(x,r)}MF\|_{WL_1(\ell_q)} \lesssim \|\chi_{B(x,r)}MF_1\|_{WL_1(\ell_q)} + \|\chi_{B(x,r)}MF_2\|_{WL_1(\ell_q)}.$$

We estimate each term.

By the weak-type Fefferman-Stein maximal inequality (15) we have

$$\|\chi_{B(x,r)}MF_1\|_{WL_1(\ell_q)} \le \|MF_1\|_{WL_1(\ell_q)} \lesssim \|F_1\|_{L_1(\ell_q)} = \|\chi_{B(x,2r)}F\|_{L_1(\ell_q)}, \tag{21}$$

where the implicit constant is independent of the vector-valued function F.

Then by (20) and (21), we obtain the inequality (18). □

We transform Lemma 4.3 to an inequality to prove Theorem 4.1.

Lemma 4.4. *Let* $1 < q \le \infty$.

1. *Let* $1 \le p < \infty$. *Then, for any ball* $B = B(x,r)$, *the inequality*

$$\|\chi_{B(x,r)}MF\|_{WL_p(\ell_q)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|\chi_{B(x,t)}F\|_{L_p(\ell_q)}\, dt \tag{22}$$

holds for all $F = \{f_j\}_{j=1}^{\infty} \subset L_{p,\mathrm{loc}}(\mathbb{R}^n)$.

2. *For any ball $B = B(x,r)$, the inequality*

$$\|\chi_{B(x,r)} MF\|_{WL_1(\ell_q)} \lesssim r^n \int_{2r}^{\infty} t^{-n-1} \|\chi_{B(x,t)} F\|_{L_1(\ell_q)}\, dt \tag{23}$$

holds for all $F = \{f_j\}_{j=1}^{\infty} \subset L_{1,\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Note that, for all $1 \le p < \infty$

$$\|\chi_{B(x,2r)} F\|_{L_p(\ell_q)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|\chi_{B(x,t)} F\|_{L_p(\ell_q)}\, dt \tag{24}$$

$$r^{\frac{n}{p}} \int_{r}^{\infty} t^{-n-1} \|\chi_{B(x,t)} F\|_{L_1(\ell_q)}\, dt \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|\chi_{B(x,t)} F\|_{L_p(\ell_q)}\, dt$$

by the Hölder inequality. We deduce (22) from (17) and (24). Likewise we deduce (23) from (18) and (24). □

With these estimates in mind, let us prove Theorem 4.1.

Proof of Theorem 4.1. Let $v_1(r) = \phi_1(x,r)^{-1} r^{-\frac{n}{p}}$, $v_2(r) = \phi_2(x,r)^{-1}$, $g(r) = \| \|F\|_{\ell_q} \|_{L_p(B(x,r))}$ and $w(r) = r^{-\frac{n}{p}-1}$. By (16), Lemma 4.4 and Theorem 3.3, we have

$$\begin{aligned}\|MF\|_{\mathcal{M}_{p,\phi_2}(l_q)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \phi_2(x,r)^{-1} \int_{2r}^{\infty} t^{-\frac{n}{p}-1} \|\chi_{B(x,t)} F\|_{L_p(\ell_q)}\, dt \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \phi_1(x,r)^{-1}\, r^{-\frac{n}{p}} \|\chi_{B(x,r)} F\|_{L_p(\ell_q)} \\ &= \|F\|_{\mathcal{M}_{p,\phi_1}(l_q)},\end{aligned}$$

if $1 < p < \infty$ and

$$\begin{aligned}\|MF\|_{W\mathcal{M}_{1,\phi_2}(l_q)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \phi_2(x,r)^{-1} \int_{2r}^{\infty} t^{-n-1} \|\chi_{B(x,t)} F\|_{L_1(\ell_q)}\, dt \\ &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \phi_1(x,r)^{-1}\, r^{-n} \|\chi_{B(x,r)} F\|_{L_1(\ell_q)} \\ &= \|F\|_{\mathcal{M}_{1,\phi_1}(l_q)}\end{aligned}$$

if $p = 1$. □

As a corollary, we can recover the results in [11,30].

Corollary 4.5. *Let $1 \le p < \infty$, $1 \le q < \infty$ and $\phi : (0,\infty) \to (0,\infty)$ be a decreasing function satisfying $\phi(t)t^{\frac{n}{p}} \lesssim \phi(T)T^{\frac{n}{p}}$ for all $0 < t < T < \infty$. Assume in addition ϕ is independent of x and that*

$$\int_r^{\infty} \phi(t) \frac{dt}{t} \lesssim \phi(r).$$

1. [30, Theorem 2.5] *Let $1 < p < \infty$. Let $F = \{f_j\}_{j=1}^{\infty}$ be a sequence in $\mathcal{M}_{p,\phi}(l_q)$. Then*
$$\|MF\|_{\mathcal{M}_{p,\phi}(l_q)} \lesssim \|F\|_{\mathcal{M}_{p,\phi}(l_q)}.$$
2. [11, Theorem 6] *Let $p = 1$. Let $F = \{f_j\}_{j=1}^{\infty}$ be a sequence in $\mathcal{M}_{1,\phi}(l_q)$. Then*
$$\|MF\|_{W\mathcal{M}_{1,\phi}(l_q)} \lesssim \|F\|_{\mathcal{M}_{1,\phi}(l_q)}.$$

5. A basic tool for the theory of generalized Nikol'skij-Besov-Triebel-Lizorkin-Morrey spaces

We now consider generalized Nikol'skij-Besov-Triebel-Lizorkin-Morrey spaces based on our maximal inequalities. For $0 < p < \infty$ and $\phi \in \mathcal{G}_p$, denote by $\mathcal{M}_{p,\phi,\Omega}(\mathbb{R}^n)$ the set of all functions $f \in \mathcal{M}_{p,\phi,\Omega}(\mathbb{R}^n) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ for which $\mathcal{F}f$ is supported on Ω. By using the Planchrel-Pólya-Nikol'skij inequality [27], we have the following estimate of the Peetre maximal operator:

Theorem 5.1. *Let $0<r\le p<\infty$ and suppose that the couple $(\phi_1,\phi_2)\in\mathcal{G}_p\times\mathcal{G}_p$ satisfies the condition*

$$\phi_1(x,t) \lesssim \phi_2(x,t), \tag{25}$$

where the implicit constant does not depend on $x \in \mathbb{R}^n$ and $t > 0$. Let Ω be a compact set, d be the diameter of Ω. Let $f \in \mathcal{M}_{p,\phi_1,\Omega}(\mathbb{R}^n)$.

(1) *If $r < p < \infty$, then $\left\| \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot-y)|}{1+|dy|^{\frac{n}{r}}} \right\|_{\mathcal{M}_{p,\phi_2}} \lesssim \|f\|_{\mathcal{M}_{p,\phi_1}}$.*

(2) *If $p = r$, then $\left\| \sup_{y\in\mathbb{R}^n} \frac{|f(\cdot-y)|}{1+|dy|^{\frac{n}{r}}} \right\|_{W\mathcal{M}_{r,\phi_2}} \lesssim \|f\|_{\mathcal{M}_{r,\phi_1}}$.*

Remark 5.2. Theorem 5.1 is proved in [36, 38] in the case of classical Morrey spaces.

We also need the following corollary to Proposition 5.1.

Corollary 5.3. *Let $1 \le p \le \infty$, $0 < q \le \infty$ and suppose that the couple (ϕ_1, ϕ_2) of the functions in $\mathcal{G}_p$ satisfies the condition* (13)*.*

1. *Let $1 < p < \infty$. Then M is bounded from $\ell_q(\mathcal{M}_{p,\phi_1}, \mathbb{R}^n)$ to $\ell_q(\mathcal{M}_{p,\phi_2}, \mathbb{R}^n)$.*
2. *Let $p = 1$. Then M is bounded from $\ell_q(\mathcal{M}_{1,\phi_1}, \mathbb{R}^n)$ to $\ell_q(W\mathcal{M}_{1,\phi_2}, \mathbb{R}^n)$.*

Let $\Omega = \{\Omega_j\}_{j=1}^{\infty}$ be a sequence of compact sets. Denote by $\ell_q(\mathcal{M}_{p,\phi_1,\Omega})$ the set of all sequences $\{f_j\}_{j=1}^{\infty}$ of the functions in $\mathcal{M}_{p,\phi_1}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$ such that $\mathrm{supp}(\mathcal{F}f_j) \subset \Omega_j$ for each $j \in \mathbb{N}$.

We have the following estimate, which is a counterpart to Theorem 5.1:

Theorem 5.4. *Let $0 < p < \infty$, $0 < q \le \infty$, $0 < r < p$ and suppose that the couple $(\phi_1, \phi_2) \in \mathcal{G}_p \times \mathcal{G}_p$ satisfies the condition* (25)*. Let $\Omega = \{\Omega_j\}_{j=1}^{\infty}$ be a sequence of compact sets, and let d_j denote the diameter of Ω_j. Then exists a positive constant C such that*

$$\left\| \left\{ \sup_{y\in\mathbb{R}^n} \frac{|f_j(\cdot-y)|}{1+|d_j y|^{\frac{n}{r}}} \right\}_{j=1}^{\infty} \right\|_{\ell_q(\mathcal{M}_{p,\phi_2})} \lesssim \left\| \{f_j\}_{j=1}^{\infty} \right\|_{\ell_q(\mathcal{M}_{p,\phi_1})}, \tag{26}$$

if we are given a collection of measurable functions $\{f_j\}_{j=1}^{\infty} \subset \ell_q(\mathcal{M}_{p,\phi_1,\Omega})$.

If we use Theorem 4.1, we can prove the following theorem, which corresponds to Theorem 5.4:

Theorem 5.5. *Let $0 < p < \infty$, $0 < q \le \infty$, $0 < r < \min\{p, q\}$ and suppose that the couple $(\phi_1, \phi_2) \in \mathcal{G}_p \times \mathcal{G}_p$ satisfies the condition* (25). *Let $\Omega = \{\Omega_j\}_{j=1}^\infty$ be a sequence of compact sets, and let d_j be the diameter of Ω_j. Then exists a positive constant C such that*

$$\left\| \left\{ \sup_{y \in \mathbb{R}^n} \frac{|f_j(\cdot - y)|}{1 + |d_j y|^{\frac{n}{r}}} \right\}_{j=1}^\infty \right\|_{\mathcal{M}_{p,\phi_2}(\ell_q)} \lesssim \left\| \{f_j\}_{j=1}^\infty \right\|_{\mathcal{M}_{p,\phi_1}(\ell_q)}, \tag{27}$$

if we are given a collection of measurable functions $\{f_j\}_{j=1}^\infty \in \mathcal{M}_{p,\phi_1,\Omega}(\ell_q)$.

The proof of this theorem is omitted since it is the same as Theorem 5.4.

Using Theorem 5.4, we can develop a theory of generalized Nikol'skij-Besov-Triebel-Lizorkin-Morrey spaces, which we define below.

Definition 5.6. Let $0 < q < \infty$, $0 < r \le \infty$, $s \in \mathbb{R}$ and $\phi \in \mathcal{G}_q$. Let θ and τ be compactly supported functions satisfying

$$0 \notin \operatorname{supp}(\tau), \quad \theta(\xi) > 0 \text{ if } \xi \in Q(2), \quad \tau(\xi) > 0 \text{ if } \xi \in Q(2) \setminus Q(1).$$

Define $\tau_k(\xi) \equiv \tau(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

1. One defines the (*nonhomogeneous*) *generalized Nikol'skij-Besov-Morrey space* $\mathcal{N}^s_{\mathcal{M}_{q,\phi},r}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\mathcal{N}^s_{\mathcal{M}_{q,\phi},r}} \equiv \begin{cases} \|\theta(D)f\|_{\mathcal{M}_{q,\phi}} + \left(\sum_{j=1}^\infty 2^{jsr} \|\tau_j(D)f\|_{\mathcal{M}_{q,\phi}}^r \right)^{\frac{1}{r}} & (r < \infty), \\ \|\theta(D)f\|_{\mathcal{M}_{q,\phi}} + \sup_{j \in \mathbb{N}} 2^{js} \|\tau_j(D)f\|_{\mathcal{M}_{q,\phi}} & (r = \infty) \end{cases} \tag{28}$$

   is finite.
2. One defines the (*nonhomogeneous*) *generalized Triebel-Lizorkin-Morrey space* $\mathcal{E}^s_{\mathcal{M}_{q,\phi},r}(\mathbb{R}^n)$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm

$$\|f\|_{\mathcal{E}^s_{\mathcal{M}_{q,\phi},r}} \equiv \begin{cases} \|\theta(D)f\|_{\mathcal{M}_{q,\phi}} + \left\| \left(\sum_{j=1}^\infty 2^{jsr} |\tau_j(D)f|^r \right)^{\frac{1}{r}} \right\|_{\mathcal{M}_{q,\phi}} & (r < \infty), \\ \|\theta(D)f\|_{\mathcal{M}_{q,\phi}} + \left\| \sup_{j \in \mathbb{N}} 2^{js} |\tau_j(D)f| \right\|_{\mathcal{M}_{q,\phi}} & (r = \infty) \end{cases} \tag{29}$$

   is finite.
3. The space $\mathcal{A}^s_{\mathcal{M}_{q,\phi},r}(\mathbb{R}^n)$ denotes either $\mathcal{N}^s_{\mathcal{M}_{q,\phi},r}(\mathbb{R}^n)$ or $\mathcal{E}^s_{\mathcal{M}_{q,\phi},r}(\mathbb{R}^n)$.

Much about generalized Nikol'skij-Besov-Triebel-Lizorkin-Morrey spaces is investigated in [25]. The difference of the definitions in [25] and this paper is that this paper extends the condition on ϕ: ϕ depends on x as well in the present paper but ϕ depends on r in [25]. Further inverstigation is left as a future work.

Let us survey the progress on Nikol'skij-Besov-Morrey spaces before we conclude this section. In 1984, Netrusov defined Nikol'skij-Besov-Morrey spaces [26]. Netrusov obtained some embedding results. Later on Nikol'skij-Besov-Morrey spaces shed light on by Kozono and Yamazaki [19] from the context of differential equations. It is Kozono and Yamazaki that applied Nikol'skij-Besov-Morrey spaces to investigate the Cauchy problem for the Navier-Stokes equation. Najafov considered Nikol'skij-Besov-Morrey spaces of the mixed derivative in [22]. Motivated by this, Tang and Xu [38] defined non-homogeneous Triebel-Lizorkin-Morrey spaces, or equivalently, non-homogeneous Morrey type Triebel-Lizorkin spaces in word of the original paper [38]. After this, Sawano and Tanaka defined homogeneous Triebel-Lizorkin-Morrey spaces [37]. Sawano and Wang obtained the trace theorem independently in [33, Theorem 1.1] and [40, Proposition 1.10], respectively. The wavelet characterization of Nikol'skij-Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces can be found in [28, 29]. Triebel-Lizorkin-Morrey spaces cover Hardy-Morrey spaces; see [31, Theorem 4.2]. We refer to [12–14, 32, 37, 38] for embedding relations of these function spaces. We refer to [3, 18] for weighted Nikol'skij-Besov-Morrey spaces and weighted Triebel-Lizorkin-Morrey spaces. See the textbooks [39, 41] for an exhaustive account of these spaces. More and more axiomatic approach to define new function spaces based on a solid function spaces was taken in [15–17, 20].

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