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Regularity of Minimizers in the Two-Phase Free Boundary Problems in Orlicz-Sobolev Spaces

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Abstract. In this paper, we consider the optimization problem of minimizing $\mathcal{J}(u) = \int_\Omega (G(|\nabla u|) + \lambda_+(u^+)^\gamma + \lambda_-(u^-)^\gamma + fu)\mathrm{d}x$ in the class of functions $W^{1,G}(\Omega)$ with $u - \varphi \in W_0^{1,G}(\Omega)$ for a given function φ, where $W^{1,G}(\Omega)$ is the class of weakly differentiable functions with $\int_\Omega G(|\nabla u|)\mathrm{d}x < \infty$. The conditions on the function G allow for a different behavior at 0 and at ∞. For $0 < \gamma \leq 1$, we prove that every minimizer u of $\mathcal{J}(u)$ is $C^{1,\alpha}_{loc}$-continuous.

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1. Introduction

Let $G(t)$ be a C^2-continuous function in $t \in [0, +\infty)$ with its derivative $g(t) = G'(t)$ nonnegative in $[0, +\infty)$. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$. Given a function $\varphi \in L^\infty(\Omega)$ with $\varphi^+ \not\equiv 0$ and $\int_\Omega G(|\nabla\varphi|)\mathrm{d}x < \infty$, and some integrable function f in Ω, we consider the optimization problem

$$\mathcal{J}(u) = \int_\Omega (G(|\nabla u|) + F_\gamma(u) + fu)\mathrm{d}x \to \min, \tag{1}$$

in the class of functions $\mathcal{K} = \{u \in L^1(\Omega) : \int_\Omega G(|\nabla u|)\mathrm{d}x < \infty, u = \varphi \text{ on } \partial\Omega\}$, where $F_\gamma(u) = \lambda_+(u^+)^\gamma + \lambda_-(u^-)^\gamma$, $\gamma \in (0, 1]$, $0 \leq \lambda_- < \lambda_+ < \infty$, and

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$u^{\pm} = \max\{\pm u, 0\}$. The Euler-Lagrange equation for such minimizers is

$$\Delta_G u := \text{div}\ \frac{g(|\nabla u|)}{|\nabla u|}\nabla u = \gamma(\lambda_+ (u^+)^{\gamma-1}\chi_{\{u>0\}} + \lambda_-(u^-)^{\gamma-1}\chi_{\{u<0\}}) + f. \quad (2)$$

The non-differentiability of $F_\gamma(u)$ impels the Euler-Lagrange equation (2) to be singular along the a priori unknown interface (so-called *free boundary*)

$$\Gamma = \partial\{u > 0\} \cup \partial\{u < 0\},$$

between the positive and negative phases of a minimizer. It is not obvious that minimizers have some C^1-regularities.

In this work we extend several regularity theories of minimizers in the optimal problems to a large class of degenerate and singular elliptic operators under the natural condition which generalizes the Ladyzhenskaya-Uraltseva operators (see [5]), for some positive constants δ, g_0

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall\ t > 0. \quad (3)$$

The operator Δ_G not only includes the case of p-Laplacian Δ_p ($\delta = g_0 = p - 1 > 0$), but also the interesting case of a variable exponent $p = p(t) > 0$:

$$\Delta_G u = \text{div}\ (|\nabla u|^{p(|\nabla u|)-2}\nabla u),$$

corresponding to set $g(t) = t^{p(t)-1}$, for which (3) holds if $\delta \leq t(\ln t)p'(t) + p(t) - 1 \leq g_0$ for all $t > 0$. Other examples of functions satisfying (3) are given by $g(t) = t^\alpha \ln(\beta t + \theta)$, with $\alpha, \beta, \theta > 0$, or by discontinuous power transitions as $g(t) = C_1 t^\alpha$, if $0 \leq t < t_0$, and $g(t) = C_2 t^\beta + C_3$, if $t \geq t_0$, where α, β, t_0 are positive numbers, C_1, C_2, C_3 are real numbers such that $g(t)$ is a C^1 function; see [2].

We recall some types of the free boundary problems. The upper case, $\gamma = 1$, relates to obstacle type problems. The intermediary problems, $0 < \gamma < 1$, can be used to model the density of certain chemical specie, in reaction with a porous catalyst pellet, and have intrigued lots of mathematicians in the past decades.

In Sobolev spaces, regularity theories for one phase ($\lambda_- = 0$) free boundary problems, particularly for p-Laplacian equations, have been well studied by a large number of mathematicians, Frehse, Stampacchia, Kinderlehrer, Caffarelli, Alt, Phillips, Weiss, Shahgholion, and others. The two phase free boundary problems governed by Laplacian equations have been also studied by lots of scholars. For more details we refer readers to the current work by Leitão, de Queiroz and Teixeira [4], where the authors provided a complete description of regularity theory for the free boundary problems governed by p-Laplacian equations ($p \geq 2$). Under the assumptions that $f \in L^q(\Omega)$ ($q > n$), the authors established $C^{1,\alpha}_{loc}$-regularity for minimizers associated with p-Laplacian equations ($p \geq 2$) when $\gamma \in (0, 1]$.

In Orlicz-Sobolev spaces, Challal, Lyaghfouri and Rodrigues considered the homogeneous one-phase obstacle problems, i.e., $f = 0, \lambda_- = 0$ in (2); see [1,2]. Under the assumption that $\frac{g(t)}{t}$ is monotone in $t > 0$, the authors obtained $C^{1,\alpha}_{loc}$-regularity for solutions of (2) and established porosity of the free boundary [1], then proved that the free boundary has locally finite $(n-1)$-Hausdorff measure [2]. Since it is not obvious that the optimization problem (1) is equivalent to the equation (2), Liu, Zheng and Zhao established the equivalence of problem (1) and (2) for $\gamma = 1$ and obtained $C^{1,\alpha}_{loc}$-regularity for minimizers under the assumption that $\frac{g(t)}{t}$ is monotone in $t > 0$; see [6]. For $\gamma \in (0, 1]$, under the assumption of non-decreasing monotonicity on $\frac{g(t)}{t}$, Zheng, Zhang and Zhao obtained $C^{1,\alpha}_{loc}$-regularity for minimizers in the inhomogeneous two-phase free boundary problems; see [8]. In the above work, the monotonicity of $\frac{g(t)}{t}$ plays an important role in the regularity theory.

In this paper, we continue the work of [4,8] considering $C^{1,\alpha}_{loc}$-regularity of minimizers in the two-phase free boundary problems in Orlicz-Sobolev spaces. Removing any monotonicity assumption on $\frac{g(t)}{t}$, we prove that any minimizer of (1) is $C^{1,\alpha}_{loc}$-continuous. Our result not only holds for $p \geq 2$ in [4], but also for the singular case $1 < p < 2$, and is also a supplementary of [8]. The main idea in this paper is inherited from [4,8].

Throughout this paper, we always assume that (3) is satisfied and

$$f \in L^\infty(\Omega),\ \varphi \in W^{1,G}(\Omega) \cap L^\infty(\Omega).$$

We close up this part by stating the existence and boundedness of a minimizer of (1) and the main result in this paper.

Proposition 1.1 ([8])**.** *For each $0 < \gamma \leq 1$, there exists a minimizer u of (1). Moreover, there exists a positive constant M, depending only on n, δ, g_0, $g^{-1}(1)$, λ_+, λ_-, $\|\varphi\|_{L^\infty(\Omega)}$, $\|\nabla\varphi\|_{L^G(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$, such that*

$$\|u\|_{L^\infty(\Omega)} + \|u\|_{W^{1,G}(\Omega)} \leq M,$$

where $g^{-1}(t)$ is the inverse function of $g(t)$.

The main result in this paper is:

Theorem 1.2. *Let u be a minimizer of the problem (1). Then $u \in C^{1,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0, 1)$. More precisely, for any $\Omega' \subset\subset \Omega$, there exists a constant $C > 0$, depending only on $n, \delta, g_0, g(1)$, λ_+, λ_-, $\|f\|_{L^\infty(\Omega)}, \|\varphi\|_{L^\infty(\Omega)}$ and Ω', such that*

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C.$$

2. Some auxiliary results

In this section, we present some results that will be used throughout the paper. We systematically use definitions and basic properties of the function G as developed in [7, Section 2].

Observe that condition (3) implies the following properties (see [7]):

(g_1) $\frac{tg(t)}{1+g_0} \leq G(t) \leq tg(t), \quad \forall t \geq 0.$

(G_1) G is convex and C^2.

(G_2) $\min\{s^{\delta+1}, s^{g_0+1}\}\frac{G(t)}{1+g_0} \leq G(st) \leq (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t), \quad \forall s,t>0.$

(G_3) $G(a+b) \leq 2^{g_0}(1+g_0)(G(a)+G(b)), \quad \forall a,b>0.$

We recall some definitions associated with Orlicz-Sobolev spaces. Define the Orlicz class $\mathcal{K}_G(\Omega) = \{u \text{ is measurable}; \int_\Omega G(|u|)\mathrm{d}x < \infty\}$. The functional $\|u\|_{L^G(\Omega)} = \inf\{k>0; \int_\Omega G(\frac{|u(x)|}{k})\mathrm{d}x \leq 1\}$ is a norm in the Orlicz space $L^G(\Omega)$ which is the linear hull of $\mathcal{K}_G(\Omega)$. Notice that this set is convex, since G is also convex. The Orlicz-Sobolev space $W^{1,G}(\Omega)$ is defined by $W^{1,G}(\Omega) = \{u \in L^G(\Omega); \nabla u \in L^G(\Omega)\}$, which is the usual subspace of $W^{1,1}(\Omega)$ associated with the norm $\|u\|_{W^{1,G}(\Omega)} = \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}$. As g is strictly increasing we can define its inverse function g^{-1} and define $\widetilde{G}(t) = \int_0^t g^{-1}(s)\mathrm{d}s$ for any $t \geq 0$. It is well known that $L^{\widetilde{G}}(\Omega)$ is the dual space of $L^G(\Omega)$, and $L^G(\Omega), W^{1,G}(\Omega)$ are reflexive.

The following result is a Poincaré-type inequality.

Lemma 2.1 ([3, Lemma 5.7]). *For any $u \in W_0^{1,G}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ in $W^{1,G}(\Omega)$, there holds $\int_\Omega G(|u|)dx \leq \int_\Omega G(c|\nabla u|)dx$, where the constant c is twice the diameter of Ω.*

The following two lemmas and the iterating formula (Lemma 2.4) will pave the way to establish $C^{1,\alpha}$ estimate for minimizers in Section 3.

Lemma 2.2 ([7, Lemma 2.7]). *Let h be a G-harmonic function in B_R, i.e., $\Delta_G h = 0$ in B_R. For every ball $B_r \subset B_R$ and every $\lambda \in (0,n)$, there exists $C = C(\lambda, n, \delta, g_0, \|h\|_{L^\infty(B_R)}) > 0$ such that*

$$\int_{B_r} G(|\nabla h|)dx \leq Cr^\lambda, \quad \forall\ 0<r\leq R.$$

Let $(u)_r = \frac{1}{|B_r|}\int_{B_r} u\mathrm{d}x$ be the average value of u on the ball B_r, we have

Lemma 2.3 (Comparison with G-harmonic functions [8, Lemma 4.1]). *Let $u \in W^{1,G}(B_R)$ and $h \in W^{1,G}(B_R)$ satisfying $\Delta_G h = 0$ in B_R. Then for some positive constant $0 < \sigma < 1$, there exists a positive constant $C = C(n,\delta,g_0)$ such that for each $0 < r \leq R$, there holds*

$$\int_{B_r} G(|\nabla u-(\nabla u)_r|)dx \leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u-(\nabla u)_R|)dx + C\int_{B_R} G(|\nabla u-\nabla h|)dx.$$

Lemma 2.4 ([4, Lemma 2.7]). *Let $\overline{\phi}(s)$ be a non-negative and non-decreasing function. Suppose that*

$$\overline{\phi}(r) \leq C_1\left(\left(\frac{r}{R}\right)^{\alpha} + \vartheta\right)\overline{\phi}(R) + C_2 R^{\beta},$$

for all $r \leq R \leq R_0$, with C_1, α, β positive constants and C_2, ϑ non-negative constants. Then, for any $\tau < \min\{\alpha, \beta\}$, there exists a constant $\vartheta_0 = \vartheta_0(C_1, \alpha, \beta, \tau)$ such that if $\vartheta < C_1, \vartheta_0$, then for all $r \leq R \leq R_0$ we have

$$\overline{\phi}(r) \leq C_3\left(\frac{r}{R}\right)^{\tau}\left(\overline{\phi}(R) + C_2 R^{\tau}\right),$$

where $C_3 = C_3(C_1, \tau - \min\{\alpha, \beta\})$ is a positive constant. In turn,

$$\overline{\phi}(r) \leq C_4 r^{\tau},$$

where $C_4 = C_4(C_2, C_3, R_0, \overline{\phi}, \tau)$ is a positive constant.

3. $C^{1,\alpha}_{loc}$-estimate of minimizers

In this section, we prove Theorem 1.2, showing that every minimizer of (1) is locally $C^{1,\alpha}$-continuous for some $\alpha \in (0,1)$.

Lemma 3.1. *Let $u \in W^{1,G}(\Omega), B_R \subset \Omega$ and h be a solution of*

$$\Delta_G h = 0 \quad \text{in } B_R, \quad h - u \in W^{1,G}_0(B_R).$$

Then for any $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|h\|_{L^\infty(B_{2R/3})}) > 0$ such that

$$\begin{aligned}\int_{B_R} G(|\nabla u - \nabla h|)\,dx &\leq C\int_{B_R}(G(|\nabla u|) - G(|\nabla h|))\,dx \\ &\quad + CR^{\frac{\lambda}{2}}\left(\int_{B_R}(G(|\nabla u|) - G(|\nabla h|))\,dx\right)^{\frac{1}{2}}.\end{aligned} \tag{4}$$

Proof. By [7, Theorem 2.3], there exists a constant $C = C(\delta, g_0) > 0$ such that

$$\begin{aligned}\int_{B_R}(G(|\nabla u|) - G(|\nabla h|))\mathrm{d}x \geq C\bigg(&\int_{A_2} G(|\nabla u - \nabla h|)\mathrm{d}x \\ &+ \int_{A_1} F(|\nabla u|)|\nabla u - \nabla h|^2\mathrm{d}x\bigg),\end{aligned} \tag{5}$$

where $A_1=\{x\in B_R; |\nabla u-\nabla h|\le 2|\nabla u|\}$, $A_2=\{x\in B_R; |\nabla u-\nabla h|>2|\nabla u|\}$ and $F(t)=\frac{g(t)}{t}$. Note that $\frac{G(t)}{t}$ is non-decreasing in $t>0$, it follows from (g_1) that

$$\begin{aligned}\int_{A_1} G(|\nabla u-\nabla h|)\mathrm{d}x &\le C\int_{A_1}\frac{G(|\nabla u|)}{|\nabla u|}|\nabla u-\nabla h|\mathrm{d}x\\ &\le C\Big(\int_{A_1}\frac{G(|\nabla u|)}{|\nabla u|^2}|\nabla u-\nabla h|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_{A_1}G(|\nabla u|)\mathrm{d}x\Big)^{\frac12}\\ &\le C\Big(\int_{A_1}F(|\nabla u|)|\nabla u-\nabla h|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_R}G(|\nabla u|)\mathrm{d}x\Big)^{\frac12}. \qquad (6)\end{aligned}$$

By (5), (6) and (G_3), we get

$$\begin{aligned}\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x &\le C\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\\ &\quad + C\Big(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_R}G(|\nabla u|)\mathrm{d}x\Big)^{\frac12}\\ &\le C\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\\ &\quad + C\Big(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_R}G(|\nabla u-\nabla h|)\mathrm{d}x\Big)^{\frac12}\\ &\quad + C\Big(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_R}G(|\nabla h|)\mathrm{d}x\Big)^{\frac12}. \qquad (7)\end{aligned}$$

One may obtain (4) by (7) and Lemma 2.2. □

Proof of Theorem 1.2. Let $B_R=B_R(x_0)$ for some $R\le R_0$, where R_0 will be chosen later. Without loss of generality, assume that $B_r\subset B_R\subset\Omega$, and B_r and B_R have the same centre. Let h be a G-harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\operatorname{div}\frac{g(|\nabla h|)}{|\nabla h|}\nabla h=0 \text{ in } B_R \quad\text{and}\quad h-u\in W_0^{1,G}(B_R).$$

By Lemma 2.3 and Lemma 3.1 we have

$$\begin{aligned}&\int_{B_r} G(|\nabla u-(\nabla u)_r|)\mathrm{d}x\\ &\le C\Big(\frac{r}{R}\Big)^{n+\sigma}\int_{B_R}G(|\nabla u-(\nabla u)_R|)\mathrm{d}x + C\int_{B_R}G(|\nabla u-\nabla h|)\mathrm{d}x\\ &\le C\Big(\frac{r}{R}\Big)^{n+\sigma}\int_{B_R}G(|\nabla u-(\nabla u)_R|)\mathrm{d}x + C\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\\ &\quad + CR^{\frac{\lambda}{2}}\Big(\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\Big)^{\frac12}, \qquad (8)\end{aligned}$$

where λ is an arbitrary constant in $(0, n)$.

We need to estimate $\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x$ in (8). Note that minimality of u implies

$$\int_{B_R}(G(|\nabla u|) - G(|\nabla h|))\mathrm{d}x \leq \int_{B_R}(F_\gamma(h) - F_\gamma(u))\mathrm{d}x + \int_{B_R} f(h-u)\mathrm{d}x. \quad (9)$$

Firstly we restrict ourselves to $\gamma \in (0,1)$. Using (4.18) on [4, page 17-18], we get

$$\begin{aligned}\int_{B_R}(F_\gamma(h) - F_\gamma(u))\mathrm{d}x &= \lambda_+\int_{B_R}((h^+)^\gamma - (u^+)^\gamma)\mathrm{d}x + \lambda_-\int_{B_R}((h^-)^\gamma - (u^-)^\gamma)\mathrm{d}x \\ &\leq C\int_{B_R}|h-u|^\gamma\mathrm{d}x \\ &\leq C\Big(\int_{B_R}|h-u|\mathrm{d}x\Big)^\gamma |B_R|^{1-\gamma}, \end{aligned} \quad (10)$$

where $C = C(\lambda_+, \lambda_-)$ is a positive constant and in the last inequality we used Hölder inequality.

We estimate the last inequality of (10). Let κ be a positive constant lying in $(0, \frac{1+\delta}{g_0})$. If $R^{-n-\kappa}\int_{B_R}|h-u|\mathrm{d}x \leq 1$, then $\int_{B_R}|h-u|\mathrm{d}x \leq R^{n+\kappa}$. Therefore, the last inequality of (10) becomes

$$C\Big(\int_{B_R}|h-u|\mathrm{d}x\Big)^\gamma |B_R|^{1-\gamma} \leq CR^{n+\kappa\gamma}. \quad (11)$$

If $R^{-n-\kappa}\int_{B_R}|h-u|\mathrm{d}x > 1$, define $\mathscr{G}(t) = G(t) - G(1)t, t \geq 1$. Since $\mathscr{G}$ is increasing in t when $t \geq 1$ due to (g_1), it follows that

$$G\Big(R^{-n-\kappa}\int_{B_R}|h-u|\mathrm{d}x\Big) \geq G(1)\Big(R^{-n-\kappa}\int_{B_R}|h-u|\mathrm{d}x\Big). \quad (12)$$

Note that convexity of G implies that $G\big(\frac{1}{|B_R|}\int_{B_R}|h-u|\mathrm{d}x\big) \leq \frac{1}{|B_R|}\int_{B_R}G(|h-u|)\mathrm{d}x$. Therefore, by (12) and (G_2) we get

$$\begin{aligned}R^{-n}\int_{B_R}|h-u|\mathrm{d}x &\leq CR^\kappa G\Big(R^{-n-\kappa}\int_{B_R}|h-u|\mathrm{d}x\Big) \\ &\leq CR^\kappa (R^{-\kappa})^{1+g_0} G\Big(R^{-n}\int_{B_R}|h-u|\mathrm{d}x\Big) \\ &\leq CR^{-\kappa g_0}R^{-n}\int_{B_R}G(|h-u|)\mathrm{d}x \\ &\leq CR^{-\kappa g_0+(1+\delta)}R^{-n}\int_{B_R}G(|\nabla h - \nabla u|)\mathrm{d}x,\end{aligned}$$

where we used Poincaré-type inequality (Lemma 2.1) in the last inequality. Then we obtain

$$\int_{B_R} |h-u|\mathrm{d}x \le CR^{-\kappa g_0+(1+\delta)} \int_{B_R} G(|\nabla h - \nabla u|)\mathrm{d}x. \tag{13}$$

By (13) and Young inequality, the last inequality of (10) becomes

$$\begin{aligned}\Big(\int_{B_R}|h-u|\mathrm{d}x\Big)^{\gamma}|B_R|^{1-\gamma} &\le C\Big(R^{-\kappa g_0+(1+\delta)}\int_{B_R} G(|\nabla h-\nabla u|)\mathrm{d}x\Big)^{\gamma}|B_R|^{1-\gamma}\\ &= C\Big(R^{(1+\delta-\kappa g_0-\theta}\int_{B_R} G(|\nabla h-\nabla u|)\mathrm{d}x\Big)^{\gamma}R^{n(1-\gamma)+\theta\gamma}\\ &\le C(\varepsilon)R^{n+\frac{\theta\gamma}{1-\gamma}} + \varepsilon R^{1+\delta-\kappa g_0-\theta}\int_{B_R} G(|\nabla h-\nabla u|)\mathrm{d}x, \end{aligned} \tag{14}$$

where we choose $\theta \in (0, 1+\delta-\kappa g_0)$. Combining (10), (11) and (14), we obtain

$$\begin{aligned}\int_{B_R}(F_\gamma(h)-F_\gamma(u))\mathrm{d}x &\le CR^{n+\kappa\gamma} + C(\varepsilon)R^{n+\frac{\theta\gamma}{1-\gamma}}\\ &\quad + \varepsilon R^{1+\delta-\kappa g_0-\theta}\int_{B_R} G(|\nabla h-\nabla u|)\mathrm{d}x. \end{aligned} \tag{15}$$

Now we estimate the last integration of (9). Arguing as above, we have

$$\int_{B_R} f(h-u)\mathrm{d}x \le CR^{n+\kappa} + CR^{-\kappa g_0+(1+\delta)}\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x, \tag{16}$$

where the positive constant C depends on $\|f\|_{L^\infty(\Omega)}$. Then by (9), (15) and (16), it follows that

$$\begin{aligned}\int_{B_R}(G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x &\le CR^{n+\kappa\gamma} + CR^{n+\frac{\theta\gamma}{1-\gamma}}\\ &\quad + CR^{1+\delta-\kappa g_0-\theta}\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x. \end{aligned} \tag{17}$$

By Lemma 3.1 and (17) we obtain

$$\begin{aligned}\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x &\le CR^{n+\kappa\gamma} + CR^{n+\frac{\theta\gamma}{1-\gamma}} + CR^{\frac{\lambda}{2}}(R^{\frac{n+\kappa\gamma}{2}} + R^{\frac{n}{2}+\frac{\theta\gamma}{2(1-\gamma)}})\\ &\quad + CR^{1+\delta-\kappa g_0-\theta}\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x\\ &\quad + CR^{\frac{\lambda}{2}+\frac{1+\delta-\kappa g_0-\theta}{2}}\Big(\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x\Big)^{\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
&\le CR^{n+\kappa\gamma} + CR^{n+\frac{\theta\gamma}{1-\gamma}} + CR^{\frac{\lambda}{2}+\frac{n+\kappa\gamma}{2}} + R^{\frac{\lambda}{2}+\frac{n}{2}+\frac{\theta\gamma}{2(1-\gamma)}} \\
&\quad + CR^{1+\delta-\kappa g_0-\theta}\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x \\
&\quad + C(\varepsilon)R^{\lambda+(1+\delta-\kappa g_0-\theta)} + \varepsilon\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x. \quad (18)
\end{aligned}
$$

Without loss of generality, we may let $0 < R \le R_0$ satisfy $CR_0^{1+\delta-\kappa g_0-\theta} \ll 1$ in (18). Thus we get

$$
\begin{aligned}
&\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x \\
&\le C\left(R^{n+\kappa\gamma} + R^{n+\frac{\theta\gamma}{1-\gamma}} + R^{\frac{\lambda}{2}+\frac{n+\kappa\gamma}{2}} + R^{\frac{\lambda}{2}+\frac{n}{2}+\frac{\theta\gamma}{2(1-\gamma)}} + R^{\lambda+(1+\delta-\kappa g_0-\theta)}\right).
\end{aligned}
$$

Note that $\gamma > 0, \kappa > 0, \theta > 0, 1+\delta-\kappa g_0-\theta > 0$ and λ is an arbitrary number in $(0,n)$. Let λ be sufficiently close to n, then there exists a positive number α_0 satisfying $\alpha_0 > n-\lambda$ such that

$$
\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x \le CR^{n+\alpha_0}. \quad (19)
$$

Finally, we get by (17) and (19)

$$
\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))\mathrm{d}x \le CR^{n+\alpha_0}. \quad (20)
$$

Putting (20) into (8), we obtain

$$
\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + CR^{n+\alpha_0} + CR^{\frac{\lambda}{2}+\frac{n+\alpha_0}{2}}.
$$

By the choice of λ, $\frac{\lambda}{2}+\frac{n+\alpha_0}{2} > n$. We conclude that there exists $\beta_0 > 0$ such that

$$
\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + CR^{n+\beta_0}.
$$

In view of Lemma 2.4, we conclude that there is a constant $\beta \in (0,1)$ such that

$$
\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \le Cr^{n+\beta}. \quad (21)
$$

Now we claim that there is a constant $\alpha \in (0,1)$ such that

$$
\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x \le Cr^{n+\alpha}, \quad (22)
$$

which and Campanato's embedding Theorem give the desired Hölder continuity of the gradient of u.

Indeed, convexity of G and (21) implies that

$$G\left(\frac{1}{|B_r|}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\right)\le\frac{1}{|B_r|}\int_{B_r}G(|\nabla u-(\nabla u)_r|)\mathrm{d}x\le Cr^{\beta}. \tag{23}$$

Arguing as estimation of the last integration of (10), let τ be a positive constant lying in $(0,\frac{\beta}{g_0})$. If $r^{-n-\tau}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\le 1$, then $\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\le r^{n+\tau}$. Therefore, (22) holds. If $r^{-n-\tau}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x>1$, then we find

$$\begin{aligned}G\left(r^{-n-\tau}\int_{B_r}|\nabla u-(\nabla h)_r|\mathrm{d}x\right)&\ge G(1)\cdot r^{-n-\tau}\int_{B_r}|\nabla u-(\nabla h)_r|\mathrm{d}x\\&\ge\frac{g(1)}{1+g_0}\cdot r^{-n-\tau}\int_{B_r}|\nabla u-(\nabla h)_r|\mathrm{d}x,\end{aligned}$$

which and (23) imply that

$$\begin{aligned}r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x&\le Cr^{\tau}G(r^{-n-\tau}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x)\\&\le Cr^{\tau}(r^{-\tau})^{1+g_0}G\left(r^{-n}\int_{B_r}|\nabla u-(\nabla u)_r|\mathrm{d}x\right)\\&\le Cr^{\beta-\tau g_0}.\end{aligned}$$

Then (22) holds with $\alpha=\beta-\tau g_0$.

As for $\gamma=1$, it suffices to note (13) and (16) still hold. One may deal with $R^{-\kappa g_0+(1+\delta)}\int_{B_R}G(|\nabla u-\nabla h|)\mathrm{d}x$ as in (14) and (18), then one may get (22). Thus the proof of Theorem 1.2 is completed for $0<\gamma\le 1$. □

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