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Stability with Initial Time Difference of Caputo Fractional Differential Equations by Lyapunov Functions

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Abstract. The stability with initial data difference for nonlinear nonautonomous Caputo fractional differential equation is introduced. This type of stability generalizes the concept of stability in the literature and it enables us to compare the behavior of two solutions when both the initial times and initial values are different. Our theory is based on a new definition of the derivative of a Lyapunov like function along the given fractional equation. Comparison results for scalar fractional differential equations are presented and sufficient conditions for stability, uniform stability and asymptotic stability with initial time difference are obtained.

Keywords. Stability, Caputo derivative, Lyapunov functions, fractional differential equations

Mathematics Subject Classification (2010). Primary 34A08, secondary 34D20, 34A34

1. Introduction

One of the main areas in the qualitative theory of differential equations is stability of solutions. The analysis of stability of fractional differential equations is more complicated than classical differential equations, since fractional derivatives are nonlocal and have weakly singular kernels. Recently, Li and Zhang [19] presented an overview on stability results of fractional differential equations. One of the approaches used to study the stability of non-linear systems is the Lyapunov second method which provides a way to analyze stability without explicitly solving the differential equation. The extension of the Lyapunov second

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method to fractional differential systems was discussed in a number of papers. For example, the Mittag-Leffler stability and the fractional Lyapunov second method were discussed in [20, 21], and a sufficient stability condition for nonlinear FrDEs is derived in [9].

In this paper we introduce and study a generalization of the classical concept of stability which involve a change in both the initial time and the initial values. This type of stability allows us to investigate the behavior of two solutions when both the initial times and initial values are different.

Recently, some types of stability with initial time difference were studied for

- ordinary differential equations ([15, 22, 22, 25, 25–27, 30, 31]);
- fractional differential equations ([7, 29]);
- functional differential equations ([1, 12]).

In studying stability for fractional differential equations, there are several approaches in the literature, one of which is the Lyapunov approach. As is noted in [28] there are several difficulties encountered when one applies the Lyapunov technique to fractional differential equations. Results on stability for nonlinear fractional differential equations in the literature via Lyapunov functions could be divided into two main groups:

a. continuously differentiable Lyapunov functions (see, for example, the papers [4, 6, 9, 21, 32]);
b. continuous Lyapunov functions (see, for example, the papers [10, 16, 17]).

In this paper we define in an appropriate way the Caputo fractional Dini fractional derivative with initial time difference of Lyapunov functions. With appropriate examples we show the natural relationship between the defined derivative and Caputo fractional derivative used in the studied equations. Several sufficient conditions for stability with initial data difference for nonlinear fractional differential equations based on Lyapunov's functions and comparison results for a nonlinear scalar fractional differential equation with a parameter are obtained. Stability, uniform stability and asymptotic uniform stability with initial time difference are studied. Some examples are given to illustrate the results.

2. Notes on fractional calculus

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [16, 23, 24] and there are several definitions of fractional derivatives and fractional integrals. In engineering, the fractional order q is often less than 1, so we restrict our attention to $q \in (0, 1)$.

1. The Riemann-Liouville (RL) fractional derivative of order $q \in (0, 1)$

of $m(t)$ is given by (see, for example, [8, 1.4.1.1], or [23])

$$ {}_{t_0}D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} m(s) ds, \quad t \geq t_0. $$

where $\Gamma(.)$ denotes the Gamma function.

2. The Caputo (C) fractional derivative of order $q \in (0,1)$ is defined by (see, for example, [8, 1.4.1.3])

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \geq t_0. \tag{1} $$

The Caputo and Riemann-Liouville formulations coincide when $m(t_0) = 0$. The properties of the Caputo derivative are quite similar to those of ordinary derivatives. Also, the initial conditions of fractional differential equations with the Caputo derivative has a clear physical meaning and as a result the Caputo derivative is usually used in real applications.

3. The Grunwald-Letnikov (GL) fractional derivative is given by (see, for example, [8, 1.4.1.2])

$$ {}_{t_0}\tilde{D}^q m(t) = \lim_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0, $$

and the Grunwald-Letnikov fractional Dini derivative by

$$ {}_{t_0}\tilde{D}_+^q m(t) = \limsup_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0, \tag{2} $$

where $qCr = \frac{q(q-1)\cdots(q-r+1)}{r!}$, r is a natural number, q is a real number, and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

The relations between the three types of fractional derivatives are given by the equalities ${}_{t_0}^{c}D^q m(t) = {}_{t_0}\tilde{D}^q [m(t) - m(t_0)]$ and ${}_{t_0}\tilde{D}^q m(t) = {}_{t_0}D^q m(t)$. Also, according to [11, Lemma 3.4] the equality ${}_{t_0}^{c}D_t^q m(t) = {}_{t_0}D_t^q m(t) - m(t_0)\frac{(t-a)^{-q}}{\Gamma(1-q)}$ holds.

From the relation between the C fractional derivative and the GL fractional derivative using (2) we define the Caputo fractional Dini derivative as

$$ {}_{t_0}^{c}D_+^q m(t) = {}_{t_0}\tilde{D}_+^q [m(t) - m(t_0)], \tag{3} $$

i.e.

$$ {}_{t_0}^{c}D_+^q m(t) = \limsup_{h \to 0+} \frac{1}{h^q} \Big[m(t) - m(t_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} (qCr) \Big(m(t-rh) - m(t_0) \Big) \Big]. \tag{4} $$

Definition 2.1. ([10]) We say $m \in C^q([t_0,T],\mathbb{R}^n)$ if $m(t)$ is differentiable (i.e. $m'(t)$ exists), the Caputo derivative ${}_{t_0}^{c}D^q m(t)$ exists and satisfies (1) for $t \in [t_0,T]$.

Remark 2.2. If $m \in C^q([t_0,T],\mathbb{R}^n)$ then ${}_{t_0}^{c}D_+^q m(t) = {}_{t_0}^{c}D^q m(t)$.

In this paper we will use the following existence result:

Proposition 2.3 ([5, Theorem 2])**.** *Let $f \in C(\mathbb{R}_+ \times \mathbb{R}^n,\mathbb{R}^n)$ be such that*

$$\|f(t,x) - f(t,y)\| = F(t)\|x-y\|$$

for all $t \geq 0, x,y \in \mathbb{R}^n$ where $F \in C(\mathbb{R}_+,\mathbb{R}_+)$. Then the integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,x(s))}{(t-s)^{1-q}}ds, \quad t \geq 0$$

has a unique solution defined in R_+.

3. Statement of the problem

Consider the following initial value problem for the system of fractional differential equations (FrDE) with a Caputo derivative for $0<q<1$

$${}_{t_0}^{c}D^q x(t) = f(t,x(t)) \quad \text{for } t > t_0, \quad x(t_0) = x_0, \tag{5}$$

where $x, x_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{R}_+,\ f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$.

The main goal of the paper is to study fractional differential equations with initial time difference. Note any change of initial time reflects not only on the initial condition but also on the fractional equation.

Let $\tau_0 \in \mathbb{R}_+,\ \tau_0 \neq t_0$ and $y_0 \in \mathbb{R}^n$ and consider the following IVP for FrDE

$${}_{\tau_0}^{c}D^q x(t) = f(t,x(t)) \quad \text{for } t > \tau_0, \quad x(\tau_0) = y_0. \tag{6}$$

We will assume in the paper that the function $f \in C[\mathbb{R}_+ \times \mathbb{R}^n,\mathbb{R}^n]$ is such that for any initial data $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ the corresponding IVP for FrDE (5) has a solution $x(t;t_0,x_0) \in C^q([t_0,\infty),\mathbb{R}^n)$. Note some sufficient conditions for global existence of solutions of (5) are given in [5, 11, 16].

Lemma 3.1. *Let the function $x \in C^q(\mathbb{R}_+,\mathbb{R}^n)$, $a \geq 0$, be a solution of the initial value problem for FrDE*

$${}_{a}^{c}D^q x(t) = f(t,x(t)) \quad \textit{for } t > a, \quad x(a) = x_0. \tag{7}$$

Then the function $\tilde{x}(t) = x(t+\eta)$ satisfies the initial value problem for the FrDE

$${}_{b}^{c}D^q \tilde{x}(t) = f(t+\eta,\tilde{x}(t)) \quad \textit{for } t > b, \quad \tilde{x}(b) = x_0 \tag{8}$$

where $b \geq 0,\ \eta = a-b$.

Proof. The function $x(t)$ is a solution of (7) iff it satisfies the Volterra fractional integral equation ([11, Lemma 6.2])

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \Big(f(s, x(s)) \Big) ds \quad \text{for } t \geq a. \tag{9}$$

The function $\tilde{x}(t)$ satisfies the initial condition of (8), i.e. $\tilde{x}(b) = x_0$.

Change the variable in the integral of (9) with $s = \xi + \eta$ and obtain $x(t) = x_0 + \frac{1}{\Gamma(q)} \int_b^{t-\eta} (t-\eta-\xi)^{q-1} \Big(f(\xi+\eta, x(\xi+\eta)) \Big) d\xi$ for $t \geq a$ or

$$x(t+\eta) = x_0 + \frac{1}{\Gamma(q)} \int_b^t (t-\xi)^{q-1} \Big(f(\xi+\eta, x(\xi+\eta)) \Big) d\xi \quad \text{for } t+\eta \geq a.$$

Therefore,

$$\tilde{x}(t) = x(t+\eta) = x_0 + \frac{1}{\Gamma(q)} \int_b^t (t-\xi)^{q-1} \Big(f(\xi+\eta, \tilde{x}(\xi)) \Big) d\xi \quad \text{for } t \geq b,$$

which proves $\tilde{x}(t)$ is a solution of the initial value problem for FrDE (8). □

As mentioned in Remark 3.3 in [21, Remark 3.3] any equilibrium point of a FrDE could by shifted to the origin via an appropriate change of variable. From Lemma 3.1 we obtain the result for the shift of any solution of a fractional differential equation.

Corollary 3.2. *Let the function $x \in C^q(\mathbb{R}_+, \mathbb{R}^n)$ be a solution of the initial value problem for FrDE*

$${}_0^c D^q x(t) = f(t+t_0, x(t)) \quad \text{for } t > 0, \quad x(0) = x_0. \tag{10}$$

Then the function $\tilde{x}(t) = x(t-t_0)$ satisfies the FrDE

$${}_{t_0}^c D^q \tilde{x}(t) = f(t, \tilde{x}(t)) \quad \text{for } t > t_0, \quad \tilde{x}(t_0) = x_0 \tag{11}$$

where $t_0 \in \mathbb{R}_+$.

Corollary 3.3. *For any solution $x(t) = x(t; t_0, x_0)$ of (5) the function $\tilde{x}(t) = x(t+t_0)$ is a solution of the initial value problem for FrDE*

$${}_0^c D^q \tilde{x}(t) = f(t+t_0, \tilde{x}(t)) \quad \text{for } t > 0, \quad \tilde{x}(0) = x_0 \tag{12}$$

and conversely, for any solution $\tilde{x}(t) = \tilde{x}(t; 0, x_0)$ of (12) the function $x(t) = \tilde{x}(t-t_0)$ is a solution of (5).

The relation between (5) and (6) is given by the following result.

Corollary 3.4. *For any solution* $x(t) = x(t;\tau_0,y_0)$ *of* (6) *the function* $\tilde{x}(t) = x(t+\eta)$ *is a solution of IVP for FrDE*

$$ {}_{t_0}^{c}D^q\tilde{x}(t) = f(t+\eta,\tilde{x}(t)) \quad \text{for } t \geq t_0, \quad \tilde{x}(t_0) = y_0, \tag{13}$$

where $\eta = \tau_0 - t_0$.

Remark 3.5. Without loss of generality we will consider the case when $\tau_0 > t_0$.

In the paper we will use the following sets:

$$\begin{aligned}
\mathcal{K} &= \{a \in C[\mathbb{R}_+,\mathbb{R}_+] : a \text{ is strictly increasing and } a(0) = 0\},\\
B(H) &= \{u \in \mathbb{R} : \ |u| < H\}, \quad H = const > 0,\\
\bar{S}(\lambda) &= \{x \in \mathbb{R}^n : \|x\| \leq \lambda\},\\
S(\lambda) &= \{x \in \mathbb{R}^n : \|x\| < \lambda\}, \quad \lambda = const > 0.
\end{aligned}$$

The goal of the paper is to study the stability with initial time difference for FrDE (5), i.e. we will study the difference between a solution of IVP for FrDE (5) and a solution of (6).

Definition 3.6. Let $x^*(t) = x(t;t_0,x_0)$ be a given solution of FrDE (5). The solution $x^*(t)$ is said to be

- *stable with initial time difference* (ITD) if for every $\epsilon > 0$ there exist $\delta = \delta(\epsilon,t_0) > 0$ and $\sigma = \sigma(\epsilon,t_0) > 0$ such that for any $y_0 \in \mathbb{R}^n$ and any $\tau_0 \in \mathbb{R}_+$ the inequalities $\|y_0 - x_0\| < \delta$ and $|\tau_0 - t_0| < \sigma$ imply $\|x(t+\eta;\tau_0,y_0) - x^*(t)\| < \epsilon$ for $t \geq t_0$ where $\eta = \tau_0 - t_0$ and $x(t;\tau_0;y_0)$ is a solution of (6);
- *attractive with initial time difference* (ITD) if there exist $\beta > 0$ and $\sigma > 0$ such that for every $\epsilon > 0$ there exists $T = T(\epsilon,t_0) > 0$ such that for any $\tau_0 \in \mathbb{R}_+, y_0 \in \mathbb{R}^n$ with $\|y_0 - x_0\| < \beta$ and $|\tau_0 - t_0| < \sigma$ the inequality $\|x(t+\eta;\tau_0,y_0) - x^*(t)\| < \epsilon$ holds for $t \geq t_0 + T$ where $\eta = \tau_0 - t_0$ and $x(t;\tau_0;y_0)$ is a solution of (6);
- *asymptotically stable with initial time difference* if the solution $x^*(t)$ is stable with ITD and attractive with ITD.

Example 3.7. Consider the initial value problem for the scalar fractional differential equation

$$ {}_{0}^{c}D^q x(t) = a\ t^{a-1} \quad x(0) = x_0 \tag{14}$$

where $a > 1$ and $x_0 \neq 0$ is fixed. The solution of (14) is given by $x^*(t) = \frac{a\Gamma(a)t^{(a+q)-1}}{\Gamma(a+q)} + x_0$. Consider the solution of (14) with another initial value $y_0 \neq x_0$ but with an unchanged initial time $t_0 = 0$. The solution of (14) with $x(0) = y_0$ is given by $\tilde{x}(t) = \frac{a\Gamma(a)t^{(a+q)-1}}{\Gamma(a+q)} + y_0$. Then, $|x^*(t) - \tilde{x}(t)| = |x_0 - y_0|$. Therefore, the solution $x^*(t)$ is stable.

Now, we consider different initial data, i.e. different initial values $y_0 \neq x_0$ and initial time $\tau_0 \neq 0$. Let $y(t)$ be a solution of the corresponding IVP for the scalar FrDE ${}_{\tau_0}^{c}D^q x(t) = a\, t^{a-1}, x(\tau_0) = y_0$. According to Lemma 3.1 the function $\tilde{y}(t) = y(t+\tau_0)$, satisfies the IVP for FrDE ${}_0^c D^q \tilde{y}(t) = a(t+\tau_0)^{a-1},\ \tilde{y}(0) = y_0$. Consider the difference of both solutions $z(t) = \tilde{y}(t) - x^*(t) = y(t+\tau_0) - x^*(t)$. The function $z(t)$ is a solution of the FrDE ${}_0^c D^q z(t) = a\big((t+\tau_0)^{a-1} - t^{a-1}\big)$ with $z(0) = y_0 - x_0$. This FrDE has no zero solution. If $a = 2$ then ${}_0^c D^q z(t) = a\tau_0$, $z(0) = y_0 - x_0$ and according to [11, Lemma 6.2],

$$z(t) = y_0 - x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} 2\tau_0 ds = y_0 - x_0 + \frac{2\tau_0 t^q}{q\Gamma(q)} \quad \text{for } t \geq 0. \tag{15}$$

Now (15) shows the solution $x^*(t)$ is not stable with ITD, i.e. the change in the initial time does not save the stability property of the solution.

Definition 3.8. The system of FrDE (5) is said to be

- *uniformly stable with initial time difference* (ITD) if for every $\epsilon > 0$ there exist $\delta = \delta(\epsilon) > 0$ and $\sigma = \sigma(\epsilon) > 0$ such that for any $x_0, y_0 \in \mathbb{R}^n$ and any $t_0, \tau_0 \in \mathbb{R}_+$ the inequalities $\|y_0 - x_0\| < \delta$ and $|\tau_0 - t_0| < \sigma$ imply $\|x(t+\eta) - x^*(t)\| < \epsilon$ for $t \geq t_0$ where $\eta = \tau_0 - t_0$ and $x^*(t) = x(t; t_0, x_0)$ and $x(t) = x(t; \tau_0, y_0)$ are solutions of initial value problems for FrDE (5) and (6) correspondingly;
- *uniformly attractive with initial time difference* (ITD) if there exists $\beta > 0$ and σ such that for every $\epsilon > 0$ there exist $T = T(\epsilon) > 0$ such that for any $t_0, \tau_0 \in \mathbb{R}_+,\ x_0, y_0 \in \mathbb{R}^n$ with $\|y_0 - x_0\| < \beta$ and $|\tau_0 - t_0| < \sigma$ the inequality $\|x(t+\eta; \tau_0, y_0) - x^*(t)\| < \epsilon$ holds for $t \geq t_0 + T$ where $\eta = \tau_0 - t_0$ and $x^*(t) = x(t; t_0, x_0)$ and $x(t) = x(t; \tau_0, y_0)$ are solutions of initial value problems for FrDE (5) and (6) correspondingly;
- *uniformly asymptotically stable with initial time difference* if FrDE (5) is uniformly stable with ITD and uniformly attractive with ITD.

Remark 3.9. The stability with initial time difference is important only in the case of nonautonomous fractional differential equations.

Consider IVP for FrDE (5) in the autonomous case, i.e. $f(t,x) \equiv f(x)$ with a given solution $x^*(t) = x(t; t_0, x_0)$. Define the function $\tilde{f} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ by $\tilde{f}(t,z) = f(z + x^*(t)) - f(x^*(t))$. Then $\tilde{f}(t,0) \equiv 0$. To study the stability with ITD of $x^*(t)$ we consider the difference $z(t) = x(t+\eta) - x^*(t)$, where $x(t) = x(t; \tau_0, y_0)$, $\eta = \tau_0 - t_0$, $t_0 \neq \tau_0$, is a solution of IVP for FrDE (6). The function $z(t)$ is a solution of FrDE ${}_{t_0}^c D^q z(t) = {}_{t_0}^c D^q x(t+\eta) - {}_{t_0}^c D^q x^*(t) = f(x(t+\eta)) - f(x^*(t)) \equiv \tilde{f}(t, z(t))$ which has a zero solution. Therefore, studying the stability with ITD is reduced to studying the stability of the zero solution of the corresponding FrDE (see Example 3.10).

The situation is not the same for nonautonomous FrDE. Similarly, define the function $\tilde{f} : \mathbb{R}_+ \times \mathbb{R}^n \times B(H) \to \mathbb{R}^n$ by $\tilde{f}(t, z, \eta) = f(t+\eta, z+x^*(t)) - f(t, x^*(t))$ where $x^*(t) = x(t; t_0, x_0)$ is a given solution of (5). In the general case $\tau_0 \neq t_0$ note $\tilde{f}(t, 0, \eta) \neq 0$ holds. Consider the difference $z(t) = x(t+\eta) - x^*(t)$, where $x(t) = x(t; \tau_0, y_0)$, $t_0 \neq \tau_0$, is a solution of IVP for FrDE (6). The function $z(t)$ is a solution of FrDE ${}_{t_0}^{c}D^q z(t) = {}_{t_0}^{c}D^q x(t+\eta) - {}_{t_0}^{c}D^q x^*(t) = f(t+\eta, x(t+\eta)) - f(t, x^*(t)) = \tilde{f}(t, z, \eta)$ which has not a zero solution (in the general case). Therefore, studying the stability with ITD could not be reduced to studying the stability of the zero solution of a corresponding FrDE (see Example 3.7).

Example 3.10. Consider the IVP for the autonomous FrDE

$$ {}_{0}^{c}D^q x(t) = x(t) + 1 \quad \text{for } t > 0, \quad x(0) = 0. \tag{16} $$

The solution of (16) is $x^*(t) = t^q E_{q,q+1}(t^q)$ (see, [13, Example 4]), where the Mittag-Leffler function is defined by $E_{q,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+\beta)}$, $q > 0$, $\beta > 0$.

Now, choose $\tau_0 > 0$ and $y_0 \neq 0$ and let $x(t) = x(t; \tau_0, y_0)$ be the solution of FrDE ${}_{\tau_0}^{c}D^q x(t) = x(t) + 1$ for $t > \tau_0$, with $x(\tau_0) = y_0$. According to Lemma 3.1 for $a = \tau_0, b = 0, \eta = \tau_0$ the function $\tilde{x}(t) = x(t + \tau_0)$ is a solution of ${}_{0}^{c}D^q x(t) = x(t) + 1, t > 0$ with $x(0) = y_0$. Consider the difference $z(t) = x(t + \tau_0) - x^*(t)$ which satisfies the IVP for FrDE ${}_{0}^{c}D^q z(t) = z(t)$, $z(0) = y_0$. This FrDE has a zero solution, so, studying the stability with initial time difference of $x(t)$ is reduced to studying the stability of the zero solution of an appropriate FrDE.

We will use comparison results for the IVP for scalar FrDE with a parameter

$$ {}_{t_0}^{c}D^q u = g\,(t, u, \eta) \quad \text{for } t > t_0, \quad u(t_0) = u_0 \tag{17} $$

where $u \in \mathbb{R}$, $g : [t_0, \infty) \times \mathbb{R} \times B(H) \to \mathbb{R}$, $g(t, 0, 0) \equiv 0$. We will assume in the paper that the function $g : [t_0, \infty) \times \mathbb{R} \times B(H) \to \mathbb{R}$ is such that for any initial data $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ and any value of the parameter $\eta^* \in B(H)$ the scalar FrDE (17) with $u(t_0) = u_0$ and $\eta = \eta^*$ has a solution $u(t; t_0, u_0, \eta^*) \in C^q([t_0, \infty), \mathbb{R})$ (we assume the existence of a maximal solution in Section 5). Also, we assume that for any compact subset $I \subset [t_0, \infty)$ and any fixed value $\eta^* \in B(H)$ of the parameter there exists a small enough number $L = L(\eta^*) > 0$ such that the corresponding FrDE ${}_{t_0}^{c}D^q u = g\,(t, u, \eta^*) + \gamma$ with $\gamma \in (0, L]$ has a solution $u(t; t_0, u_0, \eta^*) \in C^q(I, \mathbb{R})$ where $(t_0, u_0) \in I \times \mathbb{R}$.

In the definition below we assume $u(t; t_0, u_0, \eta)$ is any solution of (17) with $u(t_0) = u_0$ and a given $\eta \in B(H)$.

Definition 3.11. The zero solution of the scalar FrDE with a parameter (17) is said to be

(i) *stable w.r.t. a parameter* if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta = \delta(t_0, \epsilon) > 0$ and $\sigma = \sigma(t_0, \epsilon) > 0$ such that for any $u_0 \in \mathbb{R} : \ |u_0| < \delta$ and any $\eta \in B(\sigma)$ the inequality $|u(t; t_0, u_0, \eta)| < \epsilon$ holds for $t \geq t_0$;

(ii) *uniformly stable w.r.t. a parameter* if above δ and σ don't depend on t_0;

(iii) *attractive w.r.t. a parameter* if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exist $\delta = \delta(t_0, \epsilon) > 0$, $\sigma = \sigma(t_0, \epsilon) > 0$ and $T = T(t_0, \epsilon) > 0$ such that for any $u_0 \in \mathbb{R} : \ |u_0| < \delta$ and any $\eta \in B(\sigma)$ the inequality $|u(t; t_0, u_0, \eta)| < \epsilon$ holds for $t \geq t_0 + T$.

Example 3.12. Consider the IVP for the scalar FrDE (17) with $g(t, u, \eta) = -au + C\eta$:

$$ {}_{t_0}^{c}D^q u(t) = -Au + C\eta, \quad t \geq t_0, \quad u(t_0) = u_0 \tag{18} $$

where $A > 0, C > 0,\ \eta$ is a parameter.

From [14, Example 4.9] and [14, Equation (4.1.66)] with $\alpha = q,\ a = t_0$, $\lambda = -A,\ f(t) = C\eta,\ b = u_0$ the solution of (18) is given by

$$ u(t; t_0, u_0, \eta) = u_0 E_{q,1}(-A(t-t_0)^q) + \int_{t_0}^{t} (t-\tau)^{q-1} E_{q,q}(-A(t-\tau)^q) C\eta d\tau. \tag{19} $$

From [23, Equation (1.99)] with $\lambda = -A, \alpha = q, \beta = q, z = t - t_0$ we obtain

$$ \left| \int_0^{t-t_0} (s)^{q-1} E_{q,q}(-A(s)^q) ds \right| = (t-t_0)^q E_{q,q+1}(-A(t-t_0)^q). $$

From [11, Theorem 4.2] with $n_1 = q, n_2 = q+1, x = -A(t-t_0)^q$ we get the equality $(t-t_0)^q E_{q,q+1}((t-t_0)^q) = \frac{1}{A}\Big(\frac{1}{\Gamma(q)} - E_{q,q}(-A(t-t_0)^q)\Big)$.

Then $\left|\int_{t_0}^{t}(t-\tau)^{q-1}E_{q,q}(-A(t-\tau)^q)d\tau\right| = \left|\int_0^{t-t_0} s^{q-1}E_{q,q}(-As^q)ds\right| \leq \frac{1}{A}(1 - E_{q,q}(-A(t-t_0)^q)) \leq \frac{1}{A}$, so

$$ |u(t; t_0, u_0, \eta)| \leq |u_0| + \frac{1}{A} C |\eta| . \tag{20} $$

The inequality (20) proves the zero solution of (18) is uniformly stable w.r.t. a parameter.

In this paper we will study the connection between the stability with initial time difference of the system of FrDE (5) and the stability w.r.t. a parameter of the zero solution of the scalar FrDE (17).

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability with ITD for fractional differential equations.

Definition 3.13. Let $I \subset \mathbb{R}_+$ and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$. We will say that the function $V(t,x): I \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda(I,\Delta)$ if $V(t,x) \in C(I \times \Delta, \mathbb{R}_+)$ is locally Lipschitzian with respect to its second argument.

For fractional differential equations some authors used the Caputo fractional derivative of Lyapunov functions of the unknown solutions (for example, [4, 6, 9, 21, 32]). This approach requires the function to be smooth enough (at least continuously differentiable) and also some conditions involved are quite restrictive. Other authors use the Lakshmikantham et al. derivative of Lyapunov functions ([16, 17]) and this definition requires only the continuity of the Lyapunov function. However it can be quite restrictive (see Example 3.15).

In [2, 3] the derivative of Lyapunov functions is introduced based on the Caputo fractional Dini derivative of a function $m(t)$ given by (4). It is used to study stabiity and strict stability of Caputo fractional differential equations. Now we generalize this definition with initial time difference. We define the *Caputo fractional Dini derivative with initial time difference (ITD)* of the function $V(t,x) \in \Lambda(I,\Delta)$ along trajectories of solutions of the system FrDE (5) as follows:

$$\begin{aligned} &{}_{t_0}^{c}D_{(5)}^{q} V(t,x,y,\eta,x_0,y_0) \\ &= \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ V(t,y-x) - V(t_0,y_0-x_0) - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr \\ &\quad \times \Big[V\Big(t-rh, y-x-h^q\big(f(t+\eta,y)-f(t,x)\big)\Big) - V(t_0,y_0-x_0)\Big]\Big\}, \end{aligned} \tag{21}$$

where $t, t_0 \in I$, $y-x, y_0-x_0 \in \Delta$, and there exists $h_1 > 0$ such that $t-h \in I$, $y-x-h^q\big(f(t+\eta,y)-f(t,x)\big) \in \Delta$ for $0 < h \le h_1$ and $\eta \in B(H)$.

Lemma 3.14. *Let $n=1$, $x(t) = x(t;t_0,x_0)$ be a solution of (5) and $\tilde{x}(t) = x(t;\tau_0,y_0)$ be a solution of (6) and $V(x) = x^2$. Then ${}_{t_0}^{c}D^q V(\tilde{x}(t+\eta)-x(t)) = {}_{t_0}^{c}D_{(5)}^{q} V(x(t),\tilde{x}(t+\eta),\eta,x_0,y_0)$ where $\eta = \tau_0 - t_0$.*

Proof. Let $y(t) = \tilde{x}(t+\eta)$. Apply (3) and definition (21) and we obtain

$$\begin{aligned} &{}_{t_0}^{c}D^q \big(\tilde{x}(t+\eta) - x(t)\big)^2 \\ &= {}_{t_0}\tilde{D}_+^q \big[(\tilde{x}(t+\eta)-x(t))^2 - (y_0-x_0)^2\big] \\ &= -\limsup_{h\to 0^+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r qCr (y_0-x_0)^2 \\ &\quad + \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r qCr \Big[\Big(y(t)-x(t)-h^q\big[f(t+\eta,y(t))-f(t,x(t))\big]\Big)^2\Big]\Big\} \end{aligned}$$

$$
\begin{aligned}
&+\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr \\
&\quad\times\Big[\Big(y(t-rh)-x(t-rh)\Big)^2-\Big(y(t)-x(t)-h^q\Big[f(t+\eta,y(t))-f(t,x(t))\Big]\Big)^2\Big] \\
&= {}^{c}_{t_0}D^q_{(5)}V(x(t),y(t),\eta,x_0,y_0) \\
&\quad+\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\Big[rh\big(y'(\xi)-x'(\varsigma)\big)+h^q\big(f(t+\eta,y(t))-f(t,x(t)\big)\Big] \\
&\quad\times\Big[\big(y(t-rh)+y(t)\big)-\big(x(t-rh)+x(t)\big)-h^q\big(f(t+\eta,y(t)-f(t,x(t))\big)\Big]\Big\},
\end{aligned}
\tag{22}
$$

where $\xi,\varsigma\in(t-rh,t)$.

Use the equality $\limsup_{h\to 0^+}h^\alpha\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr=0$ for $\alpha>0$, the limit $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^r qCr=0$, where N is a natural number, and $\lim_{h\to 0^+}[\frac{t-t_0}{h}]=\infty$ and we obtain

$$
\begin{aligned}
&\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\Big[rh\big(y'(\xi)-x'(\varsigma)\big)+h^q\big(f(t+\eta,y(t))-f(t,x(t)\big)\Big] \\
&\quad\times\Big[\big(y(t-rh)+y(t)\big)-\big(x(t-rh)+x(t)\big)-h^q\big(f(t+\eta,y(t)-f(t,x(t))\big)\Big] \\
&=\limsup_{h\to 0^+}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\big(f(t+\eta,y(t))-f(t,x(t)\big) \\
&\quad\times\Big[\big(y(tc-rh)+y(t)\big)-\big(x(t-rh)+x(t)\big)\Big] \\
&=\limsup_{h\to 0^+}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\big(f(t+\eta,y(t))-f(t,x(t)\big)\big(2y(t)-2x(t)\big) \\
&=0.
\end{aligned}
\tag{23}
$$

From (22) and (23), the result follows. □

A generalization of the derivative with initial time difference of a Lyapunov function is defined and used in [10, 16, 17] :

$$
\begin{aligned}
&D^q_+V(t,y-x,\eta) \\
&=\limsup_{h\to 0+}\frac{1}{h^q}\Big(V(t,y-x)-V(t-h,y-x-h^q\big(f(t+\eta,y)-f(t,x)\big)\Big).
\end{aligned}
\tag{24}
$$

Note formula (24) is simpler than that introduced in formula (21) but it gives us quite different results than those in the literature.

Example 3.15. Let $V(t,x)=\frac{x^2}{(t+1)^2}$ for $x\in\mathbb{R},\ \ x\in B(A),\ A>0$, and consider the case $t_0\geq 0, x_0\neq y_0, \eta=\tau_0>0$.

Apply formula (24) to obtain the derivative of V, namely

$$\begin{aligned}
&D^q_+V(t,y-x,\eta)=\\
&=\limsup_{h\to 0}\Bigg\{\frac{\big(-(y-x)h^{1-q}+\big(f(t+\eta,y)-f(t,x)\big)(t+1)\big)}{(t+1)^2(t+1-h)^2}\\
&\quad\times\frac{\big(2(y-x)(t+1)-(y-x)h-h^q\big(f(t+\eta,y)-f(t,x)\big)(t+1)\big)}{(t+1)^2(t+1-h)^2}\Bigg\}\\
&=\frac{2(y-x)\big(f(t+\eta,y)-f(t,x)\big)}{(t+1)^2}.
\end{aligned}\tag{25}$$

Use formula (21) to obtain the derivative of V, namely

$$\begin{aligned}
&{}^{c}_{t_0}D^q_{(5)}V(t,x,y,\eta,x_0,y_0)\\
&=-\frac{(y_0-x_0)^2}{(1+t_0)^2}\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\\
&\quad+\frac{1}{(1t)^2}\limsup_{h\to 0^+}\frac{(y-x)^2-\big(y-x-h^q\big(f(t+\eta,y)-f(t,x)\big)\big)^2}{h^q}\\
&\quad+\limsup_{h\to 0^+}\Big(y-x-h^q\big(f(t+\eta,y)-f(t,x)\big)^2\Big)\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\frac{1}{(t+1-rh)^2}.
\end{aligned}\tag{26}$$

Apply the equality $\lim_{N\to\infty}\sum_{r=0}^{N}(-1)^r qCr=0$ for a natural number N, $\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\frac{1}{(t+1-rh)^2}={}_{t_0}\tilde{D}^q_+\big(\frac{1}{(t+1)^2}\big)={}_{t_0}D^q\big(\frac{1}{(t+1)^2}\big)$,

$$\begin{aligned}
&\limsup_{h\to 0^+}\Big(y-x-h^q\big(f(t+\eta,y)-f(t,x)\big)^2\Big)\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\frac{1}{(t+1-rh)^2}\\
&=\Big(\limsup_{h\to 0^+}(y-x-h^q\big(f(t+\eta,y)-f(t,x)\big)^2\Big)\Bigg(\limsup_{h\to 0^+}\frac{1}{h^q}\sum_{r=0}^{[\frac{t-t_0}{h}]}(-1)^r qCr\frac{1}{(t+1-rh)^2}\Bigg)\\
&=(y-x)^2\,{}_{t_0}D^q\bigg(\frac{1}{(t+1)^2}\bigg)
\end{aligned}$$

and $\limsup_{h\to 0^+}\frac{(y-x)^2-(y-x-h^q(f(t+\eta,y)-f(t,x)))^2}{h^q}=2(y-x)\big(f(t+\eta,y)-f(t,x)\big)$, from (26) we obtain the following

$$\begin{aligned}
&{}_{t_0}^{\;c}D^q_{(5)}V(t,x,y,\eta,x_0,x_0)\\
&= \frac{2(y-x)\big(f(t+\eta,y)-f(t,x)\big)}{(t+1)^2}\\
&\quad + (y-x)^2\,{}_{t_0}D^q\Big(\frac{1}{(t+1)^2}\Big) - (y_0-x_0)^2\,{}_{t_0}D^q\Big(\frac{1}{(t_0+1)^2}\Big)\\
&= \frac{2(y-x)\big(f(t+\eta,y)-f(t,x)\big)}{(t+1)^2} + {}_{t_0}D^q\Big(\frac{(y-x)^2}{(t+1)^2} - \frac{(y_0-x_0)^2}{(t_0+1)^2}\Big).
\end{aligned} \tag{27}$$

Now consider the well known case $q=1$. The Dini derivative of the Lyapunov function with respect to the ordinary differential equation $x'=f(t,x)$ is

$$\begin{aligned}
&DV(t,y-x,\eta)\\
&= \limsup_{h\to 0+}\frac{1}{h}\Big(V(t,y-x) - V(t-h,y-x-h(f(t+\eta,y)-f(t,x))\Big)\\
&= \frac{2(y-x)\big(f(t+\eta,y)-f(t,x)\big)}{(t+1)^2} - 2\frac{(y-x)^2}{(t+1)^3}\\
&= \frac{2(y-x)\big(f(t+\eta,y)-f(t,x)\big)}{(t+1)^2} + \frac{d}{dt}\Big(\frac{(y-x)^2}{(t+1)^2}\Big).
\end{aligned} \tag{28}$$

In formula (25), we obtain only the first term of $DV(t,x)$ in (28). Formulas (27) and (28) are quite similar.

4. Fractional differential inequalities with initial time difference and comparison results for scalar FrDE

Lemma 4.1 ([3, Lemma 1]). *Let $m\in C([t_0,T],\mathbb{R})$ and suppose that there exists $t^*\in(t_0,T]$, such that $m(t^*)=0$, $m(t)<0$ for $t_0\le t<t^*$ and the Caputo fractional Dini derivative ${}_{t_0}^{\;c}D^q_+m(t^*)$ exists. Then the inequality ${}_{t_0}^{\;c}D^q_+m(t^*)>0$ holds.*

Now we present a comparison result applying the Caputo fractional Dini derivative of Lyapunov functions.

Lemma 4.2 (Comparison result). *Assume the following conditions are satisfied:*

1. *The function $x^*(t)=x(t;t_0,x_0)\in C^q([t_0,t_0+T],\mathbb{R}^n)$ is a solution of* (5) *and $\tilde{x}(t)=x(t;\tau_0,y_0)\in C^q([\tau_0,\tau_0+T],\mathbb{R}^n)$ is a solution of* (6) *such that $\tilde{x}(t+\eta^*)-x^*(t)\in\Delta$ for $t\in[t_0,t_0+T]$ where $\Delta\subset\mathbb{R}^n$, $0\in\Delta$, $t_0,\tau_0\in\mathbb{R}_+:\ \eta^*=\tau_0-t_0\in B(H)$, $H,T>0$ are given constants, $x_0,y_0\in\mathbb{R}^n:\ y_0-x_0\in\Delta$.*
2. *The function $g\in C([t_0,t_0+T]\times\mathbb{R}\times B(H),\mathbb{R})$.*

3. *The function* $V \in \Lambda([t_0, t_0+T], \Delta)$ *and for* $t \in [t_0, t_0+T]$ *the inequality*

$$ {}_{t_0}^{c}D^q_{(5)} V(t, x^*(t), \tilde{x}(t+\eta^*), \eta^*, x_0, y_0) \le g(t, V(t, \tilde{x}(t+\eta^*) - x^*(t)), \eta^*) $$

holds.

4. *The function* $u^*(t) = u(t; t_0, u_0, \eta^*)$, $u^* \in C^q([t_0, t_0+T], \mathbb{R})$, *is the maximal solution of the initial value problem* (17) *for initial data* (t_0, u_0) *and parameter* $\eta = \eta^*$.
Then the inequality $V(t_0, y_0 - x_0) \le u_0$ *implies* $V(t, \tilde{x}(t+\eta) - x^*(t)) \le u^*(t)$ *for* $t \in [t_0, t_0+T]$.

Proof. Let $\gamma > 0$ be an arbitrary enough small number (i.e. $\gamma \le L = L(\eta^*)$ as described after (17)) and consider the initial value problem for the scalar FrDE

$$ {}_{t_0}^{c}D^q u = g(t, u, \eta^*) + \gamma \quad \text{for } t \in [t_0, t_0+T], \quad u(t_0) = u_0 + \gamma. \tag{29} $$

The function $u(t;\gamma) = u(t; t_0, u_0 + \gamma, \eta^*)$ is a solution of the IVP for the scalar FrDE (29) iff it satisfies the Volterra fractional integral equation [11, Lemma 6.2]

$$ u(t;\gamma) = u_0 + \gamma + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \big(g(s, u(s;\gamma), \eta^*) + \gamma \big) ds \quad \text{for } t \in [t_0, t_0+T]. \tag{30} $$

Let the function $m(t) \in C([t_0, t_0+T], \mathbb{R}_+)$ be $m(t) = V(t, \tilde{x}(t+\eta^*) - x^*(t))$. We now prove that

$$ m(t) < u(t;\gamma) \quad \text{for } t \in [t_0, t_0+T] \quad \text{and } \gamma \in (0, \min\{H, L\}). \tag{31} $$

Let $\gamma^* \le \min\{H, L\}$ be a fixed positive value. Note that the inequality (31) holds for $t = t_0$ and $\gamma = \gamma^*$ since $m(t_0) = V(t_0, x_0) \le u_0 < u(t_0; \gamma^*)$. Assume that inequality (31) is not true for $t \in (t_0, t_0+T]$ and $\gamma = \gamma^*$. Then there exists a point t^* such that $m(t^*) = u(t^*; \gamma^*)$, $m(t) < u(t; \gamma^*)$ for $t \in [t_0, t^*)$. Now Lemma 4.1 (applied to $m(t) - u(t; \gamma^*)$) yields ${}_{t_0}^{c}D^q_+ (m(t^*) - u(t^*; \gamma^*)) > 0$, i.e.

$$ {}_{t_0}^{c}D^q_+ m(t^*) > g(t^*, u(t^*; \gamma^*), \eta^*) + \gamma^* > g(t^*, m(t^*), \eta^*). \tag{32} $$

According to Corollary 3.4 the function $x^*(t+\eta)$ satisfies IVP for FrDE (13). Then from Remark 2.2, formula (4) and the IVP for FrDE (13) we obtain for $t \in (t_0, t_0+T]$ the equality

$$ \begin{aligned} &\limsup_{h \to 0+} \frac{1}{h^q} \Big[\tilde{x}(t+\eta^*) - x^*(t) - y_0 + x_0 - S(\tilde{x}(t), x^*(t), h, \eta^*) \Big] \\ &= f(t+\eta^*, \tilde{x}(t+\eta^*)) - f(t, x^*(t)) \end{aligned} $$

where

$$ S\left(\tilde{x}(t), x^*(t), h, \eta^*\right) = \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr \Big[\tilde{x}(t+\eta^*-rh) - x^*(t-rh) - y_0 + x_0 \Big]. \tag{33} $$

Therefore $S(\tilde{x}(t), x^*(t), h, \eta^*) = \tilde{x}(t+\eta^*) - x^*(t) - y_0 + x_0 - h^q\big(f(t+\eta^*, \tilde{x}(t+\eta^*)) - f(t, x^*(t))\big) - \Lambda(h^q)$ or

$$\begin{aligned} &\tilde{x}(t+\eta^*) - x^*(t) - h^q\Big(f(t+\eta^*, \tilde{x}(t+\eta^*)) - f(t, x^*(t))\Big) \\ &\quad = S\left(\tilde{x}(t), x^*(t), h, \eta^*\right) + y_0 - x_0 + \Lambda(h^q) \end{aligned} \tag{34}$$

with $\frac{\Lambda(h^q)}{h^q} \to 0$ as $h \to 0$. Then using (34) for any $t \in (t_0, t_0+T]$ we obtain

$$\begin{aligned} &{}^{c}_{t_0}D^q_+ m(t) \\ &= \Bigg\{ V(t, \tilde{x}(t+\eta^*) - x^*(t)) - V(t_0, y_0 - x_0) \\ &\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr \Big[V(t - rh, \tilde{x}(t+\eta^*) - x^*(t) \\ &\quad - h^q\Big(f(t+\eta^*, \tilde{x}(t+\eta^*)) - f(t, x^*(t))\Big) - V(t_0, y_0 - x_0) \Big] \Bigg\} \\ &\quad + \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr V\Big(t - rh, S\left(\tilde{x}(t), x^*(t), h, \eta^*\right) + y_0 - x_0 + \Lambda(h^q)\Big) \\ &\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr V\Big(t - rh, \tilde{x}(t - rh + \eta^*) - x^*(t - rh)\Big). \end{aligned} \tag{35}$$

Since V is locally Lipschitzian in its second argument with a Lipschitz constant $L > 0$ applying (33) we obtain

$$\begin{aligned} &\sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr \Big\{ V\big(t - rh, S(\tilde{x}(t), x^*(t), h, \eta^*) + y_0 - x_0 + \Lambda(h^q)\big) \\ &\quad - V\big(t - rh, \tilde{x}(t - rh + \eta^*) - x^*(t - rh)\big) \Big\} \\ &\le L \Big\| \sum_{r=1}^{[\frac{t-t_0}{h}]} qCr \sum_{j=1}^{[\frac{t-t_0}{h}]} (-1)^{j+1} qCj \left[\tilde{x}(t+\eta^* - jh) - x^*(t - jh) - y_0 + x_0\right] \\ &\quad - \sum_{r=1}^{[\frac{t-t_0}{h}]} qCr \big(\tilde{x}(t - rh + \eta^*) - x^*(t - rh) - y_0 + x_0\big) \Big\| + L\Lambda(h^q) \sum_{r=1}^{[\frac{t-t_0}{h}]} qCr \\ &= L \Big\| \Big(\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr \Big) \Big(\sum_{j=1}^{[\frac{t-t_0}{h}]} qCj \big(\tilde{x}(t+\eta^* - jh) - x^*(t - jh) - y_0 + x_0\big) \Big) \Big\| \\ &\quad + L\Lambda(h^q) \sum_{r=1}^{[\frac{t-t_0}{h}]} qCr. \end{aligned} \tag{36}$$

Substitute (36) in (35), divide both sides by h^q, take the limit as $h \to 0^+$ and $\sum_{r=0}^{\infty} qCrz^r = (1+z)^q$ if $|z| \le 1$, and we obtain for any $t \in (t_0, t_0+T]$ the inequality (note (4) and (21) and condition 3 of Lemma 4.2)

$$\begin{aligned}
&{}_{t_0}^{c}D_+^q m(t) \\
&\le {}_{t_0}^{c}D_{(5)}^q V(t, x^*(t), \tilde{x}(t+\eta^*), \eta^*, x_0, y_0) \\
&\quad + L \lim_{h\to 0+} \frac{\Lambda(h^q)}{h^q} \lim_{h\to 0+} \sum_{r=1}^{[\frac{t-t_0}{h}]} qCr + L \lim_{h\to 0^+} \sup \Big\| \Big(\sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr \Big) \\
&\quad \times \Big(\frac{1}{h^q} \sum_{j=1}^{[\frac{t-t_0}{h}]} qCj \big(\tilde{x}(t+\eta^* - jh) - x^*(t-jh) - y_0 + x_0 \big) \Big) \Big\| \\
&= {}_{t_0}^{c}D_{(5)}^q V(t, x^*(t), \tilde{x}(t+\eta^*), \eta^*, x_0, y_0) \\
&\le g(t, V(t, \tilde{x}(t+\eta^*) - x^*(t)), \eta^*) \\
&= g(t, m(t), \eta^*).
\end{aligned} \tag{37}$$

Now (37) with $t = t^*$ contradicts (32). Therefore (31) holds on $[t_0, t_0+T]$ for any arbitrary $\gamma \in (0, C]$, where $C = \min\{1, L, \eta^*)\}$.

We now show if $\gamma_2 < \gamma_1$ then

$$u(t;\gamma_2) < u(t;\gamma_1) \quad \text{for } t \in [t_0, t_0+T]. \tag{38}$$

Note that the inequality (38) holds for $t = t_0$. Assume that inequality (38) is not true. Then there exists a point $t^* \in (t_0, t_0+T]$ such that $u(t^*;\gamma_2) = u(t^*;\gamma_1)$ and $u(t;\gamma_2) < u(t;\gamma_1)$ for $t \in [t_0, t^*)$. Now Lemma 4.1 (applied to $u(t;\gamma_2) - u(t;\gamma_1)$) yields ${}_{t_0}^{c}D_+^q(u(t^*;\gamma_2) - u(t^*;\gamma_1))) > 0$. However

$${}^cD_+^q(u(t^*;\gamma_2) - u(t^*;\gamma_1))) = g(t^*, u(t^*;\gamma_2), \eta^*) + \gamma_2 - [g(t^*, u(t^*;\gamma_1), \eta^*) + \gamma_1] < 0,$$

a contradiction. Thus (38) is true.

Now $0 < \gamma \le C$, and (31) and (38) guarantee that the family of solutions $\{u(t;\gamma)\}, t \in [t_0, t_0+T]$ of (29) is uniformly bounded i.e there exists $K > 0$ with $|u(t,\gamma)| \le K$ for $(t,\gamma) \in [t_0, t_0+T] \times [0, C]$. Let $M = \sup\{|g(t,x)| : (t,x) \in [t_0, t_0+T] \times [-K, K]\}$. Take a decreasing sequence of positive numbers $\{\gamma_j\}_{j=0}^{\infty}$, $\gamma_0 \le C$, such that $\lim_{j\to\infty} \gamma_j = 0$ and consider the sequence of functions $u(t;\gamma_j)$. Now for $t_1, t_2 \in [t_0, t_0+T]$, $t_1 < t_2$, using the inequalities $a^q - b^q \le 2(a-b)^q$ for $a \ge b \ge 0$, $(t_1 - s)^q \le (t_2 - s)^q$ for $s \in [t_0, t_1]$ and

$$\int_{t_0}^{t_1} \big((t_2-s)^{q-1} - (t_1-s)^{q-1}\big) ds = \frac{1}{q}\big((t_2-t_0)^q - (t_1-t_0)^q - (t_2-t_1)^q\big) \le \frac{(t_2-t_1)^q}{q}$$

we get

$$
\begin{aligned}
&|u(t_2;\gamma_j) - u(t_1;\gamma_j)| \\
&\le \frac{1}{\Gamma(q)}\Big|\int_{t_0}^{t_1}\Big((t_2-s)^{q-1}-(t_1-s)^{q-1}\Big)\Big(g(s,u(s;\gamma_j)+\gamma_j\Big)ds\Big| \\
&\quad + \Big|\int_{t_1}^{t_2}((t_2-s)^{q-1})(g(s,u(s;\gamma_j))+\gamma_j)ds\Big| \\
&\le \frac{M+1}{\Gamma(q)}\Big\{\frac{(t_2-t_1)^q}{q}+\frac{(t_2-t_1)^q}{q}\Big\} \\
&= 2\frac{M+1}{q\Gamma(q)}(t_2-t_1)^q.
\end{aligned} \tag{39}
$$

Thus the family $\{u(t;\gamma_j)\}$ is equicontinuous on $[t_0, t_0+T]$. The Arzela-Ascoli Theorem guarantees that there exists a subsequence, $\{u(t;\gamma_{j_k})\}$ that is uniformly convergent in the interval $[t_0, t_0+T]$. Let $\lim_{k\to\infty} u(t;\gamma_{j_k}) = w(t)$. Take the limit in (30) as $k \to \infty$ and we see that $w(t)$ satisfies the initial value problem (17) for $t \in [t_0, t_0+T]$. Now from (31) we have $m(t) \le w(t) \le u^*(t)$ on $[t_0, t_0+T]$. □

If $g(t,x) \equiv 0$ in Lemma 4.2 we obtain the following result:

Corollary 4.3. *Assume the following conditions are fulfilled:*

1. *The condition* 1 *of Lemma* 4.2 *is satisfied.*
2. *The function* $V \in \Lambda([t_0, t_0+T], \Delta)$ *and the inequality*

$$
{}_{t_0}^{c}D^q_{(5)}V(t, x^*(t), \tilde{x}(t+\eta^*), \eta^*, x_0, y_0) \le 0 \quad \text{for } t \in [t_0, t_0+T]
$$

holds.

Then for $t \in [t_0, t_0+T]$ *the inequality* $V(t, \tilde{x}(t+\eta^*) - x^*(t)) \le V(t_0, y_0 - x_0)$ *holds.*

Proof. The proof follows from the fact that the Caputo fractional differential equation ${}^cD^q x = 0$ has a constant solution. Apply Lemma 4.2 with $u_0 = V(t_0, y_0 - x_0)$. □

The result of Lemma 4.2 is also true on the half line. The idea is to fix $T > 0$ and once again we have (30) and (31). Take the limit in (30) as $k \to \infty$ and we see that $\lim_{k\to\infty} u(t;\gamma_{j_k}))$ satisfies the initial value problem (17) for $t \in [t_0, t_0+T]$. We can do this argument for each $T < \infty$. This yields the following result.

Corollary 4.4. *Let the conditions in Lemma* 4.2 *be satisfied where the interval* $[t_0, t_0+T]$ *is replaced by* $[t_0, \infty)$. *Then the inequality* $V(t_0, y_0 - x_0) \le u_0$ *implies* $V(t, \tilde{x}(t+\eta^*) - x^*(t)) \le u^*(t)$ *for* $t \ge t_0$.

From Corollary 4.4 we have the following global result.

Corollary 4.5. *Assume the following conditions are fulfilled:*

1. *The condition* 1 *of Corollary* 4.3 *is satisfied.*
2. *The function* $V \in \Lambda([t_0, \infty), \Delta)$ *and the inequality*

$$ {}_{t_0}^{c}D_{(5)}^{q} V(t, x^*(t), \tilde{x}(t+\eta^*), \eta^*, x_0, y_0) \leq 0 \quad \text{for } t \geq t_0 $$

holds.

Then the inequality $V(t, \tilde{x}(t+\eta^*) - x^*(t)) \leq V(t_0, y_0 - x_0)$ *holds for* $t \geq t_0$.

If the derivative of the Lyapunov function is negative, the following result is true.

Lemma 4.6. *Assume the following conditions are satisfied:*

1. *The condition* 1 *of Lemma* 4.2 *is satisfied.*
2. *The function* $V \in \Lambda([t_0, t_0+T], \Delta)$ *and for* $t \in [t_0, t_0+T]$ *the inequality*

$$ {}_{t_0}^{c}D_{(5)}^{q} V(t, x^*(t), \tilde{x}(t+\eta^*), \eta^*, x_0, y_0) \leq -c(\|\tilde{x}(t+\eta^*) - x^*(t)\|) $$

holds where $c \in \mathcal{K}$.

Then for $t \in [t_0, t_0+T]$ *the inequality*

$$ V(t, \tilde{x}(t+\eta^*) - x^*(t)) \leq V(t_0, x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} c(\|\tilde{x}(s+\eta^*) - x^*(s)\|) ds \quad (40) $$

holds.

Proof. Define the function $m(t) \in C([t_0, t_0+T], \mathbb{R}_+)$ by $m(t) = V(t, \tilde{x}(t+\eta^*) - x^*(t))$ and the function $p \in C([t_0, t_0+T], \mathbb{R}_+)$ by $p(t) = c(\|\tilde{x}(t+\eta^*) - x^*(t)\|)$. As in the proof of (37) we have for $t \in [t_0, t_0+T]$

$$ {}_{t_0}^{c}D_{+}^{q} m(t) \leq {}_{t_0}^{c}D_{(5)}^{q} V(t, \tilde{x}(t+\eta^*) - x^*(t), \eta^*) \leq -c(\|\tilde{x}(t+\eta^*) - x^*(t)\|) = -p(t). \quad (41) $$

Let $\gamma > 0$ be arbitrary. Consider the following initial value problem for the scalar FrDE

$$ {}_{t_0}^{c}D^{q} u(t) = -p(t), \quad t \geq t_0, \quad u(t_0) = m(t_0) + \gamma. \quad (42) $$

The function $u(t;\gamma)$ is a solution of (42) iff it satisfies the following fractional integral equation

$$ u(t) = m(t_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} p(s) ds + \gamma. \quad (43) $$

According to Proposition 2.3 there exists an unique solution $u(t;\gamma) \in C^q([t_0, t_0+T], \mathbb{R})$ of (43).

We now prove that for any $\gamma > 0$

$$m(t) < u(t;\gamma), \quad t \in [t_0, t_0 + T]. \tag{44}$$

Assume the contrary. Therefore there exist $\gamma^* > 0$ and $t^* \in (t_0, t_0 + T]$ such that

$$m(t^*) = u(t^*;\gamma^*), \quad \text{and} \quad m(t) < u(t;\gamma^*) \text{ for } t \in [t_0, t^*).$$

From Lemma 4.1 (applied to $m(t) - u(t;\gamma^*)$) we obtain

$${}_{t_0}^{c}D_+^q m(t^*) > {}_{t_0}^{c}D_+^q u(t^*;\gamma^*) = {}_{t_0}^{c}D^q u(t^*;\gamma^*) = -p(t^*), \tag{45}$$

and this contradicts (41). Therefore (44) is satisfied for γ^*. From (43) and (44) since $\gamma^* > 0$ is arbitrary we obtain (40). □

If the Lyapunov function $V(t,x)$ is differentiable we could use its Caputo fractional derivative instead of the Caputo fractional Dini derivative and we obtain the following comparison result.

Lemma 4.7 (Comparison result for Caputo fractional derivative)**.** *Let the following conditions be satisfied:*

1. *The conditions* 1 *and* 2 *of Lemma* 4.2 *are satisfied.*
2. *The function $V \in \Lambda([t_0, t_0 + T], \Delta)$ is continuously differentiable and*

$${}_{t_0}^{c}D^q V(t, \tilde{x}(t+\eta^*) - x^*(t)) \leq g(t, V(t, \tilde{x}(t+\eta^*) - x^*(t))), \eta^*)$$

   for $t \in [t_0, t_0 + T]$ holds.

4. *The function $u^*(t) = u(t; t_0, u_0, \eta^*)$, $u^* \in C^q([t_0, t_0 + T], \mathbb{R})$, is the maximal solution of the initial value problem* (17) *for initial data (t_0, u_0) and parameter $\eta = \eta^*$.*

Then the inequality $V(t_0, y_0 - x_0) \leq u_0$ implies $V(t, \tilde{x}(t+\eta^*) - x^*(t)) \leq u^*(t)$ for $t \in [t_0, t_0 + T]$.

The proof of Lemma 4.6 is similar to the one in Lemma 4.2 and we omit it.

5. Main result

We will obtain sufficient conditions for stability with ITD of the system of FrDE (5) in the case $0 < q < 1$. We will consider the general case when the Lyapunov function is not differentiable and we use its Caputo fractional Dini derivative defined by (21).

Theorem 5.1. *Let the following conditions be satisfied:*

1. *The function* $x^*(t) = x(t;t_0,x_0) \in C^q([t_0,\infty),\mathbb{R}^n)$ *is a solution of the IVP for FrDE* (5) *where* $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$.
2. *The function* $g \in C([t_0,\infty)\times\mathbb{R}\times B(H),\mathbb{R})$, $g(t,0,0) \equiv 0$ *where* $H > 0$ *is a given number.*
3. *There exists a function* $V \in \Lambda([t_0,\infty),\mathbb{R}^n)$ *such that* $V(t,0) = 0$ *and*
   (i) *for any* $y, y_0 \in \mathbb{R}^n$ *and* $\eta \in B(H)$ *the inequality*
   $$ {}_{t_0}^{c}D^q_{(5)}V(t,x^*(t),y,\eta,x_0,y_0) \leq g(t,V(t,y-x^*(t)),\eta) \quad \text{for } t \geq t_0 \tag{46}$$
   holds.
   (ii) $b(\|x\|) \leq V(t,x)$ *for* $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, *where* $b \in \mathcal{K}$.
4. *The zero solution of the scalar FrDE* (17) *is stable w.r.t. a parameter (attractive w.r.t. a parameter).*

Then the solution $x^*(t)$ *is stable with ITD (attractive with ITD).*

Proof. Let $\epsilon > 0$ be given and the zero solution of the scalar FrDE (17) is stable w.r.t. a parameter. From condition 4 it follows there exist $\delta_1 = \delta_1(t_0,\epsilon) > 0$ and $\sigma = \sigma(t_0,\epsilon) > 0$ such that for any $u_0 \in \mathbb{R}:\ |u_0| < \delta_1$ and any $\eta \in B(\sigma)$ the inequality
$$|u(t;t_0,u_0,\eta)| < b(\epsilon), \quad t \geq t_0 \tag{47}$$
holds where $u(t;t_0,u_0,\eta)$ is a solution of the IVP for the scalar FrDE (17). It is clear we can choose $\sigma < H$.

Since $V(t_0,0) = 0$ there exists $\delta_2 = \delta_2(t_0,\delta_1) > 0$ such that $V(t_0,x) < \delta_1$ for $\|x\| < \delta_2$. Let $y_0 \in \mathbb{R}^n$ with $\|y_0 - x_0\| < \delta_2$. Then $V(t_0,y_0-x_0) < \delta_1$. Let $\tau_0 \in \mathbb{R}_+$ be such that $\eta^* = \tau_0 - t_0 \in B(\sigma)$.

Consider any solution $x(t) = x(t;\tau_0,y_0)$, $t \geq \tau_0$, $x \in C^q([\tau_0,\infty),\mathbb{R}^n)$ of the IVP for the FrDE (6). Now let $u_0^* = V(t_0,y_0-x_0)$. Then $u_0^* < \delta_1$ and inequality (47) holds for any solution $u(t;t_0,u_0^*,\eta^*)$ of the IVP for the scalar FrDE (17). Then from Corollary 4.3 and (47) we have
$$V(t,x(t)-x^*(t)) \leq \overline{u}(t;t_0,u_0^*,\eta^*) < b(\epsilon), \quad t \geq t_0;$$
here $\overline{u}(t;t_0,u_0^*,\eta^*)$ is the maximal solution of the IVP for the scalar FrDE (17) (with the initial point (t_0,u_0^*) and parameter $\eta = \eta^*$). Then for any $t \geq t_0$ from condition 3(ii) we obtain
$$b(\|x(t)-x^*(t)\|) \leq V(t,x(t)-x^*(t)) < b(\epsilon), \quad t \geq t_0,$$
so the result follows.

If the zero solution of the scalar FrDE (17) is attractive w.r.t. a parameter then following an argument similar to the above shows that the solution $x^*(t)$ is attractive with ITD. □

Corollary 5.2. *Suppose for any $t_0 \geq 0$ that there exist a positive $T(t_0)$ such that inequality* (46) *of Theorem* 5.1 *is satisfied for $t \geq T(t_0)$. Then the solution $x^*(t)$ of* (5) *is attractive with ITD.*

Now we present some sufficient conditions for uniform stability with ITD of FrDE in the case when the condition for the Caputo fractional Dini derivative of the Lyapunov function is satisfied only on a ball.

Theorem 5.3. *Let the following conditions be satisfied:*

1. *The function $g \in C(\mathbb{R}_+ \times \mathbb{R} \times B(H), \mathbb{R})$, $g(t, 0, 0) \equiv 0$ where $H > 0$ is a given number.*
2. *There exists a function $V \in \Lambda(\mathbb{R}_+, \bar{S}(\lambda))$ such that*
   (i) *for any $t_0 \geq 0$, $x, y, x_0, y_0 \in \mathbb{R}^n$: $y - x \in \bar{S}(\lambda), y_0 - x_0 \in \bar{S}(\lambda)$ and $\eta \in B(H)$ the inequality*
   $$ {}_{t_0}^{c}D_{(5)}^{q} V(t, x, y, \eta, x_0, y_0) \leq g(t, V(t, y - x), \eta) \quad \text{for } t \geq t_0 \tag{48}$$
   holds where $\lambda > 0$ is a given number.
   (ii) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for $t \in \mathbb{R}_+$, $x \in \bar{S}(\lambda)$, where $a, b \in \mathcal{K}$.*
3. *The zero solution of the scalar FrDE* (17) *is uniformly stable w.r.t. a parameter.*

Then the FrDE (5) *is uniformly stable with ITD.*

Proof. Let $\epsilon \in (0, \lambda]$ be given. From condition 3 of Theorem 5.3 it follows there exist $\delta_1 = \delta_1(\epsilon) > 0$ and $\sigma = \sigma(\epsilon) > 0$ such that for any $u_0 \in \mathbb{R}$: $|u_0| < \delta_1$, $t_0 \in \mathbb{R}_+$ and any $\eta \in B(\sigma)$ the inequality

$$|u(t; t_0, u_0, \eta)| < b(\epsilon), \quad t \geq t_0 \tag{49}$$

holds where $u(t; t_0, u_0, \eta)$ is any solution of the scalar FrDE (17) with initial data (t_0, u_0) and parameter η. We can choose $\sigma < H$ and $\delta_1 < \min\{\epsilon, b(\epsilon)\}$.

From $a \in \mathcal{K}$ there exists $\delta_2 = \delta_2(\epsilon) > 0$ so that $a(\delta_2) < \delta_1$. Let $\delta = \min(\epsilon, \delta_2)$. Choose the initial values $x_0, y_0 \in \mathbb{R}^n$: $y_0 - x_0 \in S(\delta)$, $t_0, \tau_0 \in \mathbb{R}_+$, $\eta^* = \tau_0 - t_0 \in B(\sigma)$. Let $x^*(t) = x(t; t_0, x_0) \in C^q([t_0, t_0 + T], \mathbb{R}^n)$ be a solution of (5) and $\tilde{x}(t) = x(t; \tau_0, y_0) \in C^q([\tau_0, \tau_0 + T], \mathbb{R}^n)$ be a solution of (6) with initial data (t_0, x_0) and (τ_0, y_0) respectively. We now prove that

$$\|\tilde{x}(t + \eta) - x^*(t)\| < \epsilon, \quad t \geq t_0. \tag{50}$$

From the choice of initial data it follows that inequality (50) holds for $t = t_0$. Assume inequality (50) is not true for all $t > t_0$. Therefore, there exists a point $t^* > t_0$ such that

$$\|\tilde{x}(t^* + \eta) - x^*(t^*)\| = \epsilon, \quad \text{and} \quad \|\tilde{x}(t + \eta) - x^*(t)\| < \epsilon, \quad t \in [t_0, t^*). \tag{51}$$

Now let $u_0^*=V(t_0,y_0-x_0)$. Then from 2(ii) we get $u_0^*\le a(\|y_0-x_0\|)<a(\delta_2)<\delta_1$. Let $u^*(t)=u(t;t_0,u_0^*,\eta^*)\in C^q([t_0,t^*],\mathbb{R})$ be the maximal solution of the scalar FrDE (17) on $[t_0,t^*]$ for the initial data (t_0,u_0^*) and parameter $\eta=\eta^*$. Since $|u_0^*|<\delta_1$ and $\eta^*\in B(\sigma)$ the maximal solution $u^*(t)$ satisfies inequality (49) for $t\in[t_0,t^*]$. From the choice of the point t^* it follows that $\tilde{x}(t+\eta)-x^*(t)\in\bar{S}(\lambda)$ for $t\in[t_0,t^*]$. Then from the reasoning in Lemma 4.6 for the interval $[t_0,t^*]$ we have

$$V(t,\tilde{x}(t+\eta)-x^*(t))\le u^*(t),\quad t\in[t_0,t^*]. \tag{52}$$

From inequalities (49), (52), the choice of t^*, and condition 2(ii) of Theorem 5.3 we obtain $b(\epsilon)>u^*(t^*)\ge V(t^*,\tilde{x}(t^*+\eta)-x^*(t^*))\ge b(\|\tilde{x}(t^*+\eta)-x^*(t^*)\|)=b(\epsilon)$. The contradiction proves (50) and therefore, the uniform stability of the FrDE (5). □

Corollary 5.4. *Let condition* 2 *of Theorem* 5.3 *be satisfied with* $g(t,u,\eta)\equiv 0$. *Then the FrDE* (5) *is uniformly stable with ITD.*

Proof. The proof follows from the fact that the Caputo fractional differential equation ${}_{t_0}^{c}D^q x=0$ has a constant solution which is uniformly stable and consequently it is uniformly stable w.r.t. a parameter. □

Corollary 5.5. *Suppose for any* $t_0\ge 0$ *that there exist a positive* $T(t_0)$ *such that inequality* (48) *of Theorem* 5.3 *is satisfied for* $t\ge T(t_0)$. *Then the FrDE* (5) *is uniformly attractive with ITD.*

Now we present some sufficient conditions for uniform asymptotic stability with ITD of the FrDE.

Theorem 5.6. *There exists a function* $V\in\Lambda(\mathbb{R}_+,\mathbb{R}^n)$ *such that*

(i) *for any* $t_0\ge 0$, $x,y,x_0,y_0\in\mathbb{R}^n$: $y-x\in\bar{S}(\lambda), y_0-x_0\in\bar{S}(\lambda)$ *and* $\eta\in B(H)$ *the inequality*

$$ {}_{t_0}^{c}D_{(5)}^{q}V(t,x,y,\eta,x_0,y_0)\le -c(\|y-x\|)\quad \text{for } t\ge t_0 \tag{53}$$

holds where $c\in\mathcal{K}$, $\lambda>0$ *is a given number;*

(ii) $b(\|x\|)\le V(t,x)\le a(\|x\|)$ *for* $t\in\mathbb{R}_+$, $x\in\mathbb{R}^n$, *where* $a,b\in\mathcal{K}$.

Then the FrDE (5) *is uniformly asymptotically stable with ITD.*

Proof. From Corollary 5.4 the FrDE (5) is uniformly stable with ITD. Therefore, for the number λ there exists $\alpha=\alpha(\lambda)\in(0,\lambda)$ and and $\sigma=\sigma(\lambda)>0$ such that for any $x_0,y_0\in\mathbb{R}^n$ and any $t_0,\tau_0\in\mathbb{R}_+$ the inequalities $\|y_0-x_0\|<\alpha$ and $|\tau_0-t_0|<\sigma$ imply

$$\|\tilde{x}(t+\eta)-x^*(t)\|<\lambda\quad \text{for } t\ge t_0 \tag{54}$$

where $\eta = \tau_0 - t_0$ and $x^*(t) = x(t; t_0, x_0)$ and $\tilde{x}(t) = x(t; \tau_0, y_0)$ are any solutions of FrDE (5) with $x(t_0) = x_0$ and $x(\tau_0) = y_0$ respectively.

Now we will prove the fractional differential equations (5) is uniformly attractive with ITD. Consider the constant $\beta > 0$ such that $a(\beta) \leq b(\alpha)$. Let $\epsilon \in (0, \lambda]$ be an arbitrary number. Choose $t_0, \tau_0 \in \mathbb{R}_+$ and $x_0, y_0 \in \mathbb{R}^n$ such that $\|y_0 - x_0\| < \beta$ and $|\tau_0 - t_0| < \sigma$. Let $x^*(t) = x(t; t_0, x_0) \in C^q([t_0, \infty), \mathbb{R}^n)$ and $\tilde{x}(t) = x(t; \tau_0, y_0) \in C^q([\tau_0, \infty), \mathbb{R}^n)$ be solutions of FrDE (5) with the chosen above initial data. Then $b(\|y_0 - x_0\|) \leq a(\|y_0 - x_0\|) < a(\beta) < b(\alpha)$, i.e. $\|y_0 - x_0\| < \alpha$ and therefore the inequality (54) is satisfied for $t \geq t_0$, i.e. $\tilde{x}(t + \eta) - x^*(t) \in S(\lambda)$ for $t \geq t_0$.

Choose constants $\gamma = \gamma(\epsilon) \in (0, \epsilon]$ and $T = T(\epsilon) > 0$ such that $a(\gamma) < b(\epsilon)$ and $T > \frac{q\Gamma(q)a(\alpha)}{c(\gamma)}$. We now prove that

$$\|\tilde{x}(t + \eta) - x^*(t)\| < \epsilon \quad \text{for } t \geq t_0 + T. \tag{55}$$

Assume

$$\|\tilde{x}(t + \eta) - x^*(t)\| \geq \gamma \quad \text{for every } t \in [t_0, t_0 + T]. \tag{56}$$

Then from Lemma 4.7 (applied to the interval $[t_0, t_0 + T]$ and $\Delta = \bar{S}(\lambda)$) we get

$$\begin{aligned}
&V(t_0 + T, \tilde{x}(t_0 + T + \eta) - x^*(t_0 + T)) \\
&\leq V(t_0, y_0 - x_0) - \frac{1}{\Gamma(q)} \int_{t_0}^{t_0+T} (t_0 + T - s)^{q-1} c(\|\tilde{x}(s + \eta) - x^*(s)\|) ds \\
&\leq a(\|y_0 - x_0\|) - \frac{c(\gamma)}{\Gamma(q)} \int_{t_0}^{t_0+T} (t_0 + T - s)^{q-1} ds \\
&= a(\|y_0 - x_0\|) - \frac{c(\gamma)}{\Gamma(q)} \frac{T}{q} \\
&< a(\alpha) - \frac{c(\gamma)}{\Gamma(q)} \frac{T}{q} \\
&< 0.
\end{aligned} \tag{57}$$

The contradiction proves there exists $t^* \in [t_0, t_0 + T]$ such that $\|\tilde{x}(t^* + \eta) - x^*(t^*)\| < \gamma$. From Corollary 4.3 applied for $t_0 = t^*$ and $\Delta = \bar{S}(\lambda)$ we obtain

$$V(t, \tilde{x}(t + \eta) - x^*(t)) \leq V(t^*, \tilde{x}(t^* + \eta) - x^*(t^*))), \quad t \geq t^*.$$

Then for any $t \geq t^*$ the inequalities $b(\|\tilde{x}(t+\eta) - x^*(t)\|) \leq V(t, \tilde{x}(t+\eta) - x^*(t)) \leq V(t^*, \tilde{x}(t^* + \eta) - x^*(t^*)) \leq a(\|\tilde{x}(t^* + \eta) - x^*(t^*)\|) \leq a(\gamma) < b(\epsilon)$ hold. Therefore (55) holds for $t \geq t^*$ (hence for $t \geq t_0 + T$). □

In the case when the Lyapunov function is differentiable, we could use the Caputo fractional derivative instead of the Caputo fractional Dini derivative and obtain the following sufficient conditions for stability with ITD.

Theorem 5.7. *Let the following conditions be satisfied:*

1. *The condition* 1 *and* 2 *of Theorem* 5.1 *are satisfied.*
2. *There exists a differentiable function* $V \in \Lambda(\mathbb{R}_+, \mathbb{R}^n)$ *such that* $V(t, 0) = 0$ *and*
   (i) *for any solution* $\tilde{x}(t) = \tilde{x}(t; \tau_0, y_0)$ *of* (6) *such that* $\eta = \tau_0 - t_0 \in B(H)$ *the inequality for the Caputo fractional derivative*
   $$ {}_{t_0}^{c}D^q V(t, \tilde{x}(t+\eta) - x^*(t)) \le g(t, V(t, \tilde{x}(t+\eta) - x^*(t)), \eta)$$
   holds.
   (ii) $b(\|x\|) \le V(t, x)$ *for* $t \in \mathbb{R}_+,\ x \in \mathbb{R}^n$, *where* $b \in \mathcal{K}$.
3. *The zero solution of the scalar FrDE* (17) *is stable w.r.t. a parameter.*

Then the solution $x^*(t)$ *of the FrDE* (5) *is stable with ITD.*

Theorem 5.8. *Let the following conditions be satisfied:*

1. *Condition* 1 *of Theorem* 5.3 *is satisfied.*
2. *There exists a differentiable function* $V \in \Lambda(\mathbb{R}_+, \bar{S}(\lambda))$ *such that*
   (i) *for any solution* $x(t) = x(t; t_0, x_0)$ *of* (5) *and any solution* $\tilde{x}(t) = \tilde{x}(t; \tau_0, y_0)$ *of* (6) *such that* $\tilde{x}(t+\eta) - x(t) \in \bar{S}(\lambda),\ t \ge t_0$ *and* $\eta = \tau_0 - t_0 \in B(H)$ *the inequality for the Caputo fractional derivative of* V
   $$ {}_{t_0}^{c}D^q V(t, \tilde{x}(t+\eta) - x(t)) \le g(t, V(t, \tilde{x}(t+\eta) - x(t)), \eta) \quad \text{for } t \ge t_0$$
   holds where $\lambda > 0$ *is a given number.*
   (ii) $b(\|x\|) \le V(t, x) \le a(\|x\|)$ *for* $t \in \mathbb{R}_+,\ x \in \bar{S}(\lambda)$, *where* $a, b \in \mathcal{K}$.
3. *The zero solution of the scalar FrDE* (17) *is uniformly stable w.r.t. a parameter.*

Then the FrDE (5) *is uniformly stable with ITD.*

The proofs of Theorem 5.7 and 5.8 are similar to those in Theorem 5.1 and 5.3 where instead of Lemma 4.6 we use Lemma 4.7.

Remark 5.9. In the case $\tau_0 = t_0,\ x^*(t) \equiv 0$ Theorem 5.7 and 5.8 gives us sufficient conditions for stability of the zero solution of a fractional differential equation which was also studied in [9, 28].

6. Applications

We will give some examples which illustrate the usefulness of the Lyapunov function to fractional differential equations for establishing stability with initial time difference.

In physical systems, the Lyapunov function used is often an expression for total energy of the system, i.e. the Lyapunov function $V(x) = x^T x$, $x \in \mathbb{R}^n$. In connection with this we will give an example of an application of the quadratic Lyapunov function to study stability with ITD.

Example 6.1. Consider the following system of fractional differential equations

$$\begin{aligned} {}_{\xi_0}^{c}D^q u_1(t) &= -u_1 - u_2 + p_1(t), \\ {}_{\xi_0}^{c}D^q u_2(t) &= -u_2 + u_1 + p_2(t) \quad \text{for } t > \xi_0 \end{aligned} \tag{58}$$

with initial condition

$$u_1(\xi_0) = u_1^0 \quad \text{and} \quad u_2(\xi_0) = u_2^0,$$

where $u_1, u_2 \in \mathbb{R}$, $\xi_0 \in \mathbb{R}_+$, $p_1, p_2 \in C(\mathbb{R}_+, \mathbb{R})$ are Lipschitz functions with constants L_1, L_2, respectively.

Let $t_0, \tau_0 \in \mathbb{R}_+ : \eta = \tau_0 - t_0 \neq 0$ and $x, y, x^0, y^0 \in \mathbb{R}^2 : x = (x_1, x_2),\ y = (y_1, y_2)$, $x^0 = (x_1^0, x_2^0),\ y^0 = (y_1^0, y_2^0),\ x^0 \neq y^0,\ y - x \in \bar{S}(\lambda),\ y_0 - x_0 \in \bar{S}(\lambda),\ \lambda > 0$ is given number.

Consider the Lyapunov function $V(x) = x_1^2 + x_2^2$ where $x \in \mathbb{R}^2$. Now condition 2(ii) of Theorem 5.3 is satisfied for $a, b \in \mathcal{K}$ with $a(s) = \frac{1}{2}s$, $b(s) = s$. Also

$$\begin{aligned} &{}_{t_0}^{c}D_{(58)}^q V(t, x, y, \eta, x_0, y_0) \\ &\leq \limsup_{h\to 0^+} \frac{1}{h^q}\Big\{(y_1 - x_1)^2 + (y_2 - x_2)^2 - \big(y_1 - x_1 - h^q\Psi\big)^2 - \big(y_2 - x_2 - h^q\Phi\big)^2 \\ &\quad + \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r qCr\Big[\big(y_1 - x_1 - h^q\Psi\big)^2 + \big(y_2 - x_2 - h^q\Phi\big)^2\Big]\Big\} \\ &\leq -\Big(2 - \frac{(t-t_0)^{-q}}{\Gamma(1-q)}\Big)\big((y_1 - x_1)^2 (y_2 - x_2)^2\big) \\ &\quad + 2(y_1 - x_1)\big(p_1(t+\eta) - p_1(t)\big) + 2(y_2 - x_2)\big(p_2(t+\eta) - p_2(t)\big) \\ &\leq -\Big(2 - \frac{(t-t_0)^{-q}}{\Gamma(1-q)}\Big)\big((y_1 - x_1)^2 (y_2 - x_2)^2\big) + C|\eta|, \end{aligned} \tag{59}$$

where $C = 2(L_1 + L_2)\lambda$, $\Psi = -y_1 - y_2 + p_1(t+\eta) + x_1 + x_2 - p_1(t)$, $\Phi = -y_2 + y_1 + p_2(t+\eta) + x_2 - x_1 - p_2(t)$.

For any point $T > t_0$ there exists $A(T) > 0$ such that $-\Big(2 - \frac{(t-t_0)^{-q}}{\Gamma(1-q)}\Big) \leq -A(T)$, $t \geq T$.

Therefore condition 2(i) of Theorem 5.3 is satisfied for $t \geq T_0$ with the function $g(t, u, \eta) = -u + C\eta$, $u \in \mathbb{R}$, $\eta > 0$. The corresponding comparison scalar fractional differential equation is

$${}_{t_0}^{c}D^q u = -2u + C\eta. \tag{60}$$

According to Example 3.12 the zero solution of (60) is uniformly stable w.r.t. a parameter and according to Corolarry 5.5 the FrDE (58) is uniformly attractive with initial time difference.

Now we give an example for an application of a non-quadratic Lyapunov function and the Caputo fractional Dini derivative with initial time difference given by (21).

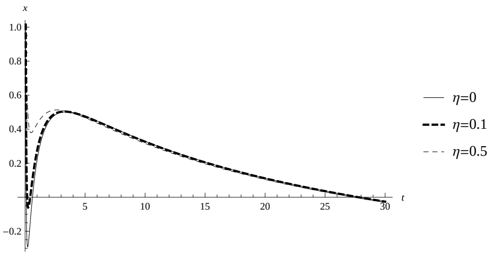

Figure 1: Graph of $2g_1(t+\eta)-\frac{t^{-q}}{\Gamma(1-q)}$ for $\eta=0,0.1,0.5$.

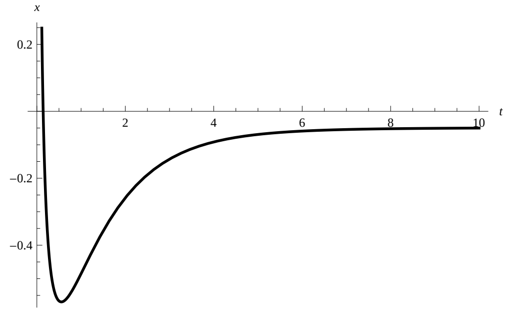

Figure 2: Graph of ${}_0D^q m(t)$–$0.5m(t) \le 0$.

Example 6.2. Consider the following IVP for the system of FrDE

$$\begin{aligned} {}_0^cD^q x_1(t) &= -g_1(t)x_1 + g_2(t)x_2, \\ {}_0^cD^q x_2(t) &= -g_2(t)x_1 - g_1(t)x_2 \quad \text{for} \quad t>0 \\ x_1(0) &= 0, \quad x_2(0)=0 \end{aligned} \tag{61}$$

where $x=(x_1,x_2)\in\mathbb{R}^2$, $f=(f_1,f_2)$, $f_1(t,x)=-g_1(t)x_1-g_2(t)x_2$, $f_2(t,x)=-g_2(t)x_1-g_1(t)x_2$, $g_1(t)=1-2\frac{t^{0.8}}{t+1}$ and $g_2\in C(\mathbb{R}_+,\mathbb{R})$ is an arbitrary function.

The IVP for the system of FrDE has a zero solution $x^*(t)\equiv 0$. Similar to that in Example 6.1 we obtain for the quadratic Lyapunov function $V(t,x)=x_1^2+x_2^2$

$$\begin{aligned} {}_0^cD^q_{(61)}V(t,0,y,\eta,0,y_0) &\le -\Big(2g_1(t+\eta)-\frac{t^{-q}}{\Gamma(1-q)}\Big)\Big((y_1)^2+(y_2)^2\Big) \\ &= -\Big(2g_1(t+\eta)-\frac{t^{-q}}{\Gamma(1-q)}\Big)V(y). \end{aligned}$$

For example, if $q=0.8$ then the function $2g_1(t+\eta)-\frac{t^{-q}}{\Gamma(1-q)}$ is negative for $t\ge 28$ (see, the graph of $2g_1(t+\eta)-\frac{t^{-q}}{\Gamma(1-q)}$, $\eta=0,0.1,0.5$ in Figure 1).

Now consider the Lyapunov function $V(t,x) = m(t)\big(x_1^2+x_2^2\big)$ where $x=(x_1,x_2)$, $m(t)=e^{-t}+0.1$, $t\ge 0$. Let $y,y_0\in\mathbb{R}^2$, $y=(y_1,y_2)$, $y_0=(y_1^{(0)},y_2^{(0)})$ and $\tau_0>0$. Then $\eta=\tau-0>0$ and according to formula (21) we obtain

$$\begin{aligned} {}_0^cD^q_{(61)}V(t,0,y,\eta,0,y_0) &= -2m(t)g_1(t+\eta)(y_1^2+y_2^2)-m(0)\Big((y_1^{(0)})^2+(y_2^{(0)})^2\Big) \\ &\quad + \big((y_1)^2+(y_2)^2\big)\limsup_{h\to 0^+}\frac{1}{h^q}\Big\{\sum_{r=0}^{[\frac{t}{h}]}(-1)^r qCrm(t-rh)\Big\}. \end{aligned}$$

Using $\limsup_{h\to 0^+} \frac{1}{h^q}\Big\{ \sum_{r=0}^{[\frac{t}{h}]} (-1)^r qCrm(t-rh)\Big\} = {}_0\tilde{D}^q_+ m(t) = {}_0\tilde{D}^q m(t) = {}_0D^q m(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\Big(\int_0^t (t-s)^{-q}\, m(s)ds\Big)$ and $g_1(t+\eta) \geq -0.25$ for $|\eta| \leq 1$ we obtain

$$ {}_0^cD^q_{(61)} V(t,0,y,\eta,0,y_0) \leq \Big({}_0 D^q m(t) + 0.5 m_1(t)\Big)\Big((y_1)^2 + (y_2)^2\Big). $$

If $q = 0.8$ we obtain ${}_0D^q m(t) = \frac{1}{\Gamma(0.2)t^{0.8}} 0.2(0.5 + 5e_1^{-t}F_1(0.2;1.2;t) + \frac{t^{0.2}}{\Gamma(0.2)}(-5e_1^{-t}F_1(0.2;1.2;t) + \frac{2.5}{3}e_1^{-t}F_1(1.2;1.2;t))$ where ${}_1F_1(a;b;z)$ is confluent hypergeometric function of the first kind given by

$$ {}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{\frac{\Gamma(a+k)}{\Gamma(a)}}{\frac{\Gamma(b+k)}{\Gamma(b)}} \frac{z^k}{k!} $$

Then ${}_0D^q m(t) - 0.5m(t) \leq 0$ for $t \geq 0.15$ (see Fig. 2).

Therefore, since the bound for the time variable 0.15 is better than the bound 28, the application of the Lyapunov function $V(t,x) = m(t)\big(x_1^2 + x_2^2\big)$ gives a better result. According to Corollary 5.2 the zero solution of the system of FrDE (61) is attractive with ITD (for $q = 0.8$).

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