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On Local Attractivity and Asymptotic Stability of Solutions of Nonlinear Volterra-Stieltjes Integral Equations in Two Variables

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Abstract. In this paper we will study the existence of solutions depending on two variables of a nonlinear integral equation of Volterra-Stieltjes type in two variables, in the space of real functions which are continuous and bounded on the set $\mathbb{R}_+ \times [0, M]$. Moreover, we will give the characterization of those solutions. In our study we will utilize the Darbo type of fixed point theorem and apply a measure of noncompactness. In the last section we present particular cases of the considered equation.

Keywords. Nonlinear Volterra-Stieltjes integral equation, fixed point theorem, measure of noncompactness, uniform local attractivity, asymptotic stability

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1. Introduction

While considering the applications of the integral equations to the description of several real world events, which appear in different aspects of branches of science and applied mathematics, e.g., in engineering, mechanics, physics etc., we can distinguish some classes of integral equations that have significant meaning (cf. [2, 12, 15, 17–21, 26, 29, 30]).

One of the mentioned classes of integral equations, is the class of the "so-called" nonlinear integral equations of fractional order. Differential and integral calculus of fractional order play a very important role in describing some real world problems. The present interest in the mathematical analysis of the fractional calculus derives from its usefulness in constructing mathematical models of various occurrences in a more precise way. For example those equations can be used to describe some problems occurring in physics, mechanics, chemistry, control theory, electricity, chaos and fractals, capacitor theory, viscoelasticity.

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In recent years numerous research papers and monographs devoted to differential and integral equations of fractional order have been published. They contain a lot of various types of existence results (cf. [1, 9, 11, 19–21, 23, 24, 28–30]).

Another important class of integral equations is the class of the "so-called" quadratic integral equations of Volterra-Chandrasekhar type. In 1950 the astrophysicist Chandrasekhar published the manuscript [14], in which he investigated the "so-called" Chandrasekhar integral equation. Since this time numerous publications have appeared, in which the equations of this type were considered (cf. [6, 13]).

The principal aim of this paper is to investigate the solvability of the nonlinear integral equations of Volterra-Stieltjes type in two variables. Using the Darbo fixed point theorem in combination with the technique of measures of noncompactness we will show that there exists a solution of the aforementioned equation in the space of functions which are continuous and bounded on the set $\mathbb{R}_+ \times [0, M]$. Such an approach is one of the most frequently applied methods in the study of the existence of solutions of various functional, differential and integral equations (cf. [3, 7, 8, 10, 17]). In the last section we will discuss a few special cases of the investigated nonlinear integral equation. The results obtained in this paper create generalization of the results obtained in [17].

2. Notation and some auxiliary facts

In this section we present a few auxiliary facts which will be applied in our further investigations. At the beginning we assume that $(E, \|\cdot\|_E)$ is a real infinite dimensional Banach space with the zero element θ. Let us denote by $B(x, r)$ the closed ball of radius r and center x. The symbol B_r stands for the ball $B(\theta, r)$. If X is the subset of E, then $\overline{X}$ denotes the closure of X and $\operatorname{Conv} X$ denotes the convex closure of X. Further, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We accept the following definition of the notion of a measure of noncompactness [8].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ = [0, \infty)$ is said to be *a measure of noncompactness* in E if it satisfies the following conditions:

1^0 The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(x) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

2^0 $X \subset Y \Rightarrow \mu(x) \leq \mu(Y)$.

3^0 $\mu(\overline{X}) = \mu(x)$.

4^0 $\mu(\operatorname{Conv} X) = \mu(x)$.

5^0 $\mu(\lambda X + (1 - \lambda) Y) \leq \lambda \mu(x) + (1 - \lambda) \mu(Y)$ for $\lambda \in [0, 1]$.

6^0 If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in 1^0 is said to be *the kernel of the measure of noncompactness* μ. We observe that the intersection set X_∞ from 6^0 belongs to $\ker \mu$. Indeed, since $\mu(X_\infty) \le \mu(X_n)$ for any n then we deduce that $\mu(X_\infty) = 0$, so $X_\infty \in \ker \mu$. This observation will be crucial in our further considerations. For other facts concerning measures of noncompactness we refer to [8] (cf. also [3, 7]).

In our considerations we will use the following fixed point theorem [8, 16].

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of the Banach space E and let $Q : \Omega \to \Omega$ be a continuous mapping, such that there exists a constant $k \in [0, 1)$ such that $\mu(QX) \le k\mu(x)$ for any nonempty subset X of Ω. Then Q has a fixed point in the set Ω.*

Remark 2.3. It can be shown [8] that the set $\operatorname{Fix} Q$ of all fixed points of the operator Q belonging to Ω, is an element of the family $\ker \mu$.

Our considerations will be placed in the space $BC = BC(\mathbb{R}_+ \times [0, M], \mathbb{R})$, where M is a positive number. This space consists of all functions $u(t, x) = u : \mathbb{R}_+ \times [0, M] \to \mathbb{R}$ defined, continuous and bounded on the set $\mathbb{R}_+ \times [0, M]$ with the supremum norm

$$\|u\|_{BC} = \sup \{|u(t, x)| : (t, x) \in \mathbb{R}_+ \times [0, M]\}.$$

For further purposes we will use the measure of noncompactness in the space $BC(\mathbb{R}_+ \times [0, M], \mathbb{R})$ described below. So, let us consider a nonempty, bounded subset U of $BC(\mathbb{R}_+ \times [0, M], \mathbb{R})$. Fix a positive number $T > 0$. For $u = u(t, x) \in U$ and $\varepsilon > 0$ let us consider *the modulus of continuity* of the function u on the set $[0, T] \times [0, M]$, defined as follows

$$\omega^T(u, \varepsilon) = \sup \left\{ |u(t_2, x_2) - u(t_1, x_1)| \;\middle|\; \begin{matrix} t_1, t_2 \in [0, T],\ x_1, x_2 \in [0, M], \\ |t_2 - t_1| \le \varepsilon,\ |x_2 - x_1| \le \varepsilon \end{matrix} \right\}.$$

Next, let us put

$$\begin{aligned} \omega^T(U, \varepsilon) &= \sup \left\{ \omega^T(u, \varepsilon) : u \in U \right\}, \\ \omega_0^T(U) &= \lim_{\varepsilon \to 0} \omega^T(U, \varepsilon), \\ \omega_0^\infty(U) &= \lim_{T \to \infty} \omega_0^T(U). \end{aligned}$$

Further, for the bounded set $U \subset BC(\mathbb{R}_+ \times [0, M], \mathbb{R})$ and for arbitrarily fixed $(t, x) \in \mathbb{R}_+ \times [0, M]$ we define

$$U(t, x) = \{u(t, x) : u \in U\}$$

and

$$\operatorname{diam} U(t,x) = \sup\{|u_2(t,x) - u_1(t,x)| : u_1, u_2 \in U\},$$
$$a(U) = \limsup_{t\to\infty} \big\{ \sup\{\operatorname{diam} U(t,x) : x \in [0,M]\}\big\}.$$

Finally, for $U \in \mathfrak{M}_{BC(\mathbb{R}_+\times[0,M],\mathbb{R})}$ consider the mapping μ defined by the formula

$$\mu(U) = \omega_0^\infty(U) + a(U). \tag{1}$$

The mapping μ is a sublinear measure of noncompactness in the space $BC = BC(\mathbb{R}_+ \times [0,M], \mathbb{R})$. The kernel $\ker\mu$ of this measure consists of all nonempty and bounded sets U such that functions from U are equicontinuous on each compact subset of the rectangle $\mathbb{R}_+ \times [0,M]$ (in particular, these functions are equicontinuous on each rectangle of the form $[0,T]\times[0,M]$) and the thickness of the slice formed by surfaces being graphs of the functions belonging to the set U and intersected in the point t ($t \in \mathbb{R}_+$), tends to zero at infinity. This property and Remark 2.3 will allow us to characterize solutions of the integral equation investigated in the next section.

In the sequel we recall the concept of *the superposition operator*. Let us consider a function $f : D \times \mathbb{R} \to \mathbb{R}$, where $D \subset \mathbb{R}^2$. Then for each function u acting from D into $\mathbb{R}$ we can assign the function Fu defined as follows

$$(Fu)(t,x) = f(t,x,u(t,x)) \quad \text{for } (t,x) \in D.$$

The operator F defined in such a way is called the superposition operator generated by the function $f = f(t,x,u)$. The theory concerning the superposition operator may be found in monographs, e.g., [5,27].

Now, we give the definitions of the concepts of global attractivity, local attractivity and asymptotic stability of solutions depending on two variables of the nonlinear integral equations in two variables in the space BC. These concepts, in case of the solutions of the equations in one variable, may be found, e.g., in the papers [1,9,10,22].

We assume that Ω is a nonempty subset of the space BC and Q is an operator acting from Ω into BC. Let us consider the following operator equation

$$u(t,x) = (Qu)(t,x), \quad (t,x) \in \mathbb{R}_+ \times [0,M]. \tag{2}$$

Definition 2.4. The solution $u = u(t,x)$ of equation (2) is said to be *globally attractive* if for each solution $v = v(t,x)$ of equation (2) we have for arbitrarily fixed $x \in [0,M]$ that

$$\lim_{t\to\infty} (u(t,x) - v(t,x)) = 0. \tag{3}$$

In other words, we may say that solutions of equation (2) are globally attractive if for arbitrary solutions $u(t,x)$ and $v(t,x)$ of this equation and for arbitrarily fixed $x \in [0,M]$ condition (3) is satisfied.

Definition 2.5. We say that solutions of equation (2) are *locally attractive* if there exists a ball $B(u_0, r)$ in the space BC such that $B(u_0, r) \cap \Omega \neq \emptyset$ and for arbitrary solutions $u(t, x)$ and $v(t, x)$ of (2) belonging to the set $B(u_0, r) \cap \Omega$ and for arbitrarily fixed $x \in [0, M]$, condition (3) does hold.
In the case when limit (3) is uniform with respect to the set $B(u_0, r) \cap \Omega$, i.e., when $x \in [0, M]$ is arbitrarily fixed and for each $\varepsilon > 0$ there exists $T > 0$ such that

$$|u(t, x) - v(t, x)| \leq \varepsilon \tag{4}$$

for all solutions $u(t, x)$, $v(t, x)$ of equation (2) from $B(u_0, r) \cap \Omega$ and for any $t \geq T$, we will say that solutions of equation (2) are *uniformly locally attractive.*

Definition 2.6. We say that solutions of equation (2) are *asymptotically stable* if there exists a ball $B(u_0, r)$ in the space BC such that $B(u_0, r) \cap \Omega \neq \emptyset$, and such that for each $\varepsilon > 0$ there exists $T > 0$ such that for arbitrary solutions $u(t, x)$ and $v(t, x)$ of (2) belonging to the set $B(u_0, r) \cap \Omega$ and for arbitrarily fixed $x \in [0, M]$, inequality (4) is satisfied for any $t \geq T$.

Let us notice that the concept of uniform local attractivity of solutions is equivalent to the concept of asymptotic stability introduced in [10]. So we can use these concepts exchangeably.

Further on, we recall some basic facts concerning functions of bounded variation (cf. [4, 25]). If f is a real function defined on the interval $[a, b]$, then the symbol $\bigvee_a^b f$ will denote the variation of the function f on the interval $[a, b]$. We say that f is of bounded variation on $[a, b]$ whenever $\bigvee_a^b f$ is finite. If we have a real function of two variables $g(w, z)$ defined on the set $A \subset \mathbb{R}^2$ with nonempty interior, then the symbol $\bigvee_{z=a}^b g(w, z)$ denotes the variation of the function $z \to g(w, z)$ on the interval $[a, b]$ contained in the domain of a function $z \to g(w, z)$, where the variable w is fixed. Analogously, we define the quantity $\bigvee_{w=p}^q g(w, z)$. For the properties of functions of bounded variation we refer to, e.g., [4, 25].

If f and g are two real functions defined on the interval $[a, b]$, then under some additional conditions we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b f(t)\, dg(t)$$

of the function f with respect to the function g. In this case we say that f is Stieltjes integrable on the interval $[a, b]$ with respect to g. Let us mention that several conditions are known which guarantee the Stieltjes integrability (cf. [4, 25]). One of the most frequently used requires that f is continuous and g is of bounded variation on $[a, b]$.

In what follows, we will use a few properties of the Stieltjes integral gathered in the below formulated lemmas [4].

Lemma 2.7. *If f is Stieltjes integrable on the interval $[a,b]$ with respect to a function g of bounded variation, then*

$$\left|\int_a^b f(t)\,dg(t)\right| \le \int_a^b |f(t)|\,d\left(\bigvee_a^t g\right).$$

Lemma 2.8. *Let f_1, f_2 be Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function g such that $f_1(t) \le f_2(t)$ for $t \in [a,b]$. Then*

$$\int_a^b f_1(t)\,dg(t) \le \int_a^b f_2(t)\,dg(t).$$

In the sequel we will also consider the double Stieltjes integral of the form

$$\int_c^d \left(\int_c^d f(t,x)\,d_y g_2(x,y)\right) d_s g_1(t,s),$$

where $g_i : [a,b] \times [c,d] \to \mathbb{R}$ $(i=1,2)$ and the symbol d_y indicates the integration with respect to the variable y (similarly we define the symbol d_s). For convenience, instead of this notation we will use the following notation

$$\int_c^d \int_c^d f(t,x)\,d_y g_2(x,y)\,d_s g_1(t,s).$$

3. Main result

We will investigate the existence of solutions of the following nonlinear quadratic integral equation of Volterra-Stieltjes type in two variables

$$u(t,x) = h(t,x) + f(t,x,u(t,x)) \int_0^t \int_0^x v(t,s,x,y,u(s,y))\,d_y g_2(x,y)\,d_s g_1(t,s), \quad (5)$$

where $t \in \mathbb{R}_+ = [0,+\infty)$ and $x \in [0,M]$ (M is a fixed positive number).
Moreover, we denote by Δ_i $(i=1,2)$ the following triangles

$$\Delta_1 = \left\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t\right\},$$
$$\Delta_2 = \left\{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le M\right\}$$

and

$$\Delta_1^T = \left\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\right\},$$

where $T > 0$ is an arbitrarily fixed number. We will investigate equation (5) under the assumptions formulated below:

(i) $h \in BC$.

(ii) The function $f(t,x,u) = f : \mathbb{R}_+ \times [0,M] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition with respect to the variable u, i.e., there exists a constant $k > 0$ such that

$$|f(t,x,u_1) - f(t,x,u_2)| \leq k\,|u_1 - u_2|$$

for all $t \in \mathbb{R}_+$, $x \in [0,M]$ and $u_1, u_2 \in \mathbb{R}$. Moreover, we assume that the function $(t,x) \to f(t,x,0)$ belongs to the space BC.

For further purposes denote $\overline{F} = \sup\{|f(t,x,0)| : t \in \mathbb{R}_+, x \in [0,M]\}$. Of course, we have that $\overline{F} < \infty$.

(iii) The function $g_i(w,z) = g_i : \Delta_i \to \mathbb{R}$ is continuous on the triangle Δ_i $(i = 1,2)$.

(iv) The function $z \to g_i(w,z)$ is of bounded variation on the interval $[0,w]$ for each fixed $w \in \mathbb{R}_+$ if $i = 1$ and for $w \in [0,M]$ if $i = 2$.

(v) a) For any $T > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ the following inequality holds

$$\bigvee_{s=0}^{t_1} [g_1(t_2,s) - g_1(t_1,s)] \leq \varepsilon.$$

b) For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x_1, x_2 \in [0,M]$ such that $x_1 < x_2$ and $x_2 - x_1 \leq \delta$ the following inequality holds

$$\bigvee_{y=0}^{x_1} [g_2(x_2,y) - g_2(x_1,y)] \leq \varepsilon.$$

(vi) $g_i(w,0) = 0$ for each $w \in \mathbb{R}_+$ if $i = 1$ and for $w \in [0,M]$ if $i = 2$.

(vii) The function $v : \Delta_1 \times \Delta_2 \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|v(t,s,x,y,u)| \leq \phi(|u|)$$

for all $(t,s) \in \Delta_1$, $(x,y) \in \Delta_2$ and $u \in \mathbb{R}$.

(viii) $\lim_{t\to\infty} \left(\bigvee_{s=0}^{t} g_1(t,s)\right) = 0$.

Before we formulate our last assumption we will provide a few lemmas, which will be needed in our investigations (cf. [11,17,30]).

Lemma 3.1. *Assume that hypotheses* (iii)–(v) *are satisfied. Then:*

a) *The function* $x \to \bigvee_{y=0}^{x} g_2(x,y)$ *is continuous on the interval* $[0,M]$.

b) *For an arbitrarily fixed number* $x_2 \in [0,M]$ *and for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $x_1 \in [0,M]$, $x_1 < x_2$ *and* $x_2 - x_1 \leq \delta$, *then*

$$\bigvee_{y=x_1}^{x_2} g_2(x_2,y) \leq \varepsilon.$$

Lemma 3.2. *Let assumptions* (iii)–(v) *be satisfied and let* $T > 0$ *be arbitrarily fixed. Then:*

a) *The function* $t \to \bigvee_{s=0}^{t} g_1(t,s)$ *is continuous on the interval* $[0,T]$.

b) *For an arbitrarily fixed number* $t_2 \in [0,T]$ *and for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $t_1 \in [0,T]$, $t_1 < t_2$ *and* $t_2 - t_1 \leq \delta$, *then*

$$\bigvee_{s=t_1}^{t_2} g_1(t_2,s) \leq \varepsilon.$$

Now, taking into account Lemmas 3.1 and 3.2, we have the following corollary.

Corollary 3.3. *There exist finite positive constants* K_1 *and* K_2 *such that*

$$K_1 = \sup\left\{\bigvee_{s=0}^{t} g_1(t,s) : t \in [0,T]\right\}, \quad \text{where } T > 0 \text{ is arbitrarily fixed}$$

$$\text{and} \quad K_2 = \sup\left\{\bigvee_{y=0}^{x} g_2(x,y) : x \in [0,M]\right\}.$$

Now we prove the following lemma.

Lemma 3.4. *Let assumptions* (iii)–(v) *and* (viii) *be satisfied and let* $k_1(t)$ *be the function defined by the formula* $k_1(t) = \bigvee_{s=0}^{t} g_1(t,s)$. *Then*

$$\overline{K_1} = \sup\{k_1(t) : t \in \mathbb{R}_+\}$$

is a finite positive constant.

Proof. Let us fix $T > 0$. In the case, when $t \in [0,T]$ we have from Corollary 3.3 that $\overline{K_1} = K_1$ is a finite positive constant.

If $t > T$, then from assumption (viii) we have that for any $\varepsilon > 0$ there exists $t_0 > 0$ such that $\bigvee_{s=0}^{t} g_1(t,s) < \varepsilon$ for all $t > t_0$. Now, from our assumptions we get that $\sup\{k_1(t) : t > T\}$ is finite. Linking these facts we conclude that the quantity

$$\overline{K_1} = \sup\{k_1(t) : t \in \mathbb{R}_+\} = \sup\left\{\bigvee_{s=0}^{t} g_1(t,s) : t \in \mathbb{R}_+\right\}$$

is a finite positive constant. □

Now we will formulate our last assumption.

(ix) There exists a positive solution r_0 of the inequality

$$\|h\|_{BC} + \left(kr + \overline{F}\right)\phi(r)\,\overline{K_1}K_2 \leq r$$

such that $\phi(r_0)\,k\overline{K_1}K_2 < 1$.

It is worthwhile noticing that although the last inequality appears to be quite complicated, it is almost always satisfied.

Based on assumption (v) we may define the functions $M_1(\varepsilon)$ and $M_2(\varepsilon)$ by the following formulas

$$M_1(\varepsilon) = \sup\left\{ \bigvee_{s=0}^{t_1} [g_1(t_2,s)-g_1(t_1,s)] \colon \ t_1,t_2\in[0,T],\ t_1<t_2,\ t_2-t_1\le\varepsilon \right\}, \tag{6}$$

where $T > 0$ is an arbitrarily fixed number,

$$M_2(\varepsilon) = \sup\left\{ \bigvee_{y=0}^{x_1} [g_2(x_2,y)-g_2(x_1,y)] \;\middle|\; \begin{array}{l} x_1,x_2\in[0,M], \\ x_1<x_2,\ x_2-x_1\le\varepsilon \end{array} \right\}. \tag{7}$$

From the assumption (v) we deduce that $M_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ for $i = 1, 2$.

Next, we define the functions $N_1(\varepsilon)$ and $N_2(\varepsilon)$ by putting

$$N_1(\varepsilon) = \sup\left\{ \bigvee_{s=t_1}^{t_2} g_1(t_2,s) \colon \ t_1,t_2\in[0,T],\ t_1<t_2,\ t_2-t_1\le\varepsilon \right\}, \tag{8}$$

where $T > 0$ is an arbitrarily fixed number,

$$N_2(\varepsilon) = \sup\left\{ \bigvee_{y=x_1}^{x_2} g_2(x_2,y) \colon \ x_1,x_2\in[0,M],\ x_1<x_2,\ x_2-x_1\le\varepsilon \right\}. \tag{9}$$

We observe that $N_i(\varepsilon) \to 0$ as $\varepsilon \to 0$ for $i = 1, 2$, which is a simple consequence of Lemmas 3.1 and 3.2.

Now we can formulate our main theorem.

Theorem 3.5. *Let assumptions* (i)–(ix) *be satisfied. Then equation* (5) *has at least one solution* $u = u(t, x) \in BC = BC(\mathbb{R}_+ \times [0, M], \mathbb{R})$. *Moreover, all solutions of this equation belonging to the ball* B_{r_0} *are uniformly locally attractive.*

Proof. For arbitrarily fixed function $u \in BC$ and $t \in \mathbb{R}_+$, $x \in [0, M]$, where M is a fixed positive number, let us put

$$(Qu)(t,x) = h(t,x)+f(t,x,u(t,x))\int_0^t\int_0^x v(t,s,x,y,u(s,y))\,d_y g_2(x,y)\,d_s g_1(t,s).$$

If we denote

$$(Fu)(t,x) = f(t,x,u(t,x))$$

and

$$(Vu)(t,x) = \int_0^t\int_0^x v(t,s,x,y,u(s,y))\,d_y g_2(x,y)\,d_s g_1(t,s),$$

we get

$$(Qu)(t,x) = h(t,x) + (Fu)(t,x)(Vu)(t,x).$$

Let us fix arbitrarily $T > 0$ and $\varepsilon > 0$. Take $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| \le \varepsilon$. Without loss of generality we can assume that $t_1 < t_2$, and then $t_2 - t_1 \le \varepsilon$. Next, choose $x_1, x_2 \in [0,M]$ such that $|x_2 - x_1| \le \varepsilon$. Without loss of generality we can assume that $x_1 < x_2$, and then $x_2 - x_1 \le \varepsilon$. Then, keeping in mind our assumptions, for an arbitrarily fixed function $u \in BC$ we obtain:

$$\begin{aligned}
&|(Vu)(t_2,x_2) - (Vu)(t_1,x_1)| \\
&\le \left| \int_0^{t_2}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right. \\
&\quad \left. - \int_0^{t_1}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right| \\
&\quad + \left| \int_0^{t_1}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right. \\
&\quad \left. - \int_0^{t_1}\int_0^{x_2} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right| \\
&\quad + \left| \int_0^{t_1}\int_0^{x_2} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right. \\
&\quad \left. - \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right| \\
&\quad + \left| \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right. \\
&\quad \left. - \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_1,y)\, d_s g_1(t_2,s) \right| \\
&\quad + \left| \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_1,y)\, d_s g_1(t_2,s) \right. \\
&\quad \left. - \int_0^{t_1}\int_0^{x_1} v(t_1,s,x_1,y,u(s,y))\, d_y g_2(x_1,y)\, d_s g_1(t_1,s) \right|.
\end{aligned} \tag{10}$$

Now, we estimate the terms occurring on the right hand side of the above inequality. In our estimations we use Lemmas 2.7 and 2.8 and our assumptions. We get:

$$\begin{aligned}
&\left| \int_0^{t_2}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right. \\
&\quad \left. - \int_0^{t_1}\int_0^{x_2} v(t_2,s,x_2,y,u(s,y))\, d_y g_2(x_2,y)\, d_s g_1(t_2,s) \right| \\
&\le \int_{t_1}^{t_2}\int_0^{x_2} |v(t_2,s,x_2,y,u(s,y))|\, d_y \left(\bigvee_{q=0}^{y} g_2(x_2,q) \right) d_s \left(\bigvee_{p=t_1}^{s} g_1(t_2,p) \right)
\end{aligned}$$

$$\leq \int_{t_1}^{t_2}\int_0^{x_2} \phi\left(|u|\right) d_y\left(\bigvee_{q=0}^{y} g_2\left(x_2,q\right)\right) d_s\left(\bigvee_{p=t_1}^{s} g_1\left(t_2,p\right)\right) \tag{11}$$
$$\leq \phi\left(\|u\|_{BC}\right)\left(\bigvee_{y=0}^{x_2} g_2(x_2,y)\right)\left(\bigvee_{s=t_1}^{t_2} g_1(t_2,s)\right)$$
$$\leq \phi\left(\|u\|_{BC}\right) K_2 N_1\left(\varepsilon\right),$$

where the function $N_1(\varepsilon)$ was defined earlier by (8). Now, using Lemmas 2.7 and 2.8 and assumption (iv), we derive the following estimate:

$$\begin{aligned}
&\left|\int_0^{t_1}\int_0^{x_2} v\left(t_2,s,x_2,y,u(s,y)\right) d_y g_2(x_2,y)\, d_s g_1(t_2,s)\right.\\
&\quad\left. - \int_0^{t_1}\int_0^{x_2} v\left(t_1,s,x_1,y,u(s,y)\right) d_y g_2(x_2,y)\, d_s g_1(t_2,s)\right|\\
&\leq \int_0^{t_1}\int_0^{x_2} \left|v\left(t_2,s,x_2,y,u(s,y)\right) - v\left(t_1,s,x_1,y,u(s,y)\right)\right|\\
&\quad d_y\left(\bigvee_{q=0}^{y} g_2(x_2,q)\right) d_s\left(\bigvee_{p=0}^{s} g_1(t_2,p)\right) \qquad (12)\\
&\leq \int_0^{t_1}\int_0^{x_2} \omega_{13}(v,\varepsilon)\, d_y\left(\bigvee_{q=0}^{y} g_2\left(x_2,q\right)\right) d_s\left(\bigvee_{p=0}^{s} g_1\left(t_2,p\right)\right)\\
&\leq \omega_{13}(v,\varepsilon)\left(\bigvee_{y=0}^{x_2} g_2(x_2,y)\right)\left(\bigvee_{s=0}^{t_2} g_1(t_2,s)\right)\\
&\leq \omega_{13}(v,\varepsilon) K_2 \overline{K_1},
\end{aligned}$$

where we put

$$\omega_{13}(v,\varepsilon)=\sup\left\{\left|v(t_2,s,x_2,y,u)-v(t_1,s,x_1,y,u)\right| \;\middle|\; \begin{array}{l}(t_1,s),(t_2,s)\in\Delta_1^T,\\ (x_1,y),(x_2,y)\in\Delta_2,\\ |t_2-t_1|\leq\varepsilon,\ |x_2-x_1|\leq\varepsilon,\\ u\in\left[-\|u\|_{BC},\|u\|_{BC}\right]\end{array}\right\}.$$

Since the function $v = v\left(t,s,x,y,u\right)$ is uniformly continuous on the set $\Delta_1^T\times\Delta_2\times\left[-\|u\|_{BC},\|u\|_{BC}\right]$, we have that $\omega_{13}(v,\varepsilon)\to 0$ as $\varepsilon\to 0$.

By proceeding similarly as when estimating the (11) and using the functions defined by (6),(7),(9) we obtain estimates of other terms occurring in (10) and finally we have:

$$\begin{aligned}
&\left|(Vu)\left(t_2,x_2\right)-(Vu)\left(t_1,x_1\right)\right|\\
&\leq \omega_{13}(v,\varepsilon)K_2\overline{K_1}+\phi\left(\|u\|_{BC}\right)\left[\left(M_2(\varepsilon)+N_2(\varepsilon)\right)\overline{K_1}+\left(M_1(\varepsilon)+N_1(\varepsilon)\right)K_2\right].
\end{aligned} \tag{13}$$

Taking into account the properties of the functions $M_1(\varepsilon)$, $M_2(\varepsilon)$, $N_1(\varepsilon)$, $N_2(\varepsilon)$ and the fact established after (12) we conclude that the function Vu is continuous on $[0,T]\times[0,M]$. The arbitrariness of T permits us to deduce that Vu is continuous on the set $\mathbb{R}_+\times[0,M]$. Hence, keeping in mind assumptions (i) and (ii) we deduce that the function Qu is continuous on the set $\mathbb{R}_+\times[0,M]$.

Subsequently, we show that for any function $u\in BC$, the function Qu is bounded on $\mathbb{R}_+\times[0,M]$. To this end let us fix $t\in\mathbb{R}_+$ and $x\in[0,M]$. Then, we get

$$\begin{aligned}
&|(Qu)(t,x)|\\
&\leq \|h\|_{BC}+\big(|f(t,x,u(t,x))-f(t,x,0)|+|f(t,x,0)|\big)\\
&\quad\times\int_0^t\int_0^x |v(t,s,x,y,u(s,y))|\; d_y\Bigg(\bigvee_{q=0}^{y} g_2(x,q)\Bigg)\, d_s\Bigg(\bigvee_{p=0}^{s} g_1(t,p)\Bigg) \qquad (14)\\
&\leq \|h\|_{BC}+\big(k\|u\|_{BC}+\overline{F}\big)\,\phi\left(\|u\|_{BC}\right)\int_0^t\int_0^x d_y\Bigg(\bigvee_{q=0}^{y} g_2(x,q)\Bigg)\, d_s\Bigg(\bigvee_{p=0}^{s} g_1(t,p)\Bigg).
\end{aligned}$$

Finally, from (14), Corollary 3.3 and in view of Lemma 3.4 we have

$$|(Qu)(t,x)|\leq \|h\|_{BC}+\big(k\,\|u\|_{BC}+\overline{F}\big)\,\phi\left(\|u\|_{BC}\right)K_2\overline{K_1}. \qquad (15)$$

From estimate (15) follows that the function Qu is bounded on the set $\mathbb{R}_+\times[0,M]$. This fact in conjunction with earlier obtained corollary concerning the continuity of Qu on $\mathbb{R}_+\times[0,M]$ allows us to deduce that Q transforms the space BC into itself. Apart from this observe that estimate (15) yields

$$\|Qu\|_{BC}\leq \|h\|_{BC}+\big(k\,\|u\|_{BC}+\overline{F}\big)\,\phi\left(\|u\|_{BC}\right)K_2\overline{K_1}. \qquad (16)$$

Further, from (16) and assumption (ix) we deduce that there exists a number $r_0>0$ such that Q transforms the ball B_{r_0} into itself. Moreover, we have that $\phi\left(r_0\right)k\overline{K_1}K_2<1$.

In the next step we show that the operator Q is continuous on the ball B_{r_0}. To this end fix $\varepsilon>0$ and $u_0\in B_{r_0}$. Next, take an arbitrary function $u\in B_{r_0}$ such that $\|u-u_0\|_{BC}\leq\varepsilon$. For arbitrarily fixed $(t,x)\in\mathbb{R}_+\times[0,M]$ we have

$$\begin{aligned}
&|\,(Qu)(t,x)-(Qu_0)(t,x)|\\
&\leq |f(t,x,u(t,x))-f(t,x,u_0(t,x))|\\
&\quad\times\int_0^t\int_0^x |v(t,s,x,y,u(s,y))|\, d_y\Bigg(\bigvee_{q=0}^{y} g_2(x,q)\Bigg) d_s\Bigg(\bigvee_{p=0}^{s} g_1(t,p)\Bigg)\\
&\quad+\big[|f(t,x,u_0(t,x))-f(t,x,0)|+|f(t,x,0)|\big]\\
&\quad\times\int_0^t\int_0^x |v(t,s,x,y,u(s,y))-v(t,s,x,y,u_0(s,y))|\, d_y\Bigg(\bigvee_{q=0}^{y} g_2(x,q)\Bigg) d_s\Bigg(\bigvee_{p=0}^{s} g_1(t,p)\Bigg)
\end{aligned}$$

$$\leq k\left\|u-u_0\right\|_{BC}\phi(\|u\|_{BC})\int_0^t\int_0^x d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)$$
$$+\left(k\left\|u_0\right\|_{BC}+\overline{F}\right)$$
$$\times\int_0^t\int_0^x\left|v(t,s,x,y,u(s,y))-v(t,s,x,y,u_0(s,y))\right|d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right).$$

Taking into account assumption (viii) we can choose $T>0$ such that

$$\bigvee_{s=0}^{t}g_1(t,s)<\varepsilon \quad \text{for all } t>T. \tag{17}$$

Now, we consider two cases.

1^0. For $t\in[0,T]$ we get

$$\left|(Qu)(t,x)-(Qu_0)(t,x)\right|\leq k\varepsilon\phi(r_0)K_2\overline{K_1}+\left(kr_0+\overline{F}\right)\omega_{r_0}(v,\varepsilon)K_2\overline{K_1},$$

where

$$\omega_{r_0}(v,\varepsilon)=\sup\left\{\left|v(t,s,x,y,u_1)-v(t,s,x,y,u_2)\right|\,\middle|\,\begin{matrix}(t,s)\in\Delta_1^T,\ (x,y)\in\Delta_2,\\ u_1,u_2\in[-r_0,r_0],\ |u_1-u_2|\leq\varepsilon\end{matrix}\right\}.$$

Because the function v is uniformly continuous on the set $\Delta_1^T\times\Delta_2\times[-r_0,r_0]$ we deduce that $\omega_{r_0}(v,\varepsilon)\to 0$ as $\varepsilon\to 0$.

2^0. For $t>T$, applying (17), we get

$$\left|(Qu)(t,x)-(Qu_0)(t,x)\right|$$
$$\leq k\left\|u-u_0\right\|_{BC}\phi\left(\|u\|_{BC}\right)\int_0^t\int_0^x d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)$$
$$+\left(k\left\|u_0\right\|_{BC}+\overline{F}\right)\int_0^t\int_0^x\Big[\left|v(t,s,x,y,u(s,y))\right|$$
$$+\left|v(t,s,x,y,u_0(s,y))\right|\Big]d_y\left(\bigvee_{q=0}^{y}g_2(x,q)\right)d_s\left(\bigvee_{p=0}^{s}g_1(t,p)\right)$$
$$\leq k\varepsilon\phi(r_0)K_2\varepsilon+\left(kr_0+\overline{F}\right)2\phi(r_0)K_2\varepsilon.$$

Now, joining cases 1^0 and 2^0 we obtain that the operator Q is continuous on the ball B_{r_0}.

In the next step we will investigate the behavior of the operator Q with respect to previously defined measure of noncompactness (1). Let us take a nonempty subset U of the ball B_{r_0}. Further on, choose an arbitrary function

$u \in U$. Next, fix $T > 0$ and $\varepsilon > 0$ and take $(t_1, x_1), (t_2, x_2) \in [0, T] \times [0, M]$ such that $|t_2 - t_1| \le \varepsilon$ and $|x_2 - x_1| \le \varepsilon$. Without loss of generality we can assume that $t_1 < t_2$ and $x_1 < x_2$. Then we get:

$$\begin{aligned} &|(Qu)(t_2, x_2) - (Qu)(t_1, x_1)| \\ &\le |h(t_2, x_2) - h(t_1, x_1)| \\ &\quad + |(Fu)(t_2, x_2)(Vu)(t_2, x_2) - (Fu)(t_2, x_2)(Vu)(t_1, x_1)| \\ &\quad + |(Fu)(t_2, x_2)(Vu)(t_1, x_1) - (Fu)(t_1, x_1)(Vu)(t_1, x_1)| \\ &\le \omega^T(h, \varepsilon) + (k\,|u(t_2, x_2)| + |f(t_2, x_2, 0)|)\,|(Vu)(t_2, x_2) - (Vu)(t_1, x_1)| \\ &\quad + |(Vu)(t_1, x_1)|\,(k\,|u(t_2, x_2) - u(t_1, x_1)| + \overline{\omega}(f, \varepsilon)) \\ &\le \omega^T(h, \varepsilon) + \left(k\,\|u\|_{BC} + \overline{F}\right)|(Vu)(t_2, x_2) - (Vu)(t_1, x_1)| \\ &\quad + |(Vu)(t_1, x_1)|\left(k\omega^T(u, \varepsilon) + \overline{\omega}(f, \varepsilon)\right), \end{aligned} \tag{18}$$

where we denoted

$$\omega^T(h, \varepsilon) = \sup\left\{ |h(t_2, x_2) - h(t_1, x_1)| \;\middle|\; \begin{array}{l} t_1, t_2 \in [0, T],\ x_1, x_2 \in [0, M], \\ |t_2 - t_1| \le \varepsilon,\ |x_2 - x_1| \le \varepsilon \end{array} \right\},$$

$$\overline{\omega}(f, \varepsilon) = \sup\left\{ |f(t_2, x_2, u) - f(t_1, x_1, u)| \;\middle|\; \begin{array}{l} t_1, t_2 \in [0, T],\ x_1, x_2 \in [0, M], \\ u \in [-r_0, r_0],\ |t_2 - t_1| \le \varepsilon,\ |x_2 - x_1| \le \varepsilon \end{array} \right\}$$

and $\omega^T(u, \varepsilon)$ was defined previously in the Section 2.

Since the functions h and u are uniformly continuous on the set $[0, T] \times [0, M]$, thus we have that $\omega^T(h, \varepsilon) \to 0$ as $\varepsilon \to 0$ and $\omega^T(u, \varepsilon) \to 0$ as $\varepsilon \to 0$. Analogously, the function f is uniformly continuous on the set $[0, T] \times [0, M] \times [-r_0, r_0]$, and hence we conclude that $\overline{\omega}(f, \varepsilon) \to 0$ as $\varepsilon \to 0$.

From estimate (13) we obtain

$$\omega^T(Vu, \varepsilon) \le \omega_{13}(v, \varepsilon) K_2 \overline{K_1} + \phi(r_0)\left[(M_2(\varepsilon) + N_2(\varepsilon))\overline{K_1} + (M_1(\varepsilon) + N_1(\varepsilon)) K_2\right]. \tag{19}$$

Subsequently, from Lemma 2.7 and assumptions (iv) and (vii), we derive

$$|(Vu)(t_1, x_1)| \le \phi(\|u\|_{BC}) \left(\bigvee_{y=0}^{x_1} g_2(x_1, y) \right) \left(\bigvee_{s=0}^{t_1} g_1(t_1, s) \right) \le \phi(r_0) K_2 \overline{K_1}. \tag{20}$$

Finally, combining estimates (18)–(20), we get

$$\begin{aligned} &\omega^T(QU, \varepsilon) \\ &\le \omega^T(h, \varepsilon) + \phi(r_0) K_2 \overline{K_1}\left(k\omega^T(U, \varepsilon) + \overline{\omega}(f, \varepsilon)\right) \\ &\quad + (kr_0 + \overline{F})\left\{\omega_{13}(v, \varepsilon) K_2 \overline{K_1} + \phi(r_0)\left[(M_2(\varepsilon) + N_2(\varepsilon))\overline{K_1} + (M_1(\varepsilon) + N_1(\varepsilon)) K_2\right]\right\}. \end{aligned}$$

Linking the properties of the functions $M_i(\varepsilon)$, $N_i(\varepsilon)$ $(i = 1, 2)$ and the functions $\varepsilon \to \omega^T(h, \varepsilon)$, $\varepsilon \to \omega_{13}(v, \varepsilon)$, $\varepsilon \to \overline{\omega}(f, \varepsilon)$, we deduce the following inequality $\omega_0^T(QU) \le \phi(r_0) K_2 \overline{K_1} k \omega_0^T(U)$. This yields

$$\omega_0^\infty (QU) \le \phi(r_0) K_2 \overline{K_1} k \omega_0^\infty (U). \tag{21}$$

In the next step, let us take the functions $u_1, u_2 \in U \subset B_{r_0}$. Then, for arbitrarily fixed $t \in \mathbb{R}_+$ and $x \in [0, M]$ we obtain:

$$\begin{aligned}
&|(Qu_2)(t,x) - (Qu_1)(t,x)| \\
&\le |(Fu_2)(t,x)||(Vu_2)(t,x) - (Vu_1)(t,x)| \\
&\quad + |(Vu_1)(t,x)||(Fu_2)(t,x) - (Fu_1)(t,x)| \\
&\le (k|u_2(t,x)| + |f(t,x,0)|)|(Vu_2)(t,x) - (Vu_1)(t,x)| \\
&\quad + |(Vu_1)(t,x)|k|u_2(t,x) - u_1(t,x)|.
\end{aligned} \tag{22}$$

Further, from our assumptions, Corollary 3.3 and in view of Lemma 3.4 we have

$$|(Vu_1)(t,x)| \le \phi(\|u_1\|_{BC}) \left(\bigvee_{y=0}^{x} g_2(x,y) \right) \left(\bigvee_{s=0}^{t} g_1(t,s) \right) \le \phi(r_0) K_2 \overline{K_1} \tag{23}$$

and

$$\begin{aligned}
&|(Vu_2)(t,x) - (Vu_1)(t,x)| \\
&\le \int_0^t \int_0^x \big(|v(t,s,x,y,u_2(s,y))| \\
&\quad + |v(t,s,x,y,u_1(s,y))| \big) \, d_y \left(\bigvee_{q=0}^{y} g_2(x,q) \right) d_s \left(\bigvee_{p=0}^{s} g_1(t,p) \right) \\
&\le 2\phi(r_0) K_2 \left(\bigvee_{s=0}^{t} g_1(t,s) \right).
\end{aligned} \tag{24}$$

Combining (22)–(24) we derive the estimate

$$\operatorname{diam}(QU)(t,x) \le (kr_0 + \overline{F})\, 2\phi(r_0) K_2 \left(\bigvee_{s=0}^{t} g_1(t,s) \right) + \phi(r_0) K_2 \overline{K_1} k \operatorname{diam} U(t,x).$$

Hence, in view of assumption (viii) we get

$$\begin{aligned}
&\limsup_{t\to\infty} \{\sup\{\operatorname{diam}(QU)(t,x) : x \in [0,M]\}\} \\
&\le \phi(r_0) K_2 \overline{K_1} k \limsup_{t\to\infty} \{\sup\{\operatorname{diam} U(t,x) : x \in [0,M]\}\}.
\end{aligned}$$

This implies

$$a(QU) \le \phi(r_0) K_2 \overline{K_1} k a(U). \tag{25}$$

Finally, linking estimates (21) and (25) we obtain $\mu(QU) \le \phi(r_0) K_2 \overline{K_1} k \mu(U)$. Taking into account Theorem 2.2 and the definition of the measure of noncompactness given by formula (1), in view of assumption (ix), we conclude that there exists at least one function $u = u(t,x)$ belonging to the ball B_{r_0}, which is the solution of equation (5). Keeping in mind the description of the kernel of the measure of noncompactness μ defined by the formula (1) in the space BC and taking into account Remark 2.3 and Definition 2.5 we deduce that solutions of equation (5) belonging to the ball B_{r_0} are uniformly locally attractive. The proof is complete. □

Remark 3.6. Notice that in our considerations the interval $[0, M]$ can be replaced by any interval $[a, b]$, $a < b$.

4. Some special cases

In this section we will discuss a few special cases of the nonlinear quadratic integral equation (5) studied in the previous section. In the paper [17] there was investigated the nonlinear quadratic integral equation of Volterra-Stieltjes type in two variables of form (5), but the domain of consideration was the bounded rectangle $[0, 1] \times [0, 1]$.

Further on, let us observe that in the case, when we have $g_1(t, s) = s$ and $g_2(x, y) = y$ and $Fu \equiv 1$, equation (5) reduces to the classical nonlinear Volterra integral equation in two variables of the form

$$u(t,x) = h(t,x) + \int_0^t \int_0^x v(t,s,x,y,u(s,y))\,dyds$$

and when we have $g_1(t, s) = s$ and $g_2(x, y) = y$ and $h(t, x) \equiv 0$ we obtain the following equation

$$u(t,x) = f(t,x,u(t,x)) \int_0^t \int_0^x v(t,s,x,y,u(s,y))\,dyds.$$

Obviously solutions of these equations should be investigated under other assumptions.

Now, we give examples of functions $g_i(w, z) = g_i : \Delta_i \to \mathbb{R}$ $(i = 1, 2)$ occurring in equation (5). First, we recall the lemma associated with these functions (cf. [11, 17, 30]).

Lemma 4.1. *Suppose that the function $g_i = g_i(w, z)$ satisfies assumptions* (iii) *and* (vi) *for $i = 1, 2$ and the following condition:*

- (v') *The function $z \to g_i(w_2, z) - g_i(w_1, z)$ $(i = 1, 2)$ is monotone on the interval $[0, w_1]$ for arbitrarily fixed w_1, w_2 such that $0 \le w_1 < w_2$. Moreover, $w_2 \le M$ in the case $i = 2$.*

Then g_i satisfies assumption (v) $(i = 1, 2)$.

As example of functions defined on the triangle $\Delta_2 = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le M\}$ we can take the functions $g_2(x,y) = g_2 : \Delta_2 \to \mathbb{R}$, $\widetilde{g_2}(x,y) = \widetilde{g_2} : \Delta_2 \to \mathbb{R}$ defined as follows

$$g_2(x,y) = \frac{1}{\alpha}\left[x^\alpha - (x-y)^\alpha\right], \quad \text{where } \alpha \in (0,1), \tag{26}$$

$$\widetilde{g_2}(x,y) = \begin{cases} x \ln \frac{x+y}{x} & \text{for } 0 < y \le x \le M \\ 0 & \text{for } x = 0 \end{cases}. \tag{27}$$

Then we get

$$d_y g_2(x,y) = \frac{1}{(x-y)^{1-\alpha}} dy \quad \text{and} \quad d_y \widetilde{g_2}(x,y) = \frac{x}{x+y} dy.$$

Of course, it can be easily verified that these functions satisfy assumptions (iii), (iv), (v)b) and (vi) (cf. [11, 17, 30]). The constant K_2 from Corollary 3.3 is finite and for the function g_2 this constant is equal $K_2 = \frac{1}{\alpha} M^\alpha$ while for the function $\widetilde{g_2}$ we have $\widetilde{K_2} = M \ln 2$.

We could not consider the functions given by the formulas (26) and (27) as the function g_1 defined on the triangle $\Delta_1 = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t\}$, because then assumption (viii) is not satisfied.

On the other hand consider the function $g_1(t,s) = g_1 : \Delta_1 \to \mathbb{R}$, defined on the triangle Δ_1 and having the form

$$g_1(t,s) = a(t)b(s),$$

where $a : \mathbb{R}_+ \to \mathbb{R}$, $b : \mathbb{R}_+ \to \mathbb{R}$. Then assumption (viii) has the form

(viii') $\lim_{t\to\infty} \left(|a(t)| \bigvee_{s=0}^{t} b(s)\right) = 0.$

In such a situation we can consider interesting cases. For example, the assumption (viii') is satisfied, when $\lim_{t\to\infty} a(t) = 0$ and $\bigvee_{s=0}^{t} b(s)$ is bounded on $\mathbb{R}_+$. If we additionally assume that the function $b(t)$ is continuous on $\mathbb{R}_+$, $b(0) = 0$ and the function $a(t)$ satisfies the Lipschitz condition on the $[0,T]$ for any fixed $T > 0$, then we get that the assumptions (iii), (iv), (v)a) and (vi) are satisfied. Moreover, the assumption (v)a) has the form:

(v') a) For any $T > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$ and $t_2 - t_1 \le \delta$ the following inequality holds

$$|a(t_2) - a(t_1)| \bigvee_{s=0}^{t_1} b(s) \le \varepsilon.$$

We will verify now the assumptions concerning the function $g_1(t,s) = g_1 : \Delta_1 \to \mathbb{R}$, if this function is given by

$$g_1(t,s) = \sum_{i=1}^{k} a_i(t) b_i(s),$$

where $a_i : \mathbb{R}_+ \to \mathbb{R}$, $b_i : \mathbb{R}_+ \to \mathbb{R}$ for $i = 1, 2, \ldots, k$. Then we have

$$\bigvee_{s=0}^{t} g_1(t,s) = \sum_{i=1}^{k} \left(|a_i(t)| \bigvee_{s=0}^{t} b_i(s) \right).$$

Let us assume that the functions $b_i = b_i(t)$ $(i = 1, 2, \ldots, k)$ are continuous on $\mathbb{R}_+$. Additionally, we require that $b_i(0) = 0$ for $i = 1, 2, \ldots, k$ and $\bigvee_{s=0}^{t} b_i(s)$ is bounded on $\mathbb{R}_+$ for $i = 1, 2, \ldots, k$. Moreover, let us assume that the functions $a_i = a_i(t)$ $(i = 1, 2, \ldots, k)$ satisfy the Lipschitz condition on $[0, T]$ for each fixed $T > 0$, i.e., for any $T > 0$ there exist the constants $L_i > 0$ $(i = 1, 2, \ldots, k)$ such that $|a_i(t_2) - a_i(t_1)| \le L_i |t_2 - t_1|$ for all $t_1, t_2 \in [0, T]$, $i = 1, 2, \ldots, k$. Apart from this we assume that $\lim_{t\to\infty} a_i(t) = 0$ for $i = 1, 2, \ldots, k$. It is not difficult to check, that when these conditions are fulfilled, then assumptions (iii), (iv), (v)a), (vi) and (viii) are satisfied.

Now, we provide an example illustrating the main result contained in Theorem 3.5.

Example 4.2. Consider the following integral equation

$$\begin{aligned} u(t,x) = \frac{\ln(x+1)}{t+2} &+ \frac{\operatorname{arctg}(x^2 + u(t,x))}{e^{t+x}+1} \\ &\times \int_0^t \int_0^x \frac{\sqrt[3]{|u(s,y)|}\, x \sin(t^2+s^2+xy+1)\left[(t+s)\ln(1+\frac{s}{t})+s\right]}{(x+y)(t+s)(t^2+1)}\, dyds, \end{aligned} \tag{28}$$

where $t \in \mathbb{R}_+ = [0, +\infty)$ and $x \in [0, 1]$ (i.e., $M = 1$). Equation (28) is a special case of equation (5), if we put

$$\begin{aligned} h(t,x) &= \frac{\ln(x+1)}{t+2}, \\ f(t,x,u) &= \frac{\operatorname{arctg}(x^2+u)}{e^{t+x}+1}, \\ v(t,s,x,y,u) &= \sqrt[3]{|u|} \sin\left(t^2+s^2+xy+1\right), \\ g_1(t,s) &= \begin{cases} \frac{s}{t^2+1} \ln\left(1+\frac{s}{t}\right) & \text{for } 0 < s \le t \\ 0 & \text{for } t = 0 \end{cases}, \\ d_s g_1(t,s) &= \frac{(t+s)\ln\left(1+\frac{s}{t}\right)+s}{(t+s)(t^2+1)} ds, \\ g_2(x,y) &= \begin{cases} x \ln\left(1+\frac{y}{x}\right) & \text{for } 0 < y \le x \le 1 \\ 0 & \text{for } x = 0 \end{cases}, \\ d_y g_2(x,y) &= \frac{x}{x+y} dy. \end{aligned}$$

It is easy to check that assumptions (i)–(viii) of the Theorem 3.5 are satisfied. Indeed, after standard calculations we obtain that

$$\|h\|_{BC} = \frac{1}{2}\ln 2,\ \overline{F} = \frac{\pi}{2},\ k = 1,\ \overline{K_1} = \frac{1}{2}\ln 2,\ K_2 = \ln 2 \text{ and } \phi(r) = \sqrt[3]{r}.$$

The inequality from assumption (ix) has the form $\|h\|_{BC}+(kr+\overline{F})\sqrt[3]{r}\,\overline{K_1}K_2 \leq r$. Keeping in mind the above indicated estimates we get $2(\ln 2)^2 r^{\frac{4}{3}} - 4r + \pi(\ln 2)^2 r^{\frac{1}{3}} + 2\ln 2 \leq 0$ and we deduce that, e.g., the number $r_0 = 1$ is a solution of this inequality, such that $\phi(r_0)\, k\overline{K_1}K_2 < 1$. Thus, in view of Theorem 3.5 we conclude that equation (28) has a solution in the space BC belonging to the ball B_1. Moreover, all solutions of equation (28), which belong to B_1, are uniformly locally attractive in the sense of Definition 2.5.

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