Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 36 (2017), 159–190
DOI: 10.4171/ZAA/1584

Boundedness of the Generalized Fractional Integral Operators on Generalized Morrey Spaces over Metric Measure Spaces

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Abstract. Our aim in this paper is to deal with the boundedness of the generalized fractional integral operators on generalized Morrey spaces $L_{p,\phi;2}(X;\mu)$ over metric measure spaces. We also discuss a necessary condition for the boundedness of the generalized fractional integral operators. As applications, we establish new results for the predual spaces.

Keywords. Sobolev's inequality, Morrey space, Riesz potential, non-doubling measure, predual spaces

Mathematics Subject Classification (2010). Primary 26A33, secondary 42B25, 42B35

1. Introduction

In the present paper, we aim to show boundedness of the generalized fractional integral operators on generalized Morrey spaces $L_{p,\phi;2}(X;\mu)$ over connected metric measure spaces (X,d,μ) assuming that the μ-measure of any ball is finite. We also discuss a necessary condition for the boundedness of the generalized fractional integral operators. Our results will extend [11, 14, 18, 22, 25, 36] and [12, Theorems 1.1–1.3]. As applications, we establish new results for the predual spaces.

Before we describe our results, let us place ourselves in the Euclidean space $\mathbb{R}^n$ and view an elementary results for Morrey spaces and fractional integral operators. For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbb{R}^n$ by

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \quad (x \in \mathbb{R}^n).$$

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The operator I_α is also called the fractional integral operator.

We denote by $B(z,r)$ the ball $\{x \in \mathbb{R}^n : |x-z| < r\}$ with center z and of radius $r > 0$, and by $|B(z,r)|$ its Lebesgue measure, i.e. $|B(z,r)| = \omega_n r^n$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

For $1 \le p < \infty$ and $\phi : (0,\infty) \to (0,\infty)$, let the generalized Morrey space $L_{p,\phi}(\mathbb{R}^n)$ be the family of all $f \in L^p_{\rm loc}(\mathbb{R}^n)$ such that $\|f\|_{L_{p,\phi}} < \infty$, where

$$\|f\|_{L_{p,\phi}} \equiv \sup_{z\in\mathbb{R}^n,\, r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(z,r)|} \int_{B(z,r)} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

When $\phi(r) \equiv r^{-\frac{\lambda}{p}}$ $(r > 0)$, $L_{p,\phi}(\mathbb{R}^n)$ coincides with $L^{p,\lambda}(\mathbb{R}^n)$ in Adams [1]. When $\phi(r) \equiv r^{-\frac{n}{q}}$, then we have the scaling relation

$$\|f(t\cdot)\|_{L_{p,\phi}} = t^{-\frac{n}{q}} \|f\|_{L_{p,\phi}}$$

for all $t > 0$ and $f \in L_{p,\phi}(\mathbb{R}^n)$.

If $\phi \equiv 1$, then $L_{p,\phi}(\mathbb{R}^n) \sim L^\infty(\mathbb{R}^n)$ with norm equivalence.

Adams [1, Theorem 3.1] showed that

$$\|I_\alpha f\|_{L^{q,\lambda}} \le C \|f\|_{L^{p,\lambda}}$$

provided that the parameters p, q, λ satisfy

$$1 < p < q < \infty,\ 0 < \lambda \le n,\ -\frac{\lambda}{p} + \alpha = -\frac{\lambda}{q}.$$

If $\lambda = n$, then this is the Hardy–Littlewood–Sobolev theorem. See also [7–9, 14, 17, 22, 26, 29, 30, 42].

For a function $\rho : (0,\infty) \to (0,\infty)$, the generalized Riesz potential $I_\rho f$ is defined by

$$I_\rho f(x) \equiv \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy \quad (x \in \mathbb{R}^n).$$

The operator I_ρ is also called the generalized fractional integral operator. If $\rho(r) = r^\alpha$ for $0 < \alpha < n$, then $I_\rho f$ coincides with the usual Riesz potential of order α. The generalized Riesz potential $I_\rho f$ was introduced in [23]. The boundedness of $I_\rho f$ of functions in $L_{p,\phi}(\mathbb{R}^n)$ was investigated in [14]. For the boundedness of $I_\rho f$, we also refer the reader to [9, 24, 36].

Now let us formulate our main results. Let X be a connected separable metric space equipped with a non-negative Radon measure μ. By $B(x,r)$ we denote the open ball centered at $x \in X$ of radius $r > 0$. We write $d(x,y)$ for the distance of the points x and y in X. We assume that $\mu(\{x\}) = 0$ and that $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$ for simplicity. We do not postulate on μ the "so called" doubling condition to show the boundedness of fractional integral operator; see Theorems 1.2, 1.5, 1.8 and 1.12. Recall that a Radon measure μ is said to be doubling, if there exists a constant $C > 0$ such that $\mu(B(x,2r)) \le C\mu(B(x,r))$ for all $x \in \operatorname{supp}(\mu)(= X)$ and $r > 0$. Otherwise μ is said to be non-doubling. In connection with the $5r$-covering lemma, the doubling condition had been a key condition in harmonic analysis. However, Nazarov, Treil and Volberg showed that the doubling condition is not necessary by using the modified maximal operator [27, 28]. See also [32–35, 39].

Let $\mathcal{G}$ be the set of all functions from $(0,\infty)$ to itself with the doubling condition; that is, there exists a constant $c_\phi \ge 1$ such that

$$\frac{1}{c_\phi} \le \frac{\phi(r)}{\phi(s)} \le c_\phi \quad \text{for} \quad r, s > 0 \quad \text{with} \quad \frac{1}{2} \le \frac{r}{s} \le 2. \tag{1}$$

We call the smallest number c_ϕ satisfying (1) the doubling constant of ϕ.

For $\phi \in \mathcal{G}$, let the generalized Morrey space $L_{p,\phi;\kappa}(X;\mu)$ be the set of all functions $f \in L^p_{\mathrm{loc}}(X;\mu)$ such that $\|f\|_{L_{p,\phi;\kappa}(X;\mu)} < \infty$, where

$$\|f\|_{L_{p,\phi;\kappa}(X;\mu)} \equiv \sup_{z\in X,\, r>0} \frac{1}{\phi(r)} \left(\frac{1}{\mu(B(z,\kappa r))} \int_{B(z,r)} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.$$

Observe that in [34, Section 2] we showed that $\|f\|_{L_{p,\phi;2}(X;\mu)}$ and $\|f\|_{L_{p,\phi;4}(X;\mu)}$ can be norms which are not equivalent. We also define the generalized Riesz potential $I_{\rho,\mu,\tau} f$ by:

$$I_{\rho,\mu,\tau} f(x) \equiv \int_X \frac{\rho(d(x,y))}{\mu(B(x,\tau d(x,y)))} f(y) \, d\mu(y) \quad (x \in X)$$

for $\tau > 0$ and a measurable function $\rho : (0,\infty) \to (0,\infty)$.

In this paper we aim to give a general version of boundedness of generalized Riesz potentials $I_{\rho,\mu,32} f$ of functions in generalized Morrey spaces $L_{p,\phi;2}(X;\mu)$ over metric measure spaces. We also discuss necessity of conditions for the boundedness of $I_{\rho,\mu,32} f$. Our results extend those in [11, 12, 14, 18, 25, 36].

We postulate the following conditions on $\rho : (0,\infty) \to (0,\infty)$: There exist constants $C_\rho > 0$ and $0 < k_1 < k_2 \le 2k_1 < \infty$ such that

$$\int_0^1 \frac{\rho(t)}{t} \, dt < \infty,$$

$$\sup_{\frac{r}{2} \le s \le r} \rho(s) \le C_\rho \int_{k_1 r}^{k_2 r} \frac{\rho(t)}{t} \, dt \quad (r > 0). \tag{2}$$

We denote the set of all such functions by $\mathcal{G}_0$.

Example 1.1. Let (X, d, μ) be the Euclidean space. In view of [2], we see that $(1-\Delta)^{-\frac{\alpha}{2}}$ falls under the scope of our main results. As $r \downarrow 0$, when $0 < \alpha < n$,

$$r^{-n}\rho(r) \sim \frac{1}{2^\alpha \pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\Gamma\left(\frac{n-\alpha}{2}\right)r^{\alpha-n},$$

when $\alpha = n$

$$r^{-n}\rho(r) \sim \frac{1}{2^{n-1}\pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\log\frac{1}{r},$$

and when $\alpha > n$,

$$r^{-n}\rho(r) \sim \frac{1}{2^n \pi^{\frac{n}{2}}}\Gamma\left(\frac{\alpha}{2}\right)^{-1}\Gamma\left(\frac{\alpha-n}{2}\right).$$

See [2, (4.2)]. Remark that when $\alpha > n$, the integral kernel is trivially integrable, so that $r^{-n}\rho(r)$ behaves like a constant function as r goes to zero. Furthermore, as $r \to \infty$,

$$r^{-n}\rho(r) \sim \frac{r^{\frac{\alpha-n-1}{2}}e^{-r}}{2^{\frac{n+\alpha-1}{2}}\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}.$$

See [2, (4.3)]. The above estimates mean that we have (2) with $k_1 = \frac{1}{4}$ and $k_2 = \frac{1}{2}$. Note that $\rho \in \mathcal{G}$ implies (2). See [38, Remark 2.2] as well as [21, Lemma 2.5].

Set

$$\tilde{\rho}(r) \equiv \int_0^r \frac{\rho(s)}{s}ds \quad \text{for } r > 0. \tag{3}$$

Let $\mathcal{G}_1$ be the set of all almost decreasing functions in $\mathcal{G}$ from $(0,\infty)$ to itself, that is, $\phi \in \mathcal{G}_1$ if and only if $\phi \in \mathcal{G}$ and there exists a constant $C > 0$ such that

$$\phi(r) \le C\phi(s) \quad \text{for all } r \ge s > 0.$$

Remark that we are given three classes $\mathcal{G}, \mathcal{G}_0, \mathcal{G}_1$. With these definitions in mind, we state the following result in the metric setting, which extends [12, Theorem 1.1]. See also research papers [3–9, 17, 22, 29, 30, 42].

Theorem 1.2. *Let $1 < p < q < \infty$, $\rho \in \mathcal{G}_0$ and $\phi \in \mathcal{G}_1$. Assume that*

$$\int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\rho(r)\phi(r) \quad (r > 0), \tag{4}$$

and that

$$\int_r^\infty \frac{\phi(t)}{t}dt \le C\phi(r) \quad (r > 0) \tag{5}$$

for some constant $C > 0$. If there exists a constant $C' > 0$ such that

$$\tilde{\rho}(r) \le C'\phi(r)^{\frac{p}{q}-1} \quad (r > 0), \tag{6}$$

then $I_{\rho,\mu,32}$ is bounded from $L_{p,\phi;2}(X;\mu)$ to $L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$.

Next we will show that condition (6) is necessary for the boundedness of $I_{\rho,\mu,32}f$; the next result extends [12, Theorem 1.1] to homogeneous spaces.

Theorem 1.3. *Let μ be a doubling measure. Let $1 < p < q < \infty$, $\rho \in \mathcal{G} \cap \mathcal{G}_0$ and $\phi \in \mathcal{G}_1$. Assume that there exists a constant $C > 0$ such that*

$$\phi(r)\mu(B(x,r))^{\frac{1}{p}} \le C\phi(r')\mu(B(x,r'))^{\frac{1}{p}} \tag{7}$$

whenever $x \in X$ and $0 < r \le r'$. If $I_{\rho,\mu,32}$ is bounded from $L_{p,\phi;2}(X;\mu)$ to $L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$, then (6) holds for some constant $C' > 0$.

Before we go further, a couple of helpful remarks may be in order.

Remark 1.4. 1. Observe that condition (6) corresponds to the scaling condition in the classical case. Indeed, if we let

$$\rho(t) = t^{\alpha},\ \phi(t) = t^{\beta} \quad (t > 0)$$

with $\alpha > 0 > \beta$, then (6) reads $\alpha = \beta\left(\frac{p}{q} - 1\right)$. In the setting of Lebesgue measure, this says that I_α is bounded from $L_{p,\phi}(\mathbb{R}^n)$ to $L_{q,\phi^{\frac{p}{q}}}(\mathbb{R}^n)$ if and only if $\alpha = \beta\left(\frac{p}{q} - 1\right)$. So, this condition can be taken as the scaling condition.

2. In Theorem 1.3, $I_{\rho,\mu,32}$ is bounded if and only if $I_{\rho,\mu,1}$ is bounded.

The following result extends [14, Theorem B] and [12, Theorem 1.2].

Theorem 1.5. *Let $1 < p < q < \infty$, $\rho \in \mathcal{G}_0$ and $\phi \in \mathcal{G}_1$. Assume (5) for some constant $C > 0$. If, in addition, there exists a constant $C' > 0$ such that*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C'\phi(r)^{\frac{p}{q}} \quad (r > 0), \tag{8}$$

then $I_{\rho,\mu,32}$ is bounded from $L_{p,\phi;2}(X;\mu)$ to $L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$.

Theorem 1.5 extends [33, Theorem 3.3] in that the underlying space and the integral kernel of the fractional integral operator are generalized.

We will show that condition (8) is necessary for the boundedness of $I_{\rho,\mu,32}f$. This result extends [12, Theorem 1.2].

Theorem 1.6. *Let μ be a doubling measure. Let $1 < p < q < \infty$, $\rho \in \mathcal{G} \cap \mathcal{G}_0$ and $\phi \in \mathcal{G}_1$. Assume (7) and that there exists a constant $C > 0$ such that*

$$\int_0^r \mu(B(x,t))^{\frac{1}{p}}\frac{\phi(t)}{t}\,dt \le C\mu(B(x,r))^{\frac{1}{p}}\phi(r) \tag{9}$$

for $x \in X$ and $r > 0$. If, in addition, $I_{\rho,\mu,32}$ is bounded from $L_{p,\phi;2}(X;\mu)$ to $L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$, then there exists a constant $C' > 0$ such that (8) holds.

Remark 1.7. We have a counterpart for the weak spaces considered in [15]. Here we omit the detail, since the proof is a minor modification of the above results.

Let $\phi \in \mathcal{G}_1$ and $1 \le p < \infty$. Define $p' = \frac{p}{p-1}$, the harmonic conjugate. We consider applications of the results above: We can pass these results to predual spaces. Following [40,45], we say that a μ-measurable function b is said to be a (p', ϕ)-block, if b is supported on a ball $B(x,r)$ such that

$$\|b\|_{L^{p'}(X;\mu)} \le \frac{1}{\mu(B(x,2r))^{\frac{1}{p}}\phi(r)}. \tag{10}$$

A function f is said to belong $\mathcal{H}_{p',\phi}(X;\mu)$, if f has an expression:

$$f = \sum_{j=1}^{\infty} \lambda_j b_j, \tag{11}$$

where each b_j is a (p', ϕ)-block and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$. The norm of such f is given by

$$\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)} = \inf \sum_{j=1}^{\infty} |\lambda_j|, \tag{12}$$

where the sequence $\{\lambda_j\}_{j=1}^{\infty}$ runs over all expressions as above. Observe that (7) guarantees the μ-a.e. absolute convergence of the right-hand side of (11). Indeed, for each (p', ϕ)-block a and $g \in L_{p,\phi;2}(X;\mu)$, we have

$$\int_X |a(x)g(x)|\, d\mu(x) \le \|g\|_{L_{p,\phi;2}(X;\mu)}$$

by virtue of the Hölder inequality. By (7), we can take $g = \chi_{B(x,r)}$ in the above. A direct consequence of this choice is that any (p', ϕ)-block has integral over $B(x,r)$ less than a constant depending only on r. Therefore, in (11), the series converges absolutely μ-a.e. About the predual space $\mathcal{H}_{p',\phi}(X;\mu)$ above, we have the following result:

Theorem 1.8. *Let $1 < p < \infty$, $\rho \in \mathcal{G}_0$ and $\phi \in \mathcal{G}_1$.*

(1) (a) *Let $f \in \mathcal{H}_{p',\phi}(X;\mu)$. Then, for all $g \in L_{p,\phi;2}(X;\mu)$, $f \cdot g \in L^1(X;\mu)$ and*

$$\int_X |f(x)g(x)|\, d\mu(x) \le \|f\|_{\mathcal{H}_{p',\phi}(X;\mu)} \|g\|_{L_{p,\phi;2}(X;\mu)}.$$

In particular, any $g \in L_{p,\phi;2}(X;\mu)$ defines a continuous linear functional L_g on $\mathcal{H}_{p',\phi}(X;\mu)$.

(b) *Conversely, any continuous linear functional L on $\mathcal{H}_{p',\phi}(X;\mu)$ can be realized as L_g by some $g \in L_{p,\phi;2}(X;\mu)$.*

(2) *The mapping* $g \in L_{p,\phi;2}(X;\mu) \mapsto L_g \in B(\mathcal{H}_{p',\phi}(X;\mu),\mathbb{C})$ *is an isomorphism, where* $B(\mathcal{H}_{p',\phi}(X;\mu),\mathbb{C})$ *denotes the dual of* $\mathcal{H}_{p',\phi}(X;\mu)$.

Theorem 1.9. *Let* $1 < p < q < \infty$, $\rho \in \mathcal{G}_0$ *and* $\phi \in \mathcal{G}_1$.

(1) *Assume* (4)–(6) *for some constant* $C > 0$ *and* $C' > 0$. *Then,* $I_{\rho,\mu,33}$ *is bounded from* $\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$ *to* $\mathcal{H}_{p',\phi}(X;\mu)$.

(2) *Assume* (5) *and* (8) *for some constant* $C > 0$ *and* $C' > 0$. *Then* $I_{\rho,\mu,33}$ *is bounded from* $\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$ *to* $\mathcal{H}_{p',\phi}(X;\mu)$.

Parts (1) and (2) of Theorem 1.9 can be considered as the dual of Theorems 1.2 and 1.5, respectively.

Theorems 1.3 and 1.6 are readily transplanted into the space $\mathcal{H}_{p',\phi}(X;\mu)$ as follows:

Theorem 1.10. *Let* μ *be a doubling measure. Let* $1 < p < q < \infty$, $\rho \in \mathcal{G} \cap \mathcal{G}_0$ *and* $\phi \in \mathcal{G}_1$. *Assume that there exists a constant* $C > 0$ *such that* (7) *holds whenever* $x \in X$ *and* $0 < r \le r'$. *If* $I_{\rho,\mu,31}$ *is bounded from* $\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$ *to* $\mathcal{H}_{p',\phi}(X;\mu)$, *then there exists a constant* $C' > 0$ *such that* (6) *holds.*

Theorem 1.11. *Let* μ *be a doubling measure. Let* $1 < p < q < \infty$, $\rho \in \mathcal{G} \cap \mathcal{G}_0$ *and* $\phi \in \mathcal{G}_1$. *Assume* (7) *and that there exists a constant* $C > 0$ *such that* (9) *holds for* $x \in X$ *and* $r > 0$. *If* $I_{\rho,\mu,31}$ *is bounded from* $\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$ *to* $\mathcal{H}_{p',\phi}(X;\mu)$, *then there exists a constant* $C' > 0$ *such that* (8) *holds.*

Theorems 1.8–1.11 supplement the earlier paper [12], that is, these four theorems for the predual spaces are new even in the $\mathbb{R}^n$ case.

Finally, we treat the case when $p = 1$, which extends [14], [12, Theorem 1.3], [36, Theorem 2], [25] and [18].

Theorem 1.12. *Let* $\rho \in \mathcal{G}_0$ *and* $\phi \in \mathcal{G}$. *If, in addition, there exists a constant* $C' > 0$ *such that*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C'\psi(r) \quad (r > 0),$$

then $I_{\rho,\mu,4}$ *is bounded from* $L_{1,\phi;2}(X;\mu)$ *to* $L_{1,\psi;4}(X;\mu)$.

In connection with Theorem 1.12 the functions supported on bounded open sets were considered in [36, Theorem 2]. Since the proof of Theorem 1.12 is very similar to that of [36, Theorem 2], we omit the proof.

We will show that condition (8) is a necessary condition for the boundedness of $I_{\rho,\mu,4}f$. This result extends [12, Theorem 1.3].

Theorem 1.13. *Let μ be a doubling measure. Let $\rho \in \mathcal{G} \cap \mathcal{G}_0$ and $\phi \in \mathcal{G}_1$. Assume* (7) *and that there exists a constant $C > 0$ such that*

$$\int_0^r \mu(B(x,t)) \frac{\phi(t)}{t}\, dt \le C\mu(B(x,r))\phi(r).$$

If, in addition, $I_{\rho,\mu,4}$ is bounded from $L_{1,\phi;2}(X;\mu)$ to $L_{1,\psi;4}(X;\mu)$, then there exists a constant $C' > 0$ such that (8) *holds.*

More and more attention is paid to analysis on measure spaces. Here we present some examples of metric measure space to which our results are applicable.

Example 1.14. In this example, we let $X = \mathbb{R}^n$ but the distance function d is distorted. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in [1,\infty)$. Define a function $F : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ by

$$F(x_1, x_2, \ldots, x_n; r) = \sum_{i=1}^n \frac{{x_i}^2}{r^{2\alpha_i}}.$$

Define a distance d by

$$d(x,y) = \inf\{r > 0 \,:\, F(x-y;r) \le 1\}.$$

The function d is indeed a distance function, since

$$(a+b)^2 \left(\frac{1}{r_1+r_2}\right)^{2\alpha_i} \le \frac{a^2 r_1}{r_1+r_2}\left(\frac{1}{r_1}\right)^{2\alpha_i} + \frac{b^2 r_2}{r_1+r_2}\left(\frac{1}{r_2}\right)^{2\alpha_i}$$

for $r_1, r_2 > 0$ and $a, b \ge 0$. Then $L_{p,\phi;2}(X;\mu)$ is a special case of the norm dealt in [41]. In [41], Softova considered the case where ϕ depends on x as well.

Example 1.15. Let p be a fixed prime number. Equip $\mathbb{Q}$ with a distance function as follows: the distance between $x, y \in \mathbb{Q}$ is

$$d_p(x,y) = p^{-m},$$

where the integers q, r, m satisfy $x - y = p^m \frac{q}{r}$ and the integers q and r are not multiples of p; see [13]. Denote by $\mathbb{Q}_p$ the completion with respect to d_p of $\mathbb{Q}$. Since $\mathbb{Q}_p$ carries the structure of a locally compact group, $\mathbb{Q}_p$ has the Haar measure μ. Then $L_{p,\phi;2}(X;\mu)$ is a special case of the norm dealt in [44]. In [44], Volosivets considered the case where ϕ depends on x as well.

A tacit understanding in this paper is that we use the letter C to denote various positive constants that may differ from line to line. Throughout this paper, we write $A \lesssim B$ to indicate that there exists a constant C independent of Morrey functions such that $A \le CB$. The symbol $A \sim B$ stands for $A \lesssim B \lesssim A$.

Finally, we shall organize the remaining part of the present paper as follows: Section 2 collects some fundamental estinates and we prove the above theorems in Section 3.

2. Preliminary lemmas

2.1. Norm estimates. We begin with the obtaining a fundamenetal estimate of the Morrey norm $\|\chi\|_{L_{p,\phi;2}(X;\mu)}$ for the indicator function of balls.

Lemma 2.1 (cf. [11, Lemma 3.1])**.** *Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_1$. Assume that there exists a constant $C' > 0$ such that (7) holds whenever $w \in X$ and $0 < r \le r'$. Then, there exists a constant $C > 0$ such that*

$$\left(\frac{\mu(B(z,R))}{\mu(B(z,2R))}\right)^{\frac{1}{p}} \phi(R)^{-1} \le \|\chi_{B(z,R)}\|_{L_{p,\phi;2}(X;\mu)} \le C\phi(R)^{-1} \tag{13}$$

for all $R > 0$ and $z \in X$. In particular, when μ is a doubling measure,

$$C^{-1}\phi(R)^{-1} \le \|\chi_{B(z,R)}\|_{L_{p,\phi;2}(X;\mu)} \le C\phi(R)^{-1}$$

for all $R > 0$ and $z \in X$.

Proof. The left inequality of (13) is a consequence of the definition

$$\|\chi_{B(z,R)}\|_{L_{p,\phi;2}(X;\mu)} \ge \frac{1}{\phi(R)} \left(\frac{\mu(B(z,R))}{\mu(B(z,2R))}\right)^{\frac{1}{p}}.$$

So we concentrate on the right inequality $\|\chi_{B(z,R)}\|_{L_{p,\phi;2}(X;\mu)} \le C\phi(R)^{-1}$. In the equality

$$\|\chi_{B(z,R)}\|_{L_{p,\phi;2}(X;\mu)} = \sup_{w \in X,\, r>0} \frac{1}{\phi(r)} \left(\frac{\mu(B(w,r) \cap B(z,R))}{\mu(B(w,2r))}\right)^{\frac{1}{p}},$$

we distinguish two cases: $r \ge 2R$ and $0 < r \le 2R$. We have

$$\begin{aligned}
\|\chi_{B(z,R)}\|_{L_{p,\phi;2}(X;\mu)} &\le \sup_{w \in X,\, 0<r\le 2R} \frac{1}{\phi(r)} \left(\frac{\mu(B(w,r) \cap B(z,R))}{\mu(B(w,2r))}\right)^{\frac{1}{p}} \\
&\quad + \sup_{w \in X,\, r \ge 2R} \frac{1}{\phi(r)} \left(\frac{\mu(B(w,r) \cap B(z,R))}{\mu(B(w,2r))}\right)^{\frac{1}{p}} \\
&\le \frac{C}{\phi(2R)} + \sup_{w \in X,\, r\ge 2R, d(z,w) \le r+R} \frac{1}{\phi(r)} \left(\frac{\mu(B(z,R))}{\mu(B(w,2r))}\right)^{\frac{1}{p}} \\
&\le \frac{C}{\phi(R)} + C \sup_{w \in X,\, r\ge 2R, d(z,w) \le r+R} \frac{1}{\phi(\frac{r}{2})} \left(\frac{\mu(B(z,R))}{\mu(B(z,\frac{r}{2}))}\right)^{\frac{1}{p}} \\
&\le \frac{C}{\phi(R)};
\end{aligned}$$

we used (7) in the last inequality. □

2.2. Lower bounds for $I_{\rho,\mu,32}$. We need to fundamental estimates for the bound of $I_{\rho,\mu,32}$ from below. The first one concerns the ball-testing. Recall here that $\tilde{\rho}$ is given by (3).

Lemma 2.2. *Let $\rho \in \mathcal{G} \cap \mathcal{G}_0$. Assume that μ is a doubling measure. Fix $z \in X$. Then there exists a constant $C > 0$ such that $\tilde{\rho}(\frac{R}{2}) \le C I_{\rho,\mu,32}\chi_{B(z,R)}(x)$ holds whenever $x \in B(z, \frac{R}{2})$ and $R > 0$.*

Remark 2.3. Note that if μ is a doubling measure, there exist constants $C > 0$ and $s \ge 1$ such that

$$\frac{\mu(B')}{\mu(B)} \ge C \left(\frac{r'}{r}\right)^s \tag{14}$$

for all balls $B = B(x, r)$ and $B' = B(x', r')$ with $x' \in B$ and $0 < r' \le r$ (see e.g. [16]).

Proof of Lemma 2.2. Let $x \in B(z, \frac{R}{2})$. Since $\rho \in \mathcal{G}$, we have

$$\begin{aligned} I_{\rho,\mu,32}\chi_{B(z,R)}(x) &\ge \int_{B(x,\frac{R}{2})} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))}\, d\mu(y) \\ &= \sum_{j=1}^{\infty} \int_{B(x,2^{-j}R)\setminus B(x,2^{-j-1}R)} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))}\, d\mu(y) \\ &\ge C \sum_{j=1}^{\infty} \int_{B(x,2^{-j}R)\setminus B(x,2^{-j-1}R)} \frac{\rho(2^{-j-1}R)}{\mu(B(x,2^{-j+5}R))}\, d\mu(y). \end{aligned}$$

Since X is connected, we can find y such that $d(x,y) = \frac{2^{-j-1}R + 2^{-j}R}{2} = 3\cdot 2^{-j-2}R$. Let $B' = B(y, 2^{-j-2}R)$. By (14), we have $\frac{\mu(B')}{\mu(B(x,2^{-j+5}R))} \ge C\left(\frac{2^{-j-2}R}{2^{-j+5}R}\right)^s = C$. Since $B' \subset B(x, 2^{-j}R) \setminus B(x, 2^{-j-1}R)$, we obtain

$$I_{\rho,\mu,32}\chi_{B(z,R)}(x) \ge C \sum_{j=1}^{\infty} \rho(2^{-j-1}R) \ge C \int_0^{\frac{R}{2}} \frac{\rho(s)}{s} ds = C\tilde{\rho}\left(\frac{R}{2}\right),$$

as required. □

Another fundamental estimate of $I_{\rho,\mu,32}$ comes from the one of the function $y \in X \setminus B(z,R) \mapsto \phi(d(y,z)) \in [0,\infty)$ with $z \in X$ and $R > 0$ fixed.

Lemma 2.4. *Let μ be a doubling measure, $\rho \in \mathcal{G} \cap \mathcal{G}_0$ and $\phi \in \mathcal{G}_0$. Fix $z \in X$. For $y \in X$, define*

$$g_R(y) \equiv \phi(d(y,z))\chi_{B(z,R)^c}(y) \tag{15}$$

and let k_1 be a constant appearing in (2). *Then there exists a constant $C > 0$ such that, for every $R > 0$,*

$$I_{\rho,\mu,32}g_R(x) \le C \int_{k_1 R}^{\infty} \frac{\rho(t)\phi(t)}{t}\, dt$$

holds, whenever $x \in B(z, \frac{R}{3})$.

Proof. Let $y \notin B(z,R)$. Since $d(x,y) \sim d(y,z)$ for $x \in B(z, \frac{R}{3})$ and $\phi \in \mathcal{G}$, we have

$$\begin{aligned}
I_{\rho,\mu,32} g_R(x) &= \int_{X \setminus B(z,R)} \frac{\rho(d(x,y))\phi(d(y,z))}{\mu(B(x, 32d(x,y)))} \, d\mu(y) \\
&\le C \int_{X \setminus B(x, 2\frac{R}{3})} \frac{\rho(d(x,y))\phi(d(x,y))}{\mu(B(x, 32d(x,y)))} \, d\mu(y) \\
&\le C \sum_{j=0}^{\infty} \int_{B(x,2^j R) \setminus B(x, 2^{j-1}R)} \frac{\rho(d(x,y))\phi(d(x,y))}{\mu(B(x, 32d(x,y)))} \, d\mu(y) \\
&\le C \sum_{j=0}^{\infty} \int_{B(x,2^j R) \setminus B(x, 2^{j-1}R)} \frac{\sup_{2^{j-1}R \le s \le 2^j R} \rho(s)\phi(2^j R)}{\mu(B(x, 2^{j+4}R))} \, d\mu(y).
\end{aligned}$$

Since $\rho \in \mathcal{G}_0$ and $\phi \in \mathcal{G}$, we obtain

$$\begin{aligned}
&I_{\rho,\mu,32} g_R(x) \\
&\le C \sum_{j=0}^{\infty} \int_{B(x,2^j R) \setminus B(x, 2^{j-1}R)} \frac{\phi(2^j R)}{\mu(B(x, 2^{j+4}R))} \left(\int_{2^j k_1 R}^{2^j k_2 R} \frac{\rho(s)}{s} \, ds \right) d\mu(y) \\
&\le C \sum_{j=0}^{\infty} \int_{B(x,2^j R) \setminus B(x, 2^{j-1}R)} \left(\int_{2^j k_1 R}^{2^j k_2 R} \frac{\rho(s)\phi(s)}{s} \, ds \right) \frac{d\mu(y)}{\mu(B(x, 2^{j+4}R))} \\
&\le C \sum_{j=0}^{\infty} \int_{2^j k_1 R}^{2^j k_2 R} \frac{\rho(s)\phi(s)}{s} \, ds \\
&\le C \int_{k_1 R}^{\infty} \frac{\rho(s)\phi(s)}{s} \, ds,
\end{aligned}$$

as required. □

We verify that the function g_R above is estimated from above by proving the next lemma:

Lemma 2.5. *Let μ be a doubling measure, $\rho \in \mathcal{G} \cap \mathcal{G}_0$ and $\phi \in \mathcal{G}$. Fix $z \in X$. Define $g_R(y)$ by (15) for $y \in X$ and $R > 0$. Then, there exists a constant $C > 0$ such that, for every $R > 0$,*

$$I_{\rho,\mu,32} g_R(x) \ge C \int_R^{\infty} \frac{\rho(t)\phi(t)}{t} \, dt \tag{16}$$

holds, whenever $x \in B(z, \frac{R}{3})$.

Proof. Let $y \notin B(z,R)$. Since $d(x,y) \sim d(y,z)$ for $x \in B(z, \frac{R}{3})$, $\rho \in \mathcal{G}$ and $\phi \in \mathcal{G}$, we have

$$\begin{aligned}
I_{\rho,\mu,32}g_R(x) &= \int_{X\setminus B(z,R)} \frac{\rho(d(x,y))\phi(d(z,y))}{\mu(B(x,32d(x,y)))}\,d\mu(y)\\
&\geq C\int_{X\setminus B(x,2R)} \frac{\rho(d(x,y))\phi(d(x,y))}{\mu(B(x,32d(x,y)))}\,d\mu(y)\\
&\geq C\sum_{j=1}^{\infty}\int_{B(x,2^{j+1}R)\setminus B(x,2^jR)} \frac{\rho(d(x,y))\phi(d(x,y))}{\mu(B(x,32d(x,y)))}\,d\mu(y)\\
&\geq C\sum_{j=1}^{\infty}\int_{B(x,2^{j+1}R)\setminus B(x,2^jR)} \frac{\rho(2^jR)\phi(2^jR)}{\mu(B(x,2^{j+6}R))}\,d\mu(y).
\end{aligned}$$

As in the proof of Lemma 2.2, in view of (14), we obtain

$$I_{\rho,\mu,32}g_R(x) \geq C\sum_{j=1}^{\infty}\rho(2^jR)\phi(2^jR) \geq C\int_R^\infty \frac{\rho(t)\phi(t)}{t}\,dt,$$

as required. □

2.3. Control of fractional integral operators by maximal operators. Now we are oriented to the control of fractional integral operators. To control them, we need a pointwise estimate of the maximal operator.

We consider the centered maximal function defined by

$$M_{16}f(x) \equiv \sup_{r>0}\frac{1}{\mu(B(x,16r))}\int_{B(x,r)}|f(y)|d\mu(y)$$

for a locally integrable function f on X.

Note that, under (5), we have

$$\lim_{r\downarrow 0}\phi(r) = \infty \tag{17}$$

and

$$\lim_{r\to\infty}\phi(r) = 0. \tag{18}$$

We "almost" have the mean value theorem for ϕ as the following lemma implies:

Lemma 2.6. *Let $\phi \in \mathcal{G}_1$ satisfy* (17) *and* (18)*. Then, for all $T \in (0,\infty)$, there exists $r_0 > 0$ such that $\phi(r_0) \sim T$.*

Proof. Define

$$r_0 = \sup\{r > 0 \,:\, \phi(r) \geq T\}.$$

Then by the definition of r_0 and sup, we have $\phi(2r_0) < T$. Again by the definiton of r_0 and sup, there exists $v \in (\frac{r_0}{2}, r_0]$ such that $\phi(v) \geq T$. Since ϕ is almost decreasing, $\phi(\frac{r_0}{2}) \geq cT$. Thus, since ϕ is doubling, we see that $\phi(r_0) \sim T$. □

Lemma 2.7. *Let $1 < p < q < \infty$, $\rho \in \mathcal{G}_0$ and $\phi \in \mathcal{G}_1$. Suppose ϕ satisfies* (17) *and* (18) *and that a μ-measurable function f satifies $\|f\|_{L_{p,\phi;2}(X;\mu)} = 1$. Assume, in addition,* (4) *holds for some constant $C' > 0$ and* (6) *holds for some constant $C'' > 0$. Then*

$$|I_{\rho,\mu,32}f(x)| \le CM_{16}f(x)^{\frac{p}{q}} \quad \text{for } x \in X. \tag{19}$$

Proof. We may assume that $M_{16}f(x) < \infty$. We write

$$U_1 \equiv \int_{B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))}|f(y)|\,d\mu(y)$$

and

$$U_2 \equiv \int_{X\setminus B(x,r)} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))}|f(y)|\,d\mu(y).$$

Then, by the triangle inequality, $|I_{\rho,\mu,32}f(x)| \le U_1 + U_2$. If $y \in B(x,2^j r) \setminus B(x,2^{j-1}r)$ and $j \in \mathbb{Z}$, then a geometric observation shows

$$\begin{aligned}
&\int_{B(x,2^j r)\setminus B(x,2^{j-1}r)} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))}|f(y)|\,d\mu(y)\\
&\le \int_{B(x,2^j r)\setminus B(x,2^{j-1}r)} \frac{\sup_{2^{j-1}r\le s\le 2^j r}\rho(s)}{\mu(B(x,2^{j+4}r))}|f(y)|\,d\mu(y).
\end{aligned}$$

If we combine this with (2), then we have

$$\begin{aligned}
&\int_{B(x,2^j r)\setminus B(x,2^{j-1}r)} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))}|f(y)|\,d\mu(y)\\
&\le C_\rho \int_{2^j k_1 r}^{2^j k_2 r} \frac{\rho(s)}{s}\,ds \times \frac{1}{\mu(B(x,2^{j+4}r))} \int_{B(x,2^j r)} |f(y)|\,d\mu(y).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
U_1 &= \sum_{j=0}^{\infty} \int_{B(x,2^{-j}r)\setminus B(x,2^{-j-1}r)} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))}|f(y)|\,d\mu(y)\\
&\le C_\rho \sum_{j=0}^{\infty} \int_{2^{-j}k_1 r}^{2^{-j}k_2 r} \frac{\rho(s)}{s}\,ds \times \frac{1}{\mu(B(x,2^{-j+4}r))} \int_{B(x,2^{-j}r)} |f(y)|\,d\mu(y)\\
&\le C_\rho \left(\int_0^{k_2 r} \frac{\rho(s)}{s}\,ds\right) M_{16}f(x)\\
&= C_\rho \tilde{\rho}(k_2 r) M_{16}f(x),
\end{aligned}$$

where $\tilde{\rho}$ is given by (3). Next note that

$$1 \geq \frac{1}{\phi(r)} \left(\frac{1}{\mu(B(z,2r))} \int_{B(z,r)} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \geq \frac{1}{\phi(r)} \frac{1}{\mu(B(z,2r))} \int_{B(z,r)} |f(x)| \, d\mu(x)$$

since $\|f\|_{L_{p,\phi;2}(X;\mu)} = 1$, so that

$$\frac{1}{\mu(B(z,2r))} \int_{B(z,r)} |f(x)| \, d\mu(x) \leq \phi(r) \tag{20}$$

for all $r > 0$. If we use (2), then we have

$$\begin{aligned} U_2 &= \sum_{j=1}^{\infty} \int_{B(x,2^j r) \setminus B(x,2^{j-1} r)} \frac{\rho(d(x,y))}{\mu(B(x,32d(x,y)))} |f(y)| \, d\mu(y) \\ &\leq C_\rho \sum_{j=1}^{\infty} \int_{2^j k_1 r}^{2^j k_2 r} \frac{\rho(s)}{s} \, ds \times \frac{1}{\mu(B(x,2^{j+4} r))} \int_{B(x,2^j r)} |f(y)| \, d\mu(y). \end{aligned}$$

If we use (20), then we have $U_2 \leq C_\rho \sum_{j=1}^{\infty} \int_{2^j k_1 r}^{2^j k_2 r} \frac{\rho(s)}{s} \, ds \times \phi(2^j r)$. Since $\phi \in \mathcal{G}_1$ and hence ϕ is doubling, we have

$$U_2 \leq C \sum_{j=1}^{\infty} \int_{2^j k_1 r}^{2^j k_2 r} \frac{\rho(s)\phi(s)}{s} ds \leq C \int_{2k_1 r}^{\infty} \frac{\rho(s)\phi(s)}{s} ds.$$

Consequently, we have by our assumption

$$|I_{\rho,\mu,32} f(x)| \leq U_1 + U_2 \leq C \tilde{\rho}(k_2 r) M_{16} f(x) + C \int_{2k_1 r}^{\infty} \frac{\rho(s)\phi(s)}{s} ds.$$

Since we are assuming that ϕ is a doubling function as well as conditions (2) and (4), we have

$$\begin{aligned} |I_{\rho,\mu,32} f(x)| &\leq C \phi(k_2 r)^{\frac{p}{q}-1} M_{16} f(x) + C \rho(2k_1 r) \phi(2k_1 r) \\ &\leq C \phi(r)^{\frac{p}{q}-1} M_{16} f(x) + C \tilde{\rho}(2k_1 k_2 r) \phi(r). \end{aligned}$$

Since $\phi \in \mathcal{G}_1$ and we assume (6),

$$\tilde{\rho}(2k_1 k_2 r) \leq C \phi(2k_1 k_2 r)^{\frac{p}{q}-1} \leq C \phi(r)^{\frac{p}{q}-1},$$

so that

$$|I_{\rho,\mu,32} f(x)| \leq C \phi(r)^{\frac{p}{q}-1} M_{16} f(x) + C \phi(r)^{\frac{p}{q}}$$

for all $r > 0$. By virtue of Lemma 2.6, we can find $R > 0$ such that $C^{-1}\phi(R) \leq M_{16} f(x) \leq C^{-1}\phi(R)$. By using this R, we can obtain (19). □

Similarly we can prove the following:

Lemma 2.8. *Let $1 < p < q < \infty$ and $\rho \in \mathcal{G}_0$. Assume that $\phi \in \mathcal{G}_1$ satisfies* (17) *and* (18) *and that there exists a constant $C' > 0$ such that* (8) *holds for any $r > 0$. Suppose, in addition, that a μ-measurable function f satisfies $\|f\|_{L_{p,\phi;2}(X;\mu)} = 1$. Then* (19) *still holds.*

2.4. Morrey boundedness of the maximal operator. Now we obtain the following estimate of the maximal operator itself:

Lemma 2.9. (cf. [19, Lemma 4.3] and [20, Lemma 2.2]) *Let $\phi \in \mathcal{G}$. Assume* (5) *holds for some constant $C' > 0$. Assume that f is a μ-measurable function on G satisfying*

$$\frac{1}{\mu(B(x,2r))} \int_{B(x,r)} |f(y)|^{p_0}\, d\mu(y) \le \phi(r)^{p_0} \tag{21}$$

for all $x \in X$ and $r > 0$. Then there exists a constant $C > 0$ such that

$$\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} M_{16} f(x)^{p_0}\, d\mu(x) \le C\phi(r)^{p_0}$$

for all $z \in X$ and $r > 0$, where the constant C is independent of f satisfying (21).

Lemma 2.9 can be regarded as the Morrey boundedness of the operator M_{16}.

Proof. Let f satisfy (21), and fix $z \in X$ and $r > 0$. Write $A_0 \equiv B(z,2r)$ and $A_j \equiv B(z,2^{j+1}r) \setminus B(z,2^j r)$ for each positive integer j. Based upon this partition $\{A_j\}_{j=1}^{\infty}$, we set

$$f_j \equiv f\chi_{A_j} \quad \text{for } j = 0,1,2,\ldots, \qquad g_0 \equiv \sum_{j=1}^{\infty} |f_j|.$$

Let us set

$$\mathrm{J}_1 \equiv \int_{B(z,r)} M_{16} f_0(x)^{p_0} d\mu(x), \quad \mathrm{J}_2 \equiv \int_{B(z,r)} M_{16} g_0(x)^{p_0} d\mu(x).$$

Then we have

$$\int_{B(z,r)} M_{16} f(x)^{p_0} d\mu(x) \le C(\mathrm{J}_1 + \mathrm{J}_2).$$

By virtue of (21) and the $L^{p_0}(X;\mu)$-boundedness of M_{16} (see [28, Lemma 3.1], [31, Theorem 1.2] and [43, Theorem 2.4]), we have

$$\mathrm{J}_1 \le C \int_X |f_0(x)|^{p_0} d\mu(x) = C \int_{B(z,2r)} |f(x)|^{p_0} d\mu(x) \le C\phi(r)^{p_0} \mu(B(z,4r)).$$

The estimate for J_1 is now valid.

Let us turn to J_2. In view of the definition of f_j and A_j, we have

$$M_{16} f_j(x) \le \sup_{t \in ((2^j-1)r,(2^{j+1}+1)r)} \frac{1}{\mu(B(x,16t))} \int_{B(x,t)} |f_j(y)| d\mu(y)$$

for $x \in B(z,r)$. For $x \in B(z,r)$, we estimate the right-hand side crudely:

$$\begin{aligned} M_{16}f_j(x) &\le \frac{1}{\mu(B(x,16(2^j-1)r)} \int_{B(x,(2^{j+1}+1)r)} |f_j(y)|d\mu(y) \\ &\le \frac{1}{\mu(B(x,16(2^j-1)r)} \int_{B(z,(2^{j+1}+2)r)} |f_j(y)|d\mu(y) \\ &\le \frac{1}{\mu(B(z,(2^{j+4}-17)r)} \int_{B(z,(2^{j+1}+2)r)} |f_j(y)|d\mu(y). \end{aligned}$$

By the Hölder inequality, we have

$$M_{16}f_j(x) \le \left(\frac{1}{\mu(B(z,(2^{j+4}-17)r))} \int_{B(z,(2^{j+1}+2)r)} |f(y)|^{p_0}d\mu(y) \right)^{\frac{1}{p_0}}.$$

Since $2^{j+4}-17 \ge 2(2^{j+1}+2)$ for any positive integer j and $\phi \in \mathcal{G}$, we see that for $x \in B(z,r)$

$$M_{16}f_j(x) \le \phi(2^{j+1}r+2r) \le C\phi(2^jr),$$

so that, adding this estimate over j, we obtain a pointwise estimate: for all $x \in B(z,r)$, $M_{16}g_0(x) \le \sum_{j=1}^\infty M_{16}f_j(x) \le C\sum_{j=1}^\infty \phi(2^jr) \le C\int_r^\infty \frac{\phi(t)}{t}dt \le C\phi(r)$. Here, for the last inequality, we used (5). Integrating the above estimate over $B(z,r)$, we obtain

$$\mathrm{J}_2 \le C\phi(r)^{p_0} \int_{B(z,r)} d\mu(x) = C\phi(r)^{p_0}\mu(B(z,r)).$$

Since $\mu(B(z,r)) \le \mu(B(z,4r))$, we deduce

$$\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} M_{16}f(x)^{p_0}d\mu \le C\phi(r)^{p_0},$$

which proves Lemma 2.9. □

2.5. Elementary results for predual spaces. To prove Theorem 1.8, we need the following lemmas.

Lemma 2.10. *If f is a non-zero $L^{p'}(X;\mu)$-function which is supported on a ball $B(x_0,r)$, then b is a (p',ϕ)-block, where*

$$b = \frac{1}{\mu(B(x_0,2r))^{\frac{1}{p}}\phi(r)\|f\|_{L^{p'}(X;\mu)}} f.$$

Hence, in particular,

$$\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)} \le \mu(B(x_0,2r))^{\frac{1}{p}}\phi(r)\|f\|_{L^{p'}(X;\mu)}. \tag{22}$$

Proof. The fact that b is a (p',ϕ)-block follows from a simple calculation and the fact that b is supported on $B(x_0,r)$;

$$\|b\|_{L^{p'}(X;\mu)} = \frac{1}{\mu(B(x_0,2r))^{\frac{1}{p}}\phi(r)\|f\|_{L^{p'}(X;\mu)}}\|f\|_{L^{p'}(X;\mu)} = \frac{1}{\mu(B(x_0,2r))^{\frac{1}{p}}\phi(r)}.$$

To prove (22), we set

$$b_1 = b, \quad \lambda_1 = \mu(B(x_0,2r))^{\frac{1}{p}}\phi(r)\|f\|_{L^{p'}(X;\mu)}, \quad b_j = 0, \quad \lambda_j = 0 \quad \text{for } j \geq 2.$$

Then each b_j is a (p',ϕ)-block and (11) holds. Thus, (22) follows from (12). □

Lemma 2.11. *Let $1 < p < \infty$ and $\phi \in \mathcal{G}_1$. The set of all $L^\infty(X;\mu)$-functions with compact support forms a dense subspace of $\mathcal{H}_{p',\phi}(X;\mu)$. In particular, the set of all $L^{p'}(X;\mu)$-functions with compact support forms a dense subspace of $\mathcal{H}_{p',\phi}(X;\mu)$.*

Proof. If we are given an expression (11) of $f \in \mathcal{H}_{p',\phi}(X;\mu)$, then the sequence $\{f_N\}_{N=1}^\infty$ approximates f in the topology of $\mathcal{H}_{p',\phi}(X;\mu)$ thanks to (12), where f_N is given by

$$f_N = \sum_{j=1}^N \lambda_j b_j.$$

So, we can choose $N \gg 1$ so that $\|f - f_N\|_{\mathcal{H}_{p',\phi}(X;\mu)} < 4^{-1}\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)}$. By the Lebesgue convergence theorem, we can choose φ_j, $j = 1,2,\ldots,N$ of the form $\varphi_j = b_j\chi_{\{|b_j|\leq R_j\}}$ with $R_j \gg 1$ so that

$$\sum_{j=1}^N |\lambda_j| \cdot \|b_j - \varphi_j\|_{L^{p'}(X;\mu)} < 4^{-1}\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)}$$

and that each φ_j is bounded and has bounded support. Thus, by setting $f^1 = \sum_{j=1}^N \lambda_j\varphi_j$, we see that

$$\|f - f^1\|_{\mathcal{H}_{p',\phi}(X;\mu)} < 2^{-1}\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)}.$$

Likewise, by applying what we have obtained to $f - f^1$, we can find $f^2 \in L^\infty(X;\mu)$ with bounded support so that

$$\|f - f^1 - f^2\|_{\mathcal{H}_{p',\phi}(X;\mu)} < 2^{-1}\|f - f^1\|_{\mathcal{H}_{p',\phi}(X;\mu)}.$$

If we repeat this procedure, then we can find $f^1, f^2, \ldots, f^M, \ldots$, so that

$$\|f - f^1 - f^2 - \cdots - f^M\|_{\mathcal{H}_{p',\phi}(X;\mu)} < 2^{-1}\|f - f^1 - f^2 - \cdots - f^{M-1}\|_{\mathcal{H}_{p',\phi}(X;\mu)}$$

for all M. Thus,

$$\|f - f^1 - f^2 - \cdots - f^M\|_{\mathcal{H}_{p',\phi}(X;\mu)} < 2^{-M}\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)},$$

as was to be shown. □

Lemma 2.12. *If $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1(\mathbb{N})$, then, for each collection $\{B_j\}_{j=1}^{\infty}$ of (p', ϕ)-blocks and $\{b_j\}_{j=1}^{\infty} \subset L^{\infty}(\mu)$ such that $|b_j| \le B_j$ for μ-a.e., we have $\sum_{j=1}^{\infty} \lambda_j b_j \in \mathcal{H}_{p',\phi}(X;\mu)$ and*

$$\left\| \sum_{j=1}^{\infty} \lambda_j b_j \right\|_{\mathcal{H}_{p',\phi}(X;\mu)} \le \sum_{j=1}^{\infty} |\lambda_j|.$$

Proof. Just observe that b_j is a (p', ϕ)-block as well. □

3. Proofs

3.1. Proofs of Theorem 1.2 and Theorem 1.5.

Proof of Theorem 1.2. By Lemma 2.7, we have

$$\frac{1}{\mu(B(z,4r))} \int_{B(z,r)} |I_{\rho,\mu,32} f(x)|^q d\mu(x) \le \frac{C}{\mu(B(z,4r))} \int_{B(z,r)} M_{16} f(x)^p d\mu(x)$$

for $x \in X$. If we devide both sides by $\phi(r)^p$, then we have by Lemma 2.9

$$\begin{aligned} &\frac{1}{\phi(r)^p \mu(B(z,4r))} \int_{B(z,r)} |I_{\rho,\mu,32} f(x)|^q d\mu(x) \\ &\le C \left(\frac{1}{\phi(r)^p \mu(B(z,4r))} \int_{B(z,r)} M_{16} f(x)^p d\mu(x) \right) \\ &\le C, \end{aligned}$$

as required. □

Proof of Theorem 1.5. With the aid of Lemmas 2.8 and 2.9, we can go through the same argument as Theorem 1.2. □

3.2. Proofs of Theorem 1.3 and Theorem 1.6.

Proof of Theorem 1.3. Fix $z \in X$. Observe from Lemma 2.2 that

$$\tilde{\rho}\left(\frac{R}{2}\right) \le C \left(\frac{1}{\mu(B(z, \frac{R}{2}))} \int_{B(z,\frac{R}{2})} I_{\rho,\mu,32} \chi_{B(z,R)}(x)^q d\mu(x) \right)^{\frac{1}{q}}. \tag{23}$$

Since $I_{\rho,\mu,32}$ is assumed bounded from $L_{p,\phi;2}(X;\mu)$ to $L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$, we note from Lemma 2.1 that

$$\|I_{\rho,\mu,32} \chi_{B(z,R)}\|_{L_{q,\phi^{\frac{p}{q}};4}(X;\mu)} \le C \|\chi_{B(z,R)}\|_{L_{p,\phi;2}(X;\mu)} \le C \phi(R)^{-1}$$

for all $R > 0$. Since $\phi \in \mathcal{G}_1$ and μ is a doubling measure, we have

$$\tilde{\rho}\left(\frac{R}{2}\right) \leq C\phi\left(\frac{R}{2}\right)^{\frac{p}{q}} \phi(R)^{-\frac{p}{q}} \left(\frac{1}{\mu(B(z,R))}\int_{B(z,\frac{R}{2})} I_{\rho,\mu,32}\chi_{B(z,R)}(x)^q d\mu(x)\right)^{\frac{1}{q}}$$

by virtue of (23). Since ϕ is doubling and $I_{\rho,\mu,32} : L_{p,\phi;2}(X;\mu) \to L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$ is bounded,

$$\tilde{\rho}\left(\frac{R}{2}\right) \leq C\phi\left(\frac{R}{2}\right)^{\frac{p}{q}} \|I_{\rho,\mu,32}\chi_{B(z,R)}\|_{L_{q,\phi^{\frac{p}{q}};4}(X;\mu)} \leq C\frac{\phi(\frac{R}{2})^{\frac{p}{q}}}{\phi(R)} \leq C\phi\left(\frac{R}{2}\right)^{\frac{p}{q}-1}$$

for all $R > 0$. Thus this theorem is proved. □

Proof of Theorem 1.6. As in the proof of Theorem 1.3, we obtain

$$\int_0^r \frac{\rho(t)}{t}\,dt \ \leq C\phi(r)^{\frac{p}{q}-1}.$$

We shall prove that $\|g_R\|_{L_{p,\phi;2}(X;\mu)} \leq C$. For the time being, let us estimate J by setting

$$\mathrm{J} \equiv \left(\frac{1}{\mu(B(x,2r))}\int_{B(x,r)\setminus B(z,R)} \phi(d(z,y))^p\,d\mu(y)\right)^{\frac{1}{p}}.$$

Suppose that $x \in B(z,\frac{R}{2})$ and $r \geq \frac{R}{2}$. Then

$$\mathrm{J} \leq \left(\frac{1}{\mu(B(z,r))}\int_{B(z,2r)\setminus B(z,R)} \phi(d(z,y))^p\,d\mu(y)\right)^{\frac{1}{p}}.$$

By the triangle inequality for $L^p(\mu)$ or the inequality $(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$, we have

$$\mathrm{J} \leq \sum_{j=0}^{\infty} \left(\frac{1}{\mu(B(z,r))}\int_{B(z,2^{-j+1}r)\setminus B(z,2^{-j}r)} \phi(d(z,y))^p\,d\mu(y)\right)^{\frac{1}{p}}.$$

Since ϕ is almost decreasing, we have $\mathrm{J} \leq C\sum_{j=0}^{\infty} \frac{\mu(B(z,2^{-j}r))^{\frac{1}{p}}}{\mu(B(z,r))^{\frac{1}{p}}}\phi(2^{-j}r)$. Finally, assuming (9), we conclude

$$\mathrm{J} \leq C\mu(B(z,r))^{-\frac{1}{p}} \int_0^r \mu(B(z,t))^{\frac{1}{p}} \frac{\phi(t)}{t}\,dt \leq C\phi(r) \tag{24}$$

as long as $x \in B(z,\frac{R}{2})$ and $r \geq \frac{R}{2}$.

Recall that g_R is given by (15). We plan to estimate

$$\|g_R\|_{L_{p,\phi;2}(X;\mu)} = \sup_{x\in X,\, r>0} \frac{1}{\phi(r)} \left(\frac{1}{\mu(B(x,2r))} \int_{B(x,r)\setminus B(z,R)} \phi(d(z,y))^p \, d\mu(y) \right)^{\frac{1}{p}}.$$

Although x and r runs over $X \times (0,\infty)$ in the above formula, there is no need to consider the case $x \in B(z, \frac{R}{2})$ and $r \le \frac{R}{2}$. Indeed, if $x \in B(z, \frac{R}{2})$ and $r \le \frac{R}{2}$, then $B(x,r) \setminus B(z,R)$ is an empty set. So, we need to distinguish the following four cases:

Case 1. $x \in B(z, \frac{R}{2})$ and $r \ge \frac{R}{2}$,

Case 2. $x \notin B(z, \frac{R}{2}) \cup B(z, 2r+2R)$ and $r \ge 4R$, or equivalently, $x \notin B(z, 2r+2R)$ and $r \ge 4R$,

Case 3. $x \in B(z, 2r+2R) \setminus B(z, \frac{R}{2})$ and $r \ge 4R$,

Case 4. $x \notin B(z, \frac{R}{2})$ and $r < 4R$.

Note that Case 1 is covered by (24). Taking into account these four cases, we obtain

$$\begin{aligned}
&\|g_R\|_{L_{p,\phi;2}(X;\mu)} \\
&\le C \sup_{\substack{x\in B(z,\frac{R}{2}) \\ r\ge \frac{R}{2}}} \frac{\phi(r)}{\phi(r)} + C \sup_{\substack{x\notin B(z,2r+2R) \\ r\ge 4R}} \frac{1}{\phi(r)} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)\setminus B(z,R)} \phi(d(z,y))^p \, d\mu(y) \right)^{\frac{1}{p}} \\
&\quad + C \sup_{\substack{x\in B(z,2r+2R)\setminus B(z,\frac{R}{2}) \\ r\ge 4R}} \frac{1}{\phi(r)} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)\setminus B(z,R)} \phi(d(z,y))^p \, d\mu(y) \right)^{\frac{1}{p}} \\
&\quad + C \sup_{x\notin B(z,\frac{R}{2}),\, r\in(0,4R)} \frac{1}{\phi(R)} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)\setminus B(z,R)} \phi(d(z,y))^p \, d\mu(y) \right)^{\frac{1}{p}}.
\end{aligned}$$

Here we used our assumption that $\phi \in \mathcal{G}_1$ to estimate the fourth term. Let $x \notin B(z, 2r+2R)$, $y \in B(x,r)$ and $r \ge 4R$. Then a geometric observation shows that $d(z,y) \ge d(x,z) - d(x,y) \ge 2r + 2R - r \ge r$. Recall again that $\phi \in \mathcal{G}_1$. So, after simplifying the first term and then estimating $\phi(d(z,y))$ in the second term and the fourth term, we obtain

$$\begin{aligned}
&\|g_R\|_{L_{p,\phi;2}(X;\mu)} \\
&\le C + C \sup_{x\notin B(z,2r+2R),\, r\ge 4R} \frac{1}{\phi(r)} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)\setminus B(z,R)} \phi(r)^p \, d\mu(y) \right)^{\frac{1}{p}} \\
&\quad + C \sup_{\substack{x\in B(z,2r+2R)\setminus B(z,\frac{R}{2}) \\ r\ge 4R}} \frac{1}{\phi(r)} \left(\frac{1}{\mu(B(z,2r))} \int_{B(z,4r)\setminus B(z,R)} \phi(d(z,y))^p \, d\mu(y) \right)^{\frac{1}{p}} \\
&\quad + C \sup_{x\notin B(z,\frac{R}{2}),\, r\in(0,4R)} \frac{1}{\phi(R)} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)\setminus B(z,R)} \phi(R)^p \, d\mu(y) \right)^{\frac{1}{p}}.
\end{aligned}$$

As in the estimate of (24), we can handle the third term to conclude

$$\begin{aligned}\|g_R\|_{L_{p,\phi;2}(X;\mu)} &\le C + C \sup_{\substack{x\in B(z,2r+2R)\setminus B(z,\frac{R}{2})\\ r\ge 4R}} \frac{1}{\phi(r)\mu(B(z,2r))^{\frac{1}{p}}} \int_0^{2r} \mu(B(z,t))^{\frac{1}{p}} \frac{\phi(t)}{t}\,dt \\ &\quad + C \sup_{x\notin B(z,\frac{R}{2}),\, r\in(0,4R)} \frac{\phi(R)}{\phi(R)} \\ &\le C + C \sup_{\substack{x\in B(z,2r+2R)\setminus B(z,\frac{R}{2})\\ r\ge 4R}} \frac{\phi(2r)}{\phi(r)} \\ &\le C.\end{aligned}$$

If we integrate (16) over the ball $B(z,\frac{R}{3})$, then we have

$$\frac{1}{\phi(R)^{\frac{p}{q}}} \int_R^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\|I_{\rho,\mu,32} g_R\|_{L_{q,\phi^{\frac{p}{q}}}(X;\mu)} \le C\|g_R\|_{L_{p,\phi;2}(X;\mu)} \le C.$$

Thus, Theorem 1.6 is proved. □

3.3. Proof of Theorem 1.8. Theorem 1.8(2) is a direct consequence of (25) and (28), which will be obtained in the proof of Theorem 1.8(1). So, we concentrate on Theorem 1.8(1).

Proof of Theorem 1.8(1). (a) Let $f \in \mathcal{H}_{p',\phi}(X;\mu)$. Then, we have an expression for all $\varepsilon > 0$:

$$f = \sum_{j=1}^\infty \lambda_j b_j,$$

where each b_j is a (p',ϕ)-block and $\{\lambda_j\}_{j=1}^\infty \in \ell^1(\mathbb{N})$ which satisfy

$$(1+\varepsilon)\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)} \ge \sum_{j=1}^\infty |\lambda_j|.$$

Then, by the triangle inequality and the Hölder inequality, we have

$$\int_X |f(x)g(x)|\,d\mu(x) \le \sum_{j=1}^\infty |\lambda_j| \int_X |b_j(x)g(x)|\,d\mu(x) \le \sum_{j=1}^\infty |\lambda_j|\, \|b_j\|_{L^{p'}(X;\mu)} \|g\|_{L^p(X;\mu)}.$$

By (10), the definition of the blocks, we have

$$\int_X |f(x)g(x)|\,d\mu(x) \le \sum_{j=1}^\infty |\lambda_j| \frac{\|g\|_{L^p(X;\mu)}}{\mu(B(x_j,2r_j))^{\frac{1}{p}}\phi(r_j)} \le \sum_{j=1}^\infty |\lambda_j|\, \|g\|_{L_{p,\phi;2}(X;\mu)}.$$

Hence,

$$\int_X |f(x)g(x)|\,d\mu(x) \le (1+\varepsilon)\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)}\|g\|_{L_{p,\phi;2}(X;\mu)}.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result:

$$\int_X |f(x)g(x)|\,d\mu(x) \le \|f\|_{\mathcal{H}_{p',\phi}(X;\mu)}\|g\|_{L_{p,\phi;2}(X;\mu)}. \tag{25}$$

(b) Let $x_0 \in X$ and $r > 0$ be fixed. If $f \in L^{p'}(X;\mu)$ is supported on $B(x_0,r)$, then

$$\|f\|_{\mathcal{H}_{p',\phi}(X;\mu)} \le \mu(B(x_0,2r))^{\frac{1}{p}}\phi(r)\|f\|_{L^{p'}(X;\mu)}$$

by virtue of Lemma 2.10, which implies that the mapping

$$f \in L^{p'}(X;\mu) \mapsto L(\chi_{B(x_0,r)}f) \in \mathbb{C}$$

has the operator norm less than $\mu(B(x_0,2r))^{\frac{1}{p}}\phi(r) \times \|L\|_{\mathcal{H}_{p',\phi}(X;\mu)\to\mathbb{C}}$.

Now we unfreeze r. Also, we pass to a discrete variable $j \in \mathbb{N}$ from a continuous variable $r > 0$. Then, by the duality $L^{p'}(X;\mu)$-$L^p(X;\mu)$, for each non-negative integer $j \in \mathbb{N}\cup\{0\}$, we can find a function $g_{B(x_0,j)}$ such that

$$\int_X g_{B(x_0,j)}(x)f(x)\,d\mu(x) = L(\chi_{B(x_0,j)}(x)f) \quad \text{for all } f \in L^{p'}(X;\mu)$$

and that $\|g_{B(x_0,j)}\|_{L^p(X;\mu)} \le \mu(B(x_0,2j))^{\frac{1}{p}}\phi(r)\|L\|_{\mathcal{H}_{p',\phi}(X;\mu)\to\mathbb{C}}$. Since such $g_{B(x_0,j)}$ is unique modulo μ-null sets,

$$g_{B(x_0,j)}(x) = \chi_{B(x_0,j)}(x)g_{B(x_0,j+1)}(x) \tag{26}$$

for μ-a.e. $x \in X$. Define

$$g(x) \equiv \lim_{j\to\infty} g_{B(x_0,j)}(x),$$

whose limit exists for μ-a.e. $x \in X$ thanks to (26). Thus, we have a μ-measurable function g such that $g_{B(x_0,j)} = \chi_{B(x_0,j)}g$ for μ-a.e.

We aim to show that $g \in L_{p,\phi;2}(X,\mu)$. To this end, for any ball $B(x,r)$ and any function $f \in L^{p'}(X;\mu)$ supported there, we observe

$$\begin{aligned}
\int_X g(x)f(x)\,d\mu(x) &= \lim_{j\to\infty}\int_X g(x)\chi_{B(x_0,j)}(x)^2 f(x)\,d\mu(x) \\
&= \lim_{j\to\infty}\int_X g_{B(x_0,j)}(x)\chi_{B(x_0,j)}(x)f(x)\,d\mu(x) \\
&= \lim_{j\to\infty} L(\chi_{B(x_0,j)}f) \\
&= L(f).
\end{aligned}$$

Hence

$$\int_X g(x)f(x)\,d\mu(x) = L(f). \tag{27}$$

Since L is bounded on $\mathcal{H}_{p',\phi}(X;\mu)$, we have

$$\left(\mu(B(x,2r))^{\frac{1}{p}}\phi(r)\right)^{-1}\left|\int_{B(x,r)} g(x)f(x)\,d\mu(x)\right| \le \|L\|_{\mathcal{H}_{p',\phi}(X;\mu)\to\mathbb{C}}\|f\|_{L^{p'}(X;\mu)}.$$

Since this formula is valid for any f with $f \in L^{p'}(X;\mu)$ supported on $B(x,r)$, it follows that

$$\left(\mu(B(x,2r))^{\frac{1}{p}}\phi(r)\right)^{-1}\|g\|_{L^p(B(x,r);\mu)} \le \|L\|_{\mathcal{H}_{p',\phi}(X;\mu)\to\mathbb{C}}, \tag{28}$$

so that we obtain $g \in L_{p,\phi;2}(X;\mu)$ with the estimate

$$\|g\|_{L_{p,\phi;2}(X,\mu)} \le \|L\|_{\mathcal{H}_{p',\phi}(X;\mu)\to\mathbb{C}}.$$

Since (27) holds and $g \in L_{p,\phi;2}(X;\mu)$ imply

$$L_g(f) = L(f)$$

for all $f \in L^{p'}(X;\mu)$ with compact support, we see that two continuous linear functionals L_g and L are the same by virtue of Lemma 2.11. □

3.4. Proofs of Theorems 1.9–1.11. To prove Theorems 1.9–1.11, we prepare the following lemmas:

Lemma 3.1. *Let $\rho \in \mathcal{G}_0$ and $\phi \in \mathcal{G}$. Assume (4) for some constant $C > 0$. If f is a bounded μ-measurable function with bounded support, then $I_{\rho,\mu,33}f \in \mathcal{H}_{p',\phi}(X;\mu)$.*

Proof. We begin with a preliminary observation:

$$\sum_{k=2}^{\infty}\left(\sup_{s\in[2^k r,2^{k+3}r]}\rho(s)\right)\phi(2^k r) < \infty \quad \text{for all } r > 0. \tag{29}$$

Since $\rho \in \mathcal{G}_0$ and $\phi \in \mathcal{G}$ and we are assuming (2), we have

$$\begin{aligned}&\sum_{k=2}^{\infty}\left(\sup_{s\in[2^k r,2^{k+3}r]}\rho(s)\right)\phi(2^k r)\\ &\le \sum_{k=2}^{\infty}\left(\sup_{s\in[2^k r,2^{k+1}r]}\rho(s) + \sup_{s\in[2^{k+1}r,2^{k+2}r]}\rho(s) + \sup_{s\in[2^{k+2}r,2^{k+3}r]}\rho(s)\right)\phi(2^k r)\end{aligned}$$

and $\sum_{k=2}^{\infty}\left(\sup_{s\in[2^kr,2^{k+3}r]}\rho(s)\right)\phi(2^kr) \le C_\rho\sum_{k=2}^{\infty}\left(\int_{2^{k+1}k_1r}^{2^{k+3}k_2r}\frac{\rho(t)}{t}\,dt\right)\phi(2^kr) \le C\sum_{k=2}^{\infty}\int_{2^{k+1}k_1r}^{2^{k+3}k_2r}\frac{\rho(t)\phi(t)}{t}\,dt \le C\int_{k_1r}^{\infty}\frac{\rho(t)\phi(t)}{t}\,dt \le C\rho(k_1r)\phi(k_1r) < \infty$ for all $r > 0$.

We suppose that f is suppored on a ball $B(x_0, r)$. We first claim that $I_{\rho,\mu,33}f$ is a bounded function on $B(x_0, 8r)$. In fact, for $x \in B(x_0, 8r)$, we have

$$\begin{aligned}|I_{\rho,\mu,33}f(x)| &\le \int_{B(x_0,r)}\frac{\rho(d(x,y))}{\mu(B(x,33d(x,y)))}|f(y)|\,d\mu(y)\\ &\le \int_{B(x,9r)}\frac{\rho(d(x,y))}{\mu(B(x,33d(x,y)))}|f(y)|\,d\mu(y)\end{aligned}$$

in view of the size of the support. Since f is bounded, we have

$$|I_{\rho,\mu,33}f(x)| \le \sup|f|\int_{B(x,9r)}\frac{\rho(d(x,y))}{\mu(B(x,33d(x,y)))}\,d\mu(y).$$

Decompose the integral dyadically and use the condition for $\mathcal{G}_0$ to obtain

$$\begin{aligned}|I_{\rho,\mu,33}f(x)| &\le \sup|f|\sum_{j=-\infty}^{4}\int_{B(x,2^jr)\setminus B(x,2^{j-1}r)}\frac{\rho(d(x,y))}{\mu(B(x,33d(x,y)))}\,d\mu(y)\\ &\le \sup|f|\sum_{j=-\infty}^{4}\sup_{2^{j-1}r\le s\le 2^jr}\rho(s)\\ &\le C\sup|f|\sum_{j=-\infty}^{4}\int_{2^jk_1r}^{2^jk_2r}\frac{\rho(t)}{t}dt\\ &\le C\sup|f|\int_0^{16k_2r}\frac{\rho(t)}{t}\,dt\\ &< \infty.\end{aligned}$$

Hence, $I_{\rho,\mu,33}f$ is a bounded function on $B(x_0, 8r)$ and we conclude

$$\chi_{B(x_0,8r)}(x)|I_{\rho,\mu,33}f(x)| \le D_1\frac{\chi_{B(x_0,8r)}(x)}{\mu(B(x_0,8r))^{\frac{1}{p'}}\mu(B(x_0,16r))^{\frac{1}{p}}\phi(8r)}$$

for all $x \in X$ and for some constant D_1 depending on f.

Let $k = 2, 3, \ldots$. If we go through a similar calculation, then we have

$$\begin{aligned}&\chi_{B(x_0,2^{k+2}r)\setminus B(x_0,2^{k+1}r)}(x)|I_{\rho,\mu,33}f(x)|\\ &\le \chi_{B(x_0,2^{k+2}r)\setminus B(x_0,2^{k+1}r)}(x)\int_{B(x_0,r)}\frac{\rho(d(x,y))|f(y)|}{\mu(B(x,33d(x,y)))}\,d\mu(y)\\ &\le \chi_{B(x_0,2^{k+2}r)\setminus B(x_0,2^{k+1}r)}(x)\sup_{s\in[2^kr,2^{k+3}r]}\rho(s)\int_{B(x_0,r)}\frac{|f(y)|}{\mu(B(x,33d(x,y)))}\,d\mu(y).\end{aligned}$$

Observe that $33d(x,y) \geq 33d(x_0,x) - 33d(x_0,y) \geq 66 \cdot 2^k r - 33r \geq 2^{k+4}r$ whenever $y \in B(x_0,r)$ and $d(x_0,x) \geq 2^{k+1}r$. Thus, it follows that

$$\begin{aligned}
&\chi_{B(x_0,2^{k+2}r)\setminus B(x_0,2^{k+1}r)}(x)|I_{\rho,\mu,33}f(x)| \\
&\leq \sup|f| \frac{\mu(B(x_0,r))}{\mu(B(x_0,2^{k+3}r))} \sup_{s\in[2^k r,2^{k+3}r]} \rho(s)\chi_{B(x_0,2^{k+2}r)}(x) \\
&\lesssim \sup|f|\mu(B(x_0,r)) \left(\sup_{s\in[2^k r,2^{k+3}r]} \rho(s)\right) \phi(2^k r) \\
&\quad \times \frac{1}{\mu(B(x_0,2^{k+2}r))^{\frac{1}{p'}}} \frac{\chi_{B(x_0,2^{k+2}r)}(x)}{\mu(B(x_0,2^{k+3}r))^{\frac{1}{p}}\phi(2^{k+2}r)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\chi_{B(x_0,2^{k+2}r)\setminus B(x_0,2^{k+1}r)}(x)|I_{\rho,\mu,33}f(x)| \\
&\leq D_2 \left(\sup_{s\in[2^k r,2^{k+3}r]} \rho(s)\right) \phi(2^k r) \frac{1}{\mu(B(x_0,2^{k+2}r))^{\frac{1}{p'}}} \frac{\chi_{B(x_0,2^{k+2}r)}(x)}{\mu(B(x_0,2^{k+3}r))^{\frac{1}{p}}\phi(2^{k+2}r)}
\end{aligned}$$

for some constant depending on f.

Consequently, if we let

$$\begin{aligned}
\lambda_1 &= D_1, \\
b_1(x) &= \frac{1}{\lambda_1}\chi_{B(x_0,8r)}(x) I_{\rho,\mu,33}f(x), \\
B_1(x) &= \frac{1}{\mu(B(x_0,8r))^{\frac{1}{p'}}} \frac{1}{\mu(B(x_0,16r))^{\frac{1}{p}}\phi(8r)} \chi_{B(x_0,8r)}(x)
\end{aligned}$$

and for $k = 2, 3, \ldots$

$$\begin{aligned}
\lambda_k &= D_2 \left(\sup_{s\in[2^k r,2^{k+3}r]} \rho(s)\right) \phi(2^k r), \\
b_k(x) &= \frac{1}{\lambda_k}\chi_{B(x_0,2^{k+2}r)\setminus B(x_0,2^{k+1}r)}(x) I_{\rho,\mu,33}f(x), \\
B_k(x) &= \frac{1}{\mu(B(x_0,2^{k+2}r))^{\frac{1}{p}}} \frac{\chi_{B(x_0,2^{k+2}r)}(x)}{\mu(B(x_0,2^{k+3}r))^{\frac{1}{p}}\phi(2^{k+2}r)},
\end{aligned}$$

then we are in the position of using Lemma 2.12 thanks to by (29) and in particular we conclude $I_{\rho,\mu,33}f$ belongs to $\mathcal{H}_{p',\phi}(X;\mu)$. □

Lemma 3.2. *Let $\tau > 1$ and f, g be positive μ-measurable functions. Then*

$$\int_X I_{\rho,\mu,\tau}f(x) \cdot g(x)\,d\mu(x) \leq \int_X f(x) \cdot I_{\rho,\mu,\tau-1}g(x)\,d\mu(x).$$

Proof. Just compare the integral kernels. We have

$$\mu(B(x,\tau d(x,y))) \geq \mu(B(y,(\tau-1)d(x,y))). \qquad \square$$

Proof of Theorem 1.9. The proof parallels, so that we deal with (1) and (2) simultaneously. Let $f \in \mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$.

Let us suppose for a while that f has bounded support and that $f \in L^\infty(X;\mu)$. By Lemma 3.1 at least, we can say that $I_{\rho,\mu,33}f$ is in $\mathcal{H}_{p',\phi}(X,\mu)$. The Hahn-Banach theorem gives us a functional $L \in B(\mathcal{H}_{p',\phi}(X;\mu),\mathbb{C})$ with norm 1 such that

$$\|I_{\rho,\mu,33}f\|_{\mathcal{H}_{p',\phi}(X;\mu)} = L(I_{\rho,\mu,33}f).$$

By Theorem 1.8, we can find $g \in L_{p,\phi;2}(X;\mu)$ with norm 1 such that $L = L_g$. Thus,

$$\|I_{\rho,\mu,33}f\|_{\mathcal{H}_{p',\phi}(X;\mu)} = L_g(I_{\rho,\mu,33}f) = \int_X I_{\rho,\mu,33}f(x)g(x)\,d\mu(x).$$

Keeping in mind that the integral kernel of $I_{\rho,\mu,33}$ is positive, we deduce

$$\|I_{\rho,\mu,33}f\|_{\mathcal{H}_{p',\phi}(X;\mu)} \leq \int_X I_{\rho,\mu,33}[|f|](x)|g(x)|\,d\mu(x).$$

If we use Lemma 3.2, then $\|I_{\rho,\mu,33}f\|_{\mathcal{H}_{p',\phi}(X;\mu)} \leq \int_X |f(x)|I_{\rho,\mu,32}[|g|](x)\,d\mu(x)$. Recall that $\|I_{\rho,\mu,32}[|g|]\|_{L_{q,\phi^{\frac{p}{q}};4}(X;\mu)} \lesssim 1$ by virtue of Theorem 1.2 and Theorem 1.5. Finally, if we use Theorem 1.8 again, we conclude

$$\|I_{\rho,\mu,33}f\|_{\mathcal{H}_{p',\phi}(X;\mu)} \leq \|f\|_{\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)}\|I_{\rho,\mu,32}[|g|]\|_{L_{q,\phi^{\frac{p}{q}};4}(X;\mu)} \lesssim \|f\|_{\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)} \quad (30)$$

for all $f \in L^\infty(X;\mu)$ with bounded support.

Let us consider the general case; just assume that $f \in \mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$. Then we have an expression:

$$f = \sum_{j=1}^\infty \lambda_j b_j, \quad \sum_{j=1}^\infty |\lambda_j| \leq 2\|f\|_{\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)}$$

where $\{\lambda_j\}_{j=1}^\infty \in \ell^1(\mathbb{N})$ and each b_j is a $(q',\phi^{\frac{p}{q}})$-block. As we did in Lemma 2.11, by changing b_j slightly, we can assume that $b_j \in L^\infty(X;\mu)$. If necessary, by removing j such that $\lambda_j = 0$ and then replacing λ_j with $|\lambda_j|$ and b_j with $\frac{\lambda_j}{|\lambda_j|}b_j$, we can assume that each λ_j is positive. Thus, if we set

$$f_N = \sum_{j=1}^N \lambda_j b_j,$$

then, we conclude that $\{f_N\}_{N=1}^\infty$ is convergent in f in $\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$ and from (30) $\{I_{\rho,\mu,33}f_N\}_{N=1}^\infty$ is convergent to a function g in $\mathcal{H}_{p',\phi}(X;\mu)$. In fact, from (30), we see that $\{I_{\rho,\mu,33}f_N\}_{N=1}^\infty$ is a Cauchy sequence.

If we set

$$f_+ = \sum_{j=1}^\infty \lambda_j \max(b_j, 0), \quad f_- = \sum_{j=1}^\infty \lambda_j \min(b_j, 0)$$

and argue as above. But we need to be careful because we need to justify the definition of $I_{\rho,\mu,33}f$; there is no guarantee of the absolute convergence of the integral defining $I_{\rho,\mu,33}f$. Observe first that

$$\|\max(b_j,0)\|_{\mathcal{H}_{q',\phi}(X;\mu)} + \|\min(b_j,0)\|_{\mathcal{H}_{q',\phi}(X;\mu)} \le 2\|b_j\|_{\mathcal{H}_{q',\phi}(X;\mu)} \le 2 < \infty$$

thanks to Lemma 2.12. Since b_j is bounded, as we have seen above,

$$\|I_{\rho,\mu,33}[\max(b_j,0)]\|_{\mathcal{H}_{p',\phi}(X;\mu)} + \|I_{\rho,\mu,33}[\min(b_j,0)]\|_{\mathcal{H}_{p',\phi}(X;\mu)} \le C < \infty,$$

where the constant C is depends on ρ. Hence,

$$\begin{aligned}&\sum_{j=1}^\infty \lambda_j \left(\|I_{\rho,\mu,33}[\max(b_j,0)]\|_{\mathcal{H}_{p',\phi}(X;\mu)} + \|I_{\rho,\mu,33}[\min(b_j,0)]\|_{\mathcal{H}_{p',\phi}(X;\mu)}\right)\\ &\lesssim \sum_{j=1}^\infty \lambda_j \\ &\lesssim \|f\|_{\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)}.\end{aligned} \tag{31}$$

Since the integral kernel is positive, we have

$$I_{\rho,\mu,33}f_+ = \sum_{j=1}^\infty \lambda_j I_{\rho,\mu,33}[\max(b_j,0)], \quad I_{\rho,\mu,33}f_- = \sum_{j=1}^\infty \lambda_j I_{\rho,\mu,33}[\min(b_j,0)]$$

and (31) shows that the convergence takes place in the topology of $\mathcal{H}_{p',\phi}(X;\mu)$ as well as in the sense of the pointwise convergence. Thus, we have

$$I_{\rho,\mu,33}f = I_{\rho,\mu,33}f_+ + I_{\rho,\mu,33}f_- = g \in \mathcal{H}_{p',\phi}(X;\mu),$$

as was to be shown. □

Proofs of Theorems 1.10 *and* 1.11. Under the assumptions of these theorems we see that $I_{\rho,\mu,32}$ is bounded from $L_{p,\phi;2}(X;\mu)$ to $L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$. Indeed, for $g \in L_{p,\phi;2}(X;\mu)$, by virtue of the duality $\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$-$L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$, we have

$$\begin{aligned}\|I_{\rho,\mu,32}g\|_{L_{q,\phi^{\frac{p}{q}};4}(X;\mu)} &= \|L_{I_{\rho,\mu,32}g}\|_{B(\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu),\mathbb{C})}\\ &= \sup\{|L_{I_{\rho,\mu,32}g}(f)| \,:\, f \in \mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu), \|f\|_{\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X,\mu)} = 1\}.\end{aligned}$$

Thus, we can find $f \in \mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$ with norm 2 such that

$$\|I_{\rho,\mu,32}g\|_{L_{q,\phi^{\frac{p}{q}};4}(X;\mu)} \le |L_f(I_{\rho,\mu,32}g)| = \left|\int_X f(x) I_{\rho,\mu,32}g(x)\,d\mu(x)\right|.$$

If we invoke Lemma 3.2, then we have

$$\|I_{\rho,\mu,32}g\|_{L_{q,\phi^{\frac{p}{q}};4}(X;\mu)} \le \int_X I_{\rho,\mu,31}[|f|](x)|g(x)|\,d\mu(x).$$

Since $I_{\rho,\mu,31}$ is bounded from $\mathcal{H}_{q',\phi^{\frac{p}{q}}}(X;\mu)$ to $\mathcal{H}_{p',\phi}(X;\mu)$, we see from Theorem 1.8 that

$$\|I_{\rho,\mu,32}g\|_{L_{q,\phi^{\frac{q}{p}};4}(X;\mu)} \le \|I_{\rho,\mu,31}[|f|]\|_{\mathcal{H}_{p',\phi}(X;\mu)}\|g\|_{L_{p,\phi;2}(X;\mu)} \le C\|g\|_{L_{p,\phi;2}(X;\mu)}.$$

This implies that $I_{\rho,\mu,32}$ is bounded from $L_{p,\phi;2}(X;\mu)$ to $L_{q,\phi^{\frac{p}{q}};4}(X;\mu)$. So, we can use Theorems 1.3 and 1.6 to conclude that (6) holds in Theorem 1.10 and that (8) holds in Theorem 1.11. □

3.5. Proof of Theorem 1.13. As in Lemmas 2.2, 2.4 and 2.5, we can prove the following lemmas:

Lemma 3.3. *Let μ be a doubling measure and let $\rho \in \mathcal{G}$. Fix $z \in X$. Then there exists a constant $C > 0$ such that $\tilde{\rho}(\frac{R}{2}) \le C I_{\rho,\mu,4}\chi_{B(z,R)}(x)$ holds whenever $x \in B(z, \frac{R}{2})$ and $R > 0$.*

Lemma 3.4. *Let μ be a doubling measure and $\phi \in \mathcal{G}$. Fix $z \in X$. Define $g_R(y)$ by (15) for $y \in X$ and $R > 0$. Let k_1 be a constant appearing in (2). Then there exists a constant $C > 0$ such that, for every $R > 0$ and $\rho \in \mathcal{G}_0$,*

$$I_{\rho,\mu,4}g_R(x) \le C\int_{k_1 R}^{\infty} \frac{\rho(t)\phi(t)}{t}\,dt$$

holds, whenever $x \in B(z, \frac{R}{3})$.

Lemma 3.5. *Let μ be a doubling measure and, let $\rho \in \mathcal{G}$ and $\phi \in \mathcal{G}$. Fix $z \in X$. Define $g_R(y)$ by (15) for $y \in X$ and $R > 0$. Then, there exists a constant $C > 0$ such that, for every $R > 0$,*

$$I_{\rho,\mu,4}g_R(x) \ge C\int_{R}^{\infty} \frac{\rho(t)\phi(t)}{t}\,dt \qquad (32)$$

holds, whenever $x \in B(z, \frac{R}{3})$.

Proof of Theorem 1.13. Fix $z \in X$. Observe from Lemma 3.3 that

$$\tilde{\rho}\left(\frac{R}{2}\right) \le C \frac{1}{\mu(B(z,\frac{R}{2}))} \int_{B(z,\frac{R}{2})} I_{\rho,\mu,4}\chi_{B(z,R)}(x)d\mu(x). \tag{33}$$

Since $I_{\rho,\mu,4}$ is assumed to be a bounded from $L_{1,\phi;2}(X;\mu)$ to $L_{1,\psi;4}(X;\mu)$, we note from Lemma 2.1 that $\|I_{\rho,\mu,4}\chi_{B(z,R)}\|_{L_{1,\psi;4}(X;\mu)} \le C\|\chi_{B(z,R)}\|_{L_{1,\phi;2}(X;\mu)} \le C\phi(R)^{-1}$ for all $R > 0$. Since $\phi \in \mathcal{G}_1$ and μ is a doubling measure, we have by (33)

$$\begin{aligned}\tilde{\rho}\left(\frac{R}{2}\right) &\le C \frac{1}{\mu(B(z,R))} \int_{B(z,\frac{R}{2})} I_{\rho,\mu,4}\chi_{B(z,R)}(x)d\mu(x)\\ &\le C\psi\left(\frac{R}{2}\right) \|I_{\rho,\mu,4}\chi_{B(z,R)}\|_{L_{1,\psi;4}(X;\mu)}\\ &\le C\frac{\psi(\frac{R}{2})}{\phi(R)}\\ &\le C\frac{\psi(\frac{R}{2})}{\phi(\frac{R}{2})}\end{aligned}$$

for all $R > 0$. Hence, we obtain $\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt \ \le C\psi(r)$.

As in the proof of Theorem 1.6, we can prove that $\|g_R\|_{L_{1,\phi;2}(X;\mu)} \le C$. If we integrate (32) over the ball $B(z,\frac{R}{3})$, then we have

$$\frac{1}{\psi(R)}\int_R^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\|I_{\rho,\mu,4}g_R\|_{L_{1,\psi;4}(X;\mu)} \le C\|g_R\|_{L_{1,\phi;2}(X;\mu)} \le C.$$

Thus, Theorem 1.13 is proved. □

Acknowledgement. The second author was partially supported by Grant-in-Aid for Scientific Research (C), No. 15K04929, Japan Society for the Promotion of Science. The authors are thankful to the anonymous reviewer(s) for their careful reading of this paper.

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Received March 20, 2015; revised October 13, 2016