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A Vanishing Theorem for Holonomic Modules with Positive Characteristic Varieties

By

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Abstract

Let M be a real analytic manifold, X a complexification of M, $\mathcal{M}$ a holonomic module over the ring $\mathcal{E}_X$ of microdifferential operators and $Char(\mathcal{M})$ its characteristic variety. We prove that if $(T_M^*X, Char(\mathcal{M}))$ is positive at $p \in T_M^*X$, then $\mathcal{E}xt^j_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)_p = 0$ for $j > 0$, where $\mathcal{C}_M$ denotes the sheaf of Sato's microfunctions.

§ 1. Preliminary

Let us recall the definition of positivity due to Melin and Sjöstrand (cf. [Me-Sj 1,2]) and a theorem of Schapira [S 1] that we shall need.

Let V be a real analytic manifold with complexification W. Denote by $I_k(V)$ the sheaf of $\mathcal{C}^\infty$ real valued functions on W vanishing up to order k (i.e: with all derivatives of order $< k$) on V. If one chooses a local coordinate system (x) on W, real on V, one can consider the morphism $v\colon W \to TV$

$$v\colon (x) \longmapsto (\mathrm{Re}\, x, \mathrm{Im}\, x). \tag{1.1}$$

If α is a 1-form on V, one proves (cf. Melin-Sjöstrand [loc. cit.]) that the function on W, $x \mapsto \langle \alpha, v(x) \rangle$ is well-defined mod $I_3(V)$ and does not depend on the choice of local coordinate system.

Now let X be a complex manifold, $\pi\colon T^*X \to X$ its cotangent bundle, and α_X the complex canonical 1-form on T^*X.

A locally closed subset Λ of T^*X will be called $\mathbb{R}^+$-conic (resp. $\mathbb{C}^\times$-conic) if it is locally a union of orbits of $\mathbb{R}^+$(resp. $\mathbb{C}^\times$) on T^*X.

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An $\mathbb{R}^+$-conic real analytic manifold Λ_0 is said to be $\mathbb{R}$-Lagrangian if Λ_0 is Lagrangian in the real symplectic space $(T^*X)^{\mathbb{R}} \simeq T^*X^{\mathbb{R}}$ (the space T^*X endowed with the 2-form 2Re $d\alpha_X$).

A real $\mathbb{R}$-Lagrangian manifold Λ_0 is said to be I-symplectic if Im $d\alpha_X|_{\Lambda_0}$ is non degenerate (i.e: is symplectic). In this case, $T^*X^{\mathbb{R}}$ is a complexification of Λ_0.

Definition 1.1. *Let Λ_0 be an $\mathbb{R}^+$-conic $\mathbb{R}$-Lagrangian and* I-*symplectic real analytic manifold in T^*X, and let Λ be an $\mathbb{R}^+$-conic subset of T^*X. One says (Λ_0, Λ) is positive at $p \in \Lambda_0$ if*

$$-\frac{1}{i} < \alpha_X|_{\Lambda_0}, v > \geqslant 0 \quad (\text{mod } I_3(\Lambda_0)) \tag{1.2}$$

on a neighborhood of p in Λ. (The function v is given by (1.1) *with $V = \Lambda_0$).*

If $(z;\zeta)$ is a system of holomorphic homogeneous symplectic coordinates with $z = x + iy$, $\zeta = \xi + i\eta$, $\alpha_X = \zeta_j dz_j$ and $\Lambda_0 = \{y = \xi = 0\}$, then (Λ_0, Λ) is positive at $p \in \Lambda_0$ iff there exists an open neighborhood U of p and a constant $C \geq 0$ such that

$$- < y, \eta > \geq - C(|y|^3 + |\xi|^3) \quad (z;\zeta) \in \Lambda \cap U. \tag{1.3}$$

When Λ is a complex Lagrangian manifold, this definition is due to Melin-Sjöstrand [loc. cit]. In the general case, it is due to Schapira [loc. cit].

We shall use the following:

Theorem 1.2 (cf. [S 1]).

Let Λ_0 be an $\mathbb{R}^+$-conic $\mathbb{R}$-Lagrangian and I-*symplectic real analytic manifold in T^*X and let Λ be a $\mathbb{C}^\times$-conic subset of T^*X. We assume that $\Lambda_0 = (T^*_{\partial\Omega}X)^+$ is the exterior conormal bundle to the real analytic boundary $\partial\Omega$ of a strictly pseudo-convex open set Ω, and that (Λ_0, Λ) is positive at $p \in \Lambda_0$. Then there exists an open neighborhood U of p such that*

$$\pi(U \cap \Lambda) \cap \Omega = \emptyset. \tag{1.4}$$

Recall that if $\Omega = \{f < 0\}$, where f is a real function on X with $df \neq 0$, then

$$(T^*_{\partial\Omega}X)^+ = \{(z;\zeta) \in T^*X; f(z) = 0, \zeta = kd'f(z), k \in \mathbb{R}^+\}. \tag{1.5}$$

Here we denote by d' the complex differential. The following result is immediately deduced from (1.3).

Lemma 1.3. *Let X_j be a complex manifold, Λ_{0j} be an $\mathbb{R}^+$-conic $\mathbb{R}$-Lagrangian and* I-*symplectic manifold in T^*X_j and let Λ_j be a $\mathbb{C}^\times$-conic subset of T^*X_j $(j = 1, 2)$. Assume $(\Lambda_{0j}, \Lambda_j)$ is positive at $p_j \in \Lambda_{oj}$ for all j.*

Then $(\Lambda_{01} \times \Lambda_{02}, \Lambda_1 \times \Lambda_2)$ is positive at $(p_1 \times p_2) \in \Lambda_{01} \times \Lambda_{02}$.

§ 2. The Vanishing Theorem

Let M be a real analytic manifold, and X a complexification of M. We set:

$$\Lambda_0 = T^*_M X. \tag{2.1}$$

Recall that Λ_0 is $\mathbb{R}$-Lagrangian and I-symplectic. Let $\mathscr{E}_X$ denote the sheaf of microdifferential operators of finite order on T^*X, and let $\mathscr{C}_M$ denote the sheaf of Sato's microfunctions on $T^*_M X$ (refer to [S-K-K], and cf. [S 2] for an exposition of the theory of $\mathscr{E}_X$-modules).

Let $\mathscr{M}$ be a left coherent $\mathscr{E}_X$-module defined on an open subset U of T^*X. We shall assume $\mathscr{M}$ is holonomic, and we denote by Λ its characteristic variety:

$$\Lambda = Char(\mathscr{M}). \tag{2.2}$$

Hence Λ is a $\mathbb{C}^\times$-conic subset of U. Let $p \in U \cap T^*_M X$.

Theorem 2.1. *We assume (Λ_0, Λ) is positive at p. Then*

$$\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}_M)_p = 0 \quad \textit{for} \quad j > 0.$$

Remark. If $p \in M$ (the zero-section of $T^*_M X$), we get

$$\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)_p = 0 \quad \text{for} \quad j > 0.$$

Proof.

We shall give the proof in several steps.

(a) By the trick of the dummy variable due to M. Kashiwara, we shall reduce the problem to the case where $p \notin T^*_X X$. Let t be a holomorphic coordinate on $\mathbb{C}$, real on $\mathbb{R}$, $q = (0; idt)$ and let δ denote the $\mathscr{D}_\mathbb{C}$-module $\mathscr{D}_\mathbb{C}/\mathscr{D}_\mathbb{C} t$.

The sequence

$$0 \longrightarrow (\mathscr{C}_M)_p \longrightarrow (\mathscr{C}_{M\times\mathbb{R}})_{(p,q)} \xrightarrow{t} (\mathscr{C}_{M\times\mathbb{R}})_{(p,q)} \longrightarrow 0,$$

is exact. Thus we get

$$R\mathscr{H}om_{\mathscr{E}_{X\times\mathbb{C}}}(\mathscr{M} \mathbin{\hat{\otimes}} \delta, \mathscr{C}_{M\times\mathbb{R}})_{(p,q)} = R\mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{C}_M)_p. \tag{2.3}$$

Since $(T^*_\mathbb{R}\mathbb{C}, T^*_{\{0\}}\mathbb{C})$ is positive at q, the positivity of (Λ_0, Λ) at p implies that of $(\Lambda_0 \times T^*_\mathbb{R}\mathbb{C}, \Lambda \times T^*_{\{0\}}\mathbb{C})$ at (p, q) on account of lemma 1.3. Thus assuming the theorem is proved outside of the zero-section, the result follows in the general case from (2.3).

(b) Now we assume $p \in \mathring{T}^*X = T^*X \backslash X$. Let X' be another copy of X, $p' \in \mathring{T}^*X'$, and let φ be a complex contact transformation which interchange (T^*X, p) and (T^*X', p').

Let $\Lambda'_0 = \varphi(\Lambda_0)$, $\Lambda' = \varphi(\Lambda)$, $\lambda_0 = T_p\Lambda_0$, $\lambda'_0 = T_{p'}\Lambda'_0$, $\lambda = T_p\Lambda$ and λ'

$= T_{p'}\Lambda'$. Denote by μ the tangent plane at (p, p') to the Lagrangian submanifold of $T^*(X \times X')$ associated to the graph of φ. Let GL denote the Lagrangian Grassmanian of $T_{(p,p')}T^*(X \times X')$, and consider the properties:

(2.4) Λ'_0 is the exterior conormal bundle of a strictly pseudo-convex open set Ω of X' in a neighborhood of p'.

(2.5) Λ' is in a generic position at p' (i.e. $\Lambda' \cap \pi^{-1}\pi(p') = \mathbb{C}^\times p'$).

Then the set of μ in GL with the properties $\lambda'_0 = \mu \circ \lambda_0$ and (2.4) is open and non void, and the set of μ in GL with $\lambda' = \mu \circ \lambda$ and (2.5) is open and dense. Thus we may find φ so that (2.4) and (2.5) are both satisfied. Here $\mu \circ \lambda_0$ or $\mu \circ \lambda$ denotes the image of λ_0 or λ by the linear contact transformation associated to μ.

(c) By quantizing φ (cf. [K-S 1,2]), we may interchange $\mathscr{C}_M$ with the sheaf $\mathscr{C}_S = j_* j^{-1} \mathscr{O}_{X'}/\mathscr{O}_{X'}$ where Ω is a strictly pseudo-convex open set with real analytic boundary $S = \partial\Omega$, $(T^*_{\partial\Omega}X')^+ = \varphi(\Lambda'_0)$, and j is the open embedding $\Omega \hookrightarrow X'$.

Now we write Λ, Λ_0, etc. instead of Λ', Λ'_0, etc. Since Λ is in a generic position, we may assume $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module by a result of Kashiwara-Kawai (Theorem 5.1.4, [K-K]). Hence we are in the following situation.

X is a complex manifold, Ω is a strictly pseudo-convex open set in X with real analytic boundary $S = \partial\Omega$. $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module with characteristic variety Λ, which satisfies (in view of Theorem 1.2):

$$\pi(\Lambda \cap \mathring{T}^*X) \cap \Omega = \emptyset. \tag{2.6}$$

The condition (2.6) implies that on Ω, $\mathscr{M}$ is locally isomorphic (as $\mathscr{D}_X$-modules) to $\mathscr{O}_X^m$ for some m by Kashiwara [K].
Thereby $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is locally constant and concentrated in degree zero, on Ω. Since $\partial\Omega$ is smooth, we get

$$H^k(R\Gamma_\Omega R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X))_{\pi(p)} = 0 \quad \text{for} \quad k > 0.$$

Hence

$$\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, j_* j^{-1} \mathscr{O}_X)_{\pi(p)} = 0 \quad \text{for} \quad k > 0.$$

To conclude, it remains to prove

$$\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)_{\pi(p)} = 0 \quad \text{for} \quad k > 1.$$

Since $Char(\mathscr{M})$ is in a generic position, there exists a 1-dimensional manifold Y passing through $\pi(p)$, and non characteristic for $\mathscr{M}$. By the Cauchy-Kowalewski-Kashiwara theorem (cf. [K]), we get:

$$\mathscr{E}xt^k_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)_{\pi(p)} \simeq \mathscr{E}xt^k_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y)_{\pi(p)} \quad \forall k.$$

Here $\mathscr{M}_Y$ is the induced system of $\mathscr{M}$ on Y. Since $Proj.dim(\mathscr{M}_Y) \leq 1$, we have

$$\mathscr{E}xt^k_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y)_{\pi(p)} = 0 \quad \text{for} \quad k > 1.$$

This completes the proof.

Examples.

(1) Let M be a real analytic manifold with complexification X and let $\{M_\alpha\}_\alpha$ be a finite set of closed submanifolds of M. Denoting by X_α a complexification of M_α, we assume $Char(\mathscr{M}) \subset \cup T^*_{X_\alpha}X$. Then $(T^*_M X, Char(\mathscr{M}))$ is positive at each $p \in T^*_M X$, and $\mathscr{E}xt^j_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)_p = 0$ for $j > 0$. Hence we recover a result of Lebeau [Le].

(2) Let $M = \mathbb{R}^{n+1}$ and $X = \mathbb{C}^{n+1}$. Denote by $(t, x_1, \dots, x_n)$ the coordinate system of X (real on M). Let $\mathscr{M}$ be the holonomic $\mathscr{E}_X$ module defined by the equations

$$\mathscr{M}: \begin{cases} P_j u = \left(2ix_j \dfrac{\partial}{\partial t} - \dfrac{\partial}{\partial x_j}\right) u = 0 \quad 1 \leq j \leq n, \\[2ex] Qu = \left(4it \dfrac{\partial^2}{\partial t^2} + \displaystyle\sum_{j=1}^n \dfrac{\partial^2}{\partial x_j^2}\right) u = 0. \end{cases}$$

Let $(t, x; \tau, \xi)$ be a coordinate system of T^*X, and $f(t, x) = t + i\sum_{j=1}^n x_j^2$. Remark that $[P_j, Q] = 4i\dfrac{\partial}{\partial t}P_j$ and $[P_j, P_k] = 0$. Moreover the following equations (#) form a regular sequence on their common zero set.

$$(\#): \begin{cases} 2ix_j\tau - \xi_j = 0 \quad 1 \leq j \leq n, \\[2ex] 4it\tau^2 + \displaystyle\sum_{j=1}^n \xi_j^2 = 0. \end{cases}$$

Thus $Char(\mathscr{M})$ is defined by equations (#), and we get:

$$Char(\mathscr{M}) = T^*_{\{f(t,x)=0\}} X \cup T^*_X X.$$

Since $f(t, x)$ is of positive type at 0 (for the definition of a positive type function, refer to [S-K-K]), $(T^*_M X, Char(\mathscr{M}))$ is positive at $(0; idt)$ (cf. [S 1]).

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