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Generalized Hardy–Morrey Spaces

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Abstract. This paper is an off-spring of the contribution [Z. Anal. Anwend. 36 (2017)(1), 17–35]. We propose a way to consider the decomposition method of generalized Hardy–Morrey spaces. Generalized Hardy–Morrey spaces emerged from generalized Morrey spaces. By means of the grand maximal operator and the norm of generalized Morrey spaces, we can define generalized Hardy–Morrey spaces. With what we have culminated for the Hardy–Littlewood maximal operator, we can easily refine the existing results. As an application, we consider bilinear estimates, which is the "so-called" Olsen inequality. In particular, our results complement the one in the 2014 paper by Iida, the fourth author and Tanaka [Z. Anal. Anwend. 33 (2014)(2), 149–170]; there were two mistakes. One lies in the decomposition result and another lies in the proof of the Olsen inequality.

Keywords. Generalized Hardy–Morrey spaces, decomposition, maximal operators, Olsen inequality

Mathematics Subject Classification (2010). Primary 41A17, secondary 42B20, 42B35

1. Introduction

We are concerned with generalized Hardy–Morrey spaces, which originate from generalized Morrey spaces, in the present paper. The generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is equipped with a function ϕ and a positive parameter $0 < p < \infty$. The generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ was defined independently by Mizuhara in 1991 [20] and Nakai in 1994 [21]. Although we can disprove that the set of all

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compactly supported smooth functions is dense in $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$, we are still able to develop a theory of the generalized Hardy–Morrey space $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$.

Let $0 < p < \infty$. Denote by $\mathcal{G}_p$ the set of all the functions $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ decreasing in the second variable such that $t \in (0,\infty) \mapsto t^{\frac{n}{p}}\phi(x,t) \in (0,\infty)$ is almost increasing uniformly over the first variable x, so that there exists a constant $C > 0$ such that

$$\phi(x,r) \le \phi(x,s), \quad C\phi(x,r)r^{\frac{n}{p}} \ge \phi(x,s)s^{\frac{n}{p}}$$

for all $x \in \mathbb{R}^n$ and $0 < s \le r < \infty$. In this paper we often assume $0 < p \le 1$.

All "cubes" in $\mathbb{R}^n$ are assumed to have their sides parallel to the coordinate axes. Denote by $\mathcal{Q} = \mathcal{Q}(\mathbb{R}^n)$ the set of all cubes. For a cube $Q \in \mathcal{Q}$, the symbol $\ell(Q)$ stands for the side-length of the cube Q; $\ell(Q) \equiv |Q|^{\frac{1}{n}}$, where $|E|$ denotes the Lebesgue measure of a measurable set E. When we are given a cube Q, we use the following abuse of notation: $\phi(Q) \equiv \phi(c(Q), \ell(Q))$, where $c(Q)$ denotes the center of Q.

Let $0 < p < \infty$ and $\phi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a function which is not necessarily in $\mathcal{G}_p$. The generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is defined as the set of all measurable functions f for which the quasi-norm

$$\|f\|_{\mathcal{M}_{p,\phi}} \equiv \sup_{Q\in\mathcal{Q}} \frac{1}{\phi(Q)} \left(\frac{1}{|Q|}\int_Q |f(y)|^p \, dy\right)^{\frac{1}{p}}$$

is finite. Seemingly the requirement $\phi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural; see [1, Theorem 2.5]. Note that many function spaces can be realized as the special case of $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$; see [1].

One of the primary aims of this paper is to prove the following decomposition result on the functions in the generalized Morrey space $\mathcal{M}_{1,\phi}(\mathbb{R}^n)$ for $\phi \in \mathcal{G}_1$:

Theorem 1.1. *Assume that $\phi, \eta \in \mathcal{G}_1$ satisfy*

$$\int_r^\infty \frac{\phi(x,s)}{\eta(x,s)s} \, ds \le C\frac{\phi(x,r)}{\eta(x,r)} \quad (x \in \mathbb{R}^n, r > 0). \tag{1}$$

Assume that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathcal{M}_{1,\eta}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill

$$\|a_j\|_{\mathcal{M}_{1,\eta}} \le \frac{1}{\eta(\ell(Q_j))}, \quad \mathrm{supp}(a_j) \subset Q_j, \quad \left\|\sum_{j=1}^\infty \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{1,\phi}} < \infty, \tag{2}$$

where χ_E denotes the indicator function of the set E. Then

$$f \equiv \sum_{j=1}^\infty \lambda_j a_j$$

converges absolutely in $L_{1,\mathrm{loc}}(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{\mathcal{M}_{1,\phi}} \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{1,\phi}}. \tag{3}$$

We shall show that (1) can not be replaced by another condition: $\phi = \eta$ in general; see Remark 3.2.

The proof of this theorem is not so difficult and it is given in an early stage of the present paper; see Section 3.1. Unlike the case when $p > 1$, when $0 < p \le 1$, $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is a nasty space as the following example shows:

Example 1.2. Suppose that $\xi \in \mathcal{G}_1$. Denote by M the Hardy–Littlewood maximal operator.

1. Let $\eta : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a function which is independent of the position x. Define $\mathcal{M}_{L\log L,\eta}$ by the following norm (see [33])

$$\|f\|_{\mathcal{M}_{L\log L,\eta}} \equiv \sup_{Q\in\mathcal{Q}} \frac{1}{\eta(\ell(Q))} \inf\left\{\lambda > 0 : \int_Q \frac{|f(x)|}{\lambda} \log\left(e + \frac{|f(x)|}{\lambda}\right) dx \le |Q|\right\}.$$

   Define $\xi(x,t) = \eta(t)$ for $x \in \mathbb{R}^n$ and $t > 0$. In [33, Lemma 3.5], we proved

$$C^{-1}\|f\|_{\mathcal{M}_{L\log L,\eta}} \le \|Mf\|_{\mathcal{M}_{1,\xi}} \le C\|f\|_{\mathcal{M}_{L\log L,\eta}} \tag{4}$$

   for all $f \in \mathcal{M}_{L\log L,\eta}(\mathbb{R}^n)$.
2. Let $0 < p < 1$. Let $\xi : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a function which is independent of the position x. Assume $\xi \in \mathcal{G}_1 = \mathcal{G}_1 \cap \mathcal{G}_p$. The space $\mathcal{M}_{p,\eta}(\mathbb{R}^n)$ is a little easier to handle than $\mathcal{M}_{1,\eta}(\mathbb{R}^n)$ in view of (4) and (5) below. In [33, Lemma 3.4], we proved

$$C^{-1}\|f\|_{\mathcal{M}_{1,\xi}} \le \|Mf\|_{\mathcal{M}_{p,\xi}} \le C\|f\|_{\mathcal{M}_{1,\xi}} \tag{5}$$

   for all $f \in \mathcal{M}_{1,\eta}(\mathbb{R}^n)$.

From these examples we see that $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ with $p \in (0,1]$ is difficult to handle. Probably, Theorem 1.1 paves the way to deal with such a nasty space.

Another method to handle these nasty spaces to use the grand maximal operator and define generalized Hardy–Morrey spaces; see (19) for the definition of the grand maximal operator. Let $t > 0$ and $f \in L_1(\mathbb{R}^n)$. Then define the heat semigroup by

$$e^{t\Delta}f(x) \equiv \int_{\mathbb{R}^n} \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)dy \quad (x \in \mathbb{R}^n).$$

Note that $e^{t\Delta}$ maps $\mathcal{S}(\mathbb{R}^n)$ to itself continuously. Using the duality, we naturally extend $e^{t\Delta}f$ to the case when $f \in \mathcal{S}'(\mathbb{R}^n)$. Let $0 < p < \infty$ and $\phi \in \mathcal{G}_p$.

The generalized Hardy–Morrey space $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\sup_{t>0} |e^{t\Delta} f(\cdot)| \in \mathcal{M}_{p,\phi}(\mathbb{R}^n)$. We equip $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ with the following norm

$$\|f\|_{H\mathcal{M}_{p,\phi}} \equiv \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{\mathcal{M}_{p,\phi}} \quad (f \in H\mathcal{M}_{p,\phi}(\mathbb{R}^n)). \tag{6}$$

It turns out that $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ and $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ are isomorphic when $1 < p < \infty$ and $\phi \in \mathcal{G}_p$; see Theorem 2.4.

We define $d_p \equiv \frac{n}{p} - n$ for $0 < p \le 1$. In addition to Theorem 1.1, we shall prove the following two theorems in the present paper:

Theorem 1.3. *Let $0 < p \le 1$ and $d \ge d_p$. Let q satisfy*

$$q \in [1,\infty] \cap (p,\infty]. \tag{7}$$

Assume that $\phi \in \mathcal{G}_p$ and $\eta \in \mathcal{G}_q$ satisfy

$$\int_r^\infty \frac{\phi(x,s)}{\eta(x,s)s}\, ds \le C \frac{\phi(x,r)}{\eta(x,r)} \tag{8}$$

for $x \in \mathbb{R}^n$ and $r > 0$. Assume in addition that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^\infty \subset \mathcal{M}_{q,\eta}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill

$$\left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^p \right)^{\frac{1}{p}} \right\|_{\mathcal{M}_{p,\phi}} < \infty$$

and

$$\|a_j\|_{\mathcal{M}_{q,\eta}} \le \frac{1}{\eta(Q_j)}, \quad \operatorname{supp}(a_j) \subset Q_j, \quad \int_{Q_j} x^\alpha a_j(x)\, dx = 0 \tag{9}$$

for all $|\alpha| \le d$. Then $f \equiv \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n)$, belongs to $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{H\mathcal{M}_{p,\phi}} \le C \left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^p \right)^{\frac{1}{p}} \right\|_{\mathcal{M}_{p,\phi}}. \tag{10}$$

Since $\eta(x,r) \ge \eta(x,s)$ for $r > s > 0$ and $x \in \mathbb{R}^n$, (1), (8) and [24, Proposition 2.7] imply

$$\int_r^\infty \frac{\phi(x,s)^p}{\eta(x,s)^p s}\, ds \le C \frac{\phi(x,r)^p}{\eta(x,r)^p} \quad (x \in \mathbb{R}^n, r > 0) \tag{11}$$

and

$$\int_r^\infty \phi(x,s) \frac{ds}{s} \le C\phi(x,r) \quad (x \in \mathbb{R}^n, r > 0). \tag{12}$$

We now present an example of the couple of functions ϕ and η satisfying (11).

Example 1.4. If there exist u and v with $v > u$ such that $\eta(x,s) = s^{-\frac{n}{v}}$ and that $\phi(x,s)s^{\frac{n}{u}} \le \phi(x,r)r^{\frac{n}{u}}$ for all $x \in \mathbb{R}^n$, $s > 0$ and $r > 0$ with $s \ge r$, then (11) is satisfied. This fact generalizes the main results in [14].

Theorem 1.3 will refine [19, p. 100 Theorem] in that we can postulate a weaker integrability condition on a_j in Theorem 1.3. We shall take its advantage in Section 4.

Theorem 1.5. *Let $L \in \mathbb{N} \cup \{0\}$, $0 < p \le 1$ and $\phi \in \mathcal{G}_1$. Assume* (12). *Then for any $f \in H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$, there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}(\mathbb{R}^n)$ and $\{a_j\}_{j=1}^\infty \subset L_\infty(\mathbb{R}^n)$ such that*

$$f = \sum_{j=1}^\infty \lambda_j a_j \tag{13}$$

in $\mathcal{S}'(\mathbb{R}^n)$, that $|a_j| \le \chi_{Q_j}$, that $\int_{\mathbb{R}^n} x^\alpha a_j(x)\,dx = 0$ for all multi-indices α with $|\alpha| \le L$ and that

$$\left\| \left(\sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}_{p,\phi}} \le C_v \|f\|_{H\mathcal{M}_{p,\phi}} \tag{14}$$

for all $v > 0$. Here the constant $C_v > 0$ is independent of f.

Unlike Orlicz spaces and variable exponent Lebesgue spaces, in general we can take a sequence $\{f_j\}_{j=1}^\infty$ of functions such that

$$f_1 \ge f_2 \ge \cdots \ge f_j \ge f_{j+1} \ge \cdots \to 0, \quad \inf_{j \in \mathbb{N}} \|f_j\|_{\mathcal{M}_{p,\phi}} > 0. \tag{15}$$

For example, when $0 < p < a$, the functions $f_j(x) \equiv \chi_{(j,\infty)}(|x|)|x|^{-\frac{n}{a}}$, $j = 1, 2, \ldots$ belong to $\mathcal{M}_{p,n-\frac{np}{a}}(\mathbb{R}^n)$ and the sequence $\{f_j\}_{j=1}^\infty$ satisfies (15). This makes it more difficult to look for a good dense space of $\mathcal{M}_{p,n-\frac{np}{a}}(\mathbb{R}^n)$. This difficulty prevents us from mimicking the proof of the decomposition of Hardy spaces described in [38]. It is not known that $H\mathcal{M}_{p,\phi}(\mathbb{R}^n) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n)$ is dense in $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$. It seems that $H\mathcal{M}_{p,\phi}(\mathbb{R}^n) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n)$ is not dense in $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ as the fact that $\mathcal{M}_{p,\phi}(\mathbb{R}^n) \cap L_{\infty,\mathrm{c}}(\mathbb{R}^n)$ is not dense in $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ implies. Recall that in [38], we resorted to density of $H_p(\mathbb{R}^n) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n)$ to obtain the atomic decomposition of $H_p(\mathbb{R}^n)$. The difficulty will cause a disability; we can prove Theorem 1.5 only when $f \in H\mathcal{M}_{p,\phi}(\mathbb{R}^n) \cap L_{1,\mathrm{loc}}(\mathbb{R}^n)$. Using a diagonal argument, we shall circumvent this problem; see (37) and (38).

Before we go further, let us recall some special cases related to generalized Morrey spaces.

Example 1.6. 1. Generalized Morrey spaces can cover $L_\infty(\mathbb{R}^n)$ spaces by letting $\phi \equiv 1$.

2. [35, Theorem 5.1] Generalized Morrey spaces arise naturally when we consider the endpoint case of the Sobolev embedding. Let $1 < p < \infty$ and $0 < \lambda < n$. Consider the norm

$$\|f\|_{\mathcal{M}_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int_{Q(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Let $\alpha \equiv \frac{n-\lambda}{p} \in (0, \infty)$. Then in [35] we showed that there exists a positive constant $C_{p,\lambda}$ such that

$$\int_B |f(x)| dx \le C_{p,\lambda} |B| (1 + |B|)^{-\frac{1}{p}} \log \left(e + \frac{1}{|B|} \right) \|(1 - \Delta)^{\alpha/2} f\|_{\mathcal{M}_{p,\lambda}} \quad (16)$$

holds for all $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ with $(1 - \Delta)^{\frac{\alpha}{2}} f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ and for all balls B and that log in (16) can not be deleted. See [4, Section 5] and [24, Proposition 7.3] for more related estimates. Meanwhile in view of the integral kernel of $(1 - \Delta)^{-\frac{\alpha}{2}}$ (see [37]) and the Adams theorem, we have

$$(1 - \Delta)^{-\frac{\alpha}{2}} : \mathcal{M}_{p,\lambda}(\mathbb{R}^n) \to \mathcal{M}_{q,\lambda}(\mathbb{R}^n) \quad (17)$$

is bounded as long as the parameters p, q, λ and α satisfy

$$1 < p < q < \infty,\ 0 < \lambda \le n,\ \frac{n-\lambda}{q} = \frac{n-\lambda}{p} - \alpha.$$

However, if $\alpha = \frac{n-\lambda}{p}$, the number q not being finite, the boundedness assertion (17) is no longer true. Hence (16) can be considered as a substitute of (17).

A passage to the Hardy type space HX from a given function space X is not a mere quest to generality. Many people have shown that Hardy spaces $H_p(\mathbb{R}^n)$ $(0 < p \le \infty)$ can be more informative than Lebesgue spaces $L_p(\mathbb{R}^n)$ when we discuss the boundedness of some operators. For example, the Riesz transform is bounded from $H_1(\mathbb{R}^n)$ to $L_1(\mathbb{R}^n)$, although they are not bounded on $L_1(\mathbb{R}^n)$. One of the earliest real variable definitions of Hardy spaces was based on the grand maximal operator, which is discussed in [38] and references therein. One can also give an equivalent definition for Hardy spaces by using the atomic decomposition. This definition states that any element of Hardy spaces can be represented as an infinite linear combination of atoms. An atom is a compactly supported function which enjoys the size condition and the cancellation moment condition. One of the advantages of the atomic decompositions in Hardy spaces is that we can prove the boundedness of some operators by verifying some estimates only for atoms. So we expect that HX is easier to handle than X itself. We seek to apply this idea when $X = \mathcal{M}_{p,\phi}(\mathbb{R}^n)$.

The concept of the atomic decomposition in Hardy spaces can be developed to other function spaces. Some of these works are on the decomposition of Hardy–Morrey spaces [12, 19], on the decomposition of Hardy spaces with variable exponent [28], and on the atomic decomposition of Morrey spaces [14]. See [10] for the martingale Hardy spaces. Motivated by these advantages that Hardy spaces enjoy, in our current research, we investigate the atomic decomposition for generalized Hardy–Morrey spaces, where we are based on the definition by means of the grand maximal operator.

There are many attempts of obtaining non-smooth atomic decompositions by using the grand maximal operator [2,9,11,14,22,23] and reducing the matters to the Hardy type spaces, where the authors handled Morrey spaces, Orlicz spaces and variable exponent Lebesgue spaces.

In addition to the notation used in [1], we adopt the following notation:

1. Let $0 < \alpha < n$. We define the fractional integral operator I_α by
$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$
for all suitable functions f on $\mathbb{R}^n$.
2. By $C^\infty_{\rm c}(\mathbb{R}^n)$ we denote the set of all compactly supported functions in $C^\infty(\mathbb{R}^n)$.
3. Let $d \in \mathbb{N}_0$. Denote by $\mathcal{P}_d(\mathbb{R}^n)$ the linear space of polynomials of degree less than or equal to d.
4. The space $L_{\infty,{\rm c}}(\mathbb{R}^n)$ denotes the set of all compactly supported essentially bounded functions.
5. For an integrable function f, define the *Fourier transform* and the *inverse Fourier transform* by:
$$\mathcal{F}f(\xi) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}\, dx,$$
$$\mathcal{F}^{-1}f(x) \equiv (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(\xi)e^{ix\cdot\xi}\, d\xi.$$

Finally, to conclude this section, we briefly describe how we organize the remaining part of this paper. Sections 2 collects preliminary facts. We prove the main theorems in Section 3 and apply these main theorems to obtain some bilinear estimates in Section 4.

2. Fundamental structure of function spaces

2.1. Structure of generalized Morrey spaces. We start with a quantitative observation for the norm of $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$.

Proposition 2.1. 1. *Let $0 < p \le 1$ and $\phi \in \mathcal{G}_p$. Then there exists a large integer N such that*

$$(1+|\cdot|)^{-N} \in \mathcal{M}_{p,\phi}(\mathbb{R}^n). \tag{18}$$

2. *Let $\phi \in \mathcal{G}_1$. Then $\mathcal{M}_{1,\phi}(\mathbb{R}^n)$ is continously embedded into $\mathcal{S}'(\mathbb{R}^n)$.*

Proof. This is a consequence of [1, Proposition 2.6]. □

2.2. Structure of generalized Hardy–Morrey spaces. The grand maximal operator characterizes Hardy–Morrey spaces defined by the norm (6). To formulate the result, we recall the following two fundamental notions.

1. Topologize $\mathcal{S}(\mathbb{R}^n)$ by norms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N(\varphi) \equiv \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|$$

for each $N \in \mathbb{N}$. Define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : p_N(\varphi) \le 1\}$.

2. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. The grand maximal operator $\mathcal{M}f$ is given by

$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\psi(t^{-1}\cdot) * f(x)| \,:\, t>0,\, \psi \in \mathcal{F}_N\} \quad (x \in \mathbb{R}^n), \tag{19}$$

where we choose and fix a large integer N.

In analogy to [22, Section 3], we can prove the following proposition:

Proposition 2.2. *Let $0 < p \le 1$ and $\phi \in \mathcal{G}_p$. Suppose that N in* (19) *is sufficiently large. Then $\|f\|_{H\mathcal{M}_{p,\phi}} \sim \|\mathcal{M}f\|_{\mathcal{M}_{p,\phi}}$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.*

From this proposition, we can use the norm $\|\mathcal{M}f\|_{\mathcal{M}_{p,\phi}}$ to define the space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$.

Lemma 2.3. *Let $0 < p \le 1$ and $\phi \in \mathcal{G}_p$. Suppose that N in* (19) *is sufficiently large. Then $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^n)$.*

Proof. Let N be a fixed large integer. Then there exists a constant $C > 0$ such that if $|y| \le 1$ and $\varphi \in \mathcal{F}_N$, then $C^{-1}\varphi(\cdot - y) \in \mathcal{F}_N$. Thus, $|\langle f, \varphi\rangle| \lesssim \inf_{|y|\le 1} \mathcal{M}f(y)$ for all $\varphi \in \mathcal{F}_N$. This implies $|\langle f,\varphi\rangle| \lesssim \|f\|_{H\mathcal{M}_{p,\phi}}$, as was to be shown. □

Going through the same argument as [22, Lemma 3.1], we obtain the following theorem, whose proof will be omitted.

Theorem 2.4. *Let $1 < p < \infty$ and $\phi \in \mathcal{G}_p$. Suppose that N in* (19) *is sufficiently large. Then $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ and $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ are isomorphic and the norms are equivalent.*

Remark 2.5. When $1 < p < \infty$ and $\phi \in \mathcal{G}_p$, M is bounded on $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ thanks to [1, Proposition 3.5]. Also, similar to [9], one can show that $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is realized as a dual space of a Banach space. By combining these facts, one can show the following fact for $f \in \mathcal{S}'(\mathbb{R}^n)$. The distribution f is represented by a function in $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ if and only if $\mathcal{M}f \in \mathcal{M}_{p,\phi}(\mathbb{R}^n)$.

3. Atomic decomposition

We return to the case where φ is independent of x and we prove the remaining theorems. Write

$$\tau \equiv \frac{n+d+1}{n} \tag{20}$$

here and below, so that $\tau p > 1$ as long as $d \geq d_p$.

3.1. Proof of Theorem 1.1. We use the following geometric observation: When we are given a dyadic cube $R = 2^{-\nu}m + [0, 2^{-\nu})^n$ for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{Z}$, we let $5R \equiv 2^{-\nu}m + [-2^{1-\nu}, 3 \times 2^{-\nu})^n$.

Lemma 3.1. *Let Q be a cube. Then there exists a dyadic cube R such that $Q \subset 5R$ and that $\ell(Q) \geq \ell(R)$. In particular, there exist 5^n dyadic cubes $R_1, R_2, \ldots, R_{5^n}$ such that*

$$\ell(Q) \geq \ell(R_1) = \ell(R_2) = \cdots = \ell(R_{5^n}), \quad \chi_Q \leq \sum_{j=1}^{5^n} \chi_{R_j} \leq \chi_{5R}.$$

Proof. The proof is a simple observation and we omit the details. □

The first step is a setup: Invoking [24, Proposition 2.7], we have

$$\int_r^\infty \frac{\sqrt{\phi(x,s)}}{s}\, ds \lesssim \sqrt{\phi(x,r)}.$$

Thus, according to [1, Corollary 4.5],

$$\|MF\|_{\mathcal{M}_{2,\sqrt{\phi}}(l_2)} \lesssim \|F\|_{\mathcal{M}_{2,\sqrt{\phi}}(l_2)}. \tag{21}$$

Next, we do some reductions. Observe that the norm will be equivalent if we take the supremum over all dyadic cubes. Thus, we need to show

$$\frac{1}{\phi(Q)|Q|}\int_Q |f(y)|\, dy \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{1,\phi}} \tag{22}$$

for any dyadic cube Q. For each j, we consider dyadic cubes $R^j_1, R^j_2, \ldots, R^j_{5^n}$ appearing in Lemma 3.1 with $Q = Q_j$. Write

$$b^j_k \equiv \chi_{R^j_k} a_j$$

for $k = 1, 2, \ldots, 5^n$ and $j = 1, 2, \ldots$. Observe that $\chi_{R^j_k} \lesssim (M\chi_{Q_j})^2$ and also that $\||f|^u\|_{\mathcal{M}_{\frac{p}{u},\phi^u}} = (\|f\|_{\mathcal{M}_{p,\phi}})^u$ for $u > 0$. Thus

$$\|b^j_k\|_{\mathcal{M}_{1,\eta}} \le \frac{1}{\eta(\ell(R^j_k))}, \quad \mathrm{supp}(b^j_k) \subset R^j_k, \quad \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{R^j_k}\right\|_{\mathcal{M}_{1,\phi}} \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{1,\phi}} < \infty$$

according to (2) and (21). This justifies that we may assume Q_j dyadic. We do another reduction: by replacing a_j with $|a_j|$ if necessary, we may assume that each a_j is non-negative.

Let us set $J_1 \equiv \{j \in \mathbb{N} : Q_j \subset Q\}$ and $J_2 \equiv \{j \in \mathbb{N} : Q_j \supset Q\}$. Using the sets J_1 and J_2, we shall show

$$\frac{1}{\phi(Q)|Q|} \int_Q \sum_{j \in J_1} \lambda_j a_j(y)\, dy \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{1,\phi}} \tag{23}$$

and

$$\frac{1}{\phi(Q)|Q|} \int_Q \sum_{j \in J_2} \lambda_j a_j(y)\, dy \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{1,\phi}}. \tag{24}$$

Once we prove (23) and (24), then we will have proved (22).

To prove (23), we observe $\frac{1}{|Q_j|}\int_{Q_j} a_j(x)\, dx \le \eta(Q_j)\|a_j\|_{\mathcal{M}_{1,\eta}} \le 1$ from the definition of the norm. Hence

$$\begin{aligned}
\frac{1}{\phi(Q)|Q|} \int_Q \sum_{j \in J_1} \lambda_j a_j(y)\, dy &= \frac{1}{\phi(Q)|Q|} \int_Q \sum_{j \in J_1} \lambda_j \left(\frac{1}{|Q_j|}\int_{Q_j} a_j(y)\, dy\right) \chi_{Q_j}(z)\, dz \\
&\lesssim \frac{1}{\phi(Q)|Q|} \int_Q \sum_{j \in J_1} \lambda_j \chi_{Q_j}(z)\, dz \\
&\le \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{1,\phi}}.
\end{aligned}$$

To prove (24), we note that there exists an increasing (possibly finite) sequence of dyadic cubes $R_1, R_2, \ldots$ such that $\{Q_j : j \in J_2\} = \{R_1, R_2, \ldots\}$. By using this sequence and (1), we have

$$\int_Q \sum_{j \in J_2} \frac{\lambda_j a_j(y)}{\phi(Q)|Q|}\, dy \le \sum_{m=1}^{\infty} \lambda_m \frac{\eta(Q)\phi(R_m)}{\phi(Q)\eta(R_m)} \|\chi_{R_m}\|_{\mathcal{M}_{1,\phi}} \lesssim \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{1,\phi}},$$

as was to be shown.

We remark that the conditions on the parameters and the functions ϕ and η are essential as the following facts show.

Remark 3.2. 1. It may be interesting to compare Theorem 1.1 in the present paper with [14, Theorem 1.1]. One can state [14, Theorem 1.1] in words of $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ as follows: Suppose that the parameters q, t, λ, ρ satisfy

$$1 < q < \infty, \quad 1 < t < \infty, \quad q < t, \quad \frac{q}{n-\lambda} < \frac{t}{n-\rho}.$$

Assume that $\{Q_j\}_{j=1}^{\infty} \subset \mathcal{Q}(\mathbb{R}^n)$, $\{a_j\}_{j=1}^{\infty} \subset \mathcal{M}_{t,\rho}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^{\infty} \subset [0,\infty)$ fulfill

$$\|a_j\|_{\mathcal{M}_{t,\rho}} \le |Q_j|^{\frac{n-\rho}{nt}}, \quad \operatorname{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{q,\lambda}} < \infty. \tag{25}$$

Then $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L_{q,\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}_{q,\lambda}} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{q,\lambda}}. \tag{26}$$

An example in [31, Section 4] shows that we can not let $q = t$. Meanwhile, when $q = 1$, Theorem 1.1 shows that we can take $t = 1$.

2. Let $n = 1$ and consider $X = \mathcal{M}_{1,\frac{1}{2}}(\mathbb{R})$ by observing that $\varphi(t) = |t|^{-\frac{1}{2}}$ belongs to X. Define

$$f_j(t) = t + 3 \cdot 4^j$$

for $t \in \mathbb{R}$. Let $E_0 = [0,1]$ and define inductively E_j by

$$E_{j+1} \equiv E_j \cup f_j(E_j), \quad j = 0, 1, 2, \ldots. \tag{27}$$

Observe that

$$1 \le \|\chi_{E_j}\|_X \le C_0 \tag{28}$$

for some constant $C_0 > 1$. Thus, $a_j = \frac{2^j}{C_0} \chi_{E_j}$ satisfies the requirement of Theorem 1.1. Note that

$$\left\| \sum_{k=1}^{j} 2^{-k} \chi_{[0,4^k]} \right\|_X \lesssim \|\varphi\|_X \sim 1 \tag{29}$$

according to [31, (4.8)] but that $\left\| \sum_{k=1}^{j} 2^{-k} a_k \right\|_X \ge \int_0^1 \sum_{k=1}^{j} 2^{-k} a_k(t)\, dt = \frac{j}{C_0}$. This implies that we can not assume $\phi = \eta$ in Theorem 1.1.

3.2. Proof of Theorem 1.3. We start with collecting an auxiliary estimate.

Lemma 3.3. *Let* p, q, η, a_j, Q_j $(j = 1, 2, \ldots)$ *be the same as Theorem* 1.3. *Then*

$$\|Ma_j\|_{\mathcal{M}_{p,\eta}} \lesssim \frac{1}{\eta(Q_j)}. \tag{30}$$

Proof. In view of (7) we have to consider two cases: $p < 1$ and $p = 1$. When $p < 1$, we use the boundedness of the Hardy–Littlewood maximal operator $M : \mathcal{M}_{1,\eta}(\mathbb{R}^n) \to \mathcal{M}_{p,\eta}(\mathbb{R}^n)$; see (5). When $p = 1$, this can be replaced by the $\mathcal{M}_{q,\eta}(\mathbb{R}^n)$-boundedness of M, namely, by using the monotonicity of the Morrey norm,

$$\|Ma_j\|_{\mathcal{M}_{p,\eta}} \le \|Ma_j\|_{\mathcal{M}_{q,\eta}} \lesssim \|a_j\|_{\mathcal{M}_{q,\eta}} \tag{31}$$

for all $f \in \mathcal{M}_{q,\eta}(\mathbb{R}^n)$. Using (5), (9) and (31), we obtain (30). □

Once we obtain (30), we can go through a similar argument as we did for the proof of [28, Theorem 1.1] using [28, Lemma 4.1].

3.3. Proof of Theorem 1.5.

Proof of Theorem 1.5 *when* $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$. We go through a similar argument as we did in [22, Theorem 4.5]. In particular, [22, (4.19)] corresponds to [29, Lemma 4.4]. In [29, Lemma 4.4], we did not assume that ϕ does not depend on $x \in \mathbb{R}^n$ but the same argument works. □

To overcome the shortcoming in the paper [14, Theorem 1.7], we use the following observation:

Remark 3.4. Let $u > 1$. Let $f \in H\mathcal{M}_{p,\phi}(\mathbb{R}^n) \cap L_u(\mathbb{R}^n)$. If one reexamines the above proof, then one learns that the convergence of (13) takes place in $L_u(\mathbb{R}^n)$. See [22, Remark 4.12] for a similar assertion.

Proof of Theorem 1.5 *for general cases.* Let $j \in \mathbb{N}$ and g_j be the good part obtained from the "Calderón–Zygmund" decomposition as we did in the proof of [22, Theorem 4.5]. Note that g_j is a locally integrable function and it satisfies $\|g_j\|_{H\mathcal{M}_{p,\phi}} \lesssim \|f\|_{H\mathcal{M}_{p,\phi}}$; see [38, pp. 102–105, pp. 110–111]. Therefore, applying the above paragraph, we see that each g_j has a decomposition; there exist a collection $\{Q_{l,j}\}_{l=1}^{\infty}$ of cubes, $\{a_{l,j}\}_{l=1}^{\infty} \subset L_\infty(\mathbb{R}^n)$, and $\{\lambda_{l,j}\}_{l=1}^{\infty} \subset [0, \infty)$ such that

$$g_j = \sum_{l=1}^{\infty} \lambda_{l,j} a_{l,j} \tag{32}$$

unconditionally in $\mathcal{S}'(\mathbb{R}^n)$, that $|a_{l,j}| \le \chi_{Q_{l,j}}$, that $\int_{\mathbb{R}^n} a_{l,j}(x) x^\alpha \, dx = 0$ for all $|\alpha| \le L$ and that

$$\left\| \left(\sum_{l=1}^{\infty} (\lambda_{l,j} \chi_{Q_{l,j}})^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}_{p,\phi}} \le C_v \|g_j\|_{H\mathcal{M}_{p,\phi}} \le C_v \|f\|_{H\mathcal{M}_{p,\phi}} \quad (j = 1, 2, \ldots). \tag{33}$$

We may assume that each $Q_{l,j}$ is realized as $5Q$ for some dyadic cube Q according to Lemma 3.1. Since $v \le 1$, by using $a^v + b^v \ge (a+b)^v$ for $a, b \ge 0$ and taking into account the case when $Q_{l,j} = Q_{l',j'}$ for some $(l,j) \ne (l',j')$, we have a decomposition there exist a family $\{a_{Q,j}\}_{Q \in \mathcal{D}} \subset L_\infty(\mathbb{R}^n)$, and a sequence $\{\lambda_{Q,j}\}_{Q \in \mathcal{D}} \subset [0,\infty)$ such that

$$g_j = \sum_{Q \in \mathcal{D}} \lambda_{Q,j} a_{Q,j} \tag{34}$$

in $\mathcal{S}'(\mathbb{R}^n)$, that

$$|a_{Q,j}| \le \chi_{5Q}, \tag{35}$$

that $\int_{\mathbb{R}^n} a_{Q,j}(x) x^\alpha \, dx = 0$ for all $|\alpha| \le L$ and that

$$\left\| \left(\sum_{Q \in \mathcal{D}} (\lambda_{Q,j}\chi_Q)^v \right)^{\frac{1}{v}} \right\|_{\mathcal{M}_{p,\phi}} \le C_v \|f\|_{H\mathcal{M}_{p,\phi}} \quad (j = 1, 2, \ldots). \tag{36}$$

Fix $Q \in \mathcal{D}$. Since $\{a_{Q,j}\}_{j=1}^\infty$ is a bounded sequence in $L_\infty(\mathbb{R}^n)$ from (35), and $\{\lambda_{Q,j}\}_{j=1}^\infty \subset [0,\infty)$ is a bounded sequence in $\mathbb{R}$ from (36), we can choose subsequences $\{a_{Q,j_k}\}_{k=1}^\infty$ and $\{\lambda_{Q,j_k}\}_{k=1}^\infty \subset [0,\infty)$ so that $\{a_{Q,j_k}\}_{k=1}^\infty$ and $\{\lambda_{Q,j_k}\}_{k=1}^\infty \subset [0,\infty)$ are convergent to a_Q and λ_Q respectively, where the convergence of $\{a_{Q,j_k}\}_{k=1}^\infty$ takes place in the weak-* topology of $L_\infty(\mathbb{R}^n)$.

Let us set

$$g \equiv \sum_{Q \in \mathcal{D}} \lambda_Q a_Q. \tag{37}$$

Then according to Theorem 1.3, we have $g \in H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$. By the Fatou lemma, we can conclude the proof once we show that

$$f = g. \tag{38}$$

To this end, we take a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Observe that g_J tends to f in $\mathcal{S}'(\mathbb{R}^n)$ as $J \to \infty$; see [29, Lemma 5.7] for the case where ϕ is independent of x. The same proof again works in our current setting. If we insert (32) to g_J, we obtain

$$\langle f, \varphi \rangle = \lim_{J \to \infty} \langle g_J, \varphi \rangle = \lim_{J \to \infty} \sum_{Q \in \mathcal{D}} \lambda_{Q,J} \langle a_{Q,J}, \varphi \rangle.$$

If we can change the order of $\lim_{J \to \infty}$ and $\sum_{Q \in \mathcal{D}}$ in the most right-hand side of the above formula, we have

$$\langle f, \varphi \rangle = \lim_{J \to \infty} \sum_{Q \in \mathcal{D}} \lambda_{Q,J} \langle a_{Q,J}, \varphi \rangle = \sum_{Q \in \mathcal{D}} \lim_{J \to \infty} \lambda_{Q,J} \langle a_{Q,J}, \varphi \rangle = \sum_{Q \in \mathcal{D}} \lambda_Q \langle a_Q, \varphi \rangle = \langle g, \varphi \rangle,$$

showing $f = g$. Thus, we are left with the task of justifying the change of the order of $\lim_{J \to \infty}$ and $\sum_{Q \in \mathcal{D}}$. Let $\varphi^\dagger \in C_c^\infty(\mathbb{R}^n)$ satisfy $\chi_{B(1)} \le \varphi^\dagger \le \chi_{B(2)}$. Since

$\varphi \in \mathcal{S}(\mathbb{R}^n)$, by decomposing $\varphi = \varphi\varphi^\dagger(R^{-1}\cdot) + \varphi(1-\varphi^\dagger(R^{-1}\cdot))$, and using the fact that $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ (defined via the grand maximal operator) is continuously embedded in $\mathcal{S}'(\mathbb{R}^n)$ as well as Theorem 1.3, we see that the contribution of the function $\varphi(1-\varphi^\dagger(R^{-1}\cdot))$ can be made as small as we wish. In fact,

$$\sum_{Q\in\mathcal{D}} |\lambda_{Q,J}\langle a_{Q,J}, \varphi(1-\varphi^\dagger(R^{-1}\cdot))\rangle| = O(R^{-1}),$$

where the implicit constants do not depend on J. This implies that we may assume that φ is supported in a compact set K.

Let $A \in \mathbb{N}$ be fixed. Suppose that K is contained in $Q(2^N)$ for some $N > 0$. Let us set

$$\begin{aligned}
\mathrm{I} &\equiv \sup_J \sum_{Q\in\mathcal{D}, Q\cap K\neq\emptyset, \ell(Q)\le 2^{-A}} |\lambda_{Q,J}\langle a_{Q,J}, \varphi\rangle|,\\
\mathrm{II} &\equiv \sup_J \sum_{Q\in\mathcal{D}, Q\cap K\neq\emptyset, 2^{-A}<\ell(Q)\le 2^{A}} |\lambda_{Q,J}\langle a_{Q,J} - a_Q, \varphi\rangle|,\\
\mathrm{III} &\equiv \sup_J \sum_{Q\in\mathcal{D}, Q\cap K\neq\emptyset, 2^{A}<\ell(Q)} |\lambda_{Q,J}\langle a_{Q,J}, \varphi\rangle|,
\end{aligned}$$

where $A > N$. Then

$$\sup_J \sum_{Q\in\mathcal{D}} |\lambda_{Q,J}\langle a_{Q,J}, \varphi\rangle| \le 2\mathrm{I} + \mathrm{II} + 2\mathrm{III}. \tag{39}$$

For $l \in \mathbb{Z}$, denote by $\mathcal{D}_l$ the set of all dyadic cubes Q such that $|Q| = 2^{-ln}$. As for I, we use

$$|\langle a_{Q,J}, \varphi\rangle| \lesssim \ell(Q)^{n+L+1} \quad (Q \in \mathcal{D}), \tag{40}$$

which is a consequence of [8, Appendix]. If $l \ge A$, we deduce from (14)

$$\frac{|\lambda_{Q,J}|}{\phi(0,2^{-l})} \le \frac{2^{\frac{ln}{p}}}{\phi(0,2^{-l})} \left\| \sum_{\overline{Q}\in\mathcal{D}_l} \lambda_{\overline{Q},J}\chi_{\overline{Q}} \right\|_{L_p(Q)} \lesssim 2^{\frac{ln}{p}} \|f\|_{H\mathcal{M}_{p,\phi}}. \tag{41}$$

Hence assuming L large enough and inserting (41) into the definition of I, we have

$$\mathrm{I} \lesssim \sum_{Q\in\mathcal{D}, Q\cap K\neq\emptyset, \ell(Q)\le 2^{-A}} \phi(0,\ell(Q)))\ell(Q)^{n+L+1} = O\left(2^{-A(n+L+1-\frac{n}{p})}\right) \tag{42}$$

as $A \to \infty$.

Next we estimate III. In view of (12), we have

$$\phi(0,2^A) \to 0 \quad (A \to \infty) \tag{43}$$

Assuming $A > N$, we learn that $[0, 2^A]^n$ is the only cube in $\mathcal{D}_{-A}$ that intersects K. Since $0 < p \le 1$, we can refine (41) to obtain

$$\sum_{Q \in \mathcal{D}, Q \cap K \neq \emptyset, \ell(Q) > 2^A} |\lambda_{Q,J}| \le \left(\frac{1}{2^{An}} \int_{[0,2^A]^n} \left| \sum_{Q \in \mathcal{D}} |\lambda_{Q,J}| \chi_Q(x) \right|^p dx \right)^{\frac{1}{p}}$$
$$\le \phi([0, 2^A]^n) \left\| \sum_{Q \in \mathcal{D}} |\lambda_{Q,J}| \chi_Q \right\|_{\mathcal{M}_{p,\phi}}.$$

Recall also that $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^n)$, so that $|\langle a_{Q,J}, \varphi \rangle| \lesssim 1$ ($Q \in \mathcal{D}$). As a consequence,

$$\mathrm{III} \lesssim \sum_{Q \in \mathcal{D}, Q \cap K \neq \emptyset, \ell(Q) > 2^A} |\lambda_{Q,J}| \lesssim \phi(0, 2^A) \left\| \sum_{Q \in \mathcal{D}} |\lambda_{Q,J}| \chi_Q \right\|_{\mathcal{M}_{p,\phi}},$$

which implies

$$\mathrm{III} \lesssim \phi(0, 2^A) \|f\|_{H\mathcal{M}_{p,\phi}}. \tag{44}$$

In view of (42)–(44), we see that I and III contribute little to the sum (39). With this in mind we use the weak-* convergence to conclude that the finite sum II converges to 0, which yields (38). □

Finally, we state a corollary to conclude this section.

Corollary 3.5. *If $\phi \in \mathcal{G}_1$ satisfies* (12), *then $H\mathcal{M}_{1,\phi}(\mathbb{R}^n)$ is embedded into $\mathcal{M}_{1,\phi}(\mathbb{R}^n)$.*

Proof. Let $f \in H\mathcal{M}_{1,\phi}(\mathbb{R}^n)$. We apply $H\mathcal{M}_{1,\phi}(\mathbb{R}^n)$ to decompose f. Under the notation of Theorem 1.5, we have

$$\sum_{j=1}^{\infty} \lambda_j |a_j| \le \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}.$$

Inequality (14) with $v = 1$ guarantees that the right-hand side belongs to $\mathcal{M}_{1,\phi}(\mathbb{R}^n)$. Hence $F(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$ converges for almost all $x \in \mathbb{R}^n$. Observe also

$$\int_{\mathbb{R}^n} |\kappa(x)| \left(\sum_{j=1}^{\infty} \lambda_j |a_j(x)| \right) dx \lesssim \int_{\mathbb{R}^n} (1 + |x|)^{-2n-1} \left(\sum_{j=1}^{\infty} \lambda_j |a_j(x)| \right) dx$$
$$\lesssim \sum_{l=1}^{\infty} (1 + l)^{-2n-1} \int_{|x| \le l} \sum_{j=1}^{\infty} \lambda_j |a_j(x)| \, dx,$$

where the implicit constant depend on κ. By the definition of the norm, we have

$$\int_{\mathbb{R}^n} |\kappa(x)| \left(\sum_{j=1}^{\infty} \lambda_j |a_j(x)| \right) dx \lesssim \sum_{l=1}^{\infty} \phi(l)(1+l)^{-\frac{2n}{p}-1} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{1,\phi}} \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{1,\phi}}.$$

Thus, f is represented by an $L_{1,\mathrm{loc}}(\mathbb{R}^n)$-function F and belongs to $\mathcal{M}_{1,\phi}(\mathbb{R}^n)$. □

With the proof above in mind, we point out a mistake in our earlier result [14, Theorem 1.3].

Remark 3.6. The function $A_{j,k}$ in [14, p. 162] is not in $L_\infty(\mathbb{R}^n)$ unless $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$. Thus, the proof of [14, Theorem 1.3] is valid when $f \in H\mathcal{M}^p_q(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$ for example, where $H\mathcal{M}^p_q(\mathbb{R}^n) = H\mathcal{M}_{q,\phi}(\mathbb{R}^n)$ with $\phi(t) = t^{-\frac{n}{p}}$. The gap was closed by the technique described above.

3.4. Applications to the boundedness of the singular integral operators. Going through the same argument as [22, Theorem 5.5] and [23, Theorem 5.5], we can prove the following theorem:

Theorem 3.7. *Let $0 < p \le 1$. Let $\phi \in \mathcal{G}_p$ satisfy (12). Let $k \in \mathcal{S}(\mathbb{R}^n)$. Write*

$$A_m \equiv \sup_{x \in \mathbb{R}^n} |x|^{n+m} |\nabla^m k(x)| \quad (m \in \mathbb{N} \cup \{0\}).$$

Define a convolution operator T by

$$Tf(x) \equiv k * f(x) \quad (f \in \mathcal{S}'(\mathbb{R}^n)).$$

Then, T, restricted to $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$, is an $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$-bounded operator and the norm depends only on $\|\mathcal{F}k\|_{L_\infty}$ and a finite number of collections $A_1, A_2, \ldots, A_N$ with N depending only on ϕ.

Once Theorem 3.7 is proved, we can obtain the Littlewood-Paley decomposition in the same way as [22, Theorem 5.7] and [23, Theorem 5.10].

Theorem 3.8. *Let $0 < p \le 1$. Let $\phi \in \mathcal{G}_p$ satisfy (12). Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a function which is supported on $B(0,4) \setminus B(0,\frac{1}{4})$ and satisfies*

$$\sum_{j=-\infty}^{\infty} |\varphi_j(\xi)|^2 > 0$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$. Then the following norm is an equivalent norm of $H\mathcal{M}_{p,\phi}(\mathbb{R}^n)$:

$$\|f\|_{\dot{\mathcal{E}}^0_{p,\phi,2}} \equiv \left\| \left(\sum_{j=-\infty}^{\infty} |\varphi_j(D) f|^2 \right)^{\frac{1}{2}} \right\|_{\mathcal{M}_{p,\phi}}, \quad f \in \mathcal{S}'(\mathbb{R}^n). \tag{45}$$

Once we obtain Theorem 3.8, we can prove the wavelet decomposition and the atomic decomposition as in [27, 34] or develop a theory of wavelets as we did in [15–18, 27].

4. Applications to the Olsen inequality

This is a bilinear estimate of I_α, which is nowadays called the Olsen inequality [25]. It is the inequality of the form

$$\|g \cdot I_\alpha f\|_Z \lesssim \|f\|_X \|g\|_Y,$$

where X, Y, Z are suitable quasi-Banach spaces. There is a vast amount of literatures on Olsen inequalities; see [3, 13, 31–33, 36, 39–42] for theoretical aspects and [5–7, 26] for applications to PDEs.

Here we will prove the following theorem:

Theorem 4.1. *Let $0 < p \le 1$ and $0 < \alpha < n$ and define q by*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}.$$

Then

$$\|I_\alpha f\|_{H\mathcal{M}_{q,\lambda}} \lesssim \|f\|_{H\mathcal{M}_{p,\lambda}}$$

for all $f \in H\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$. In particular, if $q > 1$, then

$$\|I_\alpha f\|_{\mathcal{M}_{q,\lambda}} \lesssim \|f\|_{H\mathcal{M}_{p,\lambda}}$$

for all $f \in H\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$.

Proof. Argue as we did in [30] using Theorems 2.4 and 3.8. □

Theorem 4.2. *Let $0 < p \le 1$, $0 < \lambda < n$ and $0 < \alpha < n$ and define q by*

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}.$$

Assume that $q \ge 1$. Assume in addition that

$$1 < u < \frac{n}{\alpha}.$$

Let $g \in \mathcal{M}_{1,n-\alpha}(\mathbb{R}^n)$. Then $I_\alpha f \in H\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$ and then there exists a constant $C > 0$ such that

$$\|g \cdot I_\alpha f\|_{H\mathcal{M}_{p,\lambda}} \lesssim \|g\|_{\mathcal{M}_{1,n-\alpha}} \cdot \|f\|_{H\mathcal{M}_{p,\lambda}}$$

for all $f \in H\mathcal{M}_{p,\lambda}(\mathbb{R}^n) \cap L_u(\mathbb{R}^n)$, where the implicit constant is independent of u.

Proof. Expand f according to Theorem 1.5. As we have noticed in Remark 3.4, the convergence of (13) takes place in $L_u(\mathbb{R}^n)$. Thus,

$$I_\alpha f = \sum_{j=1}^{\infty} \lambda_j I_\alpha a_j, \tag{46}$$

since I_α is bounded from $L_u(\mathbb{R}^n)$ to $L_v(\mathbb{R}^n)$ according to the Hardy–Littlewood–Sobolev theorem, where

$$\frac{1}{v} = \frac{1}{u} - \frac{\alpha}{n}.$$

Once we obtain (46), we can go through a similar argument as we did in [14, Theorem 1.7]. □

We conclude this section by giving a remedy of the mistake in the proof of [14, Theorem 1.7].

Remark 4.3. The shortcoming in the proof of [14, Theorem 1.7] is that there is no guarantee that (46) is true under the condition $f \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$, where $\mathcal{M}^{p_0}_p(\mathbb{R}^n)$ is the Morrey space in [14], Despite the mistake made in the proof of [14, Theorem 1.7], its conclusion is correct. In fact, one can assume that $f \in L_{\infty,\mathrm{c}}(\mathbb{R}^n)$ due to the Fatou property of Morrey spaces, since the integral kernel of I_α is positive and $\mathcal{M}^{p_0}_p(\mathbb{R}^n)$ is solid in the sense that $|f| \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$ whenever $f \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$. Therefore, thanks to Remark 3.4 and the Hardy–Littlewood–Sobolev theorem, one still has (46).

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