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Coproximinality for Quotient Spaces

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Abstract. In this paper we study the classical notion of coproximinality, for quotient spaces of Banach spaces. We provide a partial solution to the three space problem, analogous to a classical result of Cheney and Wulbert, by showing that for $Z \subset Y \subset X$, coproximinality of Z in X and that of Y/Z in X/Z implies the coproximinality of Y in X, when Z is an M-ideal in X. For the space $C(K)$ of continuous functions on a compact extremally disconnected set K we derive the same conclusion under the assumption that Z is an M-ideal in Y.

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1. Introduction

Let X be a real Banach space. We recall from [3, 11] that a closed subspace $Y \subset X$ is said to be coproximinal, if for any $x \in X$, there is a $y_0 \in Y$ such that for all $y \in Y$, $\|y - y_0\| \leq \|y - x\|$. y_0 is called a best coapproximation for x in Y. It is easy to see that if $P : X \to Y$ is a linear projection of norm one, onto Y, then Y is coproximinal in X. Such subspaces are called constrained subspaces. Also for closed subspaces $Z \subset Y \subset X$, if Z is coproximinal in X, then it is coproximinal in Y.

When a subspace is not the range of a projection of norm one, one studies geometric properties that mimic the effect of being the range of a projection of norm one. Among such interesting properties studied (also for any subset) in the literature are those of the notion of a sun, co-sun and the notion of best coapproximation. See [5, 11, 13]. Recently the latter condition also received attention in the work of Lewicki and Trombetta [7]. Some of these geometric conditions exhibit properties similar to the well known notion of proximinality,

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i.e., finding a best approximation in Y, for all $x \in X$ (i.e., $d(x, Y) = \|x - y_0\|$ for $y_0 \in Y$).

In this paper we first show that for closed subspaces $Z \subset Y \subset X$ if Y is coproximinal in X, then the quotient space Y/Z is a coproximinal subspace of X/Z. We give an application of our result to get new examples of coproximinal subspaces of spaces of X-valued Bochner integrable function $L^1(\mu, X)$.

A classical result of Cheney and Wulbert, [1], says that Z is proximinal in X and Y/Z is proximinal in X/Z implies that Y is proximinal in X. We consider the analogous open question, when do the conditions, Z is coproximinal in X and the quotient space Y/Z is coproximinal in X/Z, imply that Y is coproximinal in X?

Let ℓ^1 denote the space of absolutely summable series. We consider it as a dual of c_0, the space of convergent sequences. We give an example of weak*-closed subspaces, $Z \subset Z' \subset \ell^1$ such that Z is coproximinal in ℓ^1, Z' is a hyperplane in ℓ^1 and Z' is not coproximinal in ℓ^1.

Positive solutions to the 3-space problem provide a suitable technique for showing the coproximinality of a subspace $Y \subset X$, by exhibiting an appropriate closed subspace $Z \subset Y \subset X$ such that Z is coproximinal in X and Y/Z is coproximinal in X/Z and lift the solution to Y.

To give a partial positive answer we next consider a notion stronger than being a proximinal subspace. We recall from [4] that a closed subspace $Y \subset X$ is an M-ideal if there is a linear projection $P : X^* \to X^*$ such that $\ker(P) = Y^\perp$ and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. See Chapter I of [4] for several examples from classical analysis, of subspaces that are M-ideals and their geometric properties.

We show that for an M-ideal $Z \subset X$, for a closed subspace Y with $Z \subset Y \subset X$, the conditions Z is coproximinal in X and Y/Z is coproximinal in X/Z imply that Y is coproximinal in X.

We improve this result in the case of $C(K)$, the space of real-valued continuous functions on a compact extremally disconnected space K (see [6, Section 11] for properties of these spaces, and also [10]) by assuming Z to be an M-ideal only in Y.

These questions are also closely related to finding best coapproximation in sums of closed subspaces whose sum is closed, analogous to the corresponding problem in best approximations solved in [8,9]. It is not known if or under what conditions, for a finite dimensional subspace $F \subset X$ that is coproximinal and for a closed coproximinal subspace $Y \subset X$, the sum $F + Y$ is again coproximinal in X? Our investigation also has some bearing on this question. We give a positive solution in the space $C(K)$, where K is compact and extremally disconnected and the sum is an ℓ^∞-direct sum.

2. Main Results

We first show that coproximinality is preserved by quotients.

Proposition 2.1. *Let X be a Banach space and let $Z \subset Y \subset X$ be closed subspaces. If Y is coproximinal in X, then Y/Z is coproximinal in X/Z.*

Proof. Let $\pi : X \to X/Z$ be the quotient map. Let $\pi(x_0) \in X/Z$. Since Y is coproximinal in X, let $y_0 \in Y$ be such that $\|y - y_0\| \leq \|y - x_0\|$ for all $y \in Y$. For $y \in Y$,

$$\begin{aligned} \|\pi(y) - \pi(y_0)\| &= \inf\{\|y - y_0 - z\| : z \in Z\} \\ &= \inf\{\|y - z - y_0\| : z \in Z\} \\ &\leq \inf\{\|y - z - x_0\| : z \in Z\} \\ &= \|\pi(y) - \pi(x_0)\|. \end{aligned}$$

Thus Y/Z is coproximinal in X/Z. □

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, where μ is a positive measure. Let $L^1(\mu, X)$ denote the space of X-valued Bochner integrable functions, equipped with the norm, $\|f\|_1 = \int_\Omega \|f\| \, d\mu$. Let $Y \subset X$ be a closed coproximinal subspace, it is still an open problem if $L^1(\mu, Y)$ is always a coproximinal subspace of $L^1(\mu, X)$. As an application of the above proposition we have the following result. When the measure space is complete, it was recently proved in [12, Theorem 2] that if $Y \subset X$ is a closed coproximinal subspace and Y is isometric to a constrained subspace of a weakly compactly generated dual space, then $L^1(\mu, Y)$ is a coproximinal subspace of $L^1(\mu, X)$.

Corollary 2.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $Y \subset X$ be a closed subspace such that $L^1(\mu, Y)$ is a coproximinal subspace of $L^1(\mu, Y)$. Then for any closed subspace $Z \subset Y$, $L^1(\mu, Y/Z)$ is a coproximinal subspace of $L^1(\mu, X/Z)$.*

Proof. Consider the inclusion $L^1(\mu, Z) \subset L^1(\mu, Y) \subset L^1(\mu, X)$. By the above proposition, we have that $L^1(\mu, Y)/L^1(\mu, Z)$ is a coproximinal subspace of $L^1(\mu, X)/L^1(\mu, Z)$. But we have from [2, Chapter VIII, Propositions 11, 12], that $L^1(\mu, X/Z)$ is isometric to $L^1(\mu, X)/L^1(\mu, Z)$ and it is easy to see that the isometry maps $L^1(\mu, Y/Z)$ onto $L^1(\mu, Y)/L^1(\mu, Z)$. Hence the conclusion follows. □

Corollary 2.3. *Let $Z \subset X$ be a closed subspace and let $W \subset X/Z$ be a closed subspace such that $\pi^{-1}(W)$ is a coproximinal subspace of X. Then W is a coproximinal subspace of X/Z.*

Proof. Let $Y = \pi^{-1}(W)$. Clearly $Z \subset Y \subset X$ and Y is a coproximinal subspace of X. Thus by Proposition 1, $Y/Z = W$ is a coproximinal subspace of X/Z. □

Another interpretation of the result of Cheney and Wulbert is to find conditions for a closed proximinal subspace $Z \subset X$, under which any closed subspace Y with $Z \subset Y \subset X$, is again proximinal. We first note that for a coproximinal subspace $Z \subset X$ with $\dim(X/Z) \geq 3$, if for all closed subspaces $Z \subset Y \subset X$, Y/Z is coproximinal in X/Z, then X/Z is isometric to a Hilbert space. In the following lemma we use the correspondence between closed subspaces of X/Z and closed subspaces Y with $Z \subset Y \subset X$.

Lemma 2.4. *Let $Z \subset X$ be a closed coproximinal subspace such that*

$$\dim(X/Z) \geq 3.$$

Suppose for all closed subspace $Z \subset Y \subset X$, such that Y is a hyperplane in X, Y is also coproximinal in X. Then X/Z is isometric to a Hilbert space.

Proof. Since $\dim(X/Z) \geq 3$, in view of the characterization in [3], we need to show that for any $x^* \in (X/Z)^* = Z^{\perp}$, $\ker(x^*)$ is coproximinal in X/Z. But now since $Z \subset \ker(x^*) \subset X$, by hypothesis, $\ker(x^*)$ is coproximinal in X and hence by Proposition 1, $\ker(x^*)/Z$ is coproximinal in X/Z. We note that $\ker(x^*)/Z$ is precisely $\ker(x^*)$ in X/Z. □

Since any separable Banach space is isometric to a quotient of the sequence space ℓ^1, it is an appropriate domain for obtaining counterexamples related to quotient space questions. In what follows we use [7, Theorem 2.4] that shows that any coproximinal subspace of ℓ^1 is the range of a linear projection of norm one. We identify ℓ^1 as the dual of the space c_0 of sequences converging to 0.

Example 2.5. Let $Z \subset \ell^1$ be a closed coproximinal subspace of codimension greater than 3. Suppose for all closed subspaces $Z \subset Y \subset \ell^1$, such that Y is a hyperplane in ℓ^1, Y is coproximinal in ℓ^1. Then by Lemma 3, ℓ^1/Z is isometric to a Hilbert space of dimension greater than 3. By [7, Theorem 2.4], Z is the range of a projection of norm one in ℓ^1. Since $(\ell^1)^* = c_0^{**} = \ell^\infty$, using the fact that c_0 is an M-ideal in its bidual ℓ^∞, it follows from Proposition IV.1.5 and Proposition IV.1.10 from [4, Propositions IV.1.5, IV.1.10] that Z is a weak*-closed subspace of ℓ^1. Thus Z is a proximinal subspace. Also there is a closed subspace $M \subset c_0$ such that $Z = M^{\perp}$. Now $M^* = \ell^1/Z$, so that M is also isometric to a Hilbert space of dimension greater than 3. It is well known that any extreme point of a dual unit ball of a subspace has an extension to an extreme point of the dual unit ball of the space itself. Since M^* is a Hilbert space, any unit vector is an extreme point of the unit ball. Since $c_0^* = \ell^1$ has only countably many extreme points in its unit ball, we get a contradiction. This contradiction shows that for a closed subspace Y with $Z \subset Y \subset \ell^1$, Y fails to be coproximinal in ℓ^1.

By choosing the above Z to be a subspace of finite codimension bigger than 3, as Z is proximinal, we have by Cheney and Wulbert's result that Y is a proximinal subspace of ℓ^1.

We next prove a version of Cheney and Wulbert's theorem for coproximinality. We note that for $Z \subset Y \subset X$, if Z is coproximinal in X then it is also coproximinal in Y. Also being coproximinal is a transitive property. We note that in the space of convergent sequences c, $c_0 \subset c$ is an M-ideal but the constant sequence $1 \in c$, does not have any best coapproximation in c_0.

Theorem 2.6. *Let $Z \subset Y \subset X$. Suppose Z is an M-ideal in X. If Z is coproximinal in X and Y/Z is coproximinal in X/Z, then Y is coproximinal in X.*

Proof. Let $x_0 \in X$ and $x_0 \notin Z$. Since Z is coproximinal in X, let $z_0 \in Z$ be such that $\|z - z_0\| \leq \|z - x_0\|$ for all $z \in Z$. In particular $\|z_0\| \leq \|x_0\|$. Define $P : \text{span}\{x_0, Z\} \to Z$ by $P(\alpha x_0 + z) = \alpha z_0 + z$ for any $z \in Z$ and scalar α. It is easy to see that P is a linear projection onto Z. For $\alpha \neq 0$, $\|\alpha z_0 + z\| = |\alpha|\|\frac{z}{\alpha} + z_0\| \leq |\alpha|\|\frac{z}{\alpha} + x_0\| = \|\alpha x_0 + z\|$. Thus $\|P\| = 1$.

Now for $Z \subset \text{span}\{x_0, Z\} \subset X$, as Z is an M-ideal in X, by [4, Proposition I.1.17] we get that Z is an M-ideal in $\text{span}\{x_0, Z\}$. Since $P : \text{span}\{x_0, Z\} \to Z$ is a projection of norm one, by [4, Corollary I.1.3] we get that Z is an M-summand in $\text{span}\{x_0, Z\}$, i.e., $\|\alpha x_0 + z\| = \max\{|\alpha|\|x_0\|, \|z\|\}$ for any $z \in Z$ and scalar α.

Thus $d(x_0, Z) = \|x_0\| = \max\{\|x_0\|, \|z_0\|\}$ and it is easy to see that the closed ball $B(z_0, d(x_0, Z))$ in Z, is the set of best approximations in Z to x_0. It now follows from [4, Proposition II.3.2] that there is a closed subspace $Z' \subset X$ such that $X = Z \oplus_\infty Z'$ (ℓ^∞-direct sum).

Since $Z \subset Y$, we have that $Y = Z \oplus_\infty (Z' \cap Y)$. Therefore by hypothesis, under canonical isometries, $(Z' \cap Y) = Y/Z$ is a coproximinal subspace of $Z' = X/Z$.

Now for any $r_0 = s_0 + t_0$ for $s_0 \in Z$ and $t_0 \in Z'$, by the coproximinality of $(Z' \cap Y)$ in Z', there is a $t_0' \in (Y \cap Z')$ such that $\|t - t_0'\| \leq \|t - t_0\|$ for all $t \in (Z' \cap Y)$. Let $s \in Z$ and $t \in (Z' \cap Y)$.

$$\|(s+t)-(s_0+t_0')\| = \max\{\|s-s_0\|, \|t-t_0'\|\} \leq \max\{\|s-s_0\|, \|t-t_0\|\} = \|(s+t)-r_0\|.$$

Thus Y is coproximinal in X. □

Remark 2.7. We do not know if the above theorem is true if one only assumes that Z is an M-ideal in Y. In this case, as Z is also coproximinal in Y (since Z is coproximinal in X), one again has that $Y = Z \oplus_\infty Z'$ for some closed subspace $Z' \subset Y$. Now consider the canonical isometry, $z' \to \pi(z')$ of Z' with Y/Z. Let $x_0 \in X$. Since Y/Z is coproximinal in X/Z, let $y_0 \in Y$ be such that $\|\pi(y - y_0)\| \leq \|\pi(y - x_0)\|$ for all $y \in Y$. Let $y_0 = z_0 + z_0'$ for $z_0 \in Z$ and $z_0' \in Z'$. Now for any $r \in Z'$, $\|r - z_0'\| = \|\pi(r - z_0')\| \leq \|\pi(r - x_0)\| \leq \|r - x_0\|$. Thus Z' is also coproximinal in X. We do not know how to show that Y is coproximinal in X. We next solve this for a special class of spaces.

We recall from [6, Section 11] some special properties of the real-valued continuous function space $C(K)$ where K is an extremally disconnected compact space. These spaces can be abstractly characterized as those real Banach spaces X in which every family of pair-wise intersecting closed balls has non-empty intersection (see also [10]). We note that two closed balls $B(x,r), B(y,s)$ intersect if and only if $\|x-y\| \leq r+s$. For any compact extremally disconnected space K, if $C(K)$ is isometrically embedded in a Banach space B, then there is a linear onto projection $P : B \to C(K)$ such that $\|P\| = 1$. Thus in particular $C(K)$ is coproximinal in B. It is easy to see that ℓ^∞ has this property and also a closed subspace $Y \subset \ell^\infty$ is coproximinal if and only if it is isometric to a $C(K)$ space for a compact extremally disconnected set K.

Theorem 2.8. *Let K be a compact extremally disconnected space. Let $Z \subset Y \subset C(K)$ be closed subspaces such that Z is coproximinal in $C(K)$ and an M-ideal in Y and Y/Z is coproximinal in $C(K)/Z$. Then Y is coproximinal in $C(K)$.*

Proof. It follows from the proof of Theorem 3 and Remark 4 that $Y = Z \oplus_\infty Z'$ and Z' is coproximinal in $C(K)$. If $\{B(z_\alpha, r_\alpha)\}$ is any family of pair-wise intersecting closed balls in Z, then the family of larger balls $\{B_{C(K)}(z_\alpha, r_\alpha)\}$ also pair-wise intersect, hence we have an $f_0 \in C(K)$ such that $\|f_0 - z_\alpha\| \leq r_\alpha$ for all α. By coproximinality of Z, we get a $z_0 \in Z'$ such that $\|z_0 - z_\alpha\| \leq \|f_0 - z_\alpha\| \leq r_\alpha$. Thus every pairwise intersecting family of closed balls in Z has non-empty intersection. As Z' is also coproximinal in $C(K)$ we have that Z' also has the same intersection of balls property. Now it is easy to see that any family of pair-wise intersecting closed balls in $Y = Z \oplus_\infty Z'$, has non-empty intersection. Thus by the characterization theorem quoted earlier, Y is isometric to $C(K')$ for some compact extremally disconnected space K'. Therefore by the remarks made before this theorem, we see that Y is coproximinal in $C(K)$. □

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Note added in proof. In his article "Weak coproximinality for Banach spaces" [J. Nonlinear Funct. Anal. 2017 (2017), Article ID 13, 10 pp.], the author gave an example of a Banach space X and closed subspaces $Z \subset Y \subset X$ such that Z is a coproximinal subspace of X, Y/Z is a coproximinal subspace of X/Z but Y is not a coproximinal subspace of X.

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