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Two Nontrivial Solutions for the Nonhomogenous Fourth Order Kirchhoff Equation

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Dedicated Professor Boling Guo to his 80th birthday

Abstract. In this paper, we consider the following nonhomogenous fourth order Kirchhoff equation

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x,u) + g(x), \quad x \in \mathbb{R}^N,$$

where $\Delta^2 := \Delta(\Delta)$, constants $a > 0$, $b \geq 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $g \in L^2(\mathbb{R}^N)$. Under more relaxed assumptions on the nonlinear term f that are much weaker than those in L. Xu and H. Chen, using some new proof techniques especially the verification of the boundedness of Palais–Smale sequence, a new result is obtained.
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1. Introduction and main results

In this paper, we consider the following nonhomogenous Kirchhoff type problem

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = f(x,u) + g(x), \quad x \in \mathbb{R}^N, \qquad (\mathcal{P})$$

where $\Delta^2 := \Delta(\Delta)$, constants $a > 0$, $b \geq 0$, $1 < N < 8$, $V \in C(\mathbb{R}^N, \mathbb{R})$, $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $g \in L^2(\mathbb{R}^N)$ satisfy some further conditions.

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Recently, the following Schrödinger–Kirchhoff type problem with nonconstant potential

$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u),\quad x\in\mathbb{R}^N, \tag{1}$$

is studied by many mathematicians. As we know, Wu [24] is the first one consider this type of equation. In that paper, four new existence results for nontrivial solutions and a sequence of high energy solutions for problem (1) are obtained by using a symmetric mountain pass theorem. Later, many researchers generalize his results, see the references [5, 6, 9, 10, 13, 15–17, 28]. The system case also had been consider by Wu [25] and Zhou et al. [31], respectively.

Very recently, a fourth-order elliptic equation with Kirchhoff type on bounded domain is also been studied by Wang and An [21], firstly. This problem is related to the stationary analog of the evolution equation of Kirchhoff type

$$u_{tt}+\Delta^2u-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(x,u).$$

Dimensions one and two are relevant from the point of view of physics and engineering because in those situations model is considered a good approximation for describing nonlinear vibrations of beams or plates ,see the references [2,4,23]. Later, Wang et al. [22] using the mountain pass techniques and the truncation method to show the existence of nontrivial solutions for a class of fourth order elliptic equations of Kirchhoff type. Fourth order equations and fourth order Kirchhoff type equation on $\mathbb{R}^N$ had been also studied in [1, 18, 26, 27, 29, 30]. Last, using Ricceri's variational principle, the fourth order elliptic equations of Kirchhoff type had been studied in [8, 11, 12].

For the nonhomogenous Kirchhoff type problem $(\mathcal{P})$, there is little known result of the existence and multiplicity of solutions except for [27]. In [27], by applying Ekeland's variational principle and Mountain Pass Theorem, Xu and Chen studied the problem $(\mathcal{P})$ with the nonlinearity f satisfying the Ambrosetti-Rabinowitz type condition, and the existence of two solutions was obtained.

Motivated by the works of [6, 15], in the present paper, we shall consider the nonhomogeneous Kirchhoff type problem, and we are interested in looking for multiple solutions of the problem $(\mathcal{P})$. Under more relaxed assumptions on the nonlinear term f that are much weaker than those in [27], using some new proof techniques especially the verification of the boundedness of Palais–Smale sequence, a new result on multiplicity of nontrivial solutions for the problem $(\mathcal{P})$ is obtained, which sharply improves the result of [27].

To obtain the multiplicity of solutions for the nonhomogeneous Kirchhoff type problem $(\mathcal{P})$ in $\mathbb{R}^N$, we make the following assumptions:

(f1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and

$$|f(x,t)| \le C(1+|t|^{p-1}) \quad \text{for some } 4 < p < 2^* = \begin{cases} \frac{2N}{N-4}, & 8 > N > 4 \\ +\infty, & 1 < N \le 4, \end{cases}$$

where C is a positive constant.

(f3) $\frac{F(x,t)}{t^4} \to +\infty$ as $|t| \to +\infty$ uniformly in $x \in \mathbb{R}^N$.

(f4) There exists $L > 0$ and $d \in \left[0, \frac{V_0}{2}\right]$ such that

$$4F(x,t) - f(x,t)t \le d|t|^2, \qquad \text{for a.e. } x \in \mathbb{R}^N \text{ and } \forall t \ge L.$$

Now, we can state our result as follows.

Theorem 1.1. *Assume that $g \in L^2(\mathbb{R}^N)$, $g \not\equiv 0$,* (V1) *and* (f1)–(f4) *hold. Then, there exists a constant $g_0 > 0$ such that the problem $(\mathcal{P})$ has at least two different solutions whenever $\|g\|_{L^2} < g_0$, one is negative energy solution, and the other is positive energy solution.*

Xu and Chen [27] assumed the following assumptions:

(V1) $V \in C(\mathbb{R}^N, \mathbb{R})$ satisfies $\inf V(x) \ge V_0 > 0$ and for each $M > 0$, $\text{meas}\{x \in \mathbb{R}^N : V(x) \le M\} < +\infty$, where V_0 is a constant and meas denotes the Lebesgue measure in $\mathbb{R}^N$.

(f1') $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and

$$|f(x,t)| \le C(1+|t|^{p-1}) \qquad \text{for some } 2 < p < 2^* = \frac{2N}{N-2},$$

where C is a positive constant.

(f2) $f(x,t) = o(|t|)$ as $|t| \to 0$ uniformly in $x \in \mathbb{R}^N$.

(f3')

$$\inf_{x \in \mathbb{R}^N, |t|=1} F(x,t) > 0, \quad \text{where } F(x,t) = \int_0^t f(x,s)ds.$$

(f4') There exists $\mu > 4$ and $r > 0$ such that

$$\mu F(x,t) - f(x,t)t \le 0, \quad \forall x \in \mathbb{R}^N, |t| \ge r.$$

We restate the corresponding result in [27] as in the following.

Theorem 1.2 (see [27, Theorem 1.1])**.** *Assume that $g \in L^2(\mathbb{R}^N)$, $g \not\equiv 0$,* (V1), (f1'), (f2) *and* (f3'), (f4') *hold. Then there exists a constant $m_0 > 0$ such that the equation $(\mathcal{P})$ has at least two different solutions whenever $\|g\|_{L^2} < m_0$.*

Remark 1.3. Theorem 1.1 sharply improves Theorem 1.2. In fact, (f3), (f4) are much weaker than (f3'), (f4'). To be precise, by (f3'), (f4'), the inequality (2) in Remark 1.4 holds. Hence,

$$\frac{F(x,t)}{t^4} \geq c|t|^{\mu-4}, \quad \forall x \in \mathbb{R}^N \text{ and } |t| \geq 1,$$

which implies (f3). Moreover, note that $\mu > 4$, then (f4') and (2) imply $4F(x,t) - f(x,t)t = \mu F(x,t) - f(x,t)t + (4-\mu)F(x,t) \leq (4-\mu)F(x,t) \leq (4-\mu)c|t|^\mu < 0 \leq d|t|^2$ for all $x \in \mathbb{R}^N$ and $|t| \geq 1$. This shows (f4) holds by taking $L = 1$. Consequently, (f3'), (f4') imply (f3), (f4). Thus, Theorem 1.1 sharply improves Theorem 1.2.

Remark 1.4. (i) Since the problem $(\mathcal{P})$ is defined in $\mathbb{R}^N$ which is unbounded, the lack of compactness of the Sobolev embedding becomes more delicate using variational techniques. To overcome the lack of compactness, the condition (V1), which was firstly introduced by Bartsch and Wang in [3], is always assumed to preserve the compactness of embedding of the working space.

(ii) It is worth pointing out that the combination of (f3'), (f4') implies the range of p in condition (f1') should be $4 < p < 2^*$. Precisely, for any $x \in \mathbb{R}^N$, $z \in \mathbb{R}$, define

$$h(t) := F(x, t^{-1}z)t^\mu, \quad \forall t \in [1, +\infty).$$

Then, for $|z| \geq 1$ and $t \in [1, |z|]$, (f4') implies that

$$h'(t) = t^{\mu-1}\left[\mu F(x, t^{-1}z) - t^{-1}zf(x, t^{-1}z)\right] \leq 0.$$

Hence, $h(1) \geq h(|z|)$. Therefore, (f3') implies that

$$F(x,z) \geq F\left(x, \frac{z}{|z|}\right)|z|^\mu \geq c|z|^\mu, \quad \forall x \in \mathbb{R}^N \text{ and } |z| \geq 1, \tag{2}$$

where $c = \inf_{x\in\mathbb{R}^N, |t|=1} F(x,t) > 0$. If $p \leq 4$, by (f1'), we have

$$|F(x,t)| \leq \int_0^1 |f(x,st)t|ds \leq C\int_0^1 (1 + |st|^{p-1})|t|ds \leq C(|t| + |t|^p)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, which implies that

$$\limsup_{|t|\to+\infty} \frac{F(x,t)}{t^4} \leq C \quad \text{uniformaly in } x \in \mathbb{R}^N.$$

This contradicts (2). Hence $4 < p < 2^*$.

(iii) We also point out that (f1') in [27], the critical exponent 2^* should be $\frac{2N}{N-4}$.

In Theorem 1.1, we consider the case $\mu = 4$. For the case $\mu > 4$, we also have the following result about the existence of one negative energy solution, one positive energy solution for the nonhomogeneous Kirchhoff type problem $(\mathcal{P})$ in $\mathbb{R}^N$, which is a corollary of Theorem 1.1 and more general than Theorem A. To begin with, we need the following assumptions:

(f3") There exists $L' > 0$ such that

$$c' = \inf_{x\in\mathbb{R}^N, |t|=L'} F(x,t) > 0.$$

(f4") There exist $\mu > 4$ and $d' \in \left[0, \frac{c'(\mu-2)}{L'^2}\right)$ such that

$$\mu F(x,t) - f(x,t)t \le d'|t|^2, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and } \forall |t| \ge L'.$$

Now, we can state the Corollary as follows.

Corollary 1.5. *If we replace* (f3), (f4) *with* (f3"), (f4") *in Theorem* 1.1*, then the conclusion of Theorem* 1.1 *remains hold.*

2. Preliminaries

Now, we give some notations. Define the function space

$$H^2(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N),\ \Delta u \in L^2(\mathbb{R}^N)\}$$

with the norm

$$\|u\|_{H^2} = \left(\int_{\mathbb{R}^N} (||\Delta u|^2 + |\nabla u|^2 + u^2) dx\right)^{\frac{1}{2}}.$$

Denote

$$E = \left\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2) dx < +\infty\right\}$$

with the inner product and the norm

$$\langle u, v\rangle_E = \int_{\mathbb{R}^N} (\Delta u \Delta v + a\nabla u \cdot \nabla v + V(x)uv) dx, \quad \|u\|_E = \langle u, u\rangle_E.$$

Obviously, the following embedding $E \hookrightarrow L^s(\mathbb{R}^N)$, $2 \le s \le 2^*$ is continuous. Hence, for any $s \in [2, 2^*]$, there is a constant $a_s > 0$ such that

$$\|u\|_{L^s} \le a_s \|u\|_E. \tag{3}$$

It is well known that a weak solution for the problem $(\mathcal{P})$ is a critical point of the following functional I defined on E by

$$\begin{aligned} I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\Delta u|^2 + a|\nabla u|^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2 \\ + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx - \int_{\mathbb{R}^N}F(x,u)dx - \int_{\mathbb{R}^N}g(x)udx \end{aligned} \tag{4}$$

for all $u \in E$. We say that a weak solution $u \in E$ for the problem $(\mathcal{P})$ is a negative energy solution if the energy $I(u) < 0$, and a weak solution $v \in E$ for the problem $(\mathcal{P})$ is a positive energy solution if the energy $I(v) > 0$.

To apply variational techniques, we first state the key compactness result.

Lemma 2.1 ([14, Lemma 1.1])**.** *Under the assumption* (V1)*, the embedding*

$$E \hookrightarrow L^s(\mathbb{R}^N), \quad 2 \le s < 2^*$$

is compact.

The following lemma can be obtained by a similar argument as [24, Lemma 1].

Lemma 2.2. *Assume that $g \in L^2(\mathbb{R}^N)$,* (V1) *and* (f1), (f2) *hold. Then I is well defined on E, $I \in C^1(E,\mathbb{R})$ and for any u, $v \in E$,*

$$\begin{aligned} \langle I'(u), v\rangle = \int_{\mathbb{R}^N}\Delta u\Delta v dx + \left(a + b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\nabla u\cdot\nabla v dxdx \\ + \int_{\mathbb{R}^N}V(x)uv - \int_{\mathbb{R}^N}f(x,u)vdx - \int_{\mathbb{R}^N}g(x)vdx \end{aligned} \tag{5}$$

Moreover, $\Psi' : E \to E^*$ is compact, where $\Psi(u) = \int_{\mathbb{R}^N}F(x,u)dx - \int_{\mathbb{R}^N}g(x)udx$.

Ekeland's variational principle is the tool to obtain a negative energy solution, we give it here for readers' convenience.

Theorem 2.3 ([19, Theorem 4.1])**.** *Let M be a complete metric space with metric d and let $I : M \to (-\infty, +\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\epsilon > 0$ be given and $u \in M$ be such that*

$$I(u) \le \inf_M I + \epsilon.$$

Then there exists $v \in M$ such that

$$I(v) \le I(u), \quad d(u,v) \le 1,$$

and for each $w \in M$, one has

$$I(v) \le I(w) + \epsilon d(v,w).$$

Recall that, we say I satisfies the (PS) condition at the level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ condition for short) if any sequence $\{u_n\} \subset E$ along with $I(u_n) \to c$ and $I(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. If I satisfies $(\mathrm{PS})_c$ condition for each $c \in \mathbb{R}$, then we say that I satisfies the (PS) condition.

Here, we recall the classical Mountain Pass Theorem.

Theorem 2.4 ([20, Theorem 2.2]). *Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$ satisfying* (PS) *condition. Suppose $I(0) = 0$ and*

(I1) *there are constants ρ, $\alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$,*

(I2) *there is $u_1 \in X \setminus \overline{B}_\rho$ such that $I(u_1) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} I(u),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = u_1\}$.

3. Proof of the main results

Lemma 3.1. *Assume that $g \in L^2(\mathbb{R}^N)$ and* (f1), (f2) *hold. Then there exist some constants ρ, α and $\beta > 0$ such that $I(u) \geq \alpha$ whenever $\|u\|_E = \rho$ and $\|g\|_{L^2} < \beta$.*

Proof. For any $\epsilon > 0$, by (f1), (f2), there exists $C(\epsilon) > 0$ such that

$$|f(x,t)| \leq \epsilon |t| + C(\epsilon)|t|^{p-1}, \qquad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{6}$$

$$|F(x,t)| \leq \int_0^1 |f(x,st)t| ds \leq \epsilon |t|^2 + C(\epsilon)|t|^p, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{7}$$

By (3), (4), (7) and the Hölder inequality,

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx \\
&\quad - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} g(x)u dx \\
&\geq \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} g(x)u dx \\
&\geq \frac{1}{2}\|u\|_E^2 - (\epsilon\|u\|_{L^2}^2 + C(\epsilon)\|u\|_{L^p}^p) - \|g\|_{L^2}\|u\|_{L^2} \\
&\geq \frac{1}{2}\|u\|_E^2 - a_2^2\epsilon\|u\|_E^2 - a_p^p C(\epsilon)\|u\|_E^p - a_2\|g\|_{L^2}\|u\|_E \\
&= \|u\|_E \left[\left(\frac{1}{2} - a_2^2\epsilon\right)\|u\|_E - a_p^p C(\epsilon)\|u\|_E^{p-1} - a_2\|g\|_{L^2}\right].
\end{aligned}$$

Choose $\epsilon = \frac{1}{4a_2^2} > 0$, and take

$$h(t) = \frac{1}{4}t - a_p^p C(\epsilon) t^{p-1}, \quad \forall t \geq 0.$$

Note that $4 < p < 2^*$, we can conclude that there exists a constant $\rho > 0$ such that $h(\rho) = \max_{t\geq 0} h(t) > 0$. Therefore, take $\beta = \frac{1}{2a_2} h(\rho) > 0$, it follows that

$$I(u) \geq \frac{1}{2}\rho h(\rho) =: \alpha > 0$$

whenever $\|u\|_E = \rho$ and $\|g\|_{L^2} < \beta$. This completes the proof. □

Lemma 3.2. *Let assumptions* (f1)–(f3) *hold. Then there exists a function* $e \in E$ *with* $\|e\|_E > \rho$ *such that* $I(e) < 0$.

Proof. For every $M > 0$, by (f1)–(f3), there exists $C(M) > 0$ such that

$$F(x,t) \geq M|t|^4 - C(M)|t|^2, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{8}$$

Choose $\phi \in E$ with $\|\phi\|_{L^4} = 1$, then (4), (8) and the Hölder inequality imply that

$$\begin{aligned} I(t\phi) &= \frac{t^2}{2}\int_{\mathbb{R}^N} (|\Delta\phi|^2 + a|\nabla\phi|^2 + V(x)\phi^2)dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N} |\nabla\phi|^2 dx\right)^2 \\ &\quad - \int_{\mathbb{R}^N} F(x,t\phi)dx - t\int_{\mathbb{R}^N} g(x)\phi dx \\ &\leq \frac{t^2}{2}\|\phi\|_E^2 + \frac{b}{4}\|\phi\|_E^4 t^4 - M\|\phi\|_{L^4}^4 t^4 + C(M)\|\phi\|_{L^2}^2 t^2 - t\int_{\mathbb{R}^N} g(x)\phi dx \\ &\leq \left(M - \frac{b}{4}\|\phi\|_E^4\right) t^4 + \left[\frac{1}{2}\|\phi\|_E^2 + C(M)\|\phi\|_{L^2}^2\right]^2 + \|g\|_{L^2}\|\phi\|_{L^2} t, \end{aligned}$$

which implies $I(t\phi) \to -\infty$ as $t \to +\infty$ by taking $M > \frac{b}{4}\|\phi\|_E^4$. Hence, there exists $e = t_0\phi$ with t_0 large enough such that $\|e\|_E > \rho$ and $I(e) < 0$. The proof is completed. □

Lemma 3.3. *Let assumptions* (V1), (f1), (f2) *hold. Then any bounded Palais–Smale sequence of* I *has a strongly convergent subsequence in* E.

Proof. Let $\{u_n\} \subset E$ be any bounded Palais–Smale sequence of I. Then, up to a subsequence, there exists $c_1 \in \mathbb{R}$ such that

$$I(u_n) \to c_1, \quad I'(u_n) \to 0 \quad \text{and} \quad \sup_n \|u_n\|_E < +\infty. \tag{9}$$

Since the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$, $2 \leq s < 2^*$ is compact, going if necessary to a subsequence, we can assume that there is a $u \in E$ such that

$$\begin{aligned} u_n &\rightharpoonup u, && \text{weakly in } E; && (10)\\ u_n &\to u, && \text{strongly in } L^s(\mathbb{R}^N); && (11)\\ u_n(x) &\to u(x), && \text{a.e. in } \mathbb{R}^N. && (12) \end{aligned}$$

In view of (5), it has

$$\begin{aligned} &\langle I'(u_n) - I'(u), u_n - u\rangle \\ &= \int_{\mathbb{R}^N} [(\Delta u_n - \Delta u)^2 + V(x)|u_n - u|^2] dx \\ &\quad + \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla(u_n - u) dx \\ &\quad - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \int_{\mathbb{R}^N} \nabla u \cdot \nabla(u_n - u) dx \\ &\quad - \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u)](u_n - u) dx \\ &= \int_{\mathbb{R}^N} [(\Delta u_n - \Delta u)^2 + V(x)|u_n - u|^2] dx \\ &\quad + \left(a + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx \\ &\quad - \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \nabla u \cdot \nabla(u_n - u) dx \\ &\quad - \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u)](u_n - u) dx \\ &\geq \|u_n - u\|_E^2 - b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \\ &\quad - \int_{\mathbb{R}^N} \nabla u \cdot \nabla(u_n - u) dx - \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u)](u_n - u) dx. \end{aligned} \tag{13}$$

Then, (13) implies that

$$\begin{aligned} \|u_n - u\|_E^2 \leq\ & \langle I'(u_n) - I'(u), u_n - u\rangle \\ & + b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \nabla u \cdot \nabla(u_n - u) dx \\ & + \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u)](u_n - u) dx. \end{aligned} \tag{14}$$

Define the functional $h_u : E \to \mathbb{R}$ by

$$h_u(v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx, \quad \forall v \in E.$$

Obviously, h_u is a linear functional on E. Furthermore,

$$|h_u(v)| \leq \int_{\mathbb{R}^N} |\nabla u \cdot \nabla v| dx \leq \|u\|_E \|v\|_E,$$

which implies h_u is bounded on E. Hence $h_u \in E^*$. Since $u_n \rightharpoonup u$ in E, it has $\lim_{n\to\infty} h_u(u_n) = h_u(u)$, that is,$\int_{\mathbb{R}^N} \nabla u \cdot \nabla(u_n - u)dx \to 0$ as $n \to \infty$. Consequently, by (10) and the boundedness of $\{u_n\}$, it has

$$b\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u \cdot \nabla(u_n - u)dx \to 0, \quad n \to +\infty. \tag{15}$$

By (6), using the Hölder inequality, we can conclude

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} [f(x,u_n) - f(x,u)](u_n - u)dx\right| \\
&\leq [\epsilon + C(\epsilon)] \int_{\mathbb{R}^N} [|u_n| + |u| + |u_n|^{p-1} + |u|^{p-1}]|u_n - u|dx \\
&\leq [\epsilon + C(\epsilon)](\|u_n\|_{L^2} + \|u\|_{L^2})\|u_n - u\|_{L^2} + [\epsilon + C(\epsilon)](\|u_n\|_{L^p}^{p-1} + \|u\|_{L^p}^{p-1})\|u_n - u\|_{L^p}.
\end{aligned}$$

Therefore, it follows from (10) that

$$\int_{\mathbb{R}^N} [f(x,u_n) - f(x,u)](u_n - u)dx \to 0, \quad \text{as } n \to \infty. \tag{16}$$

Moreover, combining (9) with (10), then

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty. \tag{17}$$

Consequently, (14)–(17) imply that

$$u_n \to u, \quad \text{strongly in } E \text{ as } n \to \infty.$$

This completes the proof. □

Firstly, we will get a negative energy solution for the problem $(\mathcal{P})$ using the Ekeland's variational principle. We consider a minimization of I constrained in a neighborhood of zero and find a critical point of I which achieves the local minimum of I. Furthermore, the level of this local minimum is negative.

Lemma 3.4. *Assume that* $g \in L^2(\mathbb{R}^N)$, $g \not\equiv 0$ *and* (f1)–(f3) *hold. Then*

$$-\infty < \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $\overline{B}_r := \{u \in E : \|u\|_E \leq r\}$.

Proof. By (f1)–(f3), it follows from the proof of Lemma 3.2 that

$$F(x,t) \geq C_1|t|^4 - C_2|t|^2 \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where C_1 and C_2 are positive constants. Since $g(x) \in L^2(\mathbb{R}^N)$ and $g \not\equiv 0$, we can choose a function $v \in E$ such that

$$\int_{\mathbb{R}^N} g(x)v(x)dx > 0.$$

Thus,

$$\begin{aligned} I(tv) &= \frac{t^2}{2}\int_{\mathbb{R}^N}(|\Delta v|^2 + a|\nabla v|^2)dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^2 + \frac{t^2}{2}\int_{\mathbb{R}^N}V(x)v^2dx \\ &\quad - \int_{\mathbb{R}^N}F(x,tv)dx - t\int_{\mathbb{R}^N}g(x)vdx \\ &\leq \frac{t^2}{2}\|v\|_E^2 + \frac{b}{4}\|v\|_E^4t^4 - C_1\|v\|_{L^4}^4t^4 + C_2\|v\|_{L^2}^2t^2 - t\int_{\mathbb{R}^N}g(x)vdx \\ &< 0 \end{aligned}$$

for $t > 0$ small enough, which implies $\inf\{I(u) : u \in \overline{B}_\rho\} < 0$. In addition, by (3), (4), (7) and the Hölder inequality,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\Delta u|^2 + a|\nabla u|^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx \\ &\quad - \int_{\mathbb{R}^N}F(x,u)dx - \int_{\mathbb{R}^N}g(x)udx \\ &\geq -\int_{\mathbb{R}^N}F(x,u)dx - \int_{\mathbb{R}^N}g(x)udx \\ &\geq (\epsilon\|u\|_{L^2}^2 + C(\epsilon)\|u\|_{L^p}^p) - \|g\|_{L^2}\|u\|_{L^2} \\ &\geq -a_2^2\epsilon\|u\|_E^2 - a_p^pC(\epsilon)\|u\|_E^p - a_2\|g\|_{L^2}\|u\|_E, \end{aligned}$$

which implies I is bounded below in $\overline{B}_\rho$. Therefore, we obtain

$$-\infty < \inf\{I(u) : u \in \overline{B}_\rho\} < 0.$$

This completes the proof. □

Now, we could give the result of negative energy solution for the problem $(\mathcal{P})$.

Theorem 3.5. *Assume that* $g \in L^2(\mathbb{R}^N)$, $g \not\equiv 0$, (V1) *and* (f1)–(f3) *hold. Then there exists a constant* $g_0 > 0$ *such that the problem* $(\mathcal{P})$ *has a negative energy solution whenever* $\|g\|_{L^2} < g_0$, *that is, there exists a function* $u_0 \in E$ *such that* $I'(u_0) = 0$ *and* $I(u_0) < 0$.

Proof. The proof is almost the same as [7], we give it here for the completeness. By Lemmas 3.1 and 3.4, taking $g_0 = \beta > 0$, we know that

$$-\infty < \inf_{\overline{B}_\rho} I < 0 < \alpha \le \inf_{\partial B_\rho} I$$

whenever $\|g\|_{L^2} < g_0$. Set $\frac{1}{n} \in \left(0, \inf_{\partial B_\rho} I - \inf_{\overline{B}_\rho} I\right)$, $n \in \mathbb{N}$. Then, there is $u_n \in \overline{B}_\rho$ such that

$$I(u_n) \le \inf_{\overline{B}_\rho} I + \frac{1}{n}. \tag{18}$$

By Ekelands's variational principle, it follows that

$$I(u_n) \le I(u) + \frac{1}{n}\|u - u_n\|_E, \quad \forall u \in \overline{B}_\rho. \tag{19}$$

Note that $I(u_n) \le \inf_{\overline{B}_\rho} I + \frac{1}{n} < \inf_{\partial B_\rho} I$. Thus $u_n \in B_\rho$. Define $M_n\colon E \to \mathbb{R}$ by

$$M_n(u) = I(u) + \frac{1}{n}\|u - u_n\|_E.$$

By (19), we have $u_n \in B_\rho$ minimizes M_n on $\overline{B}_\rho$. Therefore, for all $\phi \in E$ with $\|\phi\|_E = 1$, taking $t > 0$ small enough such that $u_n + t\phi \in \overline{B}_\rho$, then $\frac{M_n(u_n+t\phi)-M_n(u_n)}{t} \ge 0$, which implies that

$$\frac{I(u_n + t\phi) - I(u_n)}{t} + \frac{1}{n} \ge 0.$$

Thus, $\langle I'(u_n), \phi\rangle \ge -\frac{1}{n}$. Hence,

$$\|I'(u_n)\|_E \le \frac{1}{n}. \tag{20}$$

Passing to the limit in (18) and (20), we conclude that

$$I(u_n) \to \inf_{\overline{B}_\rho} I \quad \text{and} \quad \|I'(u_n)\|_E \to 0, \quad \text{as } n \to \infty. \tag{21}$$

Note that $\|u_n\|_E \le \rho$, hence $\{u_n\} \subset E$ is a bounded Palais–Smale sequence of I. By Lemma 3.3, $\{u_n\}$ has a strongly convergent subsequence, still denoted by $\{u_n\}$ and $u_n \to u_0 \in \overline{B}_\rho$, as $n \to \infty$. Consequently, it follows from (21) that

$$I(u_0) = \inf_{\overline{B}_\rho} I < 0 \quad \text{and} \quad I'(u_0) = 0.$$

This completes the proof. □

Secondly, we get a positive energy solution for the problem $(\mathcal{P})$ with the aid of Mountain Pass Theorem.

Lemma 3.6. *Let assumptions* (V1) *and* (f1)–(f4) *hold. Then any Palais–Smale sequence of* I *is bounded.*

Proof. Let $\{u_n\} \subset E$ be any Palais–Smale sequence of I. Then, up to a subsequence, there exists $c_1 \in \mathbb{R}$ such that

$$I(u_n) \to c_1, \quad \text{and} \quad I'(u_n) \to 0. \tag{22}$$

The combination of (3), (4), (5), (22), (V1) with (f4) implies

$$\begin{aligned}
c_1+1+\|u\|_E &\geq I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{4}\int_{\mathbb{R}^N}(|\Delta u_n|^2 + a|\nabla u_n|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)u_n^2 dx + \int_{\mathbb{R}^N} \tilde{F}(x,u_n)dx \\
&\quad - \frac{3}{4}\int_{\mathbb{R}^N} g(x)u_n dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^N}(|\Delta u_n|^2 + a|\nabla u_n|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)u_n^2 dx - \frac{d}{4}\int_{\mathbb{R}^N} u_n^2 dx \\
&\quad + \int_{A_n} \tilde{F}(x,u_n)dx - \frac{3}{4}\|g\|_{L^2}\|u_n\|_{L^2} \\
&\geq \frac{1}{4}\int_{\mathbb{R}^N}(|\Delta u_n|^2 + a|\nabla u_n|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)u_n^2 dx - \frac{1}{8}\int_{\mathbb{R}^N} V_0 u_n^2 dx \\
&\quad + \int_{A_n} \tilde{F}(x,u_n)dx - \frac{3}{4}a_2\|g\|_{L^2}\|u_n\|_E \\
&\geq \frac{1}{4}\int_{\mathbb{R}^N}(|\Delta u_n|^2 + a|\nabla u_n|^2)dx + \frac{1}{4}\int_{\mathbb{R}^N} V(x)u_n^2 dx - \frac{1}{8}\int_{\mathbb{R}^N} V_0 u_n^2 dx \\
&\quad + \int_{A_n} \tilde{F}(x,u_n)dx - \frac{3}{4}a_2\|g\|_{L^2}\|u_n\|_E \\
&\geq \frac{1}{16}\|u_n\|_E^2 + \frac{1}{16}\int_{\mathbb{R}^N} V(x)u_n^2 dx + \int_{\mathbb{R}^N} \tilde{F}(x,u_n)dx - \frac{3}{4}a_2\|g\|_{L^2}\|u_n\|_E,
\end{aligned}$$

where $\tilde{F}(x,u_n) = \frac{1}{4}f(x,u_n)u_n - F(x,u_n)$ and $A_n = \{x \in \mathbb{R}^N : |u_n| \leq L\}$. Hence

$$c_1+1+\left(1+\frac{3}{4}a_2\|g\|_{L^2}\right)\|u_n\|_E \geq \frac{1}{16}\|u_n\|_E^2 + \frac{1}{16}\int_{\mathbb{R}^N} V(x)u_n^2 dx + \int_{\mathbb{R}^N} \tilde{F}(x,u_n)dx. \tag{23}$$

For $x \in \mathbb{R}^N$ and $|u_n| \leq L$, by (6) and (7), it has $|\tilde{F}(x,u_n)| \leq \frac{1}{4}|f(x,u_n)||u_n| + |F(x,u_n)| \leq \frac{5}{4}\epsilon|u_n|^2 + \frac{5}{4}C(\epsilon)|u_n|^p = \frac{5}{4}\left[\epsilon + C(\epsilon)|u_n|^{p-2}\right]|u_n|^2 \leq \frac{5}{4}\left[\epsilon + C(\epsilon)L^{p-2}\right]|u_n|^2$. Take $A > \max\{20[\epsilon + C(\epsilon)L^{p-2}], V_0\}$, then

$$\tilde{F}(x,u_n) \geq -\frac{A}{16}|u_n|^2, \quad \forall x \in \mathbb{R}^N, |u_n| \leq L. \tag{24}$$

Let $\tilde{A} = \{x \in \mathbb{R}^N : V(x) \le A\}$. By (V1) and (24), we can conclude

$$\begin{aligned}\frac{1}{16}\int_{\mathbb{R}^N} V(x)u_n^2dx + \int_{A_n} \tilde{F}(x,u_n)dx &\ge \frac{1}{16}\int_{|u_n|\le L}(V(x)-A)|u_n|^2dx \\ &\ge \frac{1}{16}\int_{\tilde{A}\cap A_n}(V(x)-A)L^2dx \\ &\ge \frac{1}{16}(V_0-A)L^2\text{meas}(\tilde{A}\cap A_n) \\ &\ge \frac{1}{16}(V_0-A)L^2\text{meas}(\tilde{A}).\end{aligned} \tag{25}$$

Note that meas$(A) < +\infty$ due to (V1), it follows from (23) and (25) that

$$c_1 + 1 + \left(1+\frac{3}{4}a_2\|g\|_{L^2}\right)\|u_n\|_E \ge \frac{1}{16}\|u_n\|_E^2 + \frac{1}{16}(V_0-A)L^2\text{meas}(\tilde{A}),$$

which implies $\{u_n\} \subset E$ is bounded in E. Hence, the proof is completed. □

Theorem 3.7. *Assume that* $g(x) \in L^2(\mathbb{R}^N)$, $g \not\equiv 0$, (V1) *and* (f1)–(f4) *hold. Then the problem* $(\mathcal{P})$ *has a positive energy solution whenever* $\|g\|_{L^2} < g_0$*, that is, there exists a function* $u_1 \in E$ *such that* $I(u_1) = 0$ *and* $I(u_1) > 0$.

Proof. We will apply Theorem 2.4 to prove Theorem 3.7. Next, we shall verify that I satisfies all the conditions of Theorem 2.4. By Theorem 3.5, we know $g_0 = \beta > 0$. Then I satisfies (I1) whenever $\|g\|_{L^2} < g_0$ by Lemma 3.1. Lemma 3.2 implies that I satisfies (I2), and I satisfies (PS) condition by virtue of Lemmas 3.3 and 3.6. Evidently, $I \in C^1(E,\mathbb{R})$ and $I(0) = 0$. Hence, applying Theorem 2.4, there exists a function $u_1 \in E$ such that $I(u_1) = 0$ and $I(u_1) \ge \alpha > 0$. The proof is completed. □

Proof of Theorem 1.1. The desired conclusion directly follows from Theorems 3.5 and 3.7. □

Proof of Corollary 1.5. It is sufficient to prove that (f3"), (f4") imply (f3), (f4) by applying Theorem 1.1. In fact, For any $(x,z) \in \mathbb{R}^N \times \mathbb{R}$, define

$$k(t) := F\left(x, \frac{z}{t}\right)t^\mu, \quad \forall t \in [1,+\infty).$$

Then for $|z| \ge L'$ and $t \in \left[1, \frac{|z|}{L'}\right]$, (f4") implies that

$$k'(t) = f(x,\frac{z}{t})\left(-\frac{z}{t^2}\right)t^\mu + \mu F\left(x,\frac{z}{t}\right)t^{\mu-1},$$
$$t^{\mu-1}\left[\mu F\left(x,\frac{z}{t}\right) - f\left(x,\frac{z}{t}\right)\frac{z}{t}\right] \le d't^{\mu-1}\left|\frac{z}{t}\right|^2 = d't^{\mu-3}|z|^2.$$

Thus,

$$k\left(\frac{|z|}{L'}\right)-k(1)=\int_1^{\frac{|z|}{L'}} k'(t)dt \le \int_1^{\frac{|z|}{L'}} d't^{\mu-3}|z|^2dt=\frac{d'|z|^\mu}{(\mu-2)L'^{\mu-2}}-\frac{d'|z|^2}{\mu-2}.$$

Hence, for any $x\in\mathbb{R}^N$ and $|z|\ge L$, by (f3), one has

$$F(x,z)=k(1)\ge k\left(\frac{|z|}{L'}\right)+\frac{d'|z|^2}{\mu-2}-\frac{d'|z|^\mu}{(\mu-2)L'^{\mu-2}},$$
$$\left[\inf_{x\in\mathbb{R}^N,|t|=L'}F(x,t)\right]\left(\frac{|z|}{L'}\right)^\mu+\frac{d'|z|^2}{\mu-2}-\frac{d'|z|^\mu}{(\mu-2)L'^{\mu-2}}\ge\left(\frac{c'}{L'^\mu}-\frac{d'}{(\mu-2)L'^{\mu-2}}\right)|z|^\mu.$$

By $d'\in\left[0,\frac{c'(\mu-2)}{L'^2}\right)$, set $C_4=\frac{c'}{L'^\mu}-\frac{d'}{(\mu-2)L'^{\mu-2}}>0$, it has $F(x,z)\ge C_4|z|^\mu$, for all $x\in\mathbb{R}^N$ and $|z|\ge L'$. Hence,

$$\frac{F(x,z)}{z^4}\ge C_4|z|^{\mu-4},\quad \forall x\in\mathbb{R}^N \text{ and } |z|\ge L'. \tag{26}$$

Note that $\mu>4$, then (26) implies (f3). Furthermore, it follows from (26) and (f4”) that

$$4F(x,z)-f(x,z)z=\mu F(x,z)-f(x,z)z+(4-z)F(x,z)\le d'|z|^2-(\mu-4)C_4|z|^\mu$$

for all $x\in\mathbb{R}^N$ and $|z|\ge L'$. This, together with $\mu>4$, shows there exists $L>0$ such that

$$4F(x,z)-f(x,z)z<0\quad \forall x\in\mathbb{R}^N \text{ and } |z|\ge L,$$

which implies (f4). Hence, the proof is completed. □

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References

[1] Ansari, H. and Vaezpour, S. M., Existence and multiplicity of solutions for fourth-order elliptic Kirchhoff equations with potential term. *Complex Var. Elliptic Equ.* 60 (2015)(5), 668 – 695.

[2] Ball, J. M., Initial-boundary value problems for an extensible beam. *J. Math. Anal. Appl.* 42 (1973), 61 – 90.

[3] Bartsch, T. and Wang, Z. Q., Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$. *Comm. Partial Diff. Equ.* 20 (1995) (9–10), 1725 – 1741.

[4] Berger, H. M., A new approach to the analysis of large deflections of plates. *J. Appl. Mech.* 22 (1955), 465 – 472.

[5] Chen, S.-J. and Li, L., Multiple solutions for the nonhomogeneous Kirchhoff equation on $\mathbb{R}^N$. *Nonlinear Anal. Real World Appl.* 14 (2013)(3), 1477 – 1486.

[6] Cheng, B., A new result on multiplicity of nontrivial solutions for the nonhomogenous Schrödinger–Kirchhoff type problem in $\mathbb{R}^n$. *Mediterr. J. Math.* 13 (2016)(3), 1099 – 1116.

[7] Cheng, B., Wu, X. and Liu, J., Multiple solutions for a class of Kirchhoff type problems with concave nonlinearity. *NoDEA Nonlinear Diff. Equ. Appl.* 19 (2012)(5), 521 – 537.

[8] Ferrara, M., Khademloo, S. and Heidarkhani, S., Multiplicity results for perturbed fourth-order Kirchhoff type elliptic problems. *Appl. Math. Comput.* 234 (2014), 316 – 325.

[9] Guo, Z., Ground states for Kirchhoff equations without compact condition. *J. Diff. Equ.* 259 (2015)(7), 2884 – 2902.

[10] He, X. and Zou, W., Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. *J. Diff. Equ.* 252 (2012)(2), 1813 – 1834.

[11] Heidarkhani, S., Ferrara, M. and Khademloo, S., Nontrivial solutions for one-dimensional fourth-order Kirchhoff-type equations. *Mediterr. J. Math.* 13 (2016)(1), 217 – 236.

[12] Heidarkhani, S., Khademloo, S. and Solimaninia, A., Multiple solutions for a perturbed fourth-order Kirchhoff type elliptic problem. *Port. Math.* 71 (2014)(1), 39 – 61.

[13] Huang, Y. and Liu, Z., On a class of Kirchhoff type problems. *Arch. Math. (Basel)* 102 (2014)(2), 127 – 139.

[14] Li, L. and Pan, W.-W., A note on nonlinear fourth-order elliptic equations on $\mathbb{R}^N$. *J. Global Optim.* 57 (2013)(4), 1319 – 1325.

[15] Li, Q. and Wu, X., A new result on high energy solutions for Schrödinger–Kirchhoff type equations in $\mathbb{R}^N$. *Appl. Math. Lett.* 30 (2014), 24 – 27.

[16] Liang, S. and Zhang, J., Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^3$. *Nonlinear Anal. Real World Appl.* 17 (2014), 126 – 136.

[17] Liu, W. and He, X., Multiplicity of high energy solutions for superlinear Kirchhoff equations. *J. Appl. Math. Comput.* 39 (2012)(1-2), 473 – 487.

[18] Ma, T. F., Existence results and numerical solutions for a beam equation with nonlinear boundary conditions (2nd International Workshop on Numerical Linear Algebra, Numerical Methods for Partial Differential Equations and Optimization (Curitiba 2001)). *Appl. Numer. Math.* 47 (2003)(2), 189 – 196.

[19] Mawhin, J. and Willem, M., *Critical Point Theory and Hamiltonian Systems.* Appl. Math. Sci. 74. New York: Springer 1989.

[20] Rabinowitz, P. H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. Math. 65. Providence (RI): Amer. Math. Soc. 1986.

[21] Wang, F. and An, Y., Existence and multiplicity of solutions for a fourth-order elliptic equation. *Bound. Value Probl.* 2012, 2012:6, 9 pp.

[22] Wang, F., Avci, M. and An, Y., Existence of solutions for fourth order elliptic equations of Kirchhoff type. *J. Math. Anal. Appl.* 409 (2014)(1), 140 – 146.

[23] Woinowsky-Krieger, S., The effect of axial force on the vibration of hinged bars. *J. Appl. Mech.* 17 (1950), 35 – 36.

[24] Wu, X., Existence of nontrivial solutions and high energy solutions for Schrödinger–Kirchhoff-type equations in $\mathbb{R}^N$. *Nonlinear Anal. Real World Appl.* 12 (2011)(2), 1278 – 1287.

[25] Wu, X., High energy solutions of systems of Kirchhoff-type equations in $\mathbb{R}^N$. *J. Math. Phys.* 53 (2012)(6), 063508, 18 pp.

[26] Xu, L. and Chen, H., Existence and multiplicity of solutions for fourth-order elliptic equations of Kirchhoff type via genus theory. *Bound. Value Probl.* 2014, 2014:212, 12 pp.

[27] Xu, L. and Chen, H., Multiplicity of nontrivial solutions for the nonhomogenous fourth order Kirchhoff equation. *Taiwanese J. Math.* 19 (2015), 1215 – 1226.

[28] Ye, Y. and Tang, C.-L., Multiple solutions for Kirchhoff-type equations in $\mathbb{R}^N$. *J. Math. Phys.* 54 (2013)(8), 081508, 16 pp.

[29] Yin, Y. and Wu, X., High energy solutions and nontrivial solutions for fourth-order elliptic equations. *J. Math. Anal. Appl.* 375 (2011)(2), 699 – 705.

[30] Zhang, W., Tang, X. and Zhang, J., Infinitely many solutions for fourth-order elliptic equations with general potentials. *J. Math. Anal. Appl.* 407 (2013)(2), 359 – 368.

[31] Zhou, F., Wu, K. and Wu, X., High energy solutions of systems of Kirchhoff-type equations on $\mathbb{R}^N$. *Comput. Math. Appl.* 66 (2013)(7), 1299 – 1305.

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