Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 36 (2017), 239–251
DOI: 10.4171/ZAA/1587

Non-Uniform Decay of the Energy of some Dissipative Evolution Systems

Kaïs Ammari, Ahmed Bchatnia and Karim El Mufti

Abstract. In this paper we consider second order evolution equations with bounded damping. We give a characterization of a non-uniform decay for the damped problem using a kind of observability estimate for the associated undamped problem.

Keywords. Bounded feedback, kind of observability estimate, non-uniform decay

Mathematics Subject Classification (2010). Primary 35B30, secondary 35B40

1. Introduction and main results

Let X be a complex Hilbert space with norm and inner product denoted respectively by $||.||_X$ and $\langle \cdot, \cdot \rangle_X$. Let A be a linear unbounded self-adjoint and strictly positive operator in X and $V = \mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$, with

$$\|x\|_V = \|A^{\frac{1}{2}} x\|_X, \quad \forall x \in V.$$

Denote by $(\mathcal{D}(A^{\frac{1}{2}}))'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ obtained by means of the inner product in X. Further, let U be a complex Hilbert space (identified to its dual) and $B \in \mathcal{L}(U, X)$. Most of the linear control problems coming from elasticity can be written as

$$\begin{cases} w''(t) + Aw(t) + Bu(t) = 0, \\ \qquad w(0) = w_0, \, w'(0) = w_1, \end{cases} \tag{1.1}$$

K. Ammari : UR Analysis and Control of PDEs, UR13ES64, Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, 5019 Monastir, Tunisia; kais.ammari@fsm.rnu.tn
A. Bchatnia: UR Analyse Non-Linéaire et Géométrie, UR13ES32, Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunisia; ahmed.bchatnia@fst.rnu.tn
K. El Mufti: UR Analysis and Control of PDEs, UR13ES64, ISCAE, University of Manouba, Tunisia; karim.elmufti@iscae.rnu.tn

where $w : [0, T] \to X$ is the state of the system, $u \in L^2(0, T; U)$ is the input function and denote the differentiation with respect to time by "$'$".

We define the energy of $w(t)$ at instant t by

$$E(w(t)) = \frac{1}{2}\Big\{||w'(t)||_X^2 + ||A^{\frac{1}{2}}w(t)||_X^2\Big\}.$$

Simple formal calculations give

$$E(w(0)) - E(w(t)) = \int_0^t \langle Bu(s), w'(s)\rangle_X \, ds, \quad \forall t \geq 0. \tag{1.2}$$

This is why, in many problems coming in particular from elasticity, the input u is given in the feedback form $u(t) = B^* w'(t)$, which obviously gives a nonincreasing energy and which corresponds to collocated actuators and sensors.

The aim of this paper is to give sufficient and necessary conditions on the conservative system (1.5) making the corresponding closed loop system

$$\begin{cases} w''(t) + Aw(t) + BB^*w'(t) = 0, \\ \qquad\qquad w(0) = w_0, \, w'(0) = w_1, \end{cases} \tag{1.3}$$

non-uniformly stable. The strategy to get such a decay rate will consist to generalize a kind of observability estimate given in [7]. Any sufficiently smooth solution of (1.3) satisfies the energy estimate

$$E(w(0)) - E(w(t)) = \int_0^t ||B^*w'(s)||_U^2 ds, \quad \forall t \geq 0. \tag{1.4}$$

In particular (1.4) implies that

$$E(w(t)) \leq E(w(0)), \quad \forall t \geq 0.$$

In the natural well-posedness space $V \times X$, the existence and uniqueness of finite energy solutions of (1.3) can be obtained by standard semi-group methods.

Denote by ϕ the solution of the associated undamped problem

$$\begin{cases} \phi''(t) + A\phi(t) = 0, \\ \phi(0) = w_0, \, \phi'(0) = w_1. \end{cases} \tag{1.5}$$

It is well known that (1.5) is well-posed in $\mathcal{D}(A) \times V$ and in $V \times X$. Our main result is stated as follows: Let $\mathcal{G}$ be a continuous positive increasing real function on $[0, +\infty)$ and define the function $\mathcal{F}$ by $\mathcal{F}(x) = x\,(\mathcal{G}(x))^2$.

Theorem 1.1. (1) *Assume that there exists $C > 0$ such that for all non-identically zero initial data $(w_0, w_1) \in V \times X$ and for all $t > 0$, the solution w of* (1.3) *satisfies:*

$$E(w(t)) \leq C||(w_0, w_1)||^2_{V \times X} \mathcal{G}^{-1}\left(\frac{1}{t}\right), \tag{1.6}$$

then there exists $C > 0$ such that the solution ϕ of (1.5) *satisfies:*

$$||(w_0, w_1)||^2_{V \times X} \leq 16 \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*\phi'(t)\|^2_U \, dt, \tag{1.7}$$

where

$$\Lambda = \frac{||(w_0, w_1)||^2_{\mathcal{D}(A) \times V}}{||(w_0, w_1)||^2_{V \times X}}.$$

(2) *Assume that $x \mapsto x\mathcal{F}^{-1}\left(\frac{1}{x}\right)$ is an increasing function and there exists $C > 0$ such that for all non-identically zero initial data $(w_0, w_1) \in \mathcal{D}(A) \times V$, the solution ϕ of* (1.5) *satisfies:*

$$||(w_0, w_1)||^2_{V \times X} \leq C \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*\phi'(t)\|^2_U dt. \tag{1.8}$$

Then there exists $C > 0$ such that for all $t > 0$, the solution w of (1.3) *satisfies:*

$$E(w(t)) \leq C||(w_0, w_1)||^2_{\mathcal{D}(A) \times V} \mathcal{F}^{-1}\left(\frac{1}{\sqrt{t}}\right). \tag{1.9}$$

Corollary 1.2. *The weak observability* [1]*, i.e. there exist $T, C > 0$ such that for all $(w_0, w_1) \in V \times X$ the solution ϕ of* (1.5) *satisfies*

$$\int_0^T ||B^*\phi'(t)||^2_U \, dt \geq C \, ||(w_0, w_1)||^2_{V \times X} \, \mathcal{G}\left(\frac{||(w_0, w_1)||^2_{X \times (\mathcal{D}(A^{\frac{1}{2}}))'}}{||(w_0, w_1)||^2_{V \times X}}\right), \tag{1.10}$$

implies in particular (1.7).

The paper is organized as follows: In Section 2 we prove our main result and in the last section we give some applications both in the linear and the nonlinear case. Decay rates for nonlinear dissipations were obtained under our generalized observability estimate. Here we mention that the literature is less provide. We cite essentially [1, 8].

[1] See [2–4] for more details.

2. Proof of Theorem 1.1

The following lemma will be very useful.

Lemma 2.1. *Let $\mathcal{H}$ (resp. $\mathcal{G}$) be a continuous positive decreasing (resp. increasing) real function on $[0,+\infty)$. Suppose that $\mathcal{H}$ is bounded by one and there exists a positive constant c such that*

$$\mathcal{H}(s) \leq \frac{c}{\big(\mathcal{G}(\mathcal{H}(s))\big)^2}\left(\mathcal{H}(s) - \mathcal{H}\left(\frac{1}{\mathcal{G}(\mathcal{H}(s))} + s\right)\right), \quad \forall s > 0. \tag{2.1}$$

Suppose that $x \mapsto x\mathcal{F}^{-1}\big(\frac{1}{x}\big)$ is an increasing function, then there exists $C > 0$ such that for any $t > 0$,

$$\mathcal{H}(t) \leq C\mathcal{F}^{-1}\left(\frac{1}{\sqrt{t}}\right). \tag{2.2}$$

The proof is similar to that of [7, Lemma B]. Since our result is more general, we give it for the reader's convenience.

Proof of Lemma 2.1. Let $t > 0$. We distinguish two cases:

- If
$$\mathcal{G}(\mathcal{H}(s)) < \frac{1}{t}, \quad \text{then} \quad \mathcal{H}(s) \leq \mathcal{G}^{-1}\left(\frac{1}{t}\right). \tag{2.3}$$

- If
$$\mathcal{G}(\mathcal{H}(s)) \geq \frac{1}{t}, \quad \text{then} \quad \frac{1}{\mathcal{G}(\mathcal{H}(s))} + s \leq t + s,$$
therefore
$$\mathcal{H}(t+s) \leq \mathcal{H}\left(\frac{1}{\mathcal{G}(\mathcal{H}(s))} + s\right) \tag{2.4}$$
and we get
$$\mathcal{F}(\mathcal{H}(s)) \leq \mathcal{H}(s) - \mathcal{H}(t+s). \tag{2.5}$$

The inequalities (2.3) and (2.5) give

$$\mathcal{H}(s) \leq \mathcal{F}^{-1}(\mathcal{H}(s) - \mathcal{H}(t+s)) + \mathcal{G}^{-1}\left(\frac{1}{t}\right), \quad \forall s, t > 0. \tag{2.6}$$

We introduce the function Ψ_t defined on $]0,+\infty[$ by

$$\Psi_t(s) = \frac{1}{\mathcal{F}^{-1}\left(\frac{t}{s}\right) + \mathcal{G}^{-1}\left(\frac{1}{t}\right)}. \tag{2.7}$$

We distinguish two cases:

- If $\mathcal{H}(s)-\mathcal{H}(t+s)<\frac{t}{t+s}$ then $\mathcal{H}(s)\le\Psi_t(t+s)$ and we deduce
$$\Psi_t(t+s)\mathcal{H}(t+s)\le 1. \tag{2.8}$$
- If $\mathcal{H}(s)-\mathcal{H}(t+s)>\frac{t}{t+s}$. Taking into account that $\mathcal{H}(s)\le 1$ we obtain
$$\frac{t}{t+s}\mathcal{H}(s)\le\frac{t}{t+s},$$
so
$$\frac{t}{t+s}\mathcal{H}(s)\le\frac{t}{t+s}<\mathcal{H}(s)-\mathcal{H}(t+s), \tag{2.9}$$
and we deduce
$$\mathcal{H}(t+s)<\mathcal{H}(s)\,\frac{s}{t+s}. \tag{2.10}$$

Consequently,
$$\begin{aligned}\Psi_t(t+s)\mathcal{H}(t+s)&<\Psi_t(t+s)\mathcal{H}(s)\,\frac{s}{t+s}\\ &=\frac{\Psi_t(t+s)}{t+s}\mathcal{H}(s)\Psi_t(s)\frac{s}{\Psi_t(s)}\\ &=\mathcal{H}(s)\Psi_t(s)\,\frac{\frac{\Psi_t(t+s)}{t+s}}{\frac{\Psi_t(s)}{s}}.\end{aligned} \tag{2.11}$$
Using the increasing property of $x\mapsto x\mathcal{F}^{-1}\left(\frac{1}{x}\right)$, we obtain
$$\Psi_t(t+s)\mathcal{H}(t+s)<\Psi_t(s)\mathcal{H}(s). \tag{2.12}$$
We have proved that for all $s,t>0$, we have either
$$\Psi_t(t+s)\mathcal{H}(t+s)\le 1 \quad\text{or}\quad \Psi_t(t+s)\mathcal{H}(t+s)<\Psi_t(s)\mathcal{H}(s).$$
In particular, we deduce that for any $t>0$ and $n\in\mathbb{N}^*$, either
$$\Psi_t((n+1)t)\mathcal{H}((n+1)t)\le 1 \quad\text{or}\quad \Psi_t((n+1)t)\mathcal{H}((n+1)t)<\Psi_{nt}(t)\mathcal{H}(nt).$$
Hence, we have
$$\Psi_t((n+1)t)\mathcal{H}((n+1)t)\le\max(1,\Psi_t(t)\mathcal{H}(t))=1. \tag{2.13}$$
Therefore, for all $t>0$ and $n\in\mathbb{N}^*$,
$$\mathcal{H}((n+1)t)\le\mathcal{F}^{-1}\left(\frac{1}{n+1}\right)+\mathcal{G}^{-1}\left(\frac{1}{t}\right). \tag{2.14}$$
Choose n such that $n+1\le t<n+2$ and make use again of the increasing property of $x\mapsto x\mathcal{F}^{-1}(\frac{1}{x})$, we get for all $t\ge 2$:
$$\mathcal{H}(t^2)\le\mathcal{F}^{-1}\left(\frac{1}{t}\right)+\mathcal{G}^{-1}\left(\frac{1}{t}\right). \tag{2.15}$$
Since $|\mathcal{G}^{-1}(x)|\le|\mathcal{F}^{-1}(x)|$ close to zero, the desired result follows. $\square$

After, we give the proof of the main result.

Proof of Theorem 1.1(1). We combine (1.6) and the following formula:

$$E(w(t)) = E(\phi(0)) - 2\int_0^t \|B^*\phi'(t)\|_U^2 ds \quad \forall t > 0, \tag{2.16}$$

to get

$$E(\phi(0)) - 2\int_0^t \|B^*\phi'(t)\|_U^2 \, ds \leq C\mathcal{G}^{-1}\left(\frac{1}{t}\right) ||(w_0, w_1)||^2_{\mathcal{D}(A)\times V}. \tag{2.17}$$

Take $t = \frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}$, we obtain $E(\phi(0)) - 2\int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*\phi'(t)\|_U^2 ds \leq \frac{1}{2}||(w_0, w_1)||^2_{V\times X}$.

We deduce that

$$||(w_0, w_1)||^2_{V\times X} \leq 4\int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|(B^*\phi)'(t)\|_U^2 ds. \tag{2.18}$$

Now, let us consider $v = \phi - w$, then v satisfies the following system:

$$\begin{cases} v''(t) + Av(t) + BB^*v'(t) = BB^\star\phi'(t), \quad t > 0, \\ (v(0), v'(0)) = (0, 0). \end{cases} \tag{2.19}$$

Multiply the first equation of (2.19) by v', and integer by parts to get

$$E(v(t)) + 2\int_0^t \|B^*v'(s)\|_U^2 \, ds = 2\int_0^t \langle B^*\phi'(s), B^*v'(s)\rangle_U \, ds. \tag{2.20}$$

Make use of Young inequality,

$$E(v(t)) + 2\int_0^t \|B^*v'(s)\|_U^2 \, ds \leq \int_0^t \left(\|B^*\phi'(s)\|_U^2 + \|B^*v'(t)\|_U^2\right) \, ds. \tag{2.21}$$

Hence,

$$E(v(t)) + \int_0^t \|B^*v'(s)\|_U^2 \, ds \leq \int_0^t \|B^*\phi'(s)\|_U^2 ds. \tag{2.22}$$

Since $\|B^*w'\|_U^2 = \|B^*\phi' - B^*v'\|_U^2 \leq 2\left(\|B^*\phi'\|_U^2 + \|B^*v'\|_U^2\right)$, therefore

$$\int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*w'(t)\|_U^2 dt \leq 2\int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \left(\|B^*\phi'\|_U^2 + \|B^*v'\|_U^2\right) \, dt. \tag{2.23}$$

Thanks to (2.22) and (2.23), we obtain

$$\int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*w'(t)\|_U^2 \, dt \leq 4\int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*\phi'(t)\|_U^2 \, dt. \tag{2.24}$$

Finally, by virtue of (2.18), we conclude the following estimate

$$||(w_0, w_1)||^2_{V\times X} \leq 16 \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*\phi'(t)\|^2_U dt. \tag{2.25}$$

This finishes the proof. □

Proof of Theorem 1.1(2). Using similar arguments as previously, the inequality (1.8) becomes

$$E(w(0)) \leq 2C \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \left(\|B^*w'(t)\|^2_U + \|B^*v'(t)\|^2_U\right) dt.$$

Multiply the first equation of (2.19) by v' and integer by parts. It follows that

$$E(v(t)) \leq \int_0^t \left(\frac{\|B^*w'(s)\|^2_U}{\varepsilon} + \varepsilon \|B^*v'(s)\|^2_U \right) ds, \quad \forall \varepsilon > 0. \tag{2.26}$$

Let $T > 0$ be fixed, for all $0 \leq t \leq T$, we have

$$\begin{aligned}
\sup_{0\leq t\leq T} E(v(t)) &\leq \int_0^T \left(\frac{\|B^*w'(t)\|^2_U}{\varepsilon} + \varepsilon \|B^*v'(t)\|^2_U \right) dt \\
&\leq \int_0^T \frac{\|B^*w'(t)\|^2_U}{\varepsilon} dt + \varepsilon C \int_0^T E(v(t))\, dt \\
&\leq \int_0^T \frac{\|B^*w'(t)\|^2_U}{\varepsilon} dt + \varepsilon CT \sup_{0\leq t\leq T} E(v(t)).
\end{aligned}$$

Choose $\varepsilon = \frac{1}{2CT}$, so we have

$$\sup_{0\leq t\leq T} E(v(t)) \leq 4CT \int_0^T \|B^*w'\|^2_U dt. \tag{2.27}$$

On the other hand, as $\|v'(t)\|^2_X \leq E(v(t))$, we integer on $[0, T]$, to get

$$\int_0^T \|v'(t)\|^2_X dt \leq \int_0^T E(v(t))\; dt \leq T \sup_{0\leq t\leq T} E(v(t)) \leq 4CT^2 \int_0^T \|B^*w'\|^2_U dt,$$

so

$$\int_0^T \|B^*v'\|^2_U dt \leq 4C^2T^2 \int_0^T \|B^*w'\|^2_U dt. \tag{2.28}$$

Hence, for $T = \frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}$ we conclude that

$$E(w(0)) \leq 2C \left(1 + 4C^2 \left(\mathcal{G}\left(\frac{1}{2C\Lambda}\right)\right)^{-2}\right) \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*w'\|^2_U \, dt. \tag{2.29}$$

One can easily verify that

$$\Lambda \leq \frac{E(w(0)) + E(w'(0))}{E(w(0))} := \tilde{\Lambda},$$

and consequently

$$E(w(0)) \leq 2C \left(1 + 4C^2 \left(\mathcal{G}\Big(\frac{1}{2C\Lambda}\Big)\right)^{-2}\right) \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\tilde{\Lambda}}\right)}} \|B^* w'(t)\|_U^2 \, dt. \tag{2.30}$$

Since $\tilde{\Lambda}$ is minimized, we get

$$E(w(0)) \leq C \left(\mathcal{G}\Big(\frac{1}{2C\Lambda}\Big)\right)^{-2} \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\tilde{\Lambda}}\right)}} \|B^* w'(t)\|_U^2 \, dt. \tag{2.31}$$

By translating the time variable and using the formula

$$E(w(t_1)) - E(w(t_2)) + \int_{t_2}^{t_1} \|B^* w'(t)\|_U^2 \, dt = 0,$$

we obtain

$$\begin{aligned}
&\frac{E(w(0))}{E(w(0)) + E(w'(0))} \\
&\leq C \left(\mathcal{G}\Big(\frac{1}{2C\Lambda}\Big)\right)^{-2} \int_s^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\tilde{\Lambda}}\right)}+s} \frac{\|B^* w'(t)\|_U^2}{E(w(0)) + E(w'(0))} dt \\
&\leq C \left(\mathcal{G}\Big(\frac{1}{2C\Lambda}\Big)\right)^{-2} \left(\frac{E(w(s))}{E(w(0)) + E(w'(0))} - \frac{E\left(w\left(\frac{1}{\mathcal{G}\left(\frac{1}{2C\tilde{\Lambda}}\right)} + s\right)\right)}{E(w(0)) + E(w'(0))}\right).
\end{aligned}$$

Put $\mathcal{H}(s) = \frac{E(w(s))}{2C(E(w(0)) + E(w'(0)))}$. Make use of the previous inequality and the decay of $\mathcal{H}$, it follows that

$$\mathcal{H}(s) \leq C\big(\mathcal{G}(\mathcal{H}(s))\big)^{-2} \left(\mathcal{H}(s) - \mathcal{H}\left(\frac{1}{\mathcal{G}(\mathcal{H}(s))} + s\right)\right), \quad \forall s > 0.$$

Thanks to (2.1) and Lemma 2.1, there exists C such that

$$\frac{E(w(t))}{E(w(0)) + E(\partial_t w(0))} \leq C\mathcal{F}^{-1}\left(\frac{1}{\sqrt{t}}\right).$$

We conclude the desired result. □

3. Some applications

We give some applications of Theorem 1.1.

3.1. The linear case.

3.1.1. Example 1. Let $\mathcal{G}$ be given by $\mathcal{G}(x) = x^p$ on $(0, r_0]$, $r_0 > 0$ and $p \in \mathbb{R} \setminus [-\frac{1}{2}, 0]$. Then the following two statements are equivalent:

i) There exists $C > 0$ such that for all non-identically zero initial data $(w_0, w_1) \in V \times X$, the solution ϕ of (1.5) satisfies:

$$||(w_0, w_1)||^2_{V\times X} \le 16 \int_0^{C\Lambda^p} \|B^*\phi'(t)\|^2_U \, dt.$$

ii) There exists $C > 0$ such that for all non-identically zero initial data $(w_0, w_1) \in \mathcal{D}(A) \times V$ and for all $t > 0$, the solution w of (1.3) satisfies

$$E(w(t)) \le \frac{C}{t^p} ||(w_0, w_1)||^2_{\mathcal{D}(A)\times V}.$$

Remark 3.1. In [7], the author construct a geometry with a trapped ray for the linear dissipative wave equation (the geometric control condition is then not fulfilled) and establish a polynomial decay rate when $(w_0, w_1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, the estimate (1.7) is satisfied for $\mathcal{G}(x) = x^\delta$, $\delta > 0$.

3.1.2. Example 2. Let $\mathcal{G}$ be given by $\mathcal{G}(x) = \frac{\exp\left(-\frac{1}{x^p}\right)}{\sqrt{x}}$ on $(0, r_0]$, $p \in \mathbb{R}_+$. The following statements hold.

i) The existence of a constant $C > 0$ such that the solution ϕ of (1.5) satisfies

$$||(w_0, w_1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} \le 16 \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda}\right)}} \|B^*\phi'(t)\|^2_U \, dt,$$

implies the existence of a constant $C_1 > 0$ such that for all non-identically zero initial data $(w_0, w_1) \in \mathcal{D}(A) \times V$ and for all $t > 0$, the solution w of (1.3) satisfies

$$E(w(t)) \le \frac{C_1}{(\ln t)^{\frac{1}{p}}} ||(w_0, w_1)||^2_{\mathcal{D}(A)\times V}.$$

ii) The existence of a constant $C_1 > 0$ such that for all non-identically $(w_0, w_1) \in \mathcal{D}(A) \times V$ and for all $t > 0$, the solution w of (1.3) satisfies

$$E(w(t)) \le \frac{C_1}{(\ln t)^{\frac{1}{p}}} ||(w_0, w_1)||^2_{\mathcal{D}(A)\times V}.$$

implies the existence of a constant $C > 0$ such that the solution ϕ of (1.5) satisfies

$$||(w_0, w_1)||^2_{V\times X} \le 16 \int_0^{\frac{1}{\mathcal{F}\left(\frac{1}{2C\Lambda}\right)}} \|B^*\phi'(t)|^2 \, dt.$$

3.2. The nonlinear case. Let Ω be a bounded connected open set of $\mathbb{R}^n$, $n > 1$ with a $\mathcal{C}^2$ boundary $\partial\Omega$. Let also $M = \left(\alpha^{ij}\right)_{1\le i,j\le n} \in \mathcal{C}^\infty\left(\bar{\Omega}; \mathbb{R}^{n\times n}\right)$ be a symmetric and uniformly positive definite matrix.

Denote by $\nabla = \left(\sum_{j=1}^n \beta^{1j}\partial_{x_j}, \dots, \sum_{j=1}^n \beta^{nj}\partial_{x_j}\right)$ and $\Delta = \sum_{i,j=1}^n \partial_{x_i}\left(\alpha^{ij}\partial_{x_j}\right)$. We deal with the following second order differential equation:

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)g(\partial_t u) = 0, & \text{in } \Omega \times (0,+\infty), \\ u = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ (u, \partial_t u)(.,0) = (u_0, u_1), & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $a = a(x) \in L^\infty(\Omega)$ is a bounded function with $a(x) \ge 0$ for all $x \in \Omega$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function with $g(0) = 0$, $sg(s) \ge 0$. We assume the additional conditions:

(i) $\exists r \in [1,\infty)$, $\exists c_1, c_2 > 0$, $|s| \le 1 \Rightarrow c_1|s|^r \le |g(s)| \le c_2|s|^{1/r}$.

(ii) $\exists k \in [0,1]$, $\exists p \in [1,\infty)$, $\exists c_3$, $c_4 > 0$, $|s| > 1 \Rightarrow c_3|s|^k \le |g(s)| \le c_4|s|^p$.

(iii) $(n-2)(1-k) \le 4r$ and $(n-2)(p-1) \le 1$.

For $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique solution

$$u \in \mathcal{C}\left([0,+\infty); H_0^1(\Omega)\right) \cap \mathcal{C}^1\left([0,+\infty); L^2(\Omega)\right).$$

For more regular initial data $(u_0, u_1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$, the solution u has the following regularity

$$u \in L^\infty\left([0,+\infty); H^2(\Omega) \cap H_0^1(\Omega)\right) \cap W^{1,\infty}\left([0,+\infty); H_0^1(\Omega)\right) \cap W^{2,\infty}\left([0,+\infty); L^2(\Omega)\right).$$

The energy of a solution is defined at instant $t \ge 0$ by

$$E(u(t)) = \frac{1}{2}\int_\Omega \left(|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2\right) dx.$$

$E(u(t))$ is a non-increasing function of time and satisfies, for all $t_2 > t_1 \ge 0$ the identity

$$E(u(t_2)) - E(u(t_1)) = -\int_{t_1}^{t_2}\int_\Omega a(x)g(\partial_t u(x,t))\partial_t u(x,t)\,dx\,dt \le 0.$$

Denote by

$$X(u_0,u_1) = E(u(0)) + E_1(u(0)) + [E_1(u(0))]^{(2p-1)} + [E_1(u(0))]^{(1+\frac{r-k}{r+1})},$$

where

$$E_1(u(0)) = \|(\Delta u_0 - ag(u_1), u_1)\|^2_{L^2(\Omega)\times H_0^1(\Omega)}.$$

We introduce u_l the solution of the linear locally damped problem:

$$\begin{cases} \partial_t^2 u_l - \Delta u_l + a(x)\partial_t u_l = 0, & \text{in } \Omega \times (0,+\infty), \\ u_l = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ (u_l, \partial_t u_l)(.,0) = (u_0,u_1) \in \left[H^2(\Omega)\times H_0^1(\Omega)\right] \cap H_0^1(\Omega), \end{cases} \tag{3.2}$$

and make the following assumption:

(A) Assume that $x \mapsto x\mathcal{F}^{-1}\left(\frac{1}{x}\right)$ is an increasing function and there exists $C > 0$ such that for all non-identically zero initial data

$$(u_0,u_1) \in \left[H^2(\Omega)\times H_0^1(\Omega)\right] \cap H_0^1(\Omega),$$

the solution ϕ of

$$\begin{cases} \partial_t^2 \phi(t) - \Delta\phi(t) = 0, \\ \phi(0) = u_0,\ \partial_t\phi(0) = u_1, \end{cases} \tag{3.3}$$

satisfies

$$||(u_0,u_1)||^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C \int_0^{\frac{1}{\mathcal{G}\left(\frac{1}{2C\Lambda_r}\right)}} \int_\Omega a(x)|\partial_t\phi(x,t)|^2 dxdt, \tag{3.4}$$

where

$$\Lambda_r = \frac{(r-1)+X(u_0,u_1)}{E(u(0))}.$$

The following result is deduced from Theorem 1.1 and [8, Proposition 3].

Proposition 3.2. *Let* **(A)** *holds. There exists $c > 0$ such that for any $(u_0,u_1) \in [H^2(\Omega)\times H_0^1(\Omega)] \cap H_0^1(\Omega)$, the solution u of* (3.1) *satisfies*

$$E(u(s)) \leq ch((r-1)+X(u_0,u_1)) + c\int_s^{s+\frac{1}{G(h)}} \int_\Omega a(x)g(\partial_t u(x,t))\partial_t u(x,t)dxdt,$$

for any $h > 0$ and any $s \geq 0$ where

$$G(h) := Ch^{(2r+1)}\mathcal{F}(h)^{4(r+1)}.$$

We have the following stabilization result for the nonlinear damped wave equation.

Theorem 3.3. *Let* **(A)** *holds and suppose that there exists c_0 such that the function G satisfies $G^{-1}(x) \geq \frac{c}{c+1}G^{-1}\big(x(c_0+1)\big)$ for all $x \geq 0$. Then the energy of the solution of* (3.1) *satisfies the estimate:*

$$E(u(t)) \leq CG^{-1}\left(\frac{c'}{t}\right)\big((r-1)+X(u_0,u_1)\big), \quad \text{for } t \text{ sufficiently large}, \tag{3.5}$$

and all non-identically zero initial data $(u^0,u^1) \in [H^2(\Omega)\cap H_0^1(\Omega)] \times H_0^1(\Omega)$, the constant C depend on the initial data (u^0,u^1).

Proof. Choosing

$$h = \frac{1}{2C\Lambda_r},$$

this implies the existence of a constant $c > 0$ such that

$$E(u(s)) \le c \int_s^{s+\frac{1}{G\left(\frac{1}{2C\Lambda_r}\right)}} \int_\Omega a(x) g(\partial_t u(x,t)) \partial_t u(x,t) dx dt. \tag{3.6}$$

Denoting by $\mathcal{H}(s) = \frac{E(u,s)}{(r-1)+X(u_0,u_1)}$, we deduce from (3.6) that

$$\mathcal{H}\left(s + \frac{1}{G(\mathcal{H}(s))}\right) \le \mathcal{H}(s) \le c\left(\mathcal{H}(s) - \mathcal{H}\left(s + \frac{1}{G(\mathcal{H}(s))}\right)\right),$$

which gives

$$\mathcal{H}\left(s + \frac{1}{G(\mathcal{H}(s))}\right) \le \frac{c}{c+1}\mathcal{H}(s).$$

- If $c_0 s \le c\frac{1}{G(\mathcal{H}(s))}$, then $\mathcal{H}(s) \le G^{-1}\left(\frac{c}{c_0 s}\right)$ and

$$\mathcal{H}((1+c_0)s) \le \mathcal{H}(s) \le G^{-1}\left(\frac{c}{c_0 s}\right). \tag{3.7}$$

- If $c_0 s > c\frac{1}{G(\mathcal{H}(s))}$, then

$$\mathcal{H}((1+c_0)s) \le \mathcal{H}\left(s + \frac{1}{G(\mathcal{H}(s))}\right) \le \frac{c}{c+1}\mathcal{H}(s). \tag{3.8}$$

By induction, we deduce from (3.7) and (3.8) that $\forall s > 0$ and $\forall n \in N^*$,

$$\mathcal{H}((1+c_0)s) \le \max\left[G^{-1}\left(\frac{c}{c_0 s}\right), \frac{c}{c+1}G^{-1}\left(\frac{c(c_0+1)}{c_0 s}\right), \dots, \right.$$
$$\left. \left(\frac{c}{c+1}\right)^n G^{-1}\left(\frac{c(c_0+1)^n}{c_0 s}\right), \left(\frac{c}{c+1}\right)^{n+1} \mathcal{H}\left(\frac{s}{(c_0+1)^{n+1}}\right)\right].$$

Now, remark that with the above hypothesis on the function G,

$$\frac{c}{c+1}G^{-1}\left(\frac{c(c_0+1)}{c_0 s}\right) \le G^{-1}\left(\frac{c}{c_0 s}\right).$$

Consequently,

$$\begin{aligned} \mathcal{H}((1+c_0)s) &\le \max\left[G^{-1}\left(\frac{c}{c_0 s}\right), \left(\frac{c}{c+1}\right)^{n+1} \mathcal{H}\left(\frac{s}{(c_0+1)^{n+1}}\right)\right], \\ &\le \max\left[G^{-1}\left(\frac{c}{c_0 s}\right), \left(\frac{c}{c+1}\right)^{n+1}\right], \quad \forall n \ge 1, \end{aligned} \tag{3.9}$$

and we conclude that

$$\mathcal{H}(s) \leq G^{-1}\left(\frac{c(1+c_0)}{c_0 s}\right), \quad \forall s > 0. \qquad \square$$

Remark 3.4. For $\mathcal{G}(x) = x^p$, we have $\mathcal{F}(x) = x^{2p+1}$ and $G(x) = x^{(4p+3)(2r+1)-1}$. The energy of the solution of (3.1) satisfies the estimate:

$$E(u(t)) \leq \frac{c}{t^{\frac{1}{(4p+3)(2r+1)-1}}}\big((r-1) + X(u_0, u_1)\big), \quad \text{for } t \text{ sufficiently large.}$$

Remark 3.5. For the wave equation with arbitrary localized nonlinear damping, we obtain in [4] a weak observability (for the optimal weak observability estimate for the linear wave equation see [6]) which implies in particular the estimate (3.4) and the logarithmic decay of the energy. At the same time, this gives a geometry where the observability estimate (3.4) is satisfied and simplify the proof of the decay result in [5].

References

[1] Alabau-Boussouira, A. and Ammari, K., Sharp energy estimates for nonlinearly locally damped PDE's via observability for the associated undamped system. *J. Funct. Anal.* 260 (2011), 2424 – 2450.

[2] Ammari, K. and Tucsnak M., Stabilization of second order evolution equations by a class of unbounded feedbacks. *ESAIM Control Optim. Calc. Var.* 6 (2001), 361 – 386.

[3] Ammari, K. and Nicaise, S., *Stabilization of Elastic Systems by Collocated Feedback*. Lect. Notes Math. 2124, Cham: Springer 2015.

[4] Ammari, K., Bchatnia, A. and El Mufti, K., Stabilization of the nonlinear damped wave equation via linear weak observability. *NoDEA Nonlinear Diff. Equ. Appl.* 23 (2016), 18 pp.

[5] Bellassoued, M., Decay of solutions of the wave equation with arbitrary localized nonlinear damping. *J. Diff. Equ.* 211 (2005), 303 – 332.

[6] Laurent, C. and Léautaud, M., Quantitative unique continuation for operators with partially analytic coefficients. Application to approximate control for waves. Preprint 2015 (available at https://arXiv.org/abs/1506.04254).

[7] Phung, K. D., Polynomial decay rate for the dissipative wave equation. *J. Diff. Equ.* 1 (2007), 92 – 124.

[8] Phung, K. D., Decay of solutions of the wave equation with localized nonlinear damping and trapped rays. *Mathem. Control Relat. Fields* 1 (2011), 251 – 265.

Received January 13, 2016; revised August 25, 2016