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Gelfand Type Elliptic Problem Involving Advection

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Abstract. We consider the following Gelfand type elliptic problem involving advection

$$-\Delta u + a(x) \cdot \nabla u = e^u \quad \text{in } \mathbb{R}^N,$$

where $a(x)$ is a smooth vector field. According to energy estimates, we obtain the nonexistence results of stable solution for this equation under some restriction conditions about $a(x)$ for $N \leq 9$. On the other hand, combining Liapunov–Schmidt reduction method, we prove that it possesses a solution for $N \geq 4$. Besides, if a is divergence free and satisfies a smallness condition, then the equation above admits a stable solution for $N \geq 11$.

Keywords. Gelfand problem, stability, nonexistence, advection

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1. Introduction

We consider the elliptic problem

$$-\Delta u + a(x) \cdot \nabla u = e^u \quad \text{in } \mathbb{R}^N, \tag{1}$$

where $a(x)$ is a smooth vector field satisfying some decay condition. If $a(x) \equiv 0$, this problem reduces to

$$-\Delta u = e^u \quad \text{in } \mathbb{R}^N, \tag{2}$$

which called Gelfand problem or Liouville problem. In dimension $N = 1, 2, 3$, Equation (2) can be derived from the thermal self-ignition model [11]. Besides, it also describes the diffusion phenomena induced by nonlinear sources [13] or a ball of isothermal gas in gravitational equilibrium as proposed by lord Kelvin [3]. Equation (2) has been considered by various authors, see [2,7,10,12,16]. One of

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the focus about Equation (2) is the classification of solution. Indeed, for every $N \geq 10$, Equation (2) admits a radial stable solution in the sense that

$$\int_{\mathbb{R}^N} e^u \phi^2 dx \leq \int_{\mathbb{R}^N} |\nabla \phi|^2 dx \quad \forall\, \phi \in C_0^\infty(\mathbb{R}^N),$$

and for $N \leq 9$, there is no stable C^2 solution of Equation (2), for details see [7,10].

Recently, elliptic problems with advection have attracted the interesting of many researchers. In particular the problem of the form

$$\begin{cases} -\Delta u + a(x) \cdot \nabla u = \lambda f(u), & \text{in } \Omega, \\ \qquad\qquad\qquad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3}$$

has been studied for various nonlinearities f. Berestycki et al. [1] used this equation to study the explosion phenomena in a flow, if $\operatorname{div} a(x) = 0$, then $a(x)$ denotes a prescribed incompressible flow. In [6,14] the authors used the generalized Hardy inequality from [4] to consider the regularity of the extremal solution of (3). The first purpose of the present paper is to consider nonexistence of stable solution of (1). In order to state our results, we first define the notion of stability as follows:

Definition 1.1. A smooth solution u of (1) is said to be stable if the principle eigenvalue of the linearized operator $-\Delta + a \cdot \nabla - e^u$ is nonnegative in $C_0^\infty(\mathbb{R}^N)$, i.e.,

$$\inf_{\varphi \in C_0^\infty(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} \left(|\nabla \varphi|^2 - e^u \varphi^2 \right) dx}{\|\varphi\|_{L^2(\mathbb{R}^N)}} \geq 0.$$

Obviously, if u is a stable solution of Equation (1), then there is some smooth positive function E such that

$$-\Delta E + a \cdot \nabla E - e^u E \geq 0 \quad \text{in } \mathbb{R}^N. \tag{4}$$

Very recently, Cowan in [5] considered the existence of stable solution of the following equation

$$\begin{cases} -\Delta u + a(x) \cdot \nabla u = u^p, & \text{in } \Omega, \\ \qquad\qquad\qquad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

and obtained

(i) Suppose $3 \leq N \leq 10$ or $N \geq 11$ and $1 < p < p_c$ with some positive critical p_c, and suppose $a(x)$ is a smooth divergence free vector field, then Equation (5) admits no positive stable solution for some small $a(x)$.

(ii) Suppose $N \geq 11, p > p_c$ and $a(x)$ is a smooth divergence free vector field, then Equation(5) has a positive stable solution for some small $a(x)$.

(iii) Suppose $N \geq 4, p > \frac{N+1}{N-3}$ and $a(x)$ is a smooth divergence free vector field, then there exists a positive solution of Equation (5) for some small $a(x)$.

In the present paper, using the techniques from [5, 7, 10], we extend the above results to Equation (5) and obtain

Theorem 1.2. *Suppose that $a(x)$ is a smooth divergence free vector (i.e., $\operatorname{div} a(x) = 0$) and satisfies*

$$|a(x)| \leq \frac{\theta}{1+|x|} \quad \text{with } 0 < \theta \text{ sufficiently small.} \tag{6}$$

Then there is no stable solution of Equation (1) for $N \leq 9$.

The second purpose of this paper is to use the approach developed in [8,9] to obtain the existence of the solution of Equation (1). Before stating our results, we sketch the ideas of the arguments. Indeed, because of smallness condition on advection a, we use the perturbation analysis (for details see [8, 15]) to look for a solution u close to a radial solution w of (2) with $w(r) = -\log \frac{2(N-2)}{r^2} + o(1)$ as $r \to +\infty$ and $w(0) = 0$. Precisely, we will look a solution of the form $u = w + \phi$, where ϕ is a lower order correction. This approach is valid, since the linearized operator $-\Delta + e^w$ is inverse in a weighted L^∞ space, for details see [9], and then we can use a fixed point argument to solve ϕ, which yields the desired solution.

Theorem 1.3. (i) *Suppose $a(x)$ satisfies (6), then Equation (1), for $N \geq 4$, admits a solution u such that*

$$u(x) = \log \frac{2(N-2)}{|x|^2} + O(1) \quad \text{as } |x| \to \infty.$$

(ii) *Suppose $a(x)$ satisfies $\operatorname{div} a(x) = 0$ and (6), then Equation (1) has a stable solution u for $N \geq 11$.*

The rest of the paper is organized as follows. In the next section we give the proof of Theorem 1.2. In the third section , we will consider the existence of solution and devote to proving Theorem 1.3.

2. Proof of Theorem 1.2

We now give the following generalized Hardy inequality from [4] as follows.

Lemma 2.1. *Suppose E is a smooth positive function on Ω and fix a constant β with $\frac{1}{2} \leq \beta \leq 1$. Then, for all $\phi \in C_0^\infty(\Omega)$ we have*

$$\beta \int_\Omega \frac{-\Delta E}{E} \phi^2 dx + (\beta - \beta^2) \int_\Omega \frac{|\nabla E|^2}{E^2} \phi^2 dx \leq \int_\Omega |\nabla \phi|^2 dx. \tag{7}$$

Combining the techniques developed in [5,7,10] with the above generalized Hardy inequality, we obtain the following energy estimate, which is a key starting point for our proof of Theorem 1.2. In the rest of this paper, by $\int \cdots$ we always denote $\int_{\mathbb{R}^N} \cdots dx$.

Proposition 2.2. *Suppose u is a smooth stable solution of Equation* (1) *and let $a(x)$ be smooth vector field such that* $\operatorname{div} a(x) = 0$*. Then for any $t \in (0, 2)$ and $0 \le \psi \in C_0^\infty(\mathbb{R}^N)$ we have*

$$\int e^{(2t+1)u}\psi^2 \le C(t) \int e^{2tu}(|\nabla\psi|^2 + |\Delta\psi^2|). \tag{8}$$

Proof. Step 1. For any $t, \delta > 0$ we claim:

$$\begin{aligned}&\int e^{2tu}|\nabla u|^2\psi^2\\ &\le \frac{\delta}{4t^3}\int a^2\psi^2 e^{2tu} + \frac{1}{\delta t}\int |\nabla\psi|^2 e^{2tu} + \frac{1}{2t}\int e^{(2t+1)u}\psi^2 + \frac{1}{4t^2}\int e^{2tu}\Delta\psi^2,\end{aligned} \tag{9}$$

where $0 \le \psi \in C_0^\infty(\mathbb{R}^N)$. Indeed, multiplying Equation (1) by $e^{2tu}\psi^2$ and integrating by parts, we see

$$2t\int e^{2tu}|\nabla u|^2\psi^2 = -\frac{1}{2t}\int \psi^2 a(x)\cdot\nabla e^{2tu} + \int e^{(2t+1)u}\psi^2 + \frac{1}{2t}\int e^{2tu}\Delta\psi^2. \tag{10}$$

Since $\operatorname{div} a(x) = 0$, an integration by parts again and an application of Young's inequality with δ, we have

$$\int \psi^2 a(x)\cdot\nabla e^{2tu} = 2\int \psi e^{2tu} a(x)\cdot\nabla\psi \le \frac{\delta}{t}\int a^2\psi^2 e^{2tu} + \frac{4t}{\delta}\int |\nabla\psi|^2 e^{2tu}.$$

Putting this equality into (10), we obtain the desired results.

Step 2. End of the Proof. Using the generalized Hardy inequality (7) and the fact E satisfies (4), we have, taking $\phi = e^{tu}\psi$,

$$\begin{aligned}&\beta\int e^{(2t+1)u}\psi^2 + (\beta-\beta^2)\int \frac{|\nabla E|^2}{E^2}e^{2tu}\psi^2 + (T-1)\int |\nabla(e^{tu}\psi)|^2\\ &\le Tt^2\int e^{2tu}\psi^2|\nabla u|^2 + \frac{T}{2}\int e^{2tu}\Delta\psi^2 + T\int e^{2tu}|\nabla\psi|^2 + \beta\int \frac{a\cdot\nabla E}{E}e^{2tu}\psi^2\end{aligned}$$

for some $T > 1$. Since by the application of Young's inequality with ϵ

$$\beta\int \frac{a\cdot\nabla E}{E}e^{2tu}\psi^2 \le \beta\epsilon\int \frac{|\nabla E|^2}{E^2}e^{2tu}\psi^2 + \frac{\beta}{4\epsilon}\int a^2 e^{2tu}\psi^2,$$

then by using (9), we see

$$\begin{aligned}&\left(\beta-\frac{Tt}{2}\right)\int e^{(2t+1)u}\psi^2+\beta(1-\beta-\epsilon)\int\frac{|\nabla E|^2}{E^2}e^{2tu}\psi^2+(T-1)\int|\nabla(e^{tu}\psi)|^2\\&\le\left(\frac{T\delta}{4t}+\frac{\beta}{4\epsilon}\right)\int a^2e^{2tu}\psi^2+T\left(1+\frac{t}{4\delta}\right)\int e^{2tu}|\nabla\psi|^2+\frac{3T}{4t}\int e^{2tu}\Delta\psi^2.\end{aligned}\tag{11}$$

We also note that, by Hardy's inequality,

$$\int a^2e^{2tu}\psi^2\le\theta^2\int\frac{e^{2tu}\psi^2}{|x|^2}\le\frac{4\theta^2}{(N-2)^2}\int|\nabla(e^{tu}\psi)|^2.$$

Putting this into (11) gives

$$\begin{aligned}&\left(\beta-\frac{Tt}{2}\right)\int e^{(2t+1)u}\psi^2+\beta(1-\beta-\epsilon)\int\frac{|\nabla E|^2}{E^2}e^{2tu}\psi^2+C_1\int|\nabla(e^{tu}\psi)|^2\\&\le C_2\int e^{2tu}(|\nabla\psi|^2+|\Delta\psi^2|),\end{aligned}$$

where

$$C_1=(T-1)-\frac{4\theta^2}{(N-2)^2}\left(\frac{T\delta}{4t}+\frac{\beta}{4\epsilon}\right),\quad C_2=\max\left\{\frac{T\delta}{4t}+\frac{\beta}{4\epsilon},\frac{3T}{4}\right\}.$$

Now for fixed $t\in(0,2)$, we first choose $0<\beta<1, T>1$ sufficiently close to 1 such that

$$\beta-\frac{Tt}{2}>0,$$

and then we pick $\epsilon>0$ small enough such that $1-\beta-\epsilon>0$, and finally we choose $\theta>0$ sufficiently small such that $C_1\ge 0$. Then we get the desired estimate and complete our proof. □

Proof of Theorem 1.2. Suppose to contrary that Equation (1) admits a stable solution for $N\le 9$. Fix $t\in(0,2)$ such that $N-2(2t+1)<0$, we will obtain a contradiction by proving $\int e^{(2t+1)u}=0$. We now consider the function $\phi\in C_0^\infty(\mathbb{R}^N)$ such that

$$\phi(x)=\begin{cases}1 & \text{for } |x|\le R\\ 0 & \text{for } |x|\ge 2R,\end{cases}$$

and $|\nabla\phi|\le\frac{C}{R}, |\Delta\phi|\le\frac{C}{R^2}$, where C is independent of R. Putting $\psi=\phi^m$ (m is large integer) into (8) gives

$$\begin{aligned}\int e^{(2t+1)u}\phi^{2m}&\le C_m\int e^{2tu}\phi^{2m-2}(|\nabla\phi|^2+|\Delta\phi|)\\&\le C\left(\int e^{(2t+1)u}\phi^{\frac{(m-1)(2t+1)}{t}}\right)^{\frac{2t}{2t+1}}\cdot\left(\int(|\nabla\phi|^2+|\Delta\phi|)^{2t+1}\right)^{\frac{1}{2t+1}}.\end{aligned}$$

Since $\frac{(m-1)(2t+1)}{t} \geq 2m$ for sufficiently large m, we have

$$\int_{B_R} e^{(2t+1)u} \leq \int e^{(2t+1)u}\phi^{2m} \leq C\int(|\nabla\phi|^2 + |\Delta\phi|)^{2t+1} \leq CR^{N-2(2t+1)} \to 0,$$

as $R \to \infty$, which is a contradiction to fact that $e^{(2t+1)u} > 0$. Then the proof of Theorem 1.2 is complete. □

3. Proof of Theorem 1.3

To fix ideas, we consider the radial solution of the problem

$$-\Delta u = e^u, \quad \text{in } \mathbb{R}^N. \tag{12}$$

It is well known that Equation (12) possess a positive radially symmetric solution $w(r)$ with $w(0) = 0$ for $N \geq 1$ and the asymptotic of w as $r \to \infty$ is given as follows (for details see [9]):

- if $3 \leq N \leq 9, w(r) = \log\frac{2(N-2)}{r^2} + O(r^{-\frac{N-2}{2}})$;
- if $N = 10$, there exist $a \in \mathbb{R}$ and $b < 0$ such that

$$w(r) = \log\frac{2(N-2)}{r^2} + ar^{-4} + br^{-4}\log r + o(r^{-4}\log r);$$

- If $N > 10$, there exist $a \in \mathbb{R}$ and $b < 0$ such that

$$w(r) = \log\frac{2(N-2)}{r^2} + ar^{m_1} + br^{m_2}\log r + o(r^{m_2}\log r),$$

  where

$$\begin{aligned} m_1 &= -\frac{N-2+\sqrt{(N-2)(N-10)}}{2} < 0, \\ m_2 &= \frac{\sqrt{(N-2)(N-10)} - (N-2)}{2} < 0. \end{aligned} \tag{13}$$

Besides, by simple calculation, $\log\frac{2(N-2)}{r^2}$ is a singular solution of Equation(12).

Since we impose a smallness condition on a, then $a(x) \cdot \nabla u$ in Equation(1) be considered as a small perturbation of term. Following the idea of [8, 9], we consider $w(r)$ as a first approximation for a solution of Equation(1), i.e., we look for a solution to Equation (1) of the form $u = w + \phi$, where ϕ is "small " compared to w at infinity. We will use a fixed point argument to find ϕ in the weighted L^∞ space. In order to use the fixed point theory, we first consider the linear equation in a suitable weighted L^∞ space

$$-\Delta\phi + e^w\phi = f, \quad \text{in } \mathbb{R}^N. \tag{14}$$

For given $1 < \sigma < 2$ and

$$0 < \beta < \begin{cases} 1, & 4 \leq N \leq 9, \\ \min\{-m_1, 1\}, & 10 \leq N, \end{cases} \tag{15}$$

where m_1 is defined as in (13), we define

$$\|\phi\|_{\tilde{X}} := \sup_{|x|\leq 1} |\phi| + \sup_{|x|\geq 1} |x|^\beta |\phi| \quad \text{and} \quad \|f\|_Y := \sup_{|x|\leq 1} |x|^\sigma |f| + \sup_{|x|\geq 1} |x|^{2+\beta} |f|.$$

Let $\tilde{X}$ and Y denote the completion of $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ under the appropriate norms. In the following, the authors (see [9]) give solvability of Equation (14) and estimates for the solution in the weighted L^∞ norm given as above.

Lemma 3.1. *Suppose $N \geq 4$, for any $f \in Y$, there exists a solution of Equation* (14)

$$\phi = T(f),$$

which defines a linear operator of f such that

$$\|\phi\|_{\tilde{X}} \leq C \|f\|_Y, \tag{16}$$

where C is a fixed constant independent of ϕ, f.

However, our problem involves advection term, we don't work directly with $\tilde{X}$, and need some revisions for norm $\|\cdot\|_X$. And so we define the norm

$$\|\phi\|_X := \sup_{|x|\leq 1} (|\phi| + |x||\nabla\phi|) + \sup_{|x|\geq 1} (|x|^\beta |\phi| + |x|^{\beta+1} |\nabla\phi|),$$

and let X denote the completion of $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ with respect to this norm.

Lemma 3.2. *Suppose $N \geq 4$, for any $f \in Y$, there exists a solution of Equation* (14)

$$\phi = T(f),$$

such that

$$\|\phi\|_X \leq C \|f\|_Y, \tag{17}$$

where C is a fixed constant independent of ϕ, f.

Proof. Suppose $f \in Y$ and let $\phi \in X$ be such that

$$-\Delta\phi - e^w \phi = f.$$

Now define the re-scaled functions $\phi_m(x) = \phi(x_m + r_m x)$, where $|x_m| > 0$, $r_m = \frac{|x_m|}{4}$. Then $\phi_m(x)$ satisfies

$$-\Delta\phi_m(x) = r_m^2 e^w \phi_m(x) + r_m^2 f(x_m + r_m x) =: g_m(x),$$

for $|x| \le 1$. According to Lemma 3.1, we have

$$\|\phi\|_{L^\infty(B_1(0))} + \sup_{|x|\ge 1} |x|^\beta |\phi| \le C\|f\|_Y.$$

Then if $|x_m| \ge 1$, we see that

$$\begin{aligned} |x_m|^\beta |g_m| = & \, r_m^2 e^w \frac{|x_m|^\beta}{|x_m + r_m x|^\beta} \cdot |x_m + r_m x|^\beta |\phi(x_m + r_m x)| \\ & + \frac{|x_m|^{\beta+2}}{|x_m + r_m x|^{\beta+2}} \cdot |x_m + r_m x|^{\beta+2} |f(x_m + r_m x)| \\ & \le C\|f\|_Y, \end{aligned}$$

for all $|x| \le 1$. Then by the elliptic estimates, for $p > N$ there is some C such that

$$\begin{aligned} |\nabla\varphi(x_m) \cdot r_m| &= |\nabla\varphi_m(0)| \\ &\le C \left(\int_{|x|\le 1} |g_m(x)|^p dx \right)^{\frac{1}{p}} + \int_{|x|\le 1} |\varphi_m(x)| dx \\ &\le C\|f\|_Y \cdot |x_m|^{-\beta}, \end{aligned}$$

here we have used the fact that $|\varphi_m(x)| \le C\|f\|_Y \cdot |x_m|^{-\beta}$ for $|x| \le 1$. And then we have for $|x_m| > 1$

$$|x_m|^{\beta+1} |\nabla\varphi(x_m)| \le C\|f\|_Y. \tag{18}$$

By the same argument as above, we see that

$$|x_m||\nabla\varphi(x_m)| \le C\|f\|_Y \quad \text{for } |x_m| \le 1 \tag{19}$$

From (18) and (19), we immediately have

$$\|\phi\|_X \le C\|f\|_Y,$$

where C is independent of f and ϕ. □

Combining Lemma 3.2, we are in a position to give the proof of Theorem 1.3(i) as follows.

Proof of Theorem 1.3(i). We look for a solution to problem (1) of the form $u = w + \phi$, which yields the following equation for ϕ

$$-\Delta\phi - e^w \phi = -a \cdot \nabla\phi - a \cdot \nabla w + e^w(e^\phi - 1 - \phi).$$

Letting T be defined as in Lemma 3.1, we are looking for a $\phi \in X$ such that

$$\phi = -T(a \cdot \nabla w) - T(a \cdot \nabla\phi) + T(e^w(e^\phi - 1 - \phi)) =: J(\phi).$$

We will use fixed point argument to find such a ϕ. For small $\rho > 0$, we define

$$F := \{\phi : \mathbb{R}^N \to \mathbb{R} \mid \|\phi\|_X \leq \rho\}.$$

We will prove that choosing $\rho > 0$ small enough, J has a fixed point in F.

Step 1. For any $\phi \in F$ and small $\rho > 0$, we have $J(\phi) \in F$. Indeed, from Lemma 3.2, there is some $C > 0$ such that

$$\|J(\phi)\|_X \leq C\|a \cdot \nabla w\|_Y + C\|a \cdot \nabla \phi\|_Y + C\|e^w(e^\phi - 1 - \phi)\|_Y,$$

we now estimate each term on the right hand side of the above inequality as follows

$$\begin{aligned}\|a \cdot \nabla w\|_Y &\leq \sup_x |a| \sup_{|x|\leq 1} |\nabla w||x|^\sigma + \sup_x |x||a| \cdot \sup_{|x|\geq 1} |x|^{\beta+1}|\nabla w(x)| \\ &\leq \sup_x((1+|x|)|a|) \cdot \|w\|_X,\end{aligned}$$

where we have used the fact that $1 \leq \sigma < 2$. By the same argument, we have $\|a \cdot \nabla \phi\|_Y \leq \sup_x((1+|x|)|a|)\|\phi\|_X$. On the other hand, by the identity $e^x = 1 + x + \int_0^x e^t(x-t)dt, \forall\, t \in \mathbb{R}$, we have

$$|e^w(e^\phi - 1 - \phi)| \leq Ce^w \cdot \phi^2 \cdot e^{|\phi|}.$$

Additionally,

$$\begin{aligned}&\sup_{|x|\leq 1} |x|^\sigma e^w \phi^2 e^{|\phi|} + \sup_{|x|\geq 1} |x|^{2+\beta} e^w \phi^2 e^{|\phi|} \\ &\leq C\|\phi\|_X^2 e^{\|\phi\|_X} + C \sup_{|x|\geq 1} |x|^2 e^w \cdot \sup_{|x|\geq 1} (|x|^\beta \phi)^2 \cdot e^{\|\phi\|_X} \\ &\leq C\|\phi\|_X^2 \cdot e^{\|\phi\|_X} \\ &\leq C\rho^2 e^\rho.\end{aligned}$$

If a satisfies $C\sup_x(1+|x|)|a| + \sup_x((1+|x|)|a|)\rho \leq \frac{\rho}{10}$ and $\rho > 0$ is fixed suitable small, we arrive at

$$\|J(\phi)\|_X \leq C\sup_x(1+|x|)|a| + \sup_x((1+|x|)|a|)\rho + C\rho^2 e^\rho \leq \rho,$$

This proves $J(F) \subseteq F$.

Step 2. J is a contraction mapping on F for a suitable ρ. Now let us take $\phi_1, \phi_2 \in F$, Then

$$\begin{aligned}\|J(\phi_1) - J(\phi_2)\|_X &\leq C\|a \cdot \nabla(\phi_1 - \phi_2)\|_Y + C\|e^w(e^{\phi_1} - e^{\phi_1} - (\phi_1 - \phi_2)\|_Y \\ &\leq C\|a \cdot \nabla(\phi_1 - \phi_2)\|_Y + C\|e^w(e^{\bar{\phi}} - 1)(\phi_1 - \phi_2)\|_Y \\ &=: I_1 + I_2,\end{aligned}$$

where $\bar{\phi} \in (\phi_1, \phi_2) \cup (\phi_2, \phi_1)$. One easily sees that

$$I_1 \le C \sup_x (1+|x|)|a| \cdot \|\phi_1 - \phi_2\|_X. \tag{20}$$

Since $|e^w(e^{\bar{\phi}} - 1)(\phi_1 - \phi_2)| \le Ce^w|\bar{\phi}|e^{\bar{\phi}}|\phi_1 - \phi_2|$, we have

$$\begin{aligned} I_2 &\le C \sup_{|x|\le 1} |x|^\sigma e^w |\bar{\phi}| e^{\bar{\phi}} |\phi_1 - \phi_2| + C \sup_{|x|\ge 1} |x|^{2+\beta} e^w |\bar{\phi}| e^{\bar{\phi}} |\phi_1 - \phi_2| \\ &\le C\rho e^\rho \|\phi_1 - \phi_2\|_X + Ce^{\|e^{\bar{\phi}}\|_X} \cdot \sup_{|x|\ge 1} |x|^2 e^w \cdot \sup_{|x|\ge 1} |x|^\beta |\bar{\phi}| \cdot \sup_{|x|\ge 1} |x|^\beta |\phi_1 - \phi_2| \\ &\le C\rho e^\rho \|\phi_1 - \phi_2\|_X. \end{aligned} \tag{21}$$

Combining (20) with (21), we obtain

$$\|J(\phi_1) - J(\phi_2)\|_X \le [C \sup_x (1+|x|)|a| + \rho e^\rho] \cdot \|\phi_1 - \phi_2\|.$$

Now we let a be such that $C \sup_x (1+|x|)|a| < \frac{1}{2}$ and fix ρ sufficiently small such that $C\rho e^\rho < \frac{1}{2}$, we have J is a contraction mapping on F in X, and hence it has a unique fixed point in this set, i.e., there is $\phi \in F$ such that

$$-\Delta(w+\phi) + a \cdot \nabla(w+\phi) = e^{w+\phi}. \qquad \square$$

In order to complete the details for the proof of Theorem 1.3(ii), we need to show the following

Lemma 3.3. *Let $N \ge 10$, then*

$$w(r) \le \log \frac{2(N-2)}{r^2}, \quad \forall\, r \in (0, +\infty).$$

Proof. It is worth to mentioning that the semilinear equation with power-type nonlinearity possess the similar result, see [16, Proposition 3.7]. Using their arguments, we can prove this Lemma. Here, we provide a simple way to prove this result.

Let $s = \log r, v(s) = w(r) + 2s - \log 2(N-2)$, then $v(s)$ satisfies

$$-(v''(s) + (N-2)v'(s)) = 2(N-2)(e^v - 1) \ge 2(N-2)(v-1),$$

then the above ODE can be factorized as follows

$$(\partial_s - m_1)(\partial_s - m_2)v(s) \le 0, \tag{22}$$

where m_1, m_2 is defined as in (13). By the definition of v, we have

$$\lim_{s\to-\infty} e^{-m_1 s} v^{(i)}(s) = 0, \quad \lim_{s\to-\infty} e^{-m_2 s} v^{(i)}(s) = 0, \quad i = 0, 1. \tag{23}$$

Multiplying (22) by $e^{-m_1 s}$ and integrating over $(-\infty, s)$, we get, by (23)

$$(\partial_s - m_2)v(s) \le 0,$$

i.e., $(e^{-m_2 s}v(s))' \le 0$. Using (23) again, we obtain $v(s) \le 0$, which is the desired result. $\square$

Proof Theorem 1.3(ii). Let $m \geq 2$ be an integer, B_m be a ball with radius m and $E = E_{m,\rho}$ satisfies

$$\begin{cases} -\Delta E + a \cdot \nabla E - e^{u_\rho} E = \mu_{m,\rho} E & \text{in } B_m \\ \qquad\qquad\qquad\qquad E = 0 & \text{on } \partial B_m, \end{cases} \tag{24}$$

where $E_{m,\rho}, \mu_{m,\rho}$ denote the first eigenfunction, first eigenvalue respectively and

$$u_\rho = w + \phi \quad \text{such that } \|\phi\|_X \leq \rho.$$

Multiply the above equation by E and integrate over B_m, we have

$$\mu_{m,\rho} = \frac{\int_{B_m} (|\nabla E|^2 - e^{u_\rho} E^2) dx}{\int_{B_m} E^2 dx}.$$

Step 1. We claim that $\mu_{m,\rho} \geq 0$ for suitably small ρ and $\lim_{m\to\infty} \mu_{m,\rho} = 0$. Indeed, for $r > 0$, there is a fixed $\epsilon > 0$ such that

$$(1+\epsilon) e^w \leq \frac{(N-2)^2}{4r^2},$$

and from Hardy's inequality and Lemma 3.3, we see that

$$\int_{B_m} |\nabla E|^2 dx \geq (1+\epsilon) \int_{B_m} e^w E^2 dx.$$

On the other hand, $u_\rho = w + \phi$ and $\|\phi\|_X \leq \rho$, we have

$$\int_{B_m} e^{u_\rho} E^2 dx = \int e^w \cdot e^\phi E^2 dx \leq e^\rho \int_{B_m} e^w E^2 dx.$$

From these facts, we see, by taking ρ small enough,

$$\mu_{m,\rho} \geq \frac{(1 + \epsilon - e^\rho) \int_{B_m} e^w E^2 dx}{\int_{B_m} E^2 dx} \geq 0.$$

Now we remain to prove $\lim_{m\to\infty} \mu_{m,\rho} \to 0$. Now let $\psi \in C_0^\infty(B_m)$ be such that

$$\psi = 1, \quad |x| \leq \frac{m}{2}, \quad |\nabla \psi| \leq \frac{C}{m},$$

where C is independent of m. Putting $E = E_{m,\rho}$ into (7) with $\beta = \frac{1}{2}$, we have, for all $\psi \in C_0^\infty(B_m)$,

$$\begin{aligned} &\int_{B_m} e^{u_\rho} \psi^2 dx + \mu_m \int_{B_m} \psi^2 dx + \frac{1}{2} \int_{B_m} \frac{|\nabla E|^2}{E^2} \psi^2 dx \\ &\leq 2 \int_{B_m} |\nabla \psi|^2 dx + \int_{B_m} \frac{a \cdot \nabla E}{E} \psi^2 dx \\ &\leq 2 \int_{B_m} |\nabla \psi|^2 dx + \epsilon \int_{B_m} \frac{|\nabla E|^2}{E^2} \psi^2 dx + \frac{1}{4\epsilon} \int_{B_m} |a|^2 \psi^2 dx \\ &\leq C(\epsilon) \int_{B_m} |\nabla \psi|^2 dx + \epsilon \int_{B_m} \frac{|\nabla E|^2}{E^2} \psi^2 dx. \end{aligned}$$

For the last inequality, we have used Hardy's inequality and the fact $|a|^2 \le \frac{\theta^2}{(1+|x|)^2}$ with θ small. Now taking $0 < \epsilon < \frac{1}{2}$, we see that

$$0 \le \mu_m \le C(\epsilon)\frac{\int_{B_m} |\nabla\psi|^2 dx}{\int_{B_m} \psi^2 dx} \to 0, \quad \text{as } m \to \infty.$$

Step 2. End of the proof. Fix $\rho > 0$, we, by suitably scaling E_m, can assume $E_m(0) = 1$. Obviously, $E_m, \mu_{m,\rho}$ satisfy

$$\begin{cases} -\Delta E_m + a \cdot \nabla E_m - e^{u_\rho} E_m = \mu_{m,\rho} E_m, & \text{in } B_m \\ E_m = 0 & \text{on } \partial B_m, \end{cases} \tag{25}$$

Now fix $k \ge 0$ and let $m \ge k+2$, by Harnack's inequality there is a $C_k > 0$ such that

$$\sup_{B_k} E_m \le C_k \inf_{B_k} E_m \le C_k,$$

for all $m \ge k+2$. Using the elliptic regularity and a diagonal argument,

$$E_m \to E \ge 0 \quad \text{in } C^{1,\beta}_{loc}(\mathbb{R}^N) \quad \text{as } m \to \infty,$$

for some $\beta > 0$ and $E(0) = 1$. And E satisfies

$$-\Delta E + a(x) \cdot \nabla E = e^{u_\rho} E \quad \text{in } \mathbb{R}^N.$$

By the strong maximum principle, one see that $E > 0$. This, by (4), shows that u_ρ is a stable solution of (1) which is the desired result. $\square$

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