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Nonlinear Dirichlet Problems with no Growth Restriction on the Reaction

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Abstract. We consider nonlinear Dirichlet problems driven by the sum of a p-Laplacian and a Laplacian and with a Carathéodory reaction which does not satisfy any global growth condition. Instead we assume that it has constant sign z-dependent zeros. Using variational methods, truncation techniques and Morse theory, we prove multiplicity theorems providing sign information for all the solutions.

Keywords. Nonlinear regularity, nonlinear maximum principle, constant sign and nodal solutions, $(p,2)$-equation, critical groups

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1. Introduction

In this paper we study the following nonlinear Dirichlet problem:

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $2 < p < +\infty$. In this problem $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. Also Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \forall u \in W_0^{1,p}(\Omega).$$

When $p = 2$, we have the usual Laplacian denoted by Δ. The reaction f is a Carathéodory function (i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is

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measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous). The interesting feature of our work here, is that we do not impose any global growth condition on $f(z,\cdot)$. Instead, we assume that $f(z,\cdot)$ admits z-dependent constant sign zeros. Hence the reaction $f(z,\cdot)$ exhibits near zero a kind of oscillatory behaviour. No condition is imposed on $f(\cdot,\zeta)$ for large values of $|\zeta|$.

Elliptic problems with a reaction having zeros, were investigated by Iturriaga–Massa–Sánchez–Ubilla [28], who considered parametric Dirichlet problems driven by the p-Laplacian and looked for positive solutions. They investigated how the positive solution depends on the parameter. There is also the recent work of Gasiński–Papageorgiou [24], who considered periodic problems driven by a nonhomogeneous differential operator and with a reaction that has zeros. They prove multiplicity results producing three solutions. We mention that equations driven by the sum of a p-Laplacian and a Laplacian arise in problems of mathematical physics. We infer the works of Benci–D'Avenia–Fortunato–Pisani [3] and Cherfils–Il′yasov [7]. Recently existence and multiplicity results for such equations were obtained by Cingolani–Degiovanni [8], Gasiński–Papageorgiou [18, 26], Mugnai–Papageorgiou [32], Papageorgiou–Rădulescu [35], Papageorgiou–Smyrlis [36] and Sun [39]. We also mention papers with a more general notion of the so-called (p,q)-Laplacian, namely Gasiński–Papageorgiou [21, 22] or its generalizations in Gasiński–Klimczak–Papageorgiou [12], Gasiński–O'Regan–Papageorgiou [13, 14] and Gasiński–Papageorgiou [19, 25].

In this paper, using variational methods based on the critical point theory together with truncation techniques and Morse theory, we prove multiplicity results providing sign information for all the solutions.

In the next section, for the convenience of the reader, we recall the main mathematical tools, which will be used in the sequel.

2. Mathematical background

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Given $\varphi \in C^1(X)$, we say that φ satisfies the *Palais–Smale condition*, if the following property holds:

> "Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and
> $$\varphi'(u_n) \longrightarrow 0 \quad \text{in } X^*,$$
> admits a strongly convergent subsequence."

This is a compactness type condition on the functional φ, which compensates for the fact that the ambient space X need not be locally compact (the space X is in general infinite dimensional). This condition on φ, leads to a deformation theorem from which one can derive the minimax theory for the critical values of φ. Prominent in that theory is the so-called "mountain pass theorem" due to Ambrosetti–Rabinowitz [1].

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the Palais–Smale condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \varrho > 0$,*

$$\max\big\{\varphi(u_0), \varphi(u_1)\big\} < \inf\big\{\varphi(u) : \ \|u - u_0\| = \varrho\big\} = m_\varrho,$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi\big(\gamma(t)\big),$$

where

$$\Gamma = \big\{\gamma \in C\big([0,1]; X\big) : \ \gamma(0) = u_0, \ \gamma(1) = u_1\big\},$$

then $c \geqslant m_\varrho$ and c is a critical value of φ.

In the analysis of problem (1), we will use the Sobolev spaces $W_0^{1,p}(\Omega)$, $H_0^1(\Omega)$ and the Banach space

$$C_0^1(\overline{\Omega}) = \big\{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\big\}.$$

The latter is an ordered Banach space with positive cone

$$C_+ = \big\{u \in C_0^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\big\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \big\{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}\big|_{\partial\Omega} < 0\big\}.$$

Here by $n(\cdot)$ we denote the outward unit normal on $\partial\Omega$.

Let $f_0 \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying

$$|f_0(z, \zeta)| \leqslant a_0(z)(1 + |\zeta|^{r-1}) \quad \text{for a.a. } z \in \Omega, \ \text{all } \zeta \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$, where

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } \ p < N, \\ +\infty & \text{if } \ N \leqslant p. \end{cases}$$

We set

$$F_0(z, \zeta) = \int_0^\zeta f_0(z, s)\, ds$$

and consider the C^1-functional $\varphi_0 \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega F_0(z, u(z)) \quad \forall u \in W_0^{1,p}(\Omega)$$

$(2 < p < +\infty)$.

The next result is a special case of a more general result of Gasiński–Papageorgiou [17, Proposition 2.6, p. 423]. The first result of this nature was proved by Brézis–Nirenberg [5].

Proposition 2.2. *If $\varphi_0 \in C^1(W_0^{1,p}(\Omega))$ is as above and $\overline{u} \in W_0^{1,p}(\Omega)$ is a local $C_0^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\varrho_0 > 0$ such that*

$$\varphi_0(\overline{u}) \leqslant \varphi_0(\overline{u} + h) \quad \forall h \in C_0^1(\overline{\Omega}),\ \|h\|_{C_0^1(\overline{\Omega})} \leqslant \varrho_0,$$

then $\overline{u} \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and $\overline{u}$ is also a local $W_0^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\varrho_1 > 0$ such that

$$\varphi_0(\overline{u}) \leqslant \varphi_0(\overline{u} + h) \quad \forall h \in W_0^{1,p}(\Omega),\ \|h\| \leqslant \varrho_1.$$

Hereafter, by $\|\cdot\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$\|u\| = \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

Let $h, \widehat{h} \in L^\infty(\Omega)$. We say that $h \prec \widehat{h}$ if and only if for every compact $K \subseteq \Omega$, we can find $\varepsilon = \varepsilon_K > 0$ such that

$$h(z) + \varepsilon \leqslant \widehat{h}(z) \quad \text{for a.a. } z \in K.$$

It is clear that, if $h, \widehat{h} \in C(\Omega)$ and $h(z) < \widehat{h}(z)$ for all $z \in \Omega$, then $h \prec \widehat{h}$.

From Gasiński–Papageorgiou [20] (see also Arcoya–Ruiz [2, Proposition 2.6]), we have the following strong comparison principle.

Proposition 2.3. *If $\xi \geqslant 0$, $h, \widehat{h} \in L^\infty(\Omega)$ with $h \prec \widehat{h}$ and $u \in C_0^1(\overline{\Omega})$, $v \in \operatorname{int} C_+$ satisfy*

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) + \xi |u(z)|^{p-2} u(z) = h(z) & \text{for a.a. } \Omega, \quad u|_{\partial\Omega} = 0, \\ \quad -\Delta_p v(z) - \Delta v(z) + \xi v(z)^{p-1} = \widehat{h}(z) & \text{for a.a. } \Omega, \quad v|_{\partial\Omega} = 0, \end{cases}$$

then $v - u \in \operatorname{int} C_+$.

In what follows by $\{\widehat{\lambda}_k(2)\}_{k \geqslant 1}$ we denote the sequence of distinct eigenvalues of $(-\Delta, H_0^1(\Omega))$. We know that $\widehat{\lambda}_1(2) > 0$, it is simple and it has eigenfunctions of constant sign. By $\widehat{u}_1(2)$ we denote the L^2-normalized (that is, $\|\widehat{u}_1(2)\|_2 = 1$), positive eigenfunction corresponding to $\widehat{\lambda}_1(2) > 0$. Standard regularity theory and the maximum principle imply that $\widehat{u}_1(2) \in \operatorname{int} C_+$. Let $E(\widehat{\lambda}_k(2))$ denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(2)$, $k \geqslant 1$. We know that for every $k \geqslant 1$, $E(\widehat{\lambda}_k(2))$ is finite dimensional, $E(\widehat{\lambda}_k(2)) \subseteq C_0^1(\overline{\Omega})$ and it has that "unique continuation property", that is, if $u \in E(\widehat{\lambda}_k(2))$ and vanishes on a set of positive measure, then $u \equiv 0$. We know that

$$H_0^1(\Omega) = \overline{\bigoplus_{k \geqslant 1} E(\widehat{\lambda}_k(2))}.$$

We have the following variational characterizations for the eigenvalues $\widehat{\lambda}_k(2)$, $k \geqslant 1$:

$$\widehat{\lambda}_1(2) = \inf\left\{\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : \ u \in H_0^1(\Omega),\ u \not\equiv 0\right\} \tag{2}$$

and for $k \geqslant 2$, we have

$$\begin{aligned}\widehat{\lambda}_k(2) &= \inf\left\{\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : \ u \in \overline{\bigoplus_{m\geqslant k} E(\widehat{\lambda}_m(2))},\ u \not\equiv 0\right\} \\ &= \sup\left\{\frac{\|\nabla u\|_2^2}{\|u\|_2^2} : \ u \in \bigoplus_{m=1}^{k} E(\widehat{\lambda}_m(2)),\ u \not\equiv 0\right\}.\end{aligned} \tag{3}$$

In (2) the infimum is realized on the one dimensional eigenspace $E(\widehat{\lambda}_1(2))$. In (3) both the infimum and the supremum are realized on $E(\widehat{\lambda}_k(2))$. For $\widehat{\lambda}_2(2)$, in addition to the variational characterization provided by (3), we have an alternative minimax characterization which we will use in the sequel.

So, let

$$\partial B_1^{L^2} = \{u \in L^2(\Omega) : \ \|u\|_2 = 1\} \quad \text{and} \quad M = H_0^1(\Omega) \cap \partial B_1^{L^2}.$$

Then we have the following minimax characterization of $\widehat{\lambda}_2(2)$ (see Motreanu–Motreanu–Papageorgiou [31, p. 246]).

Proposition 2.4. $\widehat{\lambda}_2(2) = \inf_{\widehat{\gamma}\in\widehat{\Gamma}} \max_{-1\leqslant t\leqslant 1} \|\nabla\widehat{\gamma}(t)\|_2^2$, *where*

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1]; M) : \ \widehat{\gamma}(-1) = -\widehat{u}_1(2),\ \widehat{\gamma}(1) = \widehat{u}_1(2)\}.$$

In addition to the classical eigenvalue problem for $(-\Delta, H_0^1(\Omega))$, we can also consider the following weighted version of it:

$$\begin{cases} -\Delta u(z) = \lambda\xi(z)u(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

with $\xi \in L^\infty(\Omega)$, $\xi \geqslant 0$, $\xi \not\equiv 0$. Again we have a sequence of distinct eigenvalues $\{\widehat{\lambda}_k(2,\xi)\}_{k\geqslant 1}$ increasing to $+\infty$, with $\widehat{\lambda}_1(2,\xi) > 0$ and simple. The variational characterizations provided by (2) and (3) remain valid, with denominator in the Rayleigh quotient now being $\int_\Omega \xi(z)u^2\,dz$. As before, the eigenspaces $E(\widehat{\lambda}_k(2,\xi)) \subseteq C_0^1(\overline{\Omega})$ exhibit the unique continuation property. Combining the variational characterizations with the unique continuation property, we can establish the following monotonicity properties of $\xi \longmapsto \widehat{\lambda}_k(2,\xi)$ (see [31]).

Proposition 2.5. *If* $\xi, \xi_0 \in L^\infty(\Omega)_+$ *and* $\xi(z) \leqslant \xi_0(z)$ *for almost all* $z \in \Omega$ *with strict inequality on a set of positive measure, then* $\widehat{\lambda}_k(2,\xi_0) < \widehat{\lambda}_k(2,\xi)$ *for all* $k \geqslant 1$.

If $\xi \equiv 1$, then $\widehat{\lambda}_2(2,\xi) = \widehat{\lambda}_k(2)$ for all $k \geqslant 0$. Finally we mention that $\widehat{\lambda}_1(2,\xi) > 0$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have sign changing (nodal) eigenfunctions.

Let X be a Banach space and let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$ by $H_k(Y_1, Y_2)$ we denote the k-th singular homology group with integer coefficients for the pair (Y_1, Y_2). Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned} \varphi^c &= \big\{x \in X : \ \varphi(x) \leqslant c\big\}, \\ K_\varphi &= \big\{x \in X : \ \varphi'(x) = 0\big\}, \\ K_\varphi^c &= \big\{x \in K_\varphi : \ \varphi(x) = c\big\}. \end{aligned}$$

The critical groups of φ at an isolated element $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k\big(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{u\}\big) \quad \forall k \geqslant 0,$$

with U being a neighbourhood of $u \in X$, such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology, implies that the above definition of critical groups is independent of the particular choice of the neighbourhood U.

Suppose that $\varphi \in C^1(X)$ is a functional satisfying the Palais–Smale condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \forall k \geqslant 0.$$

The second deformation theorem (see e.g., Gasiński–Papageorgiou [15, Theorem 5.1.33, p. 628]), implies that the above definition is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define:

$$M(t, u) = \sum_{k \geqslant 0} \operatorname{rank} C_k(\varphi, u) t^k \quad \forall t \in \mathbb{R},\ u \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k \geqslant 0} \dim C_k(\varphi, \infty) t^k \quad \forall t \in \mathbb{R}.$$

The Morse relation says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t) \quad \forall t \in \mathbb{R}, \tag{4}$$

where $Q(t) = \sum_{k \geqslant 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

For $1 < p < +\infty$, let $A_p\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (where $\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear map defined by

$$\langle A_p(u), h\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \forall u, h \in W_0^{1,p}(\Omega).$$

For this map we have (see Papageorgiou–Kyritsi [34, p. 314]) the following result.

Proposition 2.6. *The map $A_p\colon W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega)$ is continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is, if $u_n \longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$ and*

$$\limsup_{n\to+\infty}\langle A_p(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$. If $p = 2$, then $A_2 = A \in \mathcal{L}(H_0^1(\Omega); H^{-1}(\Omega))$.

Closing this section, let us fix our notation. For $\zeta \in \mathbb{R}$, we set $\zeta^\pm = \max\{\pm\zeta, 0\}$ and for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Also, if $h\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function (for example a Carathéodory function), then we set

$$N_h(u)(\cdot) = h\big(\cdot, u(\cdot)\big) \quad \forall u \in W_0^{1,p}(\Omega).$$

This is the Nemytskii (superposition) operator corresponding to the function $h(z,\zeta)$. Evidently, $z \longmapsto N_h(u)(z) = h(z, u(z))$ is measurable.

3. Constant sign solutions

In this section we produce constant sign solutions for problem (1). We impose the following conditions on the reaction f:

$\underline{\mathrm{H}_1}$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) for every $\varrho > 0$, there exists a function $a_\varrho \in L^\infty(\Omega)_+$, such that

$$|f(z,\zeta)| \leqslant a_\varrho(z) \quad \text{for almost all } z \in \Omega, \text{ for all } |\zeta| \leqslant \varrho;$$

(ii) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$, such that

$$\begin{aligned} w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) \quad & \forall z \in \overline{\Omega}, \\ A_p(w_-) + A(w_-) \leqslant 0 \leqslant A_p(w_+) + A(w_+) \quad & \text{in } W^{-1,p'}(\Omega), \\ f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) \quad & \text{for a.a. } z \in \Omega; \end{aligned}$$

(iii) there exists a function $\eta_0 \in L^\infty(\Omega)_+$, such that

$$\eta_0(z) \geqslant \widehat{\lambda}_1(2) \quad \text{for a.a. } z \in \Omega,$$

strictly on a set of positive measure, and

$$\widehat{\lambda}_1(2) \leqslant \eta_0(z) \leqslant \liminf_{\zeta \to 0} \frac{f(z,\zeta)}{\zeta} \quad \text{uniformly for a.a. } z \in \Omega.$$

Remark 3.1. The above hypotheses do not impose any global growth condition on $f(z,\cdot)$. Only we require that $f(z,\cdot)$ has z-dependent, constant sign zeros (see H_1(ii); compare also hypotheses in Gasiński–Papageorgiou [23]). Hypothesis H_1(iii) concerns the behaviour of $f(z,\cdot)$ near zero and incorporates in our framework also reactions which are superlinear near zero (concave terms). Note that hypothesis H_1(ii) is satisfied if there exist $c_- < 0 < c_+$ such that

$$f(z,c_+) \leqslant 0 \leqslant f(z,c_-) \quad \text{for a.a. } z \in \Omega.$$

Example 3.2. The following functional satisfies hypotheses H_1. For the sake of simplicity, we drop the z-dependence:

$$f(\zeta) = \begin{cases} \eta_0(|\zeta|^{q-2}\zeta - |\zeta|^{r-2}\zeta) & \text{if } \ |\zeta| \leqslant 1, \\ \xi(\zeta) & \text{if } \ |\zeta| > 1, \end{cases}$$

with $1 < q \leqslant 2 < r$, $\eta_0 > 0$ and $\eta_0 > \widehat{\lambda}_1(2)$ when $q = 2$ and $\xi \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying $\xi(\pm 1) = 0$.

Under these conditions on the reaction, we can produce two constant sign solutions (one positive and the other negative). In what follows we will use the order intervals.

$$[0,w_+] = \{u \in W_0^{1,p}(\Omega) : \ 0 \leqslant u(z) \leqslant w_+(z) \text{ for a.a. } z \in \Omega\},$$
$$[w_-,0] = \{u \in W_0^{1,p}(\Omega) : \ w_-(z) \leqslant u(z) \leqslant 0 \text{ for a.a. } z \in \Omega\}.$$

Proposition 3.3. *If hypotheses* H_1 *hold, then problem* (1) *admits at least two constant sign solutions*

$$u_0 \in [0,w_+] \cap \operatorname{int} C_+, \quad v_0 \in [w_-,0] \cap (-\operatorname{int} C_+).$$

Proof. First we produce the positive solution. To this end, we introduce the following truncation of the reaction $f(z,\cdot)$:

$$\widehat{f}_+(z,\zeta) = \begin{cases} 0 & \text{if } \ \zeta < 0, \\ f(z,\zeta) & \text{if } \ 0 \leqslant \zeta \leqslant w_+(z), \\ f(z,w_+(z)) & \text{if } \ w_+(z) < \zeta. \end{cases} \tag{5}$$

This is a Carathéodory function. We set

$$\widehat{F}_+(z,\zeta) = \int_0^\zeta \widehat{f}_+(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega \widehat{F}_+(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

From (5) it is clear that $\widehat{\varphi}_+$ is coercive. Also, using the Sobolev embedding theorem, we can easily see that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. Thus, by the Weierstrass theorem, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \inf\{\widehat{\varphi}_+(u) : \ u \in W_0^{1,p}(\Omega)\}. \tag{6}$$

Because of hypothesis H_1(iii), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in (0, c_+)$ such that

$$F(z,\zeta) \geqslant \frac{1}{2}(\eta_0(z) - \varepsilon)\zeta^2 \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \delta. \tag{7}$$

Here

$$F(z,\zeta) = \int_0^\zeta f(z,s)\,ds.$$

Since $\widehat{u}_1(2) \in \operatorname{int} C_+$, we can find $t \in (0,1)$ small such that

$$0 \leqslant t\widehat{u}_1(2)(z) \leqslant \delta \quad \forall z \in \overline{\Omega}. \tag{8}$$

So, we have

$$\begin{aligned} \widehat{\varphi}_+(t\widehat{u}_1(2)) &= \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\widehat{\lambda}_1(2) - \int_\Omega \widehat{F}_+(z, t\widehat{u}_1(2))\,dz \\ &\leqslant \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p + \frac{t^2}{2}\left(\int_\Omega \big(\widehat{\lambda}_1(2) - \eta_0(z) + \varepsilon\big)\widehat{u}_1(2)^2\,dz\right) \end{aligned} \tag{9}$$

(see (5), recall that $\widehat{u}_1(2) \in \operatorname{int} C_+$, $\|\widehat{u}_1(2)\|_2 = 1$ and see (7) and (8)).

Because $\widehat{u}_1(2) \in \operatorname{int} C_+$, using hypothesis H_1(iii), we have

$$\xi^* = \int_\Omega \big(\eta_0(z) - \widehat{\lambda}_1(2)\big)\widehat{u}_1(2)^2\,dz \geqslant 0.$$

Returning to (9) and choosing $\varepsilon \in (0, \xi^*)$, we have

$$\widehat{\varphi}_+(t\widehat{u}_1(2)) \leqslant \frac{t^p}{p}\|\nabla \widehat{u}_1(2)\|_p^p - \frac{t^2}{2}\xi_\varepsilon^*,$$

with $\xi_\varepsilon^* = \xi^* - \varepsilon > 0$. Since $2 < p$ and $t \in (0,1)$, by taking t even smaller if necessary, we can have $\widehat{\varphi}_+(t\widehat{u}_1(2)) < 0$ and so

$$\widehat{\varphi}_+(u_0) < 0 = \widehat{\varphi}_+(0)$$

(see (6)), hence $u_0 \neq 0$.

From (6), we have $\widehat{\varphi}'_+(u_0) = 0$, so

$$A_p(u_0) + A(u_0) = N_{\widehat{f}_+}(u_0). \tag{10}$$

On (10), first we act with $-u_0^- \in W_0^{1,p}(\Omega)$. Then $\|\nabla u_0^-\|_p^p + \|\nabla u_0^-\|_2^2 = 0$ (see (5)), so

$$u_0 \geqslant 0, \quad u_0 \not\equiv 0.$$

Next on (10) we act with $(u_0 - w_+)^+ \in W_0^{1,p}(\Omega)$ (see hypothesis H_1(ii)). Then

$$\begin{aligned}
&\langle A_p(u_0), (u_0 - w_+)^+\rangle + \langle A(u_0), (u_0 - w_+)^+\rangle \\
&= \int_\Omega f(z, w_+)(u_0 - w_+)^+\, dz \\
&\leqslant \langle A_p(w_+), (u_0 - w_+)^+\rangle + \langle A(w_+), (u_0 - w_+)^+\rangle
\end{aligned}$$

(see (5) and hypothesis H_1(ii)), so

$$\int_{\{u_0 > w_+\}} \big(|\nabla u_0|^{p-2}\nabla u_0 - |\nabla w_+|^{p-2}\nabla w_+, \nabla u_0 - \nabla w_+\big)_{\mathbb{R}^N}\, dz + \|\nabla(u_0 - w_+)^+\|_2^2 \leqslant 0,$$

thus

$$u_0 \leqslant w_+.$$

So, we have proved that $u_0 \in [0, w_+]$ and thus u_0 is a nonnegative solution of problem (1) (see (5)).

Using Lieberman [30, Theorem 1], we see that $u_0 \in C_+ \setminus \{0\}$. Hypotheses H_1(i),(iii) imply that given $\varrho > 0$, we can find $\xi_\varrho > 0$ such that

$$f(z,\zeta)\zeta + \xi_\varrho |\zeta|^p \geqslant 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \varrho. \tag{11}$$

Let $\varrho = \|u_0\|_\infty$ and let $\xi_\varrho > 0$ be as in (11). For almost all $z \in \Omega$, we have

$$-\Delta_p u_0(z) - \Delta u_0(z) + \xi_\varrho u_0(z)^{p-1} = f(z, u_0(z)) + \xi_\varrho u_0(z)^{p-1} \geqslant 0$$

so

$$\Delta_p u_0(z) + \Delta u_0(z) \leqslant \xi_\varrho u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega. \tag{12}$$

Let

$$H(t) = \frac{1}{p}t^p + \frac{1}{2}t^2 \quad \text{and} \quad H_0(t) = \frac{1}{p}t^p \quad \forall t \geqslant 0.$$

These are strictly increasing functions and $H(t) \geqslant H_0(t)$ for all $t \geqslant 0$ (recall $p > 2$). From Leoni [29, p. 6], we have

$$H^{-1}(t) \leqslant H_0^{-1}(t) \quad \forall t \geqslant 0.$$

Then for $\delta \in (0,1)$, we have

$$\int_0^\delta \frac{1}{H^{-1}(\frac{\xi_\varrho}{p}t^p)}dt \geqslant \int_0^\delta \frac{1}{H_0^{-1}(\frac{\xi_\varrho}{p}t^p)}dt = c_1 \int_0^\delta \frac{1}{t}dt = +\infty, \tag{13}$$

for some $c_1 > 0$. From (12), (13) and the nonlinear strong maximum principle of Pucci–Serrin [38, p. 111], we have

$$u_0(z) > 0 \quad \forall z \in \Omega.$$

Then we can apply the boundary point theorem of Pucci–Serrin [38, p. 120] and infer that $u_0 \in \operatorname{int} C_+$. So, finally we have $u_0 \in [0, w_+] \cap \operatorname{int} C_+$.

For the negative solution, we consider the following truncation of $f(z, \cdot)$:

$$\widehat{f}_-(z,\zeta) = \begin{cases} f(z, w_-(z)) & \text{if} \quad \zeta < w_-(z), \\ f(z,\zeta) & \text{if} \quad w_-(z) \leqslant \zeta \leqslant 0, \\ 0 & \text{if} \quad 0 < \zeta. \end{cases} \tag{14}$$

This is a Carathéodory function. We set

$$\widehat{F}_-(z,\zeta) = \int_0^\zeta \widehat{f}_-(z,s)\,ds$$

and consider the C^1-functional $\widehat{\varphi}_- \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_-(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega \widehat{F}_-(z, u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

Reasoning as above and using (14), via the direct method, we produce a negative solution $v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+)$. □

If we strengthen a little the conditions on $f(z, \cdot)$ (but without altering the overall geometry of the problem), we can improve the conclusion of Proposition 3.3. The new conditions on the reaction f are the following:

$\underline{\mathrm{H}_2:}$ $f \colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses H_2(i)–(iii) are the same as the corresponding hypotheses H_1(i)–(iii) with the addition that $w_\pm \in C^1(\Omega)$ and

(iv) for every $\varrho \in (0, \varrho_0)$ (where $\varrho_0 = \max\{\|w_-\|_\infty, \|w_+\|_\infty\}$), there exists $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta) + \xi_\varrho |\zeta|^{p-2}\zeta$ is nondecreasing on $[-\varrho, \varrho]$.

Remark 3.4. Hypothesis H_2(iv) is satisfied, if for almost all $z \in \Omega$, $f(z,\cdot) \in C^1(\mathbb{R})$ and $f'_\zeta(z,\cdot)$ is locally $L^\infty(\Omega)$-bounded.

Proposition 3.5. *If hypotheses* H_2 *hold, then problem* (1) *admits at least two constant sign solutions*

$$\begin{aligned} &u_0 \in \operatorname{int} C_+ \quad &&\textit{with}\;\; u_0(z) < w_+(z) \quad \forall z \in \overline{\Omega},\\ &v_0 \in -\operatorname{int} C_+ \quad &&\textit{with}\; w_-(z) < v_0(z) \quad \forall z \in \overline{\Omega}. \end{aligned}$$

Proof. From Proposition 3.3, we already have two constant sign solutions

$$u_0 \in [0, w_+] \cap \operatorname{int} C_+, \quad v_0 \in [w_-, 0] \cap (-\operatorname{int} C_+).$$

Let $a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$. We have $a \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ (recall that $p > 2$) and

$$\operatorname{div} a(\nabla u) = \Delta_p u + \Delta u \quad \forall u \in W_0^{1,p}(\Omega).$$

We have $\nabla a(y) = |y|^{p-2}\Big(I + (p-2)\frac{y \otimes y}{|y|^2}\Big) + I$, so

$$\big(\nabla a(y)\xi, \xi\big)_{\mathbb{R}^N} \geqslant |\xi|^2 \quad \forall y, \xi \in \mathbb{R}^N. \tag{15}$$

Hypothesis H_2(iv) implies that $f(z,\cdot)$ is Lipschitz on $[-\varrho_0, \varrho_0]$. Indeed, if $\zeta, y \in [-\varrho_0, \varrho_0]$ with $y < \zeta$, then

$$f(z,\zeta) - f(z,y) \geqslant -\xi_{\varrho_0}\big(|\zeta|^{p-2}\zeta - |y|^{p-2}y\big) \geqslant -\xi^*_{\varrho_0}(\zeta - y) \tag{16}$$

for some $\xi^*_\varrho > 0$ (see hypothesis H_2(iv) and recall that $p > 2$).

Now let $\varrho = \|u_0\|_\infty$ and let $\xi_\varrho > 0$ be as postulated by hypothesis H_2(iv). Then we have

$$\begin{aligned} A_p(u_0) + A(u_0) + \xi_\varrho u_0^{p-1} &= N_f(u_0) + \xi_\varrho u_0^{p-1}\\ &\leqslant N_f(w_+) + \xi_\varrho w_+^{p-1}\\ &\leqslant A_p(w_+) + A(w_+) + \xi_\varrho w_+^{p-1} \quad \text{in } W^{-1,p'}(\Omega) \end{aligned} \tag{17}$$

(see hypotheses H_2(iv),(ii)).

Since $w_+ \in C^1(\Omega)$ from (16) and (17) we see that we can apply the tangency principle of Pucci–Serrin [38, p. 35] and we have

$$u_0(z) < w_+(z) \quad \forall z \in \overline{\Omega}.$$

Similarly we show that $w_-(z) < v_0(z)$ for all $z \in \overline{\Omega}$. □

Next we establish the existence of extremal constant sign solutions. That is, we show that there exists a smallest positive solution u_* and a biggest negative solution v_*. Using these extremal constant sign solutions, we will produce a nodal (sign changing) solution (see Section 4).

Let S_+ (respectively S_-) be the set of nontrivial positive (respectively negative) solutions of problem (1) in the order interval $[0, w_+]$ (respectively $[w_-, 0]$). Clearly, if we find a minimal (respectively maximal) element of S_+ (respectively S_-), then we will have the extremal constant sign solutions of problem (1). From Proposition 3.3 we know that

$$S_+ \neq \emptyset, \quad S_+ \subseteq \operatorname{int} C_+ \quad \text{and} \quad S_- \neq \emptyset, \quad S_- \subseteq -\operatorname{int} C_+.$$

Proposition 3.6. *If hypotheses* H_1 *hold, then problem* (1) *admits a smallest positive solution* $u_* \in \operatorname{int} C_+$ *and a biggest negative solution* $v_* \in -\operatorname{int} C_+$.

Proof. From Dunford–Schwartz [10, p. 336] (see also Hu–Papageorgiou [27, p. 178]), we can find a sequence $\{u_n\}_{n\geqslant 1} \subseteq S_+$ such that

$$\inf S_+ = \inf_{n\geqslant 1} u_n.$$

We have

$$A_p(u_n) + A(u_n) = N_f(u_n) \quad \forall n \geqslant 1, \tag{18}$$

so the sequence $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ is bounded (recall that $u_n \in [0, w_+]$ for all $n \geqslant 1$).

So, we may assume that

$$u_n \longrightarrow u_* \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{19}$$

$$u_n \longrightarrow u_* \quad \text{in } L^p(\Omega). \tag{20}$$

On (18) we act with $u_n - u_* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (19). We obtain $\lim_{n\to+\infty} \big(\langle A_p(u_n), u_n - u_*\rangle + \langle A(u_n), u_n - u_*\rangle\big) = 0$, so

$$\limsup_{n\to+\infty} \big(\langle A_p(u_n), u_n - u_*\rangle + \langle A(u_*), u_n - u_*\rangle\big) \leqslant 0$$

(since A is monotone), thus $\limsup_{n\to+\infty}\langle A_p(u_n), u_n - u_*\rangle \leqslant 0$ and hence

$$u_n \longrightarrow u_* \quad \text{in } W_0^{1,p}(\Omega) \tag{21}$$

(see Proposition 2.6).

Therefore, if in (18) we pass to the limit as $n \to +\infty$ and use (21), then

$$A_p(u_*) + A(u_*) = N_f(u_*), \quad u_* \geqslant 0. \tag{22}$$

We need to show that $u_* \neq 0$. To this end, note that hypothesis H_1(i),(iii) imply that given $\varepsilon > 0$ and $r \geqslant 1$, we can find $c_2 = c_2(\varepsilon, r) > 0$ such that

$$f(z,\zeta)\zeta \geqslant (\eta_0(z) - \varepsilon)\zeta^2 - c_2|\zeta|^r \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \varrho_0, \tag{23}$$

where $\varrho_0 = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$. We introduce the following Carathéodory function

$$k_+(z,\zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ (\eta_0(z) - \varepsilon)\zeta - c_2\zeta^{r-1} & \text{if } 0 \leqslant \zeta \leqslant w_+(z), \\ (\eta_0(z) - \varepsilon)w_+(z) - c_2 w_+(z)^{r-1} & \text{if } w_+(z) < \zeta. \end{cases} \tag{24}$$

We consider the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = k_+(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{25}$$

We show that this problem has a unique positive solution $\overline{u} \in [0, w_+] \cap \operatorname{int} C_+$. For this purpose we set

$$K_+(z,\zeta) = \int_0^\zeta k_+(z,s)\,ds$$

and consider the C^1-functional $\psi_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega K_+(z, u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

It is clear from (24) that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W_0^{1,p}(\Omega)$ such that

$$\psi_+(\overline{u}) = \inf\{\psi_+(u) : \; u \in W_0^{1,p}(\Omega)\}. \tag{26}$$

As in the proof of Proposition 3.3, for $t \in (0,1)$ small (at least such that $t\widehat{u}_1(2) \leqslant w_+$) and choosing $\varepsilon > 0$ small, we can have $\psi_+(t\widehat{u}_1(2)) < 0$, so

$$\psi_+(\overline{u}) < 0 = \psi_+(0)$$

(see (26)), hence $\overline{u} \not\equiv 0$.

From (26), we have $\psi'_+(\overline{u}) = 0$, so

$$A_p(\overline{u}) + A(\overline{u}) = N_{k_+}(\overline{u}). \tag{27}$$

Acting on (27) with $-u_0^- \in W_0^{1,p}(\Omega)$ and with $(u_0 - w_+)^+ \in W_0^{1,p}(\Omega)$ and using (23), hypothesis H_1(ii) and the nonlinear regularity theory (see Lieberman [30, Theorem 1]) and the nonlinear maximum principle (see Pucci–Serrin [38, p. 111]), we obtain

$$\overline{u} \in [0, w_+] \cap \operatorname{int} C_+$$

(see also the proof of Proposition 3.3).

From (24) and (27) it follows that $\overline{u} \in \operatorname{int} C_+$ is a positive solution of problem (25). We show that this is the unique positive solution of (25). Let

$$G_0(t) = \frac{1}{p}t^p + \frac{1}{2}t^2 \quad \forall t > 0$$

and set

$$G(y) = G_0(|y|) \quad \forall y \in \mathbb{R}^N.$$

Note that $G \in C^1(\mathbb{R}^N)$ and $\nabla G(y) = a(y) = |y|^{p-2}y + y$ for all $y \in \mathbb{R}^N$ (see the proof of Proposition 3.5).

We consider the integral functional $\sigma_+\colon L^p(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma_+(u) = \begin{cases} \displaystyle\int_\Omega G(\nabla u^{\frac{1}{2}})\,dz & \text{if } u \geqslant 0,\ u^{\frac{1}{2}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Using Benguria–Brézis–Lieb [4, Lemma 4] (see also Díaz–Saá [9, Lemma 1]), we see that σ_+ is convex, while using Fatou's lemma we have that σ_+ is lower semicontinuous.

Suppose that $\overline{u}, \overline{y} \in W_0^{1,p}(\Omega)$ are both positive solutions of (25). From the previous part of the proof, we have that $\overline{u}, \overline{y} \in \operatorname{int} C_+$ and so $\overline{u}^2, \overline{y}^2 \in \operatorname{dom} \sigma_+ = \{u \in L^1(\Omega) :\ \sigma_+(u) < +\infty\}$ (the effective domain of σ_+). Let $h \in C_0^1(\overline{\Omega})$. For $t \in [-1, 1]$ small in absolute value, we have

$$\overline{u}^2 + th \in \operatorname{dom} \sigma_+, \quad \overline{y}^2 + th \in \operatorname{dom} \sigma_+.$$

Therefore the Gâteaux derivative of σ_+ at $\overline{u}^2$, $\overline{y}^2$ in the direction h exists and using the chain rule, we have

$$\sigma_+'(\overline{u})(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \overline{u} - \Delta \overline{u}}{\overline{u}} h\,dz, \quad \sigma_+'(\overline{y})(h) = \frac{1}{2}\int_\Omega \frac{-\Delta_p \overline{y} - \Delta \overline{y}}{\overline{y}} h\,dz.$$

The convexity of σ_+ implies the monotonicity of σ_+' and so with $h = \overline{u}^2 - \overline{y}^2 \in C_0^1(\overline{\Omega})$, we have

$$0 \leqslant \int_\Omega \left(\frac{k_+(z, \overline{u})}{\overline{u}} - \frac{k_+(z, \overline{y})}{\overline{y}} \right) (\overline{u}^2 - \overline{y}^2)\,dz = c_2 \int_\Omega (\overline{y}^{r-1} - \overline{u}^{r-1})(\overline{u}^2 - \overline{y}^2)\,dz \leqslant 0$$

(see (24), (25) and recall that $\overline{u}, \overline{y} \in [0, w_+]$), so $\overline{u} = \overline{y}$ (due to the strict monotonicity of the map $\zeta \longmapsto \zeta^{r-1}$ for $\zeta > 0$). This proves the uniqueness of the positive solutions $\overline{u} \in \operatorname{int} C_+$ of problems (25).

Claim. $\overline{u} \leqslant u$ for all $u \in S_+$.

Let $u \in S_+$ and consider the following Carathéodory function:

$$e_+(z, \zeta) = \begin{cases} 0 & \text{if } \ \zeta < 0, \\ (\eta_0(z) - \varepsilon)\zeta - c_2\zeta^{r-1} & \text{if } \ 0 \leqslant \zeta \leqslant u(z), \\ (\eta_0(z) - \varepsilon)u(z) - c_2 u(z)^{r-1} & \text{if } \ u(z) < \zeta. \end{cases} \tag{28}$$

Let

$$E_+(z,\zeta) = \int_0^\zeta e(z,s)\,ds$$

and let $\widehat{\xi}_+ \colon W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the C^1-functional defined by

$$\widehat{\xi}_+(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\|\nabla u\|_2^2 - \int_\Omega E_+(z,u(z))\,dz \quad \forall u \in W_0^{1,p}(\Omega).$$

As before, from (28) we see that $\widehat{\xi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\xi}_+(\widetilde{u}) = \inf\left\{\widehat{\xi}_+(u) :\ u \in W_0^{1,p}(\Omega)\right\}. \tag{29}$$

Again for $t \in (0,1)$ small, we have $\widehat{\xi}_+(t\widehat{u}_1(2)) < 0$ and so $\widetilde{u} \neq 0$. From (29) we have $\widehat{\xi}_+'(\widetilde{u}) = 0$, so

$$A_p(\widetilde{u}) + A(\widetilde{u}) = N_{e_+}(\widetilde{u}). \tag{30}$$

Acting on (30) with $-\widetilde{u}^- \in W_0^{1,p}(\Omega)$ and with $(\widetilde{u}-u)^+ \in W_0^{1,p}(\Omega)$, we obtain

$$\widetilde{u} \in [0,u], \quad \widetilde{u} \neq 0, \tag{31}$$

so $\widetilde{u}$ is a positive solution of (25) (recall $u \leqslant w_+$), so $\widetilde{u} = \overline{u}$ (due to the uniqueness of the positive solution $\overline{u} \in \operatorname{int} C_+$) and thus $\overline{u} \leqslant u$ (see (31)). This proves the Claim.

By virtue of the Claim, we have $\overline{u} \leqslant u_n \quad \forall n \geqslant 1$, so

$$\overline{u} \leqslant u_*$$

(see (21)) and so $u_* \neq 0$.

From (22) we see that $u_* \in S_+$. Therefore

$$u_* = \inf S_+, \quad u_* \in S_+,$$

so $u_* \in \operatorname{int} C_+$ is the smallest positive solution of (1).

For the biggest negative solution $v_* \in -\operatorname{int} C_+$ we argue similarly starting from the Carathéodory function

$$k_-(z,\zeta) = \begin{cases} (\eta_0(z)-\varepsilon)w_-(z) - c_2|w_-(z)|^{r-2}w_-(z) & \text{if } \ \zeta < w_-(z), \\ (\eta_0(z)-\varepsilon)\zeta - c_2|\zeta|^{r-2}\zeta & \text{if } \ w_-(z) \leqslant \zeta \leqslant 0, \\ 0 & \text{if } \ 0 < \zeta. \end{cases}$$

In this case the auxiliary problem

$$\begin{cases} -\Delta_p u(z) - \Delta u(z) = k_-(z,u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

has a unique negative solution $\overline{v} \in -\operatorname{int} C_+$ and $v \leqslant \overline{v}$ for all $v \in S_-$. So, we can have $\overline{v} = \sup S_-$, $\overline{v} \in S_-$. $\square$

4. Nodal solutions

In this section we produce nodal (sign changing) solutions for problem (1). To do this we need to strengthen a little our condition on $f(z,\cdot)$ near zero (see hypothesis H_1(iii)), but without altering the essential geometry of the problem. So, the new conditions on the reaction $f(z,\cdot)$ are the following:

$\underline{H_3:}$ $f\colon \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0)=0$ for almost all $z\in\Omega$, hypotheses H_3(i),(ii) are the same as the corresponding hypotheses H_1(i),(ii) and

(iii) $\widehat{\lambda}_2(2) < \liminf_{\zeta\to 0} \frac{f(z,\zeta)}{\zeta}$ uniformly for almost all $z\in\Omega$.

In what follows, $u_*\in \operatorname{int} C_+$ and $v_*\in -\operatorname{int} C_+$ are the two extremal constant sign solutions of problem (1) produced in Proposition 3.6.

Proposition 4.1. *If hypotheses* H_3 *hold, then problem* (1) *admits a nodal solution* $y_0\in[v_*,u_*]\cap C_0^1(\overline{\Omega})$.

Proof. With $u_*\in \operatorname{int} C_+$ and $v_*\in -\operatorname{int} C_+$ being the two extremal constant sign solutions of problem (1) produced in Proposition 3.6, we introduce the following truncation of $f(z,\cdot)$:

$$\beta(z,\zeta) = \begin{cases} f(z,v_*(z)) & \text{if } \zeta < v_*(z),\\ f(z,\zeta) & \text{if } v_*(z)\leqslant \zeta\leqslant u_*(z),\\ f(z,u_*(z)) & \text{if } u_*(z)<\zeta. \end{cases} \tag{32}$$

We also consider the positive and negative truncation of $\beta(z,\cdot)$, namely the functions

$$\beta_\pm(z,\zeta) = \beta(z,\pm\zeta^\pm) \quad \forall (z,\zeta)\in\Omega\times\mathbb{R}.$$

All three are Carathéodory functions. We set

$$B(z,\zeta) = \int_0^\zeta \beta(z,s)\,ds, \quad B_\pm(z,\zeta) = \int_0^\zeta \beta_\pm(z,s)\,ds$$

and consider the C^1-functionals $\widehat{\psi},\widehat{\psi}_\pm\colon W_0^{1,p}(\Omega)\longrightarrow\mathbb{R}$ defined by

$$\widehat{\psi}(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\|\nabla u\|_2^2 - \int_\Omega B(z,u(z))\,dz \quad \forall u\in W_0^{1,p}(\Omega),$$
$$\widehat{\psi}_\pm(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\|\nabla u\|_2^2 - \int_\Omega B_\pm(z,u(z))\,dz \quad \forall u\in W_0^{1,p}(\Omega).$$

Claim 1. $K_{\widehat{\psi}}\subseteq[v_*,u_*]$, $K_{\widehat{\psi}_+}=\{0,u_*\}$, $K_{\widehat{\psi}_-}=\{0,v_*\}$.

Let $u\in K_{\widehat{\psi}}$. We have

$$A_p(u)+A(u) = N_\beta(u). \tag{33}$$

On (33) we act with $(u-u_*)^+ \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned}\langle A_p(u),(u-u_*)^+\rangle+\langle A(u),(u-u_*)^+\rangle &= \int_\Omega \beta(z,u)(u-u_*)^+\,dz\\ &= \int_\Omega f(z,u_*)(u-u_*)^+\,dz\\ &= \langle A_p(u_*),(u-u_*)^+\rangle+\langle A(u_*),(u-u_*)^+\rangle\end{aligned}$$

(see (32) and since $u_* \in S_+$), so

$$\int_{\{u>u_*\}} \big(|\nabla u|^{p-2}\nabla u - |\nabla u_*|^{p-2}\nabla u_*, \nabla u - \nabla u_*\big)_{\mathbb{R}^N}\,dz + \|\nabla(u-u_*)^+\|_2^2 = 0$$

and thus $u \leqslant u_*$. Similarly, acting on (33) with $(v_*-v)^- \in W_0^{1,p}(\Omega)$, we show that $v_* \leqslant u$. Therefore

$$u \in [v_*, u_*],$$

where $[v_*,u_*] = \{w \in W_0^{1,p}(\Omega):\ v_*(z) \leqslant w(z) \leqslant u_*(z) \text{ for a.a. } z \in \Omega\}$, so

$$K_{\widehat{\psi}} \subseteq [v_*, u_*].$$

Using a similar argument, we show that $K_{\widehat{\psi}_+} \subseteq [0,u_*]$ and $K_{\widehat{\psi}_-} \subseteq [v_*,0]$. The extremality of the solutions u_* and v_* implies that

$$K_{\widehat{\psi}_+} = \{0,u_*\} \quad \text{and} \quad K_{\widehat{\psi}_-} = \{0,v_*\}.$$

This proves Claim 1.

Claim 2. $u_* \in \operatorname{int} C_+$ and $v_* \in -\operatorname{int} C_+$ are local minimizers of $\widehat{\psi}$.

From (32) it is clear that $\widehat{\psi}$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\widehat{u}_* \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\psi}_+(\widehat{u}_*) = \inf\{\widehat{\psi}_+(u):\ u \in W_0^{1,p}(\Omega)\} = \widehat{m}_+. \tag{34}$$

As in the proof of Proposition 3.3, using hypothesis H_3(iii), for $t \in (0,1)$ small (at least such that $t\widehat{u}_1(2) \leqslant u_*$, recall that $u_* \in \operatorname{int} C_+$ and see Lemma 3.3 of Filippakis-Kristaly-Papageorgiou [11], to check that this is possible), we have

$$\widehat{\psi}_+(t\widehat{u}_1(2)) < 0,$$

so $\widehat{\psi}_+(\widehat{u}_*) < 0 = \widehat{\psi}_+(0)$ (see (34)) and thus

$$\widehat{u}_* \not\equiv 0.$$

Since $\widehat{u}_* \in K_{\widehat{\psi}_+}$ (see (34)), from Claim 1, we infer that $\widehat{u}_* = u_*$. Note that $\widehat{\psi}_+|_{C_+} = \widehat{\psi}|_{C_+}$ (see (32)), so

$$u_* \in \operatorname{int} C_+ \quad \text{is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \widehat{\psi}$$

and thus

$$u_* \in \operatorname{int} C_+ \quad \text{is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \widehat{\psi}$$

(see Proposition 2.2).

Similarly, using this time the functional $\widehat{\psi}_-$, we show that $v_* \in -\operatorname{int} C_+$ is a local minimizer of $\widehat{\psi}$. This proves Claim 2.

We may assume that $K_{\widehat{\psi}}$ is finite (otherwise we already have a whole sequence of distinct nodal solutions, see Claim 1). Without any loss of generality, we may assume that $\widehat{\psi}(v_*) \leqslant \widehat{\psi}(u_*)$ (the analysis is similar if the opposite inequality holds).

Because of Claim 2, we can find $\varrho \in (0,1)$ small such that

$$\widehat{\psi}(v_*) \leqslant \widehat{\psi}(u_*) < \inf\left\{\widehat{\psi}(u): \ \|u-u_*\| = \varrho\right\} = \widehat{m}_\varrho, \quad \|v_* - u_*\| > \varrho \tag{35}$$

(see Gasiński–Papageorgiou [16]). From (32) it is clear that $\widehat{\psi}$ is coercive. So, it satisfies the Palais–Smale condition (see Papageorgiou-Winkert [37]). This fact and (35) permit the use of the mountain pass theorem (see Theorem 2.1). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\widehat{\psi}} \subseteq [v_*, u_*] \quad \text{and} \quad \widehat{m}_\varrho \leqslant \widehat{\psi}(y_0) \tag{36}$$

(see Claim 1 and (35)). From (35) and (36) we see that $y_0 \notin \{u_*, v_*\}$. We need to show that $y_0 \not\equiv 0$ and then from (36) and the extremality of the solutions u_* and v_*, we can conclude that y_0 is nodal.

From the mountain pass theorem (see Theorem 2.1), we have

$$\widehat{\psi}(y_0) = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \widehat{\psi}(\gamma(t)), \tag{37}$$

where $\Gamma = \{\gamma \in C([0,1]; W_0^{1,p}(\Omega)): \ \gamma(0) = v_*, \ \gamma(1) = u_*\}$. According to (37), in order to establish the nontriviality of y_0, it suffices to produce a path $\gamma_* \in \Gamma$ such that $\widehat{\psi}|_{\gamma_*} < 0$. In what follows, we construct such a path.

Recall that $\partial B_1^{L^2} = \{u \in L^2(\Omega): \ \|u\|_2 = 1\}$, $M = H_0^1(\Omega) \cap \partial B_1^{L^2}$. Also we introduce the Banach manifold $M_c = C_0^1(\overline{\Omega}) \cap \partial B_1^{L^2}$ and consider the following sets of paths:

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1,1]; M): \ \widehat{\gamma}(-1) = -\widehat{u}_1(2), \ \widehat{\gamma}(1) = \widehat{u}_1(2)\},$$
$$\widehat{\Gamma}_c = \{\widehat{\gamma} \in C([-1,1]; M_c): \widehat{\gamma}(-1) = -\widehat{u}_1(2), \ \widehat{\gamma}(1) = \widehat{u}_1(2)\}.$$

We know that $\widehat{\Gamma}_c$ is dense in $\widehat{\Gamma}$ (see Gasiński–O'Regan–Papageorgiou [13]). Then from Proposition 2.4 we see that given $\widehat{\delta} > 0$, we can find $\widehat{\gamma}_0 \in \widehat{\Gamma}_c$ such that

$$\|\nabla\widehat{\gamma}_0(t)\|_2^2 \leqslant \widehat{\lambda}_1(2) + \widehat{\delta} \quad \forall t \in [-1,1]. \tag{38}$$

Hypothesis H_1(iii) implies that we can find $\delta > 0$ and $c_3 > \widehat{\lambda}_2(2)$ such that

$$\frac{1}{2}c_3\zeta^2 \leqslant F(z,\zeta) \quad \text{for a.a. } z \in \Omega, \text{ all } |\zeta| \leqslant \delta. \tag{39}$$

Since $\widehat{\gamma}_0 \in \widehat{\Gamma}_c$ and $u_* \in \operatorname{int} C_+$, $v_* \in -\operatorname{int} C_+$, we can find $\tau \in (0,1)$ small such that

$$\tau\widehat{\gamma}_0(t) \in [v_*, u_*], \quad \tau|\widehat{\gamma}_0(t)(z)| \leqslant \delta \quad \forall z \in \overline{\Omega} \tag{40}$$

and

$$\tau|\nabla\widehat{\gamma}_0(t)(z)| \leqslant \delta \quad \forall t \in [-1,1] \quad \forall z \in \overline{\Omega}. \tag{41}$$

Then for all $t \in [-1,1]$, we have

$$\begin{aligned} \widehat{\psi}(\tau\gamma_0(t)) &= \frac{\tau^p}{p}\|\nabla\widehat{\gamma}_0(t)\|_p^p + \frac{\tau^2}{2}\|\nabla\widehat{\gamma}_0(t)\|_2^2 - \int_\Omega F(z,\tau\widehat{\gamma}_0(t))\,dz \\ &\leqslant \frac{\tau^p}{p}\|\nabla\widehat{\gamma}_0(t)\|_p^p + \frac{\tau^2}{2}\big(\widehat{\lambda}_2(2) + \widehat{\delta} - c_3\big) \\ &\leqslant \frac{\tau^p}{p}c_4 - \frac{\tau^2}{2}\big(c_3 - (\widehat{\lambda}_2(2) + \widehat{\delta})\big), \end{aligned} \tag{42}$$

with $c_4 \geqslant \max_{-1\leqslant t\leqslant 1} \|\nabla\widehat{\gamma}_0(t)\|_p^p > 0$ (see (38)–(41)) and recall that $\|\widehat{u}_1(2)\| = 1$).

We choose $\widehat{\delta} \in (0, c_3 - \widehat{\lambda}_2(2))$ and recall that $2 < p$ and $\tau \in (0,1)$. So, from (42) and by choosing $\tau \in (0,1)$ even smaller if necessary, we have

$$\widehat{\psi}(\tau\widehat{\gamma}_0(t)) < 0 \quad \forall t \in [-1,1]. \tag{43}$$

Let $\gamma_0 = \tau\widehat{\gamma}_0$. This is a continuous path in $H_0^1(\Omega)$ which connects $-\tau\widehat{u}_1(2)$ and $\tau\widehat{u}_1(2)$ and

$$\widehat{\psi}|_{\gamma_0} < 0 \tag{44}$$

(see (43)).

Next we construct a path in $H_0^1(\Omega)$ which connects $\tau\widehat{u}_1(2)$ and u_* and along which $\widehat{\psi}$ is strictly negative.

From (34) and the previous part of the proof, we have

$$\widehat{\psi}_+(u_*) = \widehat{m}_+ < 0 = \widehat{\psi}_+(0). \tag{45}$$

Using the second deformation theorem (see e.g., Gasiński–Papageorgiou [15, p. 628]), we can find a deformation $h\colon [0,1] \times (\widehat{\psi}_+^0 \setminus K^0_{\widehat{\psi}_+}) \longrightarrow \widehat{\psi}_+^0$ such that

$$h(0,u) = u \quad \forall u \in \widehat{\psi}_+^0 \setminus K_{\widehat{\psi}_+}, \tag{46}$$

$$h(1, \widehat{\psi}_+^0 \setminus K^0_{\widehat{\psi}_+}) \subseteq \widehat{\psi}_+^{\widehat{m}_+} \tag{47}$$

and
$$\widehat{\psi}_+(h(t,u)) \leqslant \widehat{\psi}_+(h(s,u)) \quad \forall t,s \in [0,1],\ s \leqslant t,\ u \in \widehat{\psi}_+^0 \setminus K^0_{\widehat{\psi}_+}. \tag{48}$$

Note that $\widehat{\psi}_+^{\widehat{m}_+} = \{u_*\}$ (see (34), (45) and Claim 1). Also $\widehat{\psi}_+(\tau\widehat{u}_1(2)) = \widehat{\psi}(\tau\widehat{u}_1(2)) = \widehat{\psi}(\gamma_0(1)) < 0$ (since $\widehat{\psi}_+|_{C_+} = \widehat{\psi}|_{C_+}$, see (32) and (44)), so

$$\tau\widehat{u}_1(2) \in \widehat{\psi}_+^0 \setminus K^0_{\widehat{\psi}_+} = \widehat{\psi}_+^0 \setminus \{0\}$$

(see Claim 1).

So, we can define

$$\gamma_+(t) = h(t,\tau\widehat{u}_1(2))^+ \quad \forall t \in [0,1].$$

Evidently γ_+ is a path in $H_0^1(\Omega)$ such that

$$\gamma_+(0) = h(0,\tau\widehat{u}_1(2))^+ = \tau\widehat{u}_1(2)$$

(see (46) and recall that $\tau\widehat{u}_1(2) \in \operatorname{int} C_+$) and

$$\gamma_+(1) = h(1,\tau\widehat{u}_1(2))^+ = u_*$$

(see (47) and recall that $\widehat{\psi}_+^{\widehat{m}_+} = \{u_*\}$, $u_* \in \operatorname{int} C_+$). Moreover, for all $t \in [0,1]$, we have $\widehat{\psi}(\gamma_+(t)) = \widehat{\psi}_+(\gamma_+(t)) = \widehat{\psi}_+(h(t,\tau\widehat{u}_1(2))^+) \leqslant \widehat{\psi}_+(h(t,\tau\widehat{u}_1(2))) \leqslant \widehat{\psi}_+(h(0,\tau\widehat{u}_1(2))) = \widehat{\psi}_+(\tau\widehat{u}_1(2)) = \widehat{\psi}(\tau\widehat{u}_1(2)) < 0$ (since $\widehat{\psi}|_{C_+} = \widehat{\psi}_+|_{C_+}$, see (32), (44), (46), (48) and recall that $\tau\widehat{u}_1(2) \in \operatorname{int} C_+$). Thus

$$\widehat{\psi}|_{\gamma_+} < 0. \tag{49}$$

In a similar fashion, we produce a path γ_- in $H_0^1(\Omega)$ which connect $-\tau\widehat{u}_1(2)$ and v_* and such that

$$\widehat{\psi}|_{\gamma_-} < 0. \tag{50}$$

We concatenate $\gamma_0, \gamma_+, \gamma_-$ and obtain a path $\gamma_* \in \Gamma$ such that

$$\psi|_{\gamma_*} < 0$$

(see (44), (49) and (50)), so $y_0 \not\equiv 0$ and thus $y_0 \in [v_*, u_*] \cap C_0^1(\overline{\Omega})$ is a nodal solution of (1). $\square$

We can state our first multiplicity theorem for problem (1). We stress that we provide sign information for all the solutions produced.

Theorem 4.2. *If hypotheses* H_3 *hold, then problem* (1) *has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega}) \textit{ nodal.}$$

As we did in Proposition 3.5, if we assume an additional mild condition for f (without affecting the geometry of the problem), we can improve the conclusion of Theorem 4.2. We will need this improved version in our next multiplicity theorem.

The new conditions on the reaction f are the following:

$\underline{H_4:}$ $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses H_4(i)–(iii) are the same as the corresponding hypotheses H_3(i)–(iii) and

(iv) for every $\varrho \in (0, \varrho_0]$ with $\varrho_0 = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$, there exists $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta) + \xi_\varrho |\zeta|^{p-2}\zeta$ is nondecreasing on $[-\varrho, \varrho]$.

Remark 4.3. This extra condition is automatically satisfied if for almost all $z \in \Omega$, $f(z,\cdot) \in C^1(\mathbb{R})$ and $f'_\zeta(z,\cdot)$ is locally $L^\infty(\Omega)$-bounded.

In this case we have the following multiplicity theorem.

Theorem 4.4. *If hypotheses* H_4 *hold, then problem* (1) *has at least three nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0] \textit{ nodal.}$$

Proof. From Theorem 4.2, we already have three nontrivial solutions

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \text{and} \quad y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega}) \text{ nodal.}$$

Let $\varrho_0 = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$ and let $\xi_{\varrho_0} > 0$ be as postulated by hypothesis H_4(iv). We choose $\widehat{\xi}_0 > \xi_{\varrho_0}$ and have

$$\begin{aligned} &-\Delta_p y_0(z) - \Delta y_0(z) + \widehat{\xi}_0 |y_0(z)|^{p-2} y_0(z) \\ &= f(z, y_0(z)) + \widehat{\xi}_0 |y_0(z)|^{p-2} y_0(z) \\ &\leqslant f(z, u_0(z)) + \widehat{\xi}_0 u_0(z)^{p-1} \\ &= -\Delta_p u_0(z) - \Delta u_0(z) + \widehat{\xi}_0 u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega \end{aligned} \tag{51}$$

(see hypothesis H_4(iv) and recall that $y_0 \leqslant u_0$).

Because of (15) and (16) (see the proof of Proposition 3.5), we can apply the tangency principle of Pucci–Serrin [38, p. 35] and have that

$$y_0(z) < u_0(z) \quad \forall z \in \Omega. \tag{52}$$

We consider the following two $L^\infty(\Omega)$-functions

$$\begin{aligned} h(z) &= f(z, y_0(z)) + \widehat{\xi}_0 |y_0(z)|^{p-2} y_0(z) \\ &= f(z, y_0(z)) + \xi_{\varrho_0} |y_0(z)|^{p-2} y_0(z) + (\widehat{\xi}_0 - \xi_{\varrho_0}) |y_0(z)|^{p-2} y_0(z) \\ &= h^*(z) + (\widehat{\xi}_0 + \xi_{\varrho_0}) |y_0(z)|^{p-2} y_0(z), \end{aligned}$$

with $h^*(z) = f(z, y_0(z)) + \xi_{\varrho 0}|y_0(z)|^{p-2}y_0(z)$, $h^* \in L^\infty(\Omega)$ and

$$\widehat{h}(z) = f(z, u_0(z)) + \widehat{\xi}_0 u_0(z)^{p-1} = \widehat{h}^*(z) + (\widehat{\xi} - \xi_{\varrho 0})u_0(z)^{p-1},$$

with $\widehat{h}^*(z) = f(z, u_0(z)) + \xi_{\varrho 0}u_0(z)$, $\widehat{h}^* \in L^\infty(\Omega)$.

Note that $h^* \leqslant \widehat{h}^*$ (see hypothesis H_4(iv)). Moreover, from (52) it follows that $(\widehat{\xi}_0 - \xi_{\varrho 0})|y_0|^{p-2}y_0 \prec (\widehat{\xi}_0 - \xi_{\varrho 0})u_0^{p-1}$. Therefore we have

$$h \prec \widehat{h}$$

and so because of (51) we can apply Proposition 2.3 and infer that

$$u_0 - y_0 \in \operatorname{int} C_+.$$

In a similar fashion, we show that

$$y_0 - v_0 \in \operatorname{int} C_+.$$

We conclude that $y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0]$. $\square$

Now we improve the regularity of $f(z, \cdot)$ and strengthen the condition near zero. With these new hypotheses on the reaction, we can improve the previous multiplicity theorems (Theorems 4.2 and 4.4) and produce two nodal solutions (for a total of four nontrivial solutions for problem (1)).

The new conditions on the reaction f are the following:

$\underline{H_5:}$ $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z, 0) = 0$ and $f(z, \cdot) \in C^1(\mathbb{R})$ for almost all $z \in \Omega$ and

(i) for every $\varrho > 0$ there exists $a_\varrho \in L^\infty(\Omega)_+$ such that $|f'_\zeta(z, \zeta)| \leqslant a_\varrho(z)$ for almost all $z \in \Omega$ and all $|\zeta| \leqslant \varrho$;

(ii) there exist functions $w_\pm \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} w_-(z) \leqslant c_- < 0 < c_+ \leqslant w_+(z) &\qquad \forall z \in \overline{\Omega}, \\ A_p(w_-) + A(w_-) \leqslant 0 \leqslant A_p(w_+) + A(w_+) &\qquad \text{in } W^{-1,p'}(\Omega), \\ f(z, w_+(z)) \leqslant 0 \leqslant f(z, w_-(z)) &\qquad \text{for a.a. } z \in \Omega; \end{aligned}$$

(iii) there exists an integer $m \geqslant 2$ such that $f'_\zeta(z, 0) \in [\widehat{\lambda}_m(2), \widehat{\lambda}_{m+1}(2)]$ for almost all $z \in \Omega$ and $f'_\zeta(\cdot, 0) \not\equiv \widehat{\lambda}_m(2)$, $f'_\zeta(\cdot, 0) \not\equiv \widehat{\lambda}_{m+1}(2)$.

Remark 4.5. Note that now the geometry near zero changes since $f(z, \cdot)$ is necessarily linear near there. Also, the extra regularity on $f(z, \cdot)$ implies that for every $\varrho \in (0, \varrho_0)$ (with $\varrho_0 = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$) there exists $\xi_\varrho > 0$ such that for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z, \zeta) + \xi_\varrho|\zeta|^{p-2}\zeta$ is nondecreasing on $[-\varrho, \varrho]$.

Under these conditions on the reaction f we can prove the following multiplicity theorem.

Theorem 4.6. *If hypotheses* H_5 *hold, then problem* (1) *has at least four nontrivial solutions*

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \textit{and} \quad y_0, \widehat{y} \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0] \ \textit{nodal.}$$

Proof. From Theorem 4.4 we already have three solutions

$$u_0 \in \operatorname{int} C_+, \quad v_0 \in -\operatorname{int} C_+ \quad \text{and} \quad y_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[v_0, u_0] \text{ nodal.}$$

We can always assume that u_0, v_0 are extremal (see Proposition 3.6). Let $\widehat{\psi} \in C^1(W_0^{1,p}(\Omega))$ be the functional introduced in the proof of Proposition 4.1. From Claim 2 in that proof, we know that u_0, v_0 are both local minimizers of $\widehat{\psi}$. So, we have

$$C_k(\psi, u_0) = C_k(\psi, v_0) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{53}$$

Note that $\widehat{\psi} \in C^{2-0}(W_0^{1,p}(\Omega))$ (see (32)) and so we cannot apply directly on $\widehat{\psi}$ the results of Morse theory. We need to produce a smooth modification of ψ near u_0 and v_0 where we can have nonsmoothness of $\widehat{\psi}$ (see (32)). We have

$$u_0 - y_0 \in \operatorname{int} C_+ \quad \text{and} \quad y_0 - v_0 \in \operatorname{int} C_+.$$

Fix $\widetilde{u} \in \operatorname{int} C_+$. Invoking Lemma 3.3 of Filippakis–Kristaly–Papageorgiou [11], we know that we can find $\varepsilon > 0$ small such that

$$\varepsilon\widetilde{u} \leqslant u_0 - y_0 \quad \text{and} \quad \varepsilon\widetilde{u} \leqslant y_0 - v_0.$$

We consider the multifunction $L_\varepsilon \colon \Omega \longrightarrow 2^{C^1(\mathbb{R})}$ defined by

$$L_\varepsilon(z) = \left\{ \vartheta \in C^1(\mathbb{R}) \;\middle|\; \begin{array}{l} d_{C^1(\mathbb{R})}(\beta(z,\cdot), \vartheta) < \varepsilon\widetilde{u}(z) \text{ and } \beta(z,\zeta) = \vartheta(z) \\ \text{for } |\zeta - u_0(z)| \geqslant \varepsilon\widetilde{u}(z) \text{ and for} \\ |\zeta - v_0(z)| \geqslant \varepsilon\widetilde{u}(z) \end{array} \right\}. \tag{54}$$

Evidently

$$\operatorname{Gr} L_\varepsilon = \big\{(z, \vartheta) \in \Omega \times C^1(\mathbb{R}) : \ \vartheta \in L_\varepsilon(z)\big\} \in \mathcal{L} \times \mathcal{B}(C^1(\mathbb{R})),$$

with $\mathcal{L}$ being the Lebesgue σ-field on Ω and $\mathcal{B}(C^1(\mathbb{R}))$ being the Borel σ-field on $C^1(\mathbb{R})$. Recall that $C^1(\mathbb{R})$ is a separable Fréchet space. So, we can apply the Yankov–von Neumann–Aumann selection theorem (see Hu–Papageorgiou [27, p. 158] or Gasiński–Papageorgiou [15, Theorem A.2.33, p. 906]) and obtain a Lebesgue measurable map $\widehat{\vartheta} \colon \Omega \longrightarrow C^1(\mathbb{R})$, such that

$$\widehat{\vartheta}(z) \in L_\varepsilon(z) \quad \forall z \in \Omega.$$

We set

$$\vartheta_0(z,\zeta)=\widehat{\vartheta}(z)(\zeta)\quad \forall (z,\zeta)\in\Omega\times\mathbb{R}.$$

Then $\vartheta_0(\cdot,\cdot)$ is a measurable function and for all $z\in\Omega$, $\vartheta_0(z,\cdot)\in C^1(\mathbb{R})$. We set

$$\Theta_0(z,\zeta)=\int_0^\zeta \vartheta_0(z,s)\,ds$$

and consider the C^2-functional $\widehat{\psi}_0\colon W_0^{1,p}(\Omega)\longrightarrow\mathbb{R}$ defined by

$$\widehat{\psi}_0(u)=\frac{1}{p}\|\nabla u\|_p^p+\frac{1}{2}\|\nabla u\|_2^2-\int_\Omega \Theta_0(z,u(z))\,dz\quad \forall u\in W_0^{1,p}(\Omega).$$

From (54) it is clear that $\widehat{\psi}$ and $\widehat{\psi}_0$ are nearby in $C^1(W_0^{1,p}(\Omega))$. So, by choosing $\varepsilon>0$ small and exploiting the continuity of the critical groups in the C^1-norm (see Chang [6, p. 334]), we infer that

$$C_k(\widehat{\psi},y_0)=C_k(\widehat{\psi}_0,y_0)\quad \forall k\geqslant 0. \tag{55}$$

From the proof of Proposition 4.1, we know that y_0 is a critical point of mountain pass type for the functional $\widehat{\psi}$. Therefore

$$C_1(\widehat{\psi},y_0)\neq 0$$

(see Motreanu–Motreanu–Papageorgiou [31, p. 176]), so $C_1(\widehat{\psi}_0,y_0)\neq 0$ by (55) and thus

$$C_k(\widehat{\psi}_0,y_0)=\delta_{k,1}\mathbb{Z}\quad \forall k\geqslant 0$$

(see Papageorgiou–Smyrlis [36] and recall that $\widehat{\psi}_0\in C^2(W_0^{1,p}(\Omega))$). Hence

$$C_k(\widehat{\psi},y_0)=\delta_{k,1}\mathbb{Z}\quad \forall k\geqslant 0 \tag{56}$$

(see (55)).

Let $\xi(z)=f'_\zeta(z,0)$, $\xi\in L^\infty(\Omega)_+$ (see hypotheses H_5(i),(iii)) and consider the C^2-functional $\sigma_0\colon H_0^1(\Omega)\longrightarrow\mathbb{R}$ defined by

$$\sigma_0(u)=\frac{1}{p}\|\nabla u\|_2^2-\frac{1}{2}\int_\Omega \xi(z)u(z)^2\,dz\quad \forall u\in H_0^1(\Omega).$$

Let $\widehat{\sigma}_0=\sigma_0\big|_{W_0^{1,p}(\Omega)}$ and consider the following homotopy

$$h(t,u)=t\psi(u)+(1-t)\widehat{\sigma}_0(u)\quad \forall (t,u)\in[0,1]\times W_0^{1,p}(\Omega).$$

Suppose that we can find two sequences

$$\{t_n\}_{n\geqslant 1}\subseteq[0,1]\quad\text{and}\quad \{u_n\}_{n\geqslant 1}\subseteq W_0^{1,p}(\Omega)\setminus\{0\}$$

such that

$$t_n \longrightarrow t, \quad u_n \longrightarrow 0 \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad h'_u(t_n, u_n) = 0 \quad \forall n \geqslant 1. \tag{57}$$

From the equation in (57), we have

$$t_n A_p(u_n) + A(u_n) = t_n N_\beta(u_n) + (1 - t_n)\xi u_n \quad \forall n \geqslant 1, \tag{58}$$

so

$$\begin{cases} -t_n \Delta_p u_n(z) - \Delta u_n(z) = t_n \beta(z, u_n(z)) + (1 - t_n)\xi(z) u_n(z) & \text{in } \Omega, \\ u_n|_{\partial\Omega} = 0. \end{cases} \tag{59}$$

From (59) and Lieberman [30, Theorem 1], we know that we can find $\mu \in (0, 1)$ and $c_5 > 0$ such that

$$u_n \in C_0^{1,\mu}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C_0^{1,\mu}(\overline{\Omega})} \leqslant c_5 \quad \forall n \geqslant 1. \tag{60}$$

From (57), (60) and since $C_0^{1,\mu}(\overline{\Omega})$ is embedded compactly in $C_0^1(\overline{\Omega})$, we have

$$u_n \longrightarrow 0 \quad \text{in } C_0^1(\overline{\Omega}).$$

Recall that $u_0 \in \operatorname{int} C_+$, $v_0 \in -\operatorname{int} C_+$. So, we can find $n_0 \in \mathbb{N}$ such that

$$u_n \in [v_0, u_0] \quad \forall n \geqslant n_0. \tag{61}$$

Then (58) becomes

$$t_n A_p(u_n) + A(u_n) = t_n N_f(u_n) + (1 - t_n)\xi u_n \quad \forall n \geqslant n_0 \tag{62}$$

(see (32) and (61)).

Let

$$y_n = \frac{u_n}{\|u_n\|} \quad \forall n \geqslant 1.$$

Then $\|y_n\| = 1$ for all $n \geqslant 1$ and so we may assume (at least for a subsequence) that

$$y_n \longrightarrow u \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{63}$$

$$y_n \longrightarrow u \quad \text{in } L^p(\Omega). \tag{64}$$

From (62) we have

$$t_n \|u_n\|^{p-2} A_p(y_n) + A(y_n) = t_n \frac{N_f(u_n)}{\|u_n\|} + (1 - t_n)\xi y_n \quad \forall n \geqslant n_0. \tag{65}$$

Hypotheses H_5(i),(iii) imply that the sequence $\left\{\frac{N_f(u_n)}{\|u_n\|}\right\}_{n\geqslant 1} \subseteq L^2(\Omega)$ is bounded and so for at least for a subsequence, we have

$$\frac{N_f(u_n)}{\|u_n\|} \longrightarrow \xi y \quad \text{weakly in } L^2(\Omega) \tag{66}$$

(see Gasiński–Papageorgiou [16]). So, if in (65) we pass to the limit as $n \to +\infty$ and since $\|u_n\| \to +\infty$ (see (57)), we obtain $A(y) = \xi y$, so

$$\begin{cases} -\Delta y(z) = \xi(z) y(z) & \text{in } \Omega, \\ \quad y|_{\partial\Omega} = 0. \end{cases} \tag{67}$$

From Proposition 2.5 we have

$$\widehat{\lambda}_m(2,\xi) < \widehat{\lambda}_m(2,\widehat{\lambda}_m(2)) = 1, \tag{68}$$

$$1 = \widehat{\lambda}_{m+1}(2,\widehat{\lambda}_{m+1}(2)) < \widehat{\lambda}_{m+1}(2,\xi). \tag{69}$$

From (67)–(69), it follows that $y \equiv 0$. On the other hand, from (65) as before via the nonlinear regularity result of Lieberman [30, Theorem 1], we have

$$y_n \longrightarrow y \quad \text{in } C_0^1(\Omega),$$

so $\|y\| = 1$, a contradiction.

Therefore (57) cannot occur and then from the homotopy invariance of critical groups (see Chang [6, p. 334]), we have

$$C_k(\widehat{\psi},0) = C_k(\widehat{\sigma}_0,0) \quad \forall k \geqslant 0. \tag{70}$$

Since $W_0^{1,p}(\Omega)$ is dense in $H_0^1(\Omega)$, we have

$$C_k(\widehat{\sigma}_0,0) = C_k(\widehat{\sigma},0) \quad \forall k \geqslant 0 \tag{71}$$

(see Palais [33]). Note that $u = 0$ is a nondegenerate critical point of $\widehat{\sigma}$. Hence

$$C_k(\widehat{\sigma},0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geqslant 0, \tag{72}$$

with $d_m = \dim \bigoplus_{i=1}^m E(\widehat{\lambda}_i(2)) \geqslant 2$ (see Motreanu–Motreanu–Papageorgiou [31, p. 155]). From (70)–(72) it follows that

$$C_k(\widehat{\psi},0) = \delta_{k,d_m}\mathbb{Z} \quad \forall k \geqslant 0. \tag{73}$$

Recall that $\widehat{\psi}$ is coercive (see (32)). Hence

$$C_k(\widehat{\psi},\infty) = \delta_{k,0}\mathbb{Z} \quad \forall k \geqslant 0. \tag{74}$$

Suppose that $K_{\widehat{\psi}} = \{0, u_0, v_0, y_0\}$. Then from (55), (56), (73), (74) and the Morse relation (4) with $t = -1$, we have

$$(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0,$$

so $(-1)^{d_m} = 0$, a contradiction. Thus, there exists $\widehat{y} \in K_{\widehat{\psi}}$, $\widehat{y} \notin \{0, u_0, v_0, y_0\}$. It follows (see the proof of Lieberman [30, Proposition 10]) that

$$\widehat{y} \in [v_0, y_0] \cap C_0^1(\overline{\Omega}).$$

Hence $\widehat{y}$ is nodal. Moreover, as in the proof of Theorem 4.4, we have $\widehat{y} \in \mathrm{int}_{C_0^1(\overline{\Omega})}[v_0, u_0]$. □

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