

Interior Feedback Stabilization of Wave Equations with Dynamic Boundary Delay

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Abstract. In this paper we consider an interior stabilization problem for the wave equation with dynamic boundary delay. We prove some stability results under the choice of damping operator. The proof of the main result is based on a frequency domain method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

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1. Introduction

We study the interior stabilization of a wave equation in an open bounded domain Ω of \mathbb{R}^n , $n \geq 2$. We denote by $\partial\Omega$ the boundary of Ω and we assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0, Γ_1 are closed subsets of $\partial\Omega$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover we assume $\text{meas}(\Gamma_0) > 0$. The system is given by:

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u + a u_t = 0, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, t > 0, \\ u_{tt}(x, t) = -\frac{\partial u}{\partial \nu}(x, t) - \mu u_t(x, t - \tau) & x \in \Gamma_1, t > 0, \\ u(x, 0) = u_0(x) & x \in \Omega, \\ u_t(x, 0) = u_1(x) & x \in \Omega, \\ u_t(x, t - \tau) = f_0(x, t - \tau) & x \in \Gamma_1, t \in (0, \tau), \end{array} \right. \quad (1.1)$$

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where ν stands for the unit normal vector of $\partial\Omega$ pointing towards the exterior of Ω and $\frac{\partial}{\partial\nu}$ is the normal derivative. Moreover, the constant $\tau > 0$ is the time delay, a and μ are positive numbers and the initial data u_0 , u_1 , f_0 are given functions belonging to suitable spaces that will be precised later.

Let us first review some results for particular cases which seem to us interesting.

In the absence of the delay term (i.e., $\tau = 0$) problem (1.1) becomes

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u + a u_t = 0, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, t > 0, \\ u_{tt}(x, t) = -\frac{\partial u}{\partial\nu}(x, t) - \mu u_t(x, t) & x \in \Gamma_1, t > 0, \\ u(x, 0) = u_0(x) & x \in \Omega, \\ u_t(x, 0) = u_1(x) & x \in \Omega. \end{array} \right. \quad (1.2)$$

This type of problems arise (for example) in modelling of longitudinal vibrations in a homogeneous bar in which there are viscous effects. The term au_t , indicates that the stress is proportional not only to the strain, but also to the displacement rate (see [13] for instance). From the mathematical point of view, these problems do not neglect acceleration terms on the boundary. Such type of boundary conditions are usually called *dynamic boundary conditions*. They are not only important from the theoretical point of view but also arise in several physical applications. For instance in one space dimension, problem (1.2) can modelize the dynamic evolution of a viscoelastic rod that is fixed at one end and has a tip mass attached to its free end. The dynamic boundary conditions represents the Newton's law for the attached mass (see [6, 12, 17] for more details). In the two dimension space, as showed in [49] and in the references therein, these boundary conditions arise when we consider the transverse motion of a flexible membrane Ω whose boundary may be affected by the vibrations only in a region. Also some dynamic boundary conditions as in problem (1.2) appear when we assume that Ω is an exterior domain of \mathbb{R}^3 in which homogeneous fluid is at rest except for sound waves. Each point of the boundary is subjected to small normal displacements into the obstacle (see [9] for more details). This type of dynamic boundary conditions are known as acoustic boundary conditions.

Well-posedness and longtime behavior for analogous equations as (1.1) (with out delay) on bounded domains have been investigated by many authors in recent years (see, e.g., [25, 26, 45, 46]).

Among the early results dealing with this type of boundary conditions are those of Grobbelaar-Van Dalsen [25–27] in which the author has made contributions to this field.

In [25] the author introduced a model which describes the damped longitudinal vibrations of a homogeneous flexible horizontal rod of length L when

the end $x = 0$ is rigidly fixed while the other end $x = L$ is free to move with an attached load. This yields to a system of two second order equations of the form

$$\left\{ \begin{array}{ll} u_{tt} - u_{xx} - u_{txx} = 0, & x \in (0, L), \ t > 0, \\ u(0, t) = u_t(0, t) = 0, & t > 0, \\ u_{tt}(L, t) = -[u_x + u_{tx}](L, t), & t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x) & x \in (0, L), \\ u(L, 0) = \eta, \quad u_t(L, 0) = \mu. & \end{array} \right. \quad (1.3)$$

By rewriting problem (1.3) within the framework of the abstract theories of the so-called B -evolution theory, an existence of a unique solution in the strong sense has been shown. An exponential decay result was also proved in [27] for a problem related to (1.3), which describe the weakly damped vibrations of an extensible beam. See [27] for more details.

Subsequently, Zang and Hu [51], considered the problem

$$\left\{ \begin{array}{ll} u_{tt} - p(u_x)_{xt} - q(u_x)_x = 0, & x \in (0, 1), \ t > 0, \\ u(0, t) = 0, & t \geq 0, \\ p(u_x)_t + q(u_x)(1, t) + ku_{tt}(1, t) = 0, & t \geq 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in (0, 1). \end{array} \right.$$

By using the Nakao inequality, and under appropriate conditions on p and q , they established both exponential and polynomial decay rates for the energy depending on the form of the terms p and q .

Similarly, and always in the absence of the delay term, Pellicer and Sol-Morales [46] considered the one dimensional problem of (1.1) as an alternative model for the classical spring-mass damper system, and by using the dominant eigenvalues method, they proved that for small values of the parameter a the partial differential equations in problem (1.2) has the classical second order differential equation

$$m_1 u''(t) + d_1 u'(t) + k_1 u(t) = 0,$$

as a limit, where the parameter m_1 , d_1 and k_1 are determined from the values of the spring-mass damper system. Thus, the asymptotic stability of the model has been determined as a consequence of this limit. But they did not obtain any rate of convergence. This result was followed by recent works [45, 47]. In particular in [47], the authors considered a one dimensional nonlocal nonlinear strongly damped wave equation with dynamical boundary conditions. In other

words, they looked to the following problem:

$$\left\{ \begin{array}{l} u_{tt} - u_{xx} - \alpha u_{txx} + \varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right) = 0, \\ u(0, t) = 0, \\ u_{tt}(1, t) + \varepsilon [u_x + \alpha u_{tx} + r u_t](1, t) + \varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right) = 0, \end{array} \right. \quad (1.4)$$

with $x \in (0, 1)$, $t > 0$, $r, \alpha > 0$ and $\varepsilon \geq 0$. The above system models a spring-mass-damper system, where the term $\varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right)$ represents a control acceleration at $x = 1$. By using the invariant manifold theory, the authors proved that for small values of the parameter ε , the solutions of (1.4) are attracted to a two dimensional invariant manifold. See [47], for further details.

The main difficulty of the problem considered is related to the non ordinary boundary conditions defined on Γ_1 . Very little attention has been paid to this type of boundary conditions. We mention only a few particular results in the one dimensional space [23, 29, 32, 46].

A related problem to (1.2) is the following:

$$\begin{aligned} u_{tt} - \Delta u + g(u_t) &= f && \text{in } \Omega \times (0, T) \\ \frac{\partial u}{\partial \nu} + K(u)u_{tt} + h(u_t) &= 0 && \text{on } \partial\Omega \times (0, T) \\ u(x, 0) &= u_0(x) && \text{in } \Omega \\ u_t(x, 0) &= u_1(x) && \text{in } \Omega \end{aligned}$$

where the boundary term $h(u_t) = |u_t|^\rho u_t$ arises when one studies flows of gas in a channel with porous walls. The term u_{tt} on the boundary appears from the internal forces, and the nonlinearity $K(u)u_{tt}$ on the boundary represents the internal forces when the density of the medium depends on the displacement. This problem has been studied in [22, 23]: by using the Fadeo–Galerkin approximations and a compactness argument the authors proved the global existence and the exponential decay of the solution of the problem.

We recall some results related to the interaction of an elastic medium with rigid mass. By using the classical semigroup theory, Littman and Markus [35] established a uniqueness result for a particular Euler–Bernoulli beam rigid body structure. They also proved the asymptotic stability of the structure by using the feedback boundary damping. In [36] the authors considered the Euler–Bernoulli beam equation which describes the dynamics of clamped elastic beam in which one segment of the beam is made with viscoelastic material and the other of elastic material. By combining the frequency domain method with the multiplier technique, they proved the exponential decay for the transversal motion but not for the longitudinal motion of the model, when the Kelvin–Voigt

damping is distributed only on a subinterval of the domain. In relation with this point, see also the work by Chen et al. [16] concerning the Euler–Bernoulli beam equation with the global or local Kelvin–Voigt damping. Also models of vibrating strings with local viscoelasticity and Boltzmann damping, instead of the Kelvin–Voigt one, were considered in [37] and an exponential energy decay rate was established. Recently, Grobbelaar-Van Dalsen [28] considered an extensible thermo-elastic beam which is hanged at one end with rigid body attached to its free end, i.e., one dimensional hybrid thermoelastic structure, and showed that the method used in [44] is still valid to establish an uniform stabilization of the system. Concerning the controllability of the hybrid system we refer to the work by Castro and Zuazua [14], in which they considered flexible beams connected by point mass and the model takes account of the rotational inertia.

The purpose of this paper is to study problem (1.1), in which a delay term acted in the dynamic boundary conditions. In recent years one very active area of mathematical control theory has been the investigation of the delay effect in the stabilization of hyperbolic systems and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays (see [2, 3, 21, 39, 40, 42] for more details).

As it has been proved by Datko [19, Example 3.5], systems of the form

$$\begin{cases} w_{tt} - w_{xx} - aw_{xxt} = 0, & x \in (0, 1), t > 0, \\ w(0, t) = 0, & t > 0 \\ w_x(1, t) = -kw_t(1, t - \tau), & t > 0, \end{cases} \quad (1.5)$$

where a, k and τ are positive constants become unstable for an arbitrarily small values of τ and any values of a and k . In (1.5) and even in the presence of the strong damping $-aw_{xxt}$, without any other damping, the overall structure can be unstable. This was one of the main motivations for considering problem (1.1) (of course the structure of problem (1.1) and (1.5) are different due to the nature of the boundary conditions in each problem).

Subsequently, Datko et al. [21] treated the following one dimensional problem:

$$\begin{cases} u_{tt}(x, t) - u_{xx}(x, t) + 2au_t(x, t) + a^2u(x, t) = 0, & 0 < x < 1, t > 0, \\ u(0, t) = 0, & t > 0, \\ u_x(1, t) + ku_t(1, t - \tau) = 0, & t > 0, \end{cases} \quad (1.6)$$

which models the vibrations of a string clamped at one end and free at the other end, where $u(x, t)$ is the displacement of the string. Also, the string is controlled by a boundary control force (with a delay) at the free end. They showed that, if the positive constants a and k satisfy

$$k \frac{e^{2a} + 1}{e^{2a} - 1} < 1,$$

then the delayed feedback system (1.6) is stable for all sufficiently small delays. On the other hand if

$$k \frac{e^{2a} + 1}{e^{2a} - 1} > 1,$$

then there exists a dense open set D in $(0, \infty)$ such that for each $\tau \in D$, system (1.6) admits exponentially unstable solutions.

It is well known that in the absence of delay in (1.6), that is for $\tau = 0$, system (1.6) is uniformly asymptotically stable under the condition $a^2 + k^2 > 0$ and the total energy of the solution satisfies for all $t > 0$,

$$E(t, u) = \int_0^1 (u_t^2 + u_x^2 + a^2 u^2) dx \leq CE(0, u) e^{-\alpha t}$$

for some positive constant α . See [15] for more details.

Recently, Ammari et al. [5] have treated the N -dimensional wave equation

$$\left\{ \begin{array}{ll} u_{tt}(x, t) - \Delta u(x, t) + au_t(x, t - \tau) = 0, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial \nu}(x, t) + ku(x, t) = 0, & x \in \Gamma_1, t > 0, \\ u(x, 0) = u_0(x) & x \in \Omega \\ u_t(x, 0) = u_1(x) & x \in \Omega \\ u_t(x, t - \tau) - f_0(x, t - \tau) = 0 & x \in \Gamma_1, t \in (0, \tau) \end{array} \right.$$

where Ω is an open bounded domain of \mathbb{R}^N , $N \geq 2$ with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Under the usual geometric condition on the domain Ω , they showed an exponential stability result, provided that the delay coefficient a is sufficiently small.

In [39] the authors examined a system of wave equation with a linear boundary damping term with a delay. Namely, they looked to the following system

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u = 0, & x \in \Omega, t > 0 \\ u(x, t) = 0, & x \in \Gamma_0, t > 0 \\ \frac{\partial u}{\partial \nu}(x, t) - \mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau) = 0 & x \in \Gamma_1, t > 0 \\ u(x, 0) = u_0(x) & x \in \Omega \\ u_t(x, 0) = u_1(x) & x \in \Omega \\ u_t(x, t - \tau) - g_0(x, t - \tau) = 0 & x \in \Omega, \tau \in (0, 1) \end{array} \right. \tag{1.7}$$

and proved under the assumption

$$\mu_2 < \mu_1 \tag{1.8}$$

(which means that the weight of the feedback with delay is smaller than the one without delay) that null stationary state is exponentially stable. On the contrary, if (1.8) does not hold, they found a sequence of delays for which the corresponding solution of (1.7) will be unstable. The main approach used in [39], is an observability inequality obtained with a Carleman estimate.

The case of time-varying delay (i.e., $\tau = \tau(t)$ is a function depending on t) has been studied by Nicaise, Valein and Fridman [43] in one space dimension. In their work, an exponential stability result was given under the condition:

$$\mu_2 < \sqrt{1-d} \mu_1,$$

where d is a constant such that

$$\tau'(t) \leq d < 1, \quad \forall t > 0.$$

Delay effects arise in many applications and practical problems and it is well-known that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in absence of delay (see, e.g., [18, 21, 33, 34, 52]).

The stability of (1.1) with $\tau = 0, a = 0$ has been studied in [1] where it has been shown that the system is stable under some geometric condition on Γ_1 (as in [7]). Moreover, if $\mu = 0$, that is in absence of delay, the above problem for any $a > 0$ is exponentially stable even. On the contrary, in presence of a delay term there are instability phenomena probably, as in [39].

Let us also cite the recent work of Ammari and Nicaise, [4], in which the authors performed a complete study of the stabilisation of elastic systems by collocated feedback with or without delay.

In this paper the idea is to contrast the effect of the time delay by using the dissipative feedback (i.e., by giving the control in the feedback form $au_t(x, t)$ or $-a\Delta u_t(x, t)$, $x \in \Omega$, $t > 0$).

In the next section, we will show the global existence of problem (1.1) by transforming the delay term and by using a semigroup approach. The natural question is then the stability of problem (1.1). This is the goal of Section 3. We will show that a “shifted” problem is asymptotically stable with a polynomial decay rate and we cannot answer the question of the stability of problem (1.1). In fact, in the last section, numerical experiments in 1D shows that under certain conditions, problem (1.1) is unstable. To stabilize problem (1.1), we will see that a Kelvin–Voigt damping is efficient. This is done in Section 4.

Lastly, we will conduct some numerical examples in 1D to illustrate these stability or instability results.

2. Well-posedness of problem (1.1)

In this section we will first transform the delay boundary conditions by adding a new unknown. Then we will use a semigroup approach and the Lumer–Phillips’ theorem to prove the existence and uniqueness of the solution of the problem (1.1).

We point out that the well-posedness in evolution equations with delay is not always obtained. Recently, Dreher, Quintilla and Racke have shown some ill-posedness results for a wide range of evolution equations with a delay term [24].

2.1. Setup and notations. We present here some material that we shall use in order to prove the local existence of the solution of problem (1.1). We denote

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) \mid u_{\Gamma_0} = 0\}.$$

We set γ_1 the trace operator from $H_{\Gamma_0}^1(\Omega)$ on $L^2(\Gamma_1)$ and

$$H^{\frac{1}{2}}(\Gamma_1) = \gamma_1(H_{\Gamma_0}^1(\Omega)).$$

By (\cdot, \cdot) we denote the scalar product in $L^2(\Omega)$, i.e., $(u, v)(t) = \int_{\Omega} u(x, t)v(x, t)dx$. Also we mean by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$, and by $\|\cdot\|_{q, \Gamma_1}$ the $L^q(\Gamma_1)$ norm.

Let $T > 0$ be a real number and X a Banach space endowed with norm $\|\cdot\|_X$. $L^p(0, T; X)$, $1 \leq p < \infty$ denotes the space of functions f which are L^p over $(0, T)$ with values in X , which are measurable and $\|f\|_X \in L^p(0, T)$. This space is a Banach space endowed with the norm

$$\|f\|_{L^p(0, T; X)} = \left(\int_0^T \|f\|_X^p dt \right)^{\frac{1}{p}}.$$

$L^\infty(0, T; X)$ denotes the space of functions $f :]0, T[\rightarrow X$ which are measurable and $\|f\|_X \in L^\infty(0, T)$. This space is a Banach space endowed with the norm:

$$\|f\|_{L^\infty(0, T; X)} = \text{ess sup}_{0 < t < T} \|f\|_X.$$

We recall that if X and Y are two Banach spaces such that $X \hookrightarrow Y$ (continuous embedding), then

$$L^p(0, T; X) \hookrightarrow L^p(0, T; Y), \quad 1 \leq p \leq \infty.$$

2.2. Semigroup formulation of the problem. In this section, we will prove the global existence and the uniqueness of the solution of problem (1.1). We will first transform the problem (1.1) to the problem (2.2) by making the change of variables (2.1), and then we use the semigroup approach to prove the existence of the unique solution of problem (2.2).

To overcome the problem of the boundary delay, we introduce the new variable:

$$z(x, \rho, t) = u_t(x, t - \tau\rho), \quad x \in \Gamma_1, \quad \rho \in (0, 1), \quad t > 0. \quad (2.1)$$

Then, we have

$$\tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad \text{in } \Gamma_1 \times (0, 1) \times (0, +\infty).$$

Therefore, problem (1.1) is equivalent to:

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u + a u_t = 0, & x \in \Omega, t > 0 \\ u(x, t) = 0, & x \in \Gamma_0, t > 0 \\ u_{tt}(x, t) + \frac{\partial u}{\partial \nu}(x, t) + \mu z(x, 1, t) = 0, & x \in \Gamma_1, t > 0 \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, & x \in \Gamma_1, \rho \in (0, 1), t > 0 \\ z(x, 0, t) = u_t(x, t) & x \in \Gamma_1, t > 0 \\ u(x, 0) = u_0(x) & x \in \Omega \\ u_t(x, 0) = u_1(x) & x \in \Omega \\ z(x, \rho, 0) - f_0(x, -\tau\rho) = 0 & x \in \Gamma_1, \rho \in (0, 1). \end{array} \right. \quad (2.2)$$

The first natural question is the existence of solutions of the problem (2.2). In this section we will give a sufficient condition that guarantees that this problem is well-posed.

For this purpose we will use a semigroup formulation of the initial-boundary value problem (2.2). If we denote $V := (u, u_t, \gamma_1(u_t), z)^T$, we define the energy space:

$$\mathcal{H} = H_{\Gamma_0}^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1 \times (0, 1)).$$

Clearly, \mathcal{H} is a Hilbert space with respect to the inner product

$$\langle V_1, V_2 \rangle_{\mathcal{H}} = \int_{\Omega} \nabla u_1 \cdot \nabla u_2 dx + \int_{\Omega} v_1 v_2 dx + \int_{\Gamma_1} w_1 w_2 d\sigma + \xi \int_{\Gamma_1} \int_0^1 z_1 z_2 d\rho d\sigma \quad (2.3)$$

for $V_1 = (u_1, v_1, w_1, z_1)^T$, $V_2 = (u_2, v_2, w_2, z_2)^T$ and $\xi > 0$ a nonnegative real number defined later.

Therefore, if $V_0 \in \mathcal{H}$ and $V \in \mathcal{H}$, the problem (2.2) is formally equivalent to the following abstract evolution equation in the Hilbert space \mathcal{H} :

$$\begin{cases} V'(t) = \mathcal{A}V(t), & t > 0, \\ V(0) = V_0, \end{cases} \quad (2.4)$$

where $'$ denotes the derivative with respect to time t ,

$$V_0 := (u_0, u_1, \gamma_1(u_1), f_0(\cdot, -\tau)) ^T$$

and the operator \mathcal{A} is defined by:

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta u - a v \\ -\frac{\partial u}{\partial \nu} - \mu z(\cdot, 1) \\ -\frac{1}{\tau} z_\rho \end{pmatrix}.$$

The domain of \mathcal{A} is the set of $V = (u, v, w, z)^T$ such that:

$$\begin{aligned} (u, v, w, z)^T &\in (H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)) \times H^1_{\Gamma_0}(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0, 1)), \\ w = \gamma_1(v) = z(\cdot, 0) &\quad \text{on } \Gamma_1. \end{aligned} \tag{2.5}$$

Let us finally define $\xi^* = \mu\tau$. For all $\xi > \xi^*$, we also define $\mu_1 = \frac{\xi}{2\tau} + \frac{\mu}{2}$ and $\mathcal{A}_d = \mathcal{A} - \mu_1 I$.

The well-posedness of problem (2.2) is ensured by:

Theorem 2.1. *Let $V_0 \in \mathcal{H}$, then there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of problem (2.4). Moreover, if $V_0 \in \mathcal{D}(\mathcal{A})$, then*

$$V \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

Proof. To prove Theorem 2.1, we first prove that there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of the shifted problem:

$$\begin{cases} V'(t) = \mathcal{A}_d V(t), & t > 0, \\ V(0) = V_0, \end{cases} \tag{2.6}$$

Then as $\mathcal{A} = \mathcal{A}_d + \mu_1 I$, there will exist $V \in C(\mathbb{R}_+; \mathcal{H})$ solution of problem (2.4).

In order to prove the existence and uniqueness of the solution of problem (2.6) we use the semigroup approach and the Lumer–Phillips’ theorem.

Indeed, let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$. By definition of the operator \mathcal{A} and the scalar product of \mathcal{H} , we have:

$$\begin{aligned} \langle \mathcal{A}V, V \rangle_{\mathcal{H}} &= \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} v \Delta u \, dx - \int_{\Omega} a|v(x)|^2 \, dx \\ &\quad + \int_{\Gamma_1} w \left(-\frac{\partial u}{\partial \nu} - \mu z(\sigma, 1) \right) \, d\sigma - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z z_{\rho} \, d\rho \, d\sigma. \end{aligned}$$

By Green’s formula we obtain:

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} = - \int_{\Omega} a|v(x)|^2 \, dx - \mu \int_{\Gamma_1} z(\sigma, 1) w \, d\sigma - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_{\rho} z \, d\rho \, dx. \tag{2.7}$$

But we have:

$$\frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_{\rho} z(\sigma, \rho) \, d\rho \, d\sigma = \frac{\xi}{2\tau} \int_{\Gamma_1} \int_0^1 \frac{\partial}{\partial \rho} z^2(\sigma, \rho) \, d\rho \, d\sigma = \frac{\xi}{2\tau} \int_{\Gamma_1} (z^2(\sigma, 1) - z^2(\sigma, 0)) \, d\sigma.$$

Thus from the compatibility condition (2.5), we get $-\frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_{\rho} z \, d\rho \, d\sigma = \frac{\xi}{2\tau} \int_{\Gamma_1} (v^2 - z^2(\sigma, 1)) \, d\sigma$. Therefore equation (2.7) becomes:

$$\begin{aligned} \langle \mathcal{A}V, V \rangle_{\mathcal{H}} &= - \int_{\Omega} a|v(x)|^2 \, dx - \frac{\xi}{2\tau} \int_{\Gamma_1} \int_0^1 z^2(\sigma, 1) \, d\sigma \\ &\quad + \frac{\xi}{2\tau} \int_{\Gamma_1} |v|^2(\sigma) \, d\sigma - \mu \int_{\Gamma_1} z(\sigma, 1) w \, d\sigma. \end{aligned}$$

To treat the last term in the preceding equation, Young's inequality gives:

$$-\int_{\Gamma_1} v(\sigma)z(\sigma, 1) d\sigma \leq \frac{1}{2} \int_{\Gamma_1} z^2(\sigma, 1) d\sigma + \frac{1}{2} \int_{\Gamma_1} v^2(\sigma) d\sigma.$$

Therefore, we firstly get:

$$\begin{aligned} & \langle \mathcal{A}V, V \rangle_{\mathcal{H}} + \int_{\Omega} a|v(x)|^2 dx - \left(\frac{\xi}{2\tau} + \frac{\mu}{2}\right) \int_{\Gamma_1} |v(\sigma)|^2 d\sigma + \left(\frac{\xi}{2\tau} - \frac{\mu}{2}\right) \int_{\Gamma_1} z^2(\sigma, 1) d\sigma \\ & \leq 0 \end{aligned}$$

which writes

$$\langle \mathcal{A}V, V \rangle_{\mathcal{H}} + \int_{\Omega} a|v(x)|^2 dx - \mu_1 \int_{\Gamma_1} |v(\sigma)|^2 d\sigma + \left(\frac{\xi}{2\tau} - \frac{\mu}{2}\right) \int_{\Gamma_1} z^2(\sigma, 1) d\sigma \leq 0.$$

From the preceding inequality, we get:

$$\left\langle \left(\mathcal{A} - \mu_1 I\right)V, V \right\rangle_{\mathcal{H}} \leq - \int_{\Omega} a|v(x)|^2 dx - \left(\frac{\xi}{2\tau} - \frac{\mu}{2}\right) \int_{\Gamma_1} z^2(\sigma, 1) d\sigma. \quad (2.8)$$

As $\left(\frac{\xi}{2\tau} - \frac{\mu}{2}\right) > 0$ for all $\xi > \xi^*$, we finally get:

$$\left\langle \left(\mathcal{A} - \mu_1 I\right)V, V \right\rangle_{\mathcal{H}} \leq 0.$$

Thus the operator $\mathcal{A}_d = \mathcal{A} - \mu_1 I$ is dissipative.

Now we want to show that $\forall \lambda > 0, \forall \xi > \xi^*, \lambda I - \mathcal{A}_d$ is surjective. To prove that, it is clear that it suffices to show that $\lambda I - \mathcal{A}$ is surjective for all $\lambda > 0$.

For $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})V = F,$$

which is:

$$\lambda u - v = f_1, \tag{2.9}$$

$$\lambda v - \Delta u + av = f_2, \tag{2.10}$$

$$\lambda w + \frac{\partial u}{\partial \nu} + \mu z(\cdot, 1) = f_3, \tag{2.11}$$

$$\lambda z + \frac{1}{\tau} z_{\rho} = f_4. \tag{2.12}$$

To find $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of the system (2.9)–(2.12), we suppose u is determined with the appropriate regularity. Then from (2.9), we get:

$$v = \lambda u - f_1. \tag{2.13}$$

Therefore, from the compatibility condition on Γ_1 , (2.5), we determine $z(\cdot, 0)$ by:

$$z(x, 0) = v(x) = \lambda u(x) - f_1(x), \quad \text{for } x \in \Gamma_1.$$

Thus, from (2.12), z is the solution of the linear Cauchy problem:

$$\begin{cases} z_\rho = \tau \left(f_4(x) - \lambda z(x, \rho) \right), & \text{for } x \in \Gamma_1, \rho \in (0, 1), \\ z(x, 0) = \lambda u(x) - f_1(x). \end{cases} \quad (2.14)$$

The solution of the Cauchy problem (2.14) is given by:

$$z(x, \rho) = \lambda u(x)e^{-\lambda\rho\tau} - f_1e^{-\lambda\rho\tau} + \tau e^{-\lambda\rho\tau} \int_0^\rho f_4(x, \sigma)e^{\lambda\sigma\tau} d\sigma \quad (2.15)$$

for $x \in \Gamma_1$, $\rho \in (0, 1)$. So, we have at the point $\rho = 1$,

$$z(x, 1) = \lambda u(x)e^{-\lambda\tau} + z_1(x), \quad \text{for } x \in \Gamma_1 \quad (2.16)$$

with

$$z_1(x) = -f_1e^{-\lambda\tau} + \tau e^{-\lambda\tau} \int_0^1 f_4(x, \sigma)e^{\lambda\sigma\tau} d\sigma, \quad \text{for } x \in \Gamma_1.$$

Since $f_1 \in H_{\Gamma_0}^1(\Omega)$ and $f_4 \in L^2(\Gamma_1 \times (0, 1))$, then $z_1 \in L^2(\Gamma_1)$.

Consequently, knowing u , we may deduce v by (2.13), z by (2.15) and using (2.16), we deduce w by (2.11). We recall that since $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ we automatically get $w = \gamma_1(v)$.

From equations (2.10), (2.11), and (2.13), u must satisfy:

$$\lambda(\lambda + a)u - \Delta u = f_2 + (\lambda + a)f_1, \quad \text{in } \Omega \quad (2.17)$$

with the boundary conditions

$$\begin{aligned} u &= 0, & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} &= f_3 - \lambda z(\cdot, 0) - \mu z(\cdot, 1) & \text{on } \Gamma_1. \end{aligned} \quad (2.18)$$

Using the preceding expression of $z(\cdot, 1)$ and the expression of v given by (2.13), we have:

$$\frac{\partial u}{\partial \nu} = -(\lambda^2 + \mu\lambda e^{-\lambda\tau})u + f(x), \quad \text{for } x \in \Gamma_1 \quad (2.19)$$

with

$$f(x) = f_3(x) + \lambda f_1(x) - \mu z_1(x), \quad \text{for } x \in \Gamma_1.$$

From the regularity of f_3 , f_2 , z_1 , we get $f \in L^2(\Gamma_1)$.

The variational formulation of problem (2.17), (2.18),(2.19) is to find $u \in H_{\Gamma_0}^1(\Omega)$ such that:

$$\begin{aligned} & \int_{\Omega} \lambda(\lambda + a)u\omega + \nabla u \nabla \omega dx + \int_{\Gamma_1} (\lambda^2 + \mu\lambda e^{-\lambda\tau}) u(\sigma)\omega(\sigma)d\sigma, \\ & = \int_{\Omega} (f_2 + (\lambda + a)f_1)\omega dx + \int_{\Gamma_1} f(\sigma)\omega(\sigma)d\sigma, \end{aligned} \quad (2.20)$$

for any $\omega \in H_{\Gamma_0}^1(\Omega)$. Since $\lambda > 0$, $\mu > 0$, the left hand side of (2.20) defines a coercive bilinear form on $H_{\Gamma_0}^1(\Omega)$. Thus by applying the Lax-Milgram theorem, there exists a unique $u \in H_{\Gamma_0}^1(\Omega)$ solution of (2.20). Now, choosing $\omega \in \mathcal{C}_c^\infty$, u is a solution of (2.17) in the sense of distribution and therefore $u \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$. Thus using the Green's formula and exploiting the equation (2.17) on Ω , we obtain finally:

$$\int_{\Gamma_1} (\lambda^2 + \mu\lambda e^{-\lambda\tau})u(\sigma)\omega(\sigma)d\sigma + \left\langle \frac{\partial u}{\partial \nu}; \omega \right\rangle_{\Gamma_1} = \int_{\Gamma_1} f(\sigma)\omega(\sigma)d\sigma \quad \forall \omega \in H_{\Gamma_0}^1(\Omega).$$

So $u \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$ verifies (2.19) and we recover u and v and thus by (2.15), we obtain z and finally setting $w = \gamma_1(v)$, we have found $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A})$ solution of $(I - \mathcal{A})V = F$.

Thus from the Lumer–Phillips' theorem, there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of the shifted problem (2.6). This completes the proof of Theorem 2.1. \square

Remark 2.2. According to the above the operator \mathcal{A}_d generates a C_0 semigroup of contractions $e^{t\mathcal{A}_d}$ on \mathcal{H} .

3. Asymptotic behavior

In this section, we show that if $\xi > \xi^*$, the semigroup $e^{t\mathcal{A}_d}$ decays to the null steady state with a polynomial decay rate for regular initial data. To obtain this, our technique is based on a frequency domain method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

Theorem 3.1. *Let $\xi > \xi^*$. Denote $\mu_1 = \frac{\xi}{2\tau} + \frac{\mu}{2}$ and $\mathcal{A}_d = \mathcal{A} - \mu_1 I$. Then there exists a constant $C > 0$ such that, for all $V_0 \in \mathcal{D}(\mathcal{A}_d)$, the semigroup $e^{t\mathcal{A}_d}$ satisfies the following estimate*

$$\|e^{t\mathcal{A}_d}V_0\|_{\mathcal{H}} \leq \frac{C}{\sqrt{t}} \|V_0\|_{\mathcal{D}(\mathcal{A}_d)}, \quad \forall t > 0. \quad (3.1)$$

Remark 3.2. Let us notice that although the semigroup $e^{t\mathcal{A}_d}$ generates a polynomial stability, we cannot conclude on the stability of the semigroup $e^{t\mathcal{A}}$.

Indeed let us consider $V_0 \in \mathcal{D}(\mathcal{A})$ and $\tilde{V}_0 = V_0 - \mu_1 V_0$, and the two following problems:

$$\begin{cases} V' &= \mathcal{A}V \\ V_{t=0} &= V_0 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{V}' &= \mathcal{A}_d \tilde{V} \\ \tilde{V}_{t=0} &= \tilde{V}_0. \end{cases}$$

Given $V_0 = (u_0, u_1, \gamma_1(u_1), z_0)^T \in \mathcal{D}(\mathcal{A})$, the second problem writes in term of $V = (u, u_t, \gamma_1(u_t), z)^T$:

$$\left\{ \begin{array}{l} u_{tt} - \frac{1}{1 + \mu_1} \Delta u + \frac{a + \mu_1}{1 + \mu_1} u_t = 0, \quad x \in \Omega, t > 0 \\ u(x, t) = 0, \quad x \in \Gamma_0, t > 0 \\ u_{tt}(x, t) + \frac{\partial u}{\partial \nu}(x, t) + \mu z(x, 1, t) - \mu u_t(x, t) = 0, \quad x \in \Gamma_1, t > 0 \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) + \mu_1 \tau z(x, \rho, t) = 0, \quad x \in \Gamma_1, \rho \in (0, 1), t > 0 \\ z(x, 0, t) - u_t(x, t) = 0, \quad x \in \Gamma_1, t > 0 \\ u(x, 0) - (1 - \mu_1)u_0(x) = 0, \quad x \in \Omega \\ u_t(x, 0) - (1 - \mu_1)u_1(x) = 0, \quad x \in \Omega \\ z(x, \rho, 0) - (1 - \mu_1)f_0(x, -\tau\rho) = 0, \quad x \in \Gamma_1, \rho \in (0, 1). \end{array} \right. \quad (3.2)$$

We call this problem the “shifted” problem.

By Duhamel’s formula, we get:

$$e^{-\mu_1 t} V(t) = \frac{-e^{-\mu_1 t}}{1 + \mu_1} V_0 + \frac{\tilde{V}(t)}{1 + \mu_1} - \frac{\mu_1}{1 + \mu_1} \int_0^t e^{-\mu_1(t-s)} \tilde{V}(s) ds, \quad \forall t > 0. \quad (3.3)$$

The first two terms of the right hand side of equation (3.3) tends to zero as t tends to infinity. So we obtain:

$$V(t) \simeq \int_0^t e^{\mu_1 s} \tilde{V}(s) ds.$$

We only know at this stage that $\|\tilde{V}(s)\|_{\mathcal{D}(\mathcal{A}_d)}$ tends to zero at least as $s^{-\frac{1}{2}}$, and thus $\|V(t)\|_{\mathcal{D}(\mathcal{A})}$ may tend to zero or blow-up in infinite time. We will illustrate this behavior by numerical examples in 1D in the last section of this work.

Proof of Theorem 3.1. We will use the following frequency domain theorem for polynomial stability from [10] (see also [8,38] for weaker variants) of a C_0 semi-group of contractions on a Hilbert space:

Lemma 3.3. *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space \mathcal{H} satisfies*

$$\|e^{t\mathcal{L}}U_0\|_{\mathcal{H}} \leq \frac{C}{t^{\frac{\theta}{2}}} \|U_0\|_{\mathcal{D}(\mathcal{L})}$$

for some constant $C > 0$ and for $\theta > 0$ if and only if

$$\rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$\limsup_{|\beta| \rightarrow \infty} \frac{1}{\beta^{\theta}} \|(i\beta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (3.4)$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator \mathcal{L} .

Remark 3.4. In view of this theorem we need to identify the spectrum of \mathcal{A}_d lying on the imaginary axis. Unfortunately, as the embedding of $L^2(\Gamma_1, H^1(0, 1))$ into $L^2(\Gamma_1 \times (0, 1))$ is not compact, \mathcal{A}_d has not a compact resolvent. Therefore its spectrum $\sigma(\mathcal{A}_d)$ does not consist only of eigenvalues of \mathcal{A}_d . We have then to show that :

1. if β is a real number, then $i\beta I - \mathcal{A}_d$ is injective and
2. if β is a real number, then $i\beta I - \mathcal{A}_d$ is surjective.

It is the objective of the two following lemmas.

First we look at the point spectrum of \mathcal{A}_d .

Lemma 3.5. *If β is a real number, then $i\beta$ is not an eigenvalue of \mathcal{A}_d .*

Proof. We will show that the equation

$$\mathcal{A}_d Z = i\beta Z \quad (3.5)$$

with $Z = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}_d)$ and $\beta \in \mathbb{R}$ has only the trivial solution. Equation (3.5) writes :

$$\begin{aligned} (i\beta + \mu_1)u - v &= 0, \\ (i\beta + \mu_1)v - \Delta u + av &= 0, \\ (i\beta + \mu_1)w + \frac{\partial u}{\partial \nu} + \mu z(\cdot, 1) &= 0, \\ (i\beta + \mu_1)z + \frac{1}{\tau} z_{\rho} &= 0. \end{aligned} \quad (3.6)$$

$$(3.7)$$

By taking the inner product of (3.5) with Z and using (2.8), we get:

$$\Re(\langle \mathcal{A}_d Z, Z \rangle_{\mathcal{H}}) \leq - \int_{\Omega} a |v(x)|^2 dx - \left(\frac{\xi}{2\tau} - \frac{\mu}{2} \right) \int_{\Gamma_1} |z(\sigma, 1)|^2 d\sigma. \quad (3.8)$$

Thus we firstly obtain that:

$$v = 0 \quad \text{and} \quad z(., 1) = 0.$$

Next, according to (3.6), we have $v = (i\beta + \mu_1)u$. Thus we have $u = 0$; since $w = \gamma_1(v) = z(., 0)$, we obtain also $w = 0$ and $z(., 0) = 0$. Moreover as z satisfies (3.7) by integration, we obtain:

$$z(., \rho) = z(., 0) e^{-\tau(i\beta + \mu_1)\rho}.$$

But as $z(., 0) = 0$, we finally have $z = 0$. Thus the only solution of (3.5) is the trivial one. \square

Next, we show that \mathcal{A}_d has no continuous spectrum on the imaginary axis.

Lemma 3.6. *Let $\xi > \xi^*$. If β is a real number, then $i\beta$ belongs to the resolvent set $\rho(\mathcal{A}_d)$ of \mathcal{A}_d .*

Proof. In view of Lemma 3.5 it is enough to show that $i\beta I - \mathcal{A}_d$ is surjective. For $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}_d)$ solution of

$$(i\beta I - \mathcal{A}_d)V = F,$$

which is:

$$(i\beta + \mu_1)u - v = f_1, \tag{3.9}$$

$$(i\beta + \mu_1)v - \Delta u + av = f_2, \tag{3.10}$$

$$(i\beta + \mu_1)w + \frac{\partial u}{\partial \nu} + \mu z(., 1) = f_3, \tag{3.11}$$

$$(i\beta + \mu_1)z + \frac{1}{\tau}z_\rho = f_4. \tag{3.12}$$

To find $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}_d)$ solution of system (3.9)–(3.12), we suppose u is determined with the appropriate regularity. Then from (3.9), we get:

$$v = (i\beta + \mu_1)u - f_1. \tag{3.13}$$

Therefore, from the compatibility condition on Γ_1 , equation (2.5), we determine $z(., 0)$ by:

$$z(x, 0) = v(x) = (i\beta + \mu_1)u(x) - f_1(x), \quad \text{for } x \in \Gamma_1.$$

Thus, from (3.12), z is the solution of the linear Cauchy problem:

$$\begin{cases} z_\rho = \tau(f_4(x) - (i\beta + \mu_1)z(x, \rho)), & \text{for } x \in \Gamma_1, \rho \in (0, 1), \\ z(x, 0) = (i\beta + \mu_1)u(x) - f_1(x). \end{cases} \tag{3.14}$$

The solution of the Cauchy problem (3.14) is given by: for $x \in \Gamma_1$, $\rho \in (0, 1)$

$$\begin{aligned} z(x, \rho) &= (i\beta + \mu_1)u(x)e^{-(i\beta + \mu_1)\rho\tau} - f_1e^{-(i\beta + \mu_1)\rho\tau} \\ &\quad + \tau e^{-(i\beta + \mu_1)\rho\tau} \int_0^\rho f_4(x, \sigma)e^{(i\beta + \mu_1)\sigma\tau} d\sigma. \end{aligned} \quad (3.15)$$

So, we have at the point $\rho = 1$,

$$z(x, 1) = (i\beta + \mu_1)u(x)e^{-(i\beta + \mu_1)\tau} + z_1(x), \quad \text{for } x \in \Gamma_1 \quad (3.16)$$

with

$$z_1(x) = -f_1e^{-(i\beta + \mu_1)\tau} + \tau e^{-(i\beta + \mu_1)\tau} \int_0^1 f_4(x, \sigma)e^{(i\beta + \mu_1)\sigma\tau} d\sigma, \quad \text{for } x \in \Gamma_1.$$

Since $f_1 \in H_{\Gamma_0}^1(\Omega)$ and $f_4 \in L^2(\Gamma_1) \times L^2(0, 1)$, then $z_1 \in L^2(\Gamma_1)$.

Consequently, knowing u , we may deduce v by (3.13), z by (3.15) and using (3.16), we deduce w by (3.11). We recall that since $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}_d)$ we automatically get $w = \gamma_1(v)$.

From equations (3.10) and (3.11), u must satisfy:

$$(i\beta + \mu_1)(i\beta + \mu_1 + a)u - \Delta u = f_2 + (i\beta + \mu_1 + a)f_1, \quad \text{in } \Omega \quad (3.17)$$

with the boundary conditions

$$\begin{aligned} u &= 0, & \text{on } \Gamma_0 & \quad (3.18) \\ \frac{\partial u}{\partial \nu} &= f_3 - (i\beta + \mu_1)z(\cdot, 0) - \mu z(\cdot, 1), & \text{on } \Gamma_1. \end{aligned}$$

Using the preceding expression of $z(\cdot, 1)$ and the expression of v given by (3.13), we have:

$$\frac{\partial u}{\partial \nu} = -((i\beta + \mu_1)^2 + \mu(i\beta + \mu_1)e^{-(i\beta + \mu_1)\tau})u + f(x), \quad \text{for } x \in \Gamma_1 \quad (3.19)$$

with

$$f(x) = f_3(x) + (i\beta + \mu_1)f_1(x) - \mu z_1(x), \quad \text{for } x \in \Gamma_1.$$

From the regularity of f_3 , f_2 , z_1 , we get $f \in L^2(\Gamma_1)$.

The variational formulation of problem (3.17), (3.18), (3.19) is to find $u \in H_{\Gamma_0}^1(\Omega)$ such that for any $\omega \in H_{\Gamma_0}^1(\Omega)$:

$$\begin{aligned} &\int_{\Omega} \left((i\beta + \mu_1)(i\beta + \mu_1 + a)u\bar{\omega} + \nabla u \nabla \bar{\omega} \right) dx \\ &\quad + \int_{\Gamma_1} \left((i\beta + \mu_1)^2 + \mu(i\beta + \mu_1)e^{-(i\beta + \mu_1)\tau} \right) u(\sigma)\bar{\omega}(\sigma) d\sigma \\ &= \int_{\Omega} (f_2 + (i\beta + \mu_1 + a)f_1)\bar{\omega} dx + \int_{\Gamma_1} f(\sigma)\bar{\omega}(\sigma) d\sigma. \end{aligned} \quad (3.20)$$

Multiplying the preceding equation by $-i\beta + \mu_1$ leads to, for any $\omega \in H_{\Gamma_0}^1(\Omega)$:

$$\begin{aligned} & \int_{\Omega} \left(|i\beta + \mu_1|^2 (i\beta + \mu_1 + a) u \bar{\omega} + (-i\beta + \mu_1) \nabla u \nabla \bar{\omega} \right) dx \\ & + \int_{\Gamma_1} |i\beta + \mu_1|^2 (i\beta + \mu_1 + \mu e^{-(i\beta + \mu_1)\tau}) u(\sigma) \bar{\omega}(\sigma) d\sigma \\ & = \int_{\Omega} (-i\beta + \mu_1) (f_2 + (i\beta + \mu_1 + a) f_1) \bar{\omega} dx + \int_{\Gamma_1} (-i\beta + \mu_1) f(\sigma) \bar{\omega}(\sigma) d\sigma. \end{aligned} \tag{3.21}$$

Since $\mu_1 > \mu > 0$, the left hand side of (3.21) defines a coercive sesquilinear form on $H_{\Gamma_0}^1(\Omega)$. Thus by applying the Lax-Milgram theorem, there exists a unique $u \in H_{\Gamma_0}^1(\Omega)$ solution of (3.20). Now, choosing $\omega \in \mathcal{C}_c^\infty$, u is a solution of (3.17) in the sense of distribution. Using the regularity of f_1 and f_2 , we finally have $u \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$. Thus using the Green's formula and exploiting the equation (3.17) on Ω , we obtain finally: $\forall \omega \in H_{\Gamma_0}^1(\Omega)$,

$$\int_{\Gamma_1} \left((i\beta + \mu_1)^2 + \mu(i\beta + \mu_1) e^{-(i\beta + \mu_1)\tau} \right) u(\sigma) \omega(\sigma) d\sigma + \left\langle \frac{\partial u}{\partial \nu}; \omega \right\rangle_{\Gamma_1} = \int_{\Gamma_1} f(\sigma) \omega(\sigma) d\sigma.$$

So $u \in H^2(\Omega) \cap H_{\Gamma_0}^1(\Omega)$ verifies (3.19). Then we recover v by equation (3.9) and by equation (3.15), we obtain z . Finally setting $w = \gamma_1(v)$, we have found $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}_d)$ solution of $(i\beta I - \mathcal{A}_d)V = F$. \square

The following lemma shows that (3.4) holds with $\mathcal{L} = \mathcal{A}_d$ and $\theta = 2$.

Lemma 3.7. *The resolvent operator of \mathcal{A}_d satisfies condition (3.4) for $\theta = 2$.*

Proof. Suppose that condition (3.4) is false with $\theta = 2$. By the Banach–Steinhaus Theorem (see [11]), there exist a sequence of real numbers $\beta_n \rightarrow +\infty$ and a sequence of vectors $Z_n = (u_n, v_n, w_n, z_n)^t \in \mathcal{D}(\mathcal{A}_d)$ with $\|Z_n\|_{\mathcal{H}} = 1$ such that

$$\|\beta_n^2 (i\beta_n I - \mathcal{A}_d) Z_n\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{3.22}$$

i.e.,

$$\begin{aligned} \beta_n ((i\beta_n + \mu_1) u_n - v_n) &\equiv f_n \rightarrow 0 && \text{in } H_{\Gamma_0}^1(\Omega), \\ \beta_n (i\beta_n v_n - \Delta u_n + (\mu_1 + a) v_n) &\equiv g_n \rightarrow 0 && \text{in } L^2(\Omega), \\ \beta_n \left((i\beta_n + \mu_1) w_n + \frac{\partial u_n}{\partial \nu} + \mu z_n(\cdot, 1) \right) &\equiv h_n \rightarrow 0 && \text{in } L^2(\Gamma_1), \end{aligned} \tag{3.23}$$

$$\beta_n \left((i\beta_n + \mu_1) z_n + \frac{1}{\tau} \partial_\rho z_n \right) \equiv k_n \rightarrow 0 \quad \text{in } L^2(\Gamma_1 \times (0, 1)) \tag{3.24}$$

since $\beta_n \leq \beta_n^2$.

Our goal is to derive from (3.22) that $\|Z_n\|_{\mathcal{H}}$ converges to zero, thus there is a contradiction.

We first notice that we have

$$\|\beta_n^2(i\beta_n I - \mathcal{A}_d)Z_n\|_{\mathcal{H}} \geq |\Re(\langle \beta_n^2(i\beta_n I - \mathcal{A}_d)Z_n, Z_n \rangle_{\mathcal{H}})|.$$

Then, by (3.8) and (3.22),

$$\beta_n v_n \rightarrow 0 \text{ in } L^2(\Omega), \quad \beta_n z_n(\cdot, 1) \rightarrow 0 \text{ in } L^2(\Gamma_1),$$

and

$$u_n \rightarrow 0, \quad \Delta u_n \rightarrow 0 \text{ in } L^2(\Omega) \Rightarrow u_n \rightarrow 0 \text{ in } H_{\Gamma_0}^1(\Omega).$$

This further leads, by (3.23) and the trace theorem, to

$$w_n \rightarrow 0 \text{ in } L^2(\Gamma_1). \quad (3.25)$$

Moreover, since $Z_n \in \mathcal{D}(\mathcal{A}_d)$, we have, by (3.25),

$$z_n(\cdot, 0) \rightarrow 0 \text{ in } L^2(\Gamma_1). \quad (3.26)$$

We have

$$z_n(\cdot, \rho) = z_n(\cdot, 0) e^{-(i\beta_n + \mu_1)\tau\rho} + \int_0^\rho e^{-(i\beta_n + \mu_1)\tau(\rho-s)} \frac{\tau \hat{k}_n(s)}{\beta_n} ds. \quad (3.27)$$

This implies, according to (3.27), (3.26) and (3.24), that $z_n \rightarrow 0$ in $L^2(\Gamma_1 \times (0, 1))$ and clearly contradicts $\|Z_n\|_{\mathcal{H}} = 1$. \square

The two hypotheses of Lemma 3.3 are proved by Lemma 3.7 and Lemma 3.6. Then (3.1) holds. The proof of Theorem 3.1 is then finished. \square

4. Changing the damping law

Let us consider now the same system as (1.1) but with a Kelvin–Voigt damping. The system is given by:

$$\left\{ \begin{array}{l} u_{tt} - \Delta u - a \Delta u_t = 0, \quad x \in \Omega, \quad t > 0 \\ u(x, t) = 0, \quad x \in \Gamma_0, \quad t > 0 \\ u_{tt}(x, t) + \frac{\partial u}{\partial \nu}(x, t) + a \frac{\partial u_t}{\partial \nu} + \mu u_t(x, t - \tau) = 0, \quad x \in \Gamma_1, \quad t > 0 \\ u(x, 0) - u_0(x) = 0, \quad x \in \Omega \\ u_t(x, 0) - u_1(x) = 0, \quad x \in \Omega \\ u_t(x, t - \tau) - f_0(x, t - \tau) = 0, \quad x \in \Gamma_1, \quad t \in (0, \tau). \end{array} \right. \quad (4.1)$$

Which, as above, is equivalent to:

$$\left\{ \begin{array}{l} u_{tt} - \Delta u - a \Delta u_t = 0, \quad x \in \Omega, t > 0 \\ u(x, t) = 0, \quad x \in \Gamma_0, t > 0 \\ u_{tt}(x, t) + \frac{\partial u}{\partial \nu}(x, t) + a \frac{\partial u_t}{\partial \nu}(x, t) + \mu z(x, 1, t) = 0, \quad x \in \Gamma_1, t > 0 \\ \tau z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \quad x \in \Gamma_1, \rho \in (0, 1), t > 0 \\ z(x, 0, t) - u_t(x, t) = 0, \quad x \in \Gamma_1, t > 0 \\ u(x, 0) - u_0(x) = 0, \quad x \in \Omega \\ u_t(x, 0) - u_1(x) = 0, \quad x \in \Omega \\ z(x, \rho, 0) - f_0(x, -\tau\rho) = 0, \quad x \in \Gamma_1, \rho \in (0, 1) \end{array} \right. \quad (4.2)$$

Let the operator \mathcal{A}_{kv} defined by:

$$\mathcal{A}_{kv} \begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ \Delta u + a \Delta v \\ -\frac{\partial u}{\partial \nu} - a \frac{\partial v}{\partial \nu} - \mu z(\cdot, 1) \\ -\frac{1}{\tau} z_\rho \end{pmatrix}.$$

The domain of \mathcal{A}_{kv} is the set of $V = (u, v, w, z)^T$ such that:

$$(u, v, w, z)^T \in (H_{\Gamma_0}^1(\Omega) \cap H^2(\Omega)) \times H_{\Gamma_0}^1(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1; H^1(0, 1)) \quad (4.3)$$

$$\frac{\partial v}{\partial \nu} \in L^2(\Gamma_1)$$

$$w = \gamma_1(v) = z(\cdot, 0) \quad \text{on } \Gamma_1. \quad (4.4)$$

Notations: For $c \in \mathbb{R}$, we define:

$$C_\Omega(c) = \inf_{u \in H_{\Gamma_0}^1(\Omega)} \frac{\|\nabla u\|_2^2 + c \|u\|_{2, \Gamma_1}^2}{\|u\|_2^2}. \quad (4.5)$$

$C_\Omega(c)$ is the first eigenvalue of the operator $-\Delta$ under the Dirichlet–Robin boundary conditions:

$$\left\{ \begin{array}{l} u(x) = 0 \quad x \in \Gamma_0 \\ \frac{\partial u}{\partial \nu}(x) + cu(x) = 0 \quad x \in \Gamma_1. \end{array} \right. \quad (4.6)$$

From Kato’s perturbation theory [30] (see also [31, Theorem 1.3.1]), $C_\Omega(c)$ is a continuous increasing function. From Poincaré inequality and the continuity of the trace operator γ_1 , we have $C_\Omega(0) > 0$ and $C_\Omega(c) \rightarrow -\infty$ as $c \rightarrow -\infty$. Thus it exists a unique $c^* < 0$ such that:

$$C_\Omega(c^*) = 0. \quad (4.7)$$

In the following, we fix $\xi = \mu\tau$ in the norm (2.3). We will see in the next result why this choice is well adapted.

Theorem 4.1. *Suppose that a and μ satisfy the following assumption:*

$$\mu < |c^*|a. \quad (4.8)$$

Then, the operator \mathcal{A}_{kv} generates a C_0 semigroup of contractions on \mathcal{H} . If, in particular, $V_0 \in \mathcal{H}$, then there exists a unique solution $V \in C(\mathbb{R}_+; \mathcal{H})$ of problem (4.1). Moreover, if $V_0 \in \mathcal{D}(\mathcal{A}_{kv})$, then

$$V \in C(\mathbb{R}_+; \mathcal{D}(\mathcal{A}_{kv})) \cap C^1(\mathbb{R}_+; \mathcal{H}).$$

Proof. To prove Theorem 4.1, we use again the semigroup approach and the Lumer–Phillips’ theorem.

For this purpose, we show firstly that the operator \mathcal{A}_{kv} is dissipative. Indeed, let $V = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}_{kv})$. By definition of the operator \mathcal{A}_{kv} and the scalar product of \mathcal{H} , we have:

$$\begin{aligned} \langle \mathcal{A}_{kv}V, V \rangle_{\mathcal{H}} &= \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} v (\Delta u + a\Delta v) dx \\ &\quad + \int_{\Gamma_1} w \left(-\frac{\partial u}{\partial \nu} - a\frac{\partial v}{\partial \nu} - \mu z(\sigma, 1) \right) d\sigma - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z z_{\rho} d\rho d\sigma. \end{aligned}$$

Applying Green’s formula and the compatibility condition $w = \gamma_1(v)$, we obtain:

$$\langle \mathcal{A}_{kv}V, V \rangle_{\mathcal{H}} = -\mu \int_{\Gamma_1} z(\sigma, 1) w d\sigma - a \int_{\Omega} |\nabla v|^2 dx - \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_{\rho} z d\rho dx. \quad (4.9)$$

But we have:

$$\begin{aligned} \frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_{\rho} z(\sigma, \rho) d\rho d\sigma &= \frac{\xi}{2\tau} \int_{\Gamma_1} \int_0^1 \frac{\partial}{\partial \rho} z^2(\sigma, \rho) d\rho d\sigma \\ &= \frac{\xi}{2\tau} \int_{\Gamma_1} (z^2(\sigma, 1) - z^2(\sigma, 0)) d\sigma. \end{aligned}$$

Thus from the compatibility condition (4.4), we get:

$$-\frac{\xi}{\tau} \int_{\Gamma_1} \int_0^1 z_{\rho} z d\rho d\sigma = \frac{\xi}{2\tau} \int_{\Gamma_1} (v^2 - z^2(\sigma, 1)) d\sigma.$$

Therefore equation (4.9) becomes:

$$\begin{aligned} \langle \mathcal{A}_{kv}V, V \rangle_{\mathcal{H}} &= -a \int_{\Omega} |\nabla v|^2 dx + \frac{\xi}{2\tau} \int_{\Gamma_1} v^2 d\sigma \\ &\quad - \frac{\xi}{2\tau} \int_{\Gamma_1} \int_0^1 z^2(\sigma, 1) d\sigma - \mu \int_{\Gamma_1} v(\sigma) z(\sigma, 1) d\sigma. \end{aligned}$$

To treat the last term in the preceding equation, Young’s inequality gives:

$$- \int_{\Gamma_1} v(\sigma) z(\sigma, 1) d\sigma \leq \frac{1}{2} \int_{\Gamma_1} z^2(\sigma, 1) d\sigma + \frac{1}{2} \int_{\Gamma_1} v^2(\sigma) d\sigma.$$

Therefore, we firstly get:

$$\langle \mathcal{A}_{kv}V, V \rangle_{\mathcal{H}} + a \int_{\Omega} |\nabla v|^2 dx - \left(\frac{\xi}{2\tau} + \frac{\mu}{2} \right) \int_{\Gamma_1} v^2 d\sigma + \left(\frac{\xi}{2\tau} - \frac{\mu}{2} \right) \int_{\Gamma_1} z^2(\sigma, 1) d\sigma \leq 0$$

At this point, as $\xi = \mu\tau$, the previous inequality becomes:

$$\langle \mathcal{A}_{kv}V, V \rangle_{\mathcal{H}} + a \int_{\Omega} |\nabla v|^2 dx - \mu \int_{\Gamma_1} v^2 d\sigma \leq 0.$$

Denoting now $c = -\frac{\mu}{a}$, we get $\langle \mathcal{A}_{kv}V, V \rangle_{\mathcal{H}} + a \left(\int_{\Omega} |\nabla v|^2 dx + c \int_{\Gamma_1} v^2 d\sigma \right) \leq 0$. By definition (4.5), we thus get:

$$\langle \mathcal{A}_{kv}V, V \rangle_{\mathcal{H}} + a C_{\Omega}(c) \|v\|_2^2 \leq 0. \tag{4.10}$$

From assumption (4.8), $C_{\Omega}(c) > 0$. This inequality proves that the operator \mathcal{A}_{kv} is dissipative. To show that $\lambda I - \mathcal{A}_{kv}$ is surjective for all $\lambda > 0$, we easily adapt the proof of Theorem 2.1.

The proof of Theorem 4.1, follows from the Lumer–Phillips’ theorem. \square

Moreover the semigroup operator $e^{t\mathcal{A}_{kv}}$ is exponential stable on \mathcal{H} . We have the following result.

Theorem 4.2. *Suppose that the assumption (4.8) is satisfied. Then, there exists $C, \omega > 0$ such that for all $t > 0$ we have*

$$\|e^{t\mathcal{A}_{kv}}\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{-\omega t}.$$

Remark 4.3. We note here that, in this case where the damping operator is sufficiently unbounded for controlling the delay one, we obtain the exponential stability result.

Note that without internal damping (i.e., if $a = 0$, the previous model is destabilized for arbitrarily small delays for every value $\mu > 0$, see [20]. Thus, the internal damping $-a\Delta u_t$ makes the system robust with respect to time delays in the boundary condition if the coefficient a is sufficiently large with respect to μ .

Remark 4.4. In the recent work of Nicaise and Pignotti, [41], they studied the existence and stability of a problem closely related to problem (4.1). They obtain a slightly different condition to ensure the existence and the stability. Namely, let us define C_P , a sort of Poincaré constant, by:

$$C_P = \sup_{u \in H_{\Gamma_0}^1(\Omega)} \frac{\|u\|_{2, \Gamma_1}^2}{\|\nabla u\|_2^2}.$$

By using the semigroup approach and the Lumer–Phillips’ theorem they proved the global existence of the solution and by using an observability inequality they proved the exponential stability under the condition:

$$\mu C_P < a.$$

A simple argument shows that $\frac{1}{C_P} \geq |c^*|$. Thus the bound for the damping coefficient a obtained in this work is of the same type that the one obtained by Nicaise and Pignotti.

Proof of Theorem 4.2. We will employ the following frequency domain theorem for exponential stability from [48] of a C_0 semigroup of contractions on a Hilbert space:

Lemma 4.5. *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space \mathcal{H} satisfies, for all $t > 0$,*

$$\|e^{t\mathcal{L}}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{-\omega t}$$

for some constant $C, \omega > 0$ if and only if

$$\rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$\limsup_{|\beta| \rightarrow \infty} \|(i\beta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{4.11}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator \mathcal{L} .

The proof of Theorem 4.2 is based on the following lemmas. For the same reason as before (see Remark 3.4), we have to show that there is no eigenvalue lying on the imaginary axis and that \mathcal{A}_{kv} has no continuous spectrum on the imaginary axis.

We first look at the point spectrum.

Lemma 4.6. *If β is a real number, then $i\beta$ is not an eigenvalue of \mathcal{A}_{kv} .*

Proof. We will show that the equation

$$\mathcal{A}_{kv} Z = i\beta Z \tag{4.12}$$

with $Z = (u, v, w, z)^T \in \mathcal{D}(\mathcal{A}_{kv})$ and $\beta \in \mathbb{R}$ has only the trivial solution. System (4.12) writes:

$$v = i\beta u \tag{4.13}$$

$$\Delta u + a \Delta v = i\beta v \tag{4.14}$$

$$-\frac{\partial u}{\partial \nu} - a \frac{\partial v}{\partial \nu} - \mu z(\cdot, 1) = i\beta w \tag{4.15}$$

$$-\frac{1}{\tau} z_\rho = i\beta z. \tag{4.16}$$

Let us firstly treat the case where $\beta = 0$. From (4.13), we deduce that $v = 0$ and from the compatibility condition (4.4), we deduce that $w = 0$ and $z(\cdot, 0) = 0$. From (4.16), we obtain that $z(\cdot, \rho) = 0$. Replacing v and z in (4.15) by 0 and using (4.14) with $v = 0$, since $u \in H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)$, we finally deduce that $u = 0$. Thus in the case where $\beta = 0$, the only solution of (4.12) is the trivial one.

Let us suppose now that $\beta \neq 0$. Denoting now $c = -\frac{\mu}{a}$, by taking the inner product of (4.12) with Z , using the inequality (4.10) we get:

$$\Re \langle \mathcal{A}_{kv} Z, Z \rangle_{\mathcal{H}} \leq -a C_{\Omega}(c) \|v\|_2^2. \tag{4.17}$$

From assumption (4.8), $C_{\Omega}(c) > 0$ and thus we obtain that $v = 0$. Thus from (4.13), we deduce that $u = 0$.

Next, since we have $w = \gamma_1(v)$, we also have $w = 0$. Moreover as we have $w = z(\cdot, 0)$, we get: $z(\cdot, 0) = 0$. From (4.15) we also have $z(\cdot, 1) = 0$. As z satisfies (4.16), we get the following identity:

$$z(\cdot, \rho) = e^{-i\beta\tau\rho} z(\cdot, 0).$$

Thus the only solution of (4.12) is the trivial one. □

By the same way, as in Lemma 3.6, we show that \mathcal{A}_{kv} has no continuous spectrum on the imaginary axis.

Lemma 4.7. *If λ is a real number, then $i\lambda$ belongs to the resolvent set $\rho(\mathcal{A}_{kv})$ of \mathcal{A}_{kv} .*

Lemma 4.8. *The resolvent operator of \mathcal{A}_{kv} satisfies condition (4.11).*

Proof. Suppose that condition (4.11) is false. By the Banach–Steinhaus Theorem (see [11]), there exist a sequence of real numbers $\beta_n \rightarrow +\infty$ and a sequence of vectors $Z_n = (u_n, v_n, w_n, z_n)^T \in \mathcal{D}(\mathcal{A}_{kv})$ with $\|Z_n\|_{\mathcal{H}} = 1$ such that

$$\|(i\beta_n I - \mathcal{A}_{kv}) Z_n\|_{\mathcal{H}} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{4.18}$$

i.e.,

$$(i\beta_n u_n - v_n) \equiv f_n \rightarrow 0 \quad \text{in } H^1_{\Gamma_0}(\Omega), \tag{4.19}$$

$$(i\beta_n v_n - \Delta u_n - a\Delta v_n) \equiv g_n \rightarrow 0 \quad \text{in } L^2(\Omega), \tag{4.20}$$

$$\left(i\beta_n w_n + \frac{\partial u_n}{\partial \nu} + a \frac{\partial v_n}{\partial \nu} + \mu z_n(\cdot, 1) \right) \equiv h_n \rightarrow 0 \quad \text{in } L^2(\Gamma_1), \tag{4.21}$$

$$\left(i\beta_n z_n + \frac{1}{\tau} \partial_{\rho} z_n \right) \equiv k_n \rightarrow 0 \quad \text{in } L^2(\Gamma_1 \times (0, 1)). \tag{4.22}$$

Our goal is to derive from (4.18) that $\|Z_n\|_{\mathcal{H}}$ converges to zero, thus there is a contradiction. We first notice that we have

$$\|(i\beta_n I - \mathcal{A}_{kv})Z_n\|_{\mathcal{H}} \geq |\Re(\langle (i\beta_n I - \mathcal{A}_{kv})Z_n, Z_n \rangle_{\mathcal{H}})|.$$

Thus by (4.17) and (4.18), $v_n \rightarrow 0$ in $L^2(\Omega)$. From (4.19), $u_n \rightarrow 0$ in $L^2(\Omega)$. But we also have,

$$\Delta u_n \rightarrow 0 \text{ in } L^2(\Omega) \quad \text{and} \quad \Delta v_n \rightarrow 0 \text{ in } L^2(\Omega).$$

Thus we firstly obtain:

$$u_n \rightarrow 0 \text{ in } H_{\Gamma_0}^1(\Omega) \quad \text{and} \quad v_n \rightarrow 0 \text{ in } H_{\Gamma_0}^1(\Omega). \quad (4.23)$$

By the trace theorem, we have:

$$w_n = \gamma(v_n) \rightarrow 0 \quad \text{in } L^2(\Gamma_1). \quad (4.24)$$

Moreover, since $Z_n \in \mathcal{D}(\mathcal{A}_{kv})$, $w_n = z_n(\cdot, 0)$. Thus we get:

$$z_n(\cdot, 0) \rightarrow 0 \quad \text{in } L^2(\Gamma_1). \quad (4.25)$$

Now from (4.21), we also have:

$$z_n(\cdot, 1) \rightarrow 0 \quad \text{in } L^2(\Gamma_1).$$

As the following identity $z_n(\cdot, \rho) = z_n(\cdot, 0)e^{-i\tau\beta_n\rho} + \tau \int_0^\rho e^{-i\tau\beta_n(\rho-s)} k_n(\cdot, s) ds$ holds, according to (4.22), (4.25) we finally have:

$$z_n \rightarrow 0 \quad \text{in } L^2(\Gamma_1 \times (0, 1)). \quad (4.26)$$

Identities (4.23), (4.24) and (4.26) clearly contradicts the fact that, for all $n \in \mathbb{N}$, $\|Z_n\|_{\mathcal{H}} = 1$. \square

The two hypotheses of Lemma 4.5 are proved. The proof of Theorem 4.2 is then finished. \square

5. Comments and numerical illustrations

To illustrate numerically the results presented in this paper, we present numerical simulations for problem (1.1) and for the Kelvin–Voigt damping, namely problem (4.1), in 1D. So let us consider $\Omega = (0, 1)$, $\Gamma_0 = \{0\}$, $\Gamma_1 = \{1\}$.

To solve numerically problem (1.1) (resp. problem (4.1)), we have to consider its equivalent formulation, namely problem (2.2) (resp. problem (4.2)),

which writes in the present case:

$$\left\{ \begin{array}{ll} u_{tt} - u_{xx} + a u_t = 0, & x \in (0, 1), t > 0 \\ u(0, t) = 0, & t > 0 \\ u_{tt}(1, t) + u_x(1, t) + \mu z(x, 1, t) = 0 & t > 0 \\ \tau z_t(1, \rho, t) + z_\rho(1, \rho, t) = 0, & \rho \in (0, 1), t > 0, \\ z(1, 0, t) = u_t(1, t) & t > 0 \\ u(x, 0) = u_0(x) & x \in (0, 1) \\ u_t(x, 0) = u_1(x) & x \in (0, 1) \\ z(1, \rho, 0) - f_0(1, -\tau\rho) = 0 & \rho \in (0, 1). \end{array} \right.$$

A 1D formulation of the “shifted” problem (3.2) as well as the Kelvin–Voigt damping problem, problem (4.2) is of the same type.

As the stability result that we have presented in this work, namely Theorem 3.1, is a stability result for the “shifted” problem (3.2), we have to perform numerical simulations for both problems: the original one, problem (1.1) and the “shifted” problem (3.2).

For this sake, to avoid a CFL condition between the mesh size and the time step, we decided to discretize the different problems by implicit first order in time, and finite difference method in space. For every simulations the numerical parameters are the following:

$$\tau = 2, \quad \xi = 2\xi^*, \quad \Delta x = \frac{1}{20}, \quad \Delta \rho = \frac{1}{20}, \quad \Delta t = 0.1 \\ u_0(x) = u_1(x) = xe^{10x}, \quad f_0(1, \rho) = e^\rho e^{10}.$$

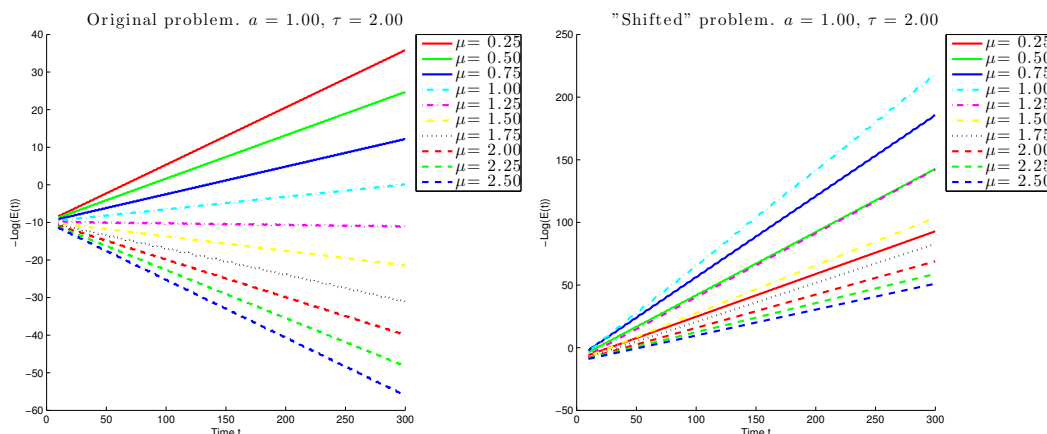
For every time $t > 0$, we denote $E(t) = \left\| (u(\cdot, t), u_t(\cdot, t), u_t(1, t), z(1, \cdot, t)) \right\|_{\mathcal{H}}^T$. The choice of u_0, u_1, f_0 ensures a large initial energy.

In Figure 1 and Figure 2, we present the resulting simulation for the original problem and the “shifted” one.

Let us first notice that the convergence rate for the shifted problem is better than the one expected: we have proved a polynomial decay rate whereas numerically, we observe an exponential decay rate. This is probably due two facts:

1. the particular case of the dimension 1 as proved recently by G. Q. Xu, S. P. Yung and L. Kwan Li [50].
2. the numerical diffusion participates to the exponential stability as for the Kelvin–Voigt damping.

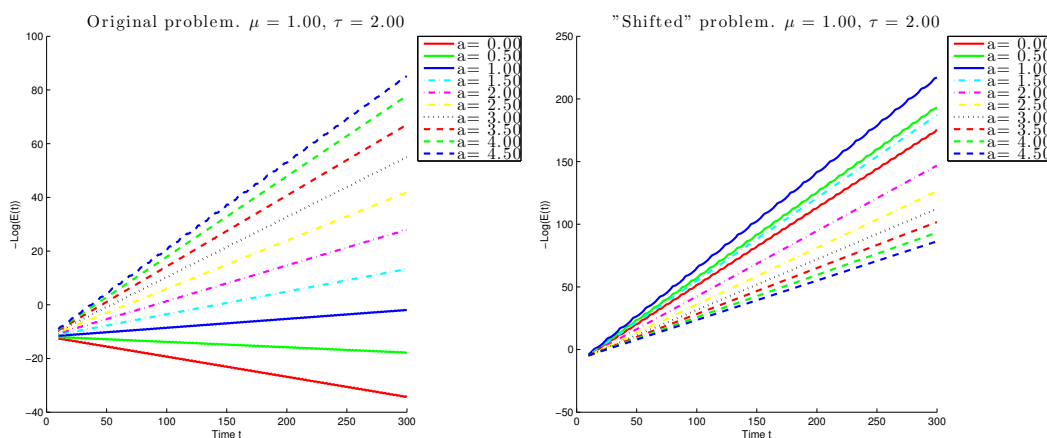
Moreover, as it was conjectured in Remark 3.2, the “shifted” problem converges for a large set of parameters a and μ whereas the original problem does not and even worse, it exhibits an exponential growth.



(a) Original problem

(b) Shifted problem

Figure 1: Energy (in -log scale) versus time: influence of μ .



(a) Original problem

(b) Shifted problem

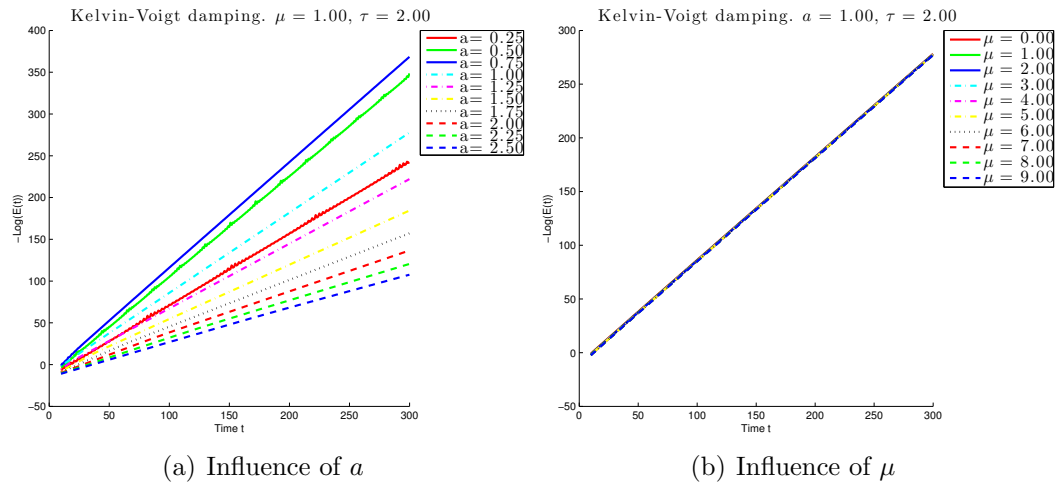
Figure 2: Energy (in -log scale) versus time: influence of a .

In Figure 3, we present the simulations for the case of the Kelvin–Voigt damping for which we have proved that under the condition $\mu < |c^*|a$, we have an exponential decay rate.

From equations (4.6) and (4.7), the constant c^* must satisfy:

$$\begin{cases} u_{xx} = 0, & x \in (0, 1), \\ u(0) = 0, & u_x(1) + c^*u(1) = 0. \end{cases}$$

Thus we obtain $c^* = -1$. Let us first notice that even though the condition between a and μ is not fulfilled, we have an exponential decay of the solution. This is also probably due to the particular case of the dimension 1 as well. Secondly it seems that numerically the convergence rate does not depend on the parameter μ .

Figure 3: Energy (in $-\log$ scale) versus time.

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