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Kernel Estimates for Schrödinger Type Operators with Unbounded Diffusion and Potential Terms

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Abstract. We prove that the heat kernel associated to the Schrödinger type operator $A := (1+|x|^\alpha)\Delta - |x|^\beta$ satisfies the estimate

$$k(t,x,y) \le c_1 e^{\lambda_0 t} e^{c_2 t^{-b}} \frac{(|x||y|)^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}}{1+|y|^\alpha} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}$$

for $t>0, |x|,|y| \ge 1$, where c_1, c_2 are positive constants and $b = \frac{\beta-\alpha+2}{\beta+\alpha-2}$ provided that $N>2$, $\alpha \ge 2$ and $\beta > \alpha - 2$. We also obtain an estimate of the eigenfunctions of A.

Keywords. Schrödinger type operator, semigroup, heat kernel estimates

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1. Introduction

In this paper we consider the operator

$$Au(x) = (1+|x|^\alpha)\Delta u(x) - |x|^\beta u(x), \quad x \in \mathbb{R}^N,$$

for $N>2$, $\alpha \ge 2$ and $\beta > \alpha - 2$. We propose to study the behaviour of the associated heat kernel and associated eigenfunctions.

Recently several paper have dealt with elliptic operators with polynomially growing diffusion coefficients (see for example [3, 4, 6, 7, 9–11, 13–16]).

In [11] (resp. [3]) it is proved that the realization A_p of A in $L^p(\mathbb{R}^N)$ for $1<p<\infty$ with domain

$$D_p(A) = \left\{ u \in W^{2,p}(\mathbb{R}^N) \;\middle|\; (1+|x|^\alpha)|D^2u|, (1+|x|^\alpha)^{\frac{1}{2}}\nabla u, |x|^\beta u \in L^p(\mathbb{R}^N) \right\}$$

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generates a strongly continuous and analytic semigroup $T_p(\cdot)$ for $\alpha \in [0,2]$ and $\beta > 0$ (resp. $\alpha > 2$ and $\beta > \alpha - 2$). This semigroup is also consistent, irreducible and ultracontractive. For the case $\beta = 0$ we refer to [7, 13].

Since the coefficients of the operator A are locally regular it follows that the semigroup $T_p(\cdot)$ admits an integral representation through a heat kernel $k(t,x,y)$

$$T_p(t)u(x) = \int_{\mathbb{R}^N} k(t,x,y)u(y)dy, \quad t > 0,\ x \in \mathbb{R}^N,$$

for all $u \in L^p(\mathbb{R}^N)$ (cf. [2, 12]).

In [11] estimates of the kernel $k(t,x,y)$ for $\alpha \in [0,2)$ and $\beta > 2$ were obtained. Our contribution in this paper is to show similar upper bounds for the case $\alpha \geq 2$ and $\beta > \alpha - 2$. Our techniques consist in providing upper and lower estimates for the ground state of A_p corresponding to the largest eigenvalue λ_0 and adapting the arguments used in [5].

The paper is structured as follows. In Section 2 we prove that the eigenfunction $\psi(x)$ associated to the largest eigenvalue λ_0 can be estimated from below and above by the function

$$|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr} \quad \text{for } |x| \geq 1.$$

In Section 3 we introduce the measure $d\mu = (1+|x|^\alpha)^{-1}dx$ for which the operator A is symmetric and generates an analytic semigroup (which is a Markov semigroup) with kernel

$$k_\mu(t,x,y) = (1+|x|^\alpha)k(t,x,y).$$

Adapting the arguments used in [5, 11], we show the following intrinsic ultracontractivity

$$k_\mu(t,x,y) \leq c_1 e^{\lambda_0 t} e^{c_2 t^{-b}} \psi(x)\psi(y), \quad t > 0,\ x,y \in \mathbb{R}^N,$$

where c_1, c_2 are positive constant, $b = \frac{\beta-\alpha+2}{\beta+\alpha-2}$, provided that $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$. So one deduces the heat kernel estimate

$$k(t,x,y) \leq c_1 e^{\lambda_0 t} e^{c_2 t^{-b}} \left(|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}\right)\left(\frac{|y|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}}{1+|y|^\alpha} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}\right)$$

for $t > 0$, $|x|, |y| \geq 1$. As an application we obtain the behaviour of all eigenfunctions of A_p at infinity. With respect to t we prove the following sharp estimates

$$k_\mu(t,x,y) \leq Ct^{-\frac{N}{2}} (1+|x|^\alpha)^{\frac{2-N}{4}} (1+|y|^\alpha)^{\frac{2-N}{4}}$$

for $0 < t \leq 1$ and $x, y \in \mathbb{R}^N$. Here we use the results in [14] and weighted Nash inequalities introduced in [1]. We end this section by giving a brief description of how to extend the heat kernel estimates to a more general class of elliptic operators in divergence form.

In the sequel we denote by $B_R \subset \mathbb{R}^N$ the open ball, centered at 0 with radius $R > 0$.

2. Estimate of the ground state ψ

We begin by estimating the eigenfunction ψ corresponding to the largest eigenvalue λ_0 of A. First we recall some spectral properties obtained in [3, 11].

Proposition 2.1. *Assume $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$ then*

(i) *the resolvent of A_p is compact in $L^p(\mathbb{R}^N)$;*

(ii) *the spectrum of A_p consists of a sequence of negative real eigenvalues which accumulates at $-\infty$. Moreover, $\sigma(A_p)$ is independent of p;*

(iii) *the semigroup $T_p(\cdot)$ is irreducible, the eigenspace corresponding to the largest eigenvalue λ_0 of A_p is one-dimensional and is spanned by strictly positive functions ψ, which is radial, belongs to $C_b^{1+\nu}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N)$ for any $\nu \in (0, 1)$ and tends to 0 when $|x| \to \infty$.*

We can now prove upper and lower estimates for ψ. We note here that the proof of [11, Proposition 3.1] cannot be adapted to our situation. So, we propose to use another technique to estimate ψ.

Proposition 2.2. *Let $\lambda_0 < 0$ be the largest eigenvalue of A_p and ψ be the corresponding eigenfunction. If $N > 2$, $\alpha \geq 2$ and $\beta > \alpha - 2$ then*

$$C_1|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \leq \psi(x) \leq C_2|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}$$

for any $x \in \mathbb{R}^N \setminus B_1$ and some positive constants C_1, C_2.

Proof. Since the eigenfunction is radial, we have to study the asymptotic behavior of the solution of an ordinary differential equation. We follow the idea of the WKB method (see [17]), but since the error function is not bounded we need to compute it directly.

Let $f_{\alpha,\beta,\lambda}$ be the function

$$f_{\alpha,\beta,\lambda}(x) = |x|^{-\frac{N-1}{2}} h^{-\frac{1}{4}}(|x|) \exp\left\{-\int_R^{|x|} h^{\frac{1}{2}}(s)ds - \int_R^{|x|} v_\lambda(s)ds\right\},$$

where $\lambda \in \mathbb{R}$, $h(r) = \frac{r^\beta}{1+r^\alpha}$, and v_λ is a smooth function to be chosen later on. If we set

$$w(r) = r^{\frac{N-1}{2}} f_{\alpha,\beta,\lambda}(r), \tag{1}$$

then

$$w' = w\left(-\frac{h'}{4h} - h^{\frac{1}{2}} - v_\lambda\right) \quad \text{and} \quad w'' = w(g+m+h), \tag{2}$$

where

$$g = \frac{5}{16}\left(\frac{h'}{h}\right)^2 - \frac{h''}{4h} + v_\lambda^2 + v_\lambda\left(\frac{h'}{2h} + 2h^{\frac{1}{2}}\right) - v_\lambda' - m \tag{3}$$

and

$$m(r) := \frac{(N-1)(N-3)}{4r^2}.$$

On the other hand, taking in mind (1) we also obtain

$$w''(r) = r^{\frac{N-1}{2}}\left(f''_{\alpha,\beta,\lambda} + \frac{N-1}{r}f'_{\alpha,\beta,\lambda} + \frac{(N-1)(N-3)}{4r^2}f_{\alpha,\beta,\lambda}\right). \tag{4}$$

Comparing (2) and (4) we get

$$f''_{\alpha,\beta,\lambda} + \frac{N-1}{r}f'_{\alpha,\beta,\lambda} = \frac{r^\beta}{1+r^\alpha}f_{\alpha,\beta,\lambda} + g f_{\alpha,\beta,\lambda}.$$

That is

$$\Delta f_{\alpha,\beta,\lambda}(x) - \frac{|x|^\beta}{1+|x|^\alpha}f_{\alpha,\beta,\lambda}(x) = g(|x|)f_{\alpha,\beta,\lambda}(x).$$

To evaluate the function g we set $\xi = \frac{\beta-\alpha}{2}+1$, which is positive by the condition $\beta > \alpha - 2$. We have

$$\frac{h'}{h} = \frac{1}{r}(\beta-\alpha) + \frac{1}{r}O(r^{-\alpha}), \quad \frac{h''}{h} = \frac{1}{r^2}(\beta-\alpha)(\beta-\alpha-1) + \frac{1}{r^2}O(r^{-\alpha}).$$

Then (3) is reduced to

$$\begin{aligned} &g(r) \\ &= -v_\lambda' + \frac{v_\lambda}{r}\left(\xi - 1 + O(r^{-\alpha}) + 2r^\xi\sqrt{\frac{r^\alpha}{1+r^\alpha}}\right) + v_\lambda^2 + \frac{c_0}{r^2} + \frac{1}{r^2}\big(O(r^{-\alpha}) + O(r^{-2\alpha})\big) \\ &= -v_\lambda{}' + \frac{v_\lambda}{r}\left(\xi - 1 + O(r^{-\alpha}) + 2r^\xi - 2r^\xi\frac{(1+r^\alpha)^{\frac{1}{2}} - r^{\frac{\alpha}{2}}}{(1+r^\alpha)^{\frac{1}{2}}}\right) + v_\lambda^2 + \frac{c_0}{r^2} + \frac{1}{r^2}O(r^{-\alpha}) \\ &= -v_\lambda{}' + \frac{v_\lambda}{r}\big(\xi - 1 + 2r^\xi + (1+r^\xi)O(r^{-\alpha})\big) + v_\lambda^2 + \frac{c_0}{r^2} + \frac{1}{r^2}O(r^{-\alpha}), \end{aligned} \tag{5}$$

where $c_0 = c_0(\xi) = \left(\frac{\xi-1}{2}\right)^2 + \frac{\xi-1}{2} - \frac{(N-1)(N-3)}{4}$. So, if we take in (5)

$$v_\lambda(r) = \sum_{i=1}^{k} c_i \frac{1}{r^{i\xi+1}}, \quad r \geq 1,$$

we obtain

$$\begin{aligned} r^2 g(r) &= \sum_{i=1}^{k} c_i(i\xi+1)\frac{1}{r^{i\xi}} + (\xi-1)\sum_{i=1}^{k} c_i\frac{1}{r^{i\xi}} + 2\sum_{i=0}^{k-1} c_{i+1}\frac{1}{r^{i\xi}} \\ &\quad + \left(\sum_{i=1}^{k} c_i\frac{1}{r^{i\xi}} + \sum_{i=0}^{k-1} c_{i+1}\frac{1}{r^{i\xi}}\right) O(r^{-\alpha}) + \sum_{i,j=1}^{k} c_i c_j \frac{1}{r^{(i+j)\xi}} + c_0 + O(r^{-\alpha}) \\ &= \sum_{i=2}^{k-1}\left[c_i\xi(i+1) + 2c_{i+1} + \sum_{j+s=i} c_j c_s\right]\frac{1}{r^{i\xi}} + (2c_1\xi + 2c_2)r^{-\xi} \\ &\quad + c_k\xi(k+1)\frac{1}{r^{k\xi}} + 2c_1 + \sum_{i+j\geq k}\frac{c_i c_j}{r^{(i+j)\xi}} + c_0 + O(r^{-\alpha}). \end{aligned}$$

We can choose $c_1, \ldots, c_k$ such that

$$2c_1 + c_0 = \lambda,\ 2c_1\xi + 2c_2 = 0 \quad \text{and} \quad \left[\xi(i+1)c_i + 2c_{i+1} + \sum_{j+s=i} c_j c_s\right] = 0$$

for $i = 2, \ldots, k-1$ and obtain

$$r^2 g(r) = \lambda + c_k\xi(k+1)\frac{1}{r^{k\xi}} + \sum_{i+j\geq k}\frac{c_i c_j}{r^{(i+j)\xi}} + O(r^{-\alpha}).$$

Thus,

$$g(r) = O\left(\frac{1}{r^{k\xi+2}}\right) + O\left(\frac{1}{r^{\alpha+2}}\right) + \frac{\lambda}{r^2}.$$

Since $\xi > 0$ there exists $k \in \mathbb{N}$ such that $k\xi + 2 - \alpha > 0$. So we have

$$(1+|x|^\alpha)\Delta f_{\alpha,\beta,\lambda}(x) - |x|^\beta f_{\alpha,\beta,\lambda}(x) = o(1) f_{\alpha,\beta,\lambda}(x) + \lambda\frac{1+|x|^\alpha}{|x|^2} f_{\alpha,\beta,\lambda}(x). \tag{6}$$

We prove first the upper bound. For ψ we know that

$$\Delta\psi - \frac{|x|^\beta}{1+|x|^\alpha}\psi - \frac{\lambda_0}{1+|x|^\alpha}\psi = 0\ . \tag{7}$$

Since $\alpha - 2 \geq 0$ and $\lambda_0 < 0$, for $|x|$ large enough we have $o(1) + 2\lambda_0\frac{1+|x|^\alpha}{|x|^2} < \lambda_0$. Then, by (6), it follows that

$$(1+|x|^\alpha)\Delta f_{\alpha,\beta,2\lambda_0}(x) - |x|^\beta f_{\alpha,\beta,2\lambda_0}(x) < \lambda_0 f_{\alpha,\beta,2\lambda_0}.$$

Thus,

$$\Delta f_{\alpha,\beta,2\lambda_0}(x) - \frac{|x|^\beta}{1+|x|^\alpha} f_{\alpha,\beta,2\lambda_0}(x) - \frac{\lambda_0}{1+|x|^\alpha} f_{\alpha,\beta,2\lambda_0}(x) < 0, \tag{8}$$

in $\mathbb{R}^N \setminus B_R$ for some $R > 0$. Comparing (7) and (8), in $\mathbb{R}^N \setminus B_R$ we have

$$\Delta(f_{\alpha,\beta,2\lambda_0} - C\psi) < \frac{\lambda_0 + |x|^\beta}{1 + |x|^\alpha}(f_{\alpha,\beta,2\lambda_0} - C\psi) \quad \text{for any constant } C > 0.$$

Since $\beta > 0$, we have

$$W(x) := \frac{\lambda_0 + |x|^\beta}{1 + |x|^\alpha} > 0$$

for $|x|$ large enough. Since both $f_{\alpha,\beta,2\lambda_0}$ and ψ tend to 0 as $|x| \to \infty$ and since there exists C_2 such that $\psi \leq C_2 f_{\alpha,\beta,2\lambda_0}$ on ∂B_R, we can apply the maximum principle to the problem

$$\begin{cases} \Delta g(x) - W(x)g(x) < 0 & \text{in } \mathbb{R}^N \setminus B_R, \\ g(x) \geq 0 & \text{in } \partial B_R, \\ \lim_{|x| \to \infty} g(x) = 0, & \end{cases}$$

where $g := f_{\alpha,\beta,2\lambda_0} - C_2^{-1}\psi$, to obtain $\psi \leq C_2 f_{\alpha,\beta,2\lambda_0}$ in $\mathbb{R}^N \setminus B_R$. Here one has to note that since $\lim_{|x|\to\infty} g(x) = 0$, one can see that the classical maximum principle on bounded domains can be applied, cf. [8, Theorem 3.5]. Then,

$$\begin{aligned} \psi(x) &\leq C_2 |x|^{-\frac{N-1}{2} - \frac{1}{4}(\beta-\alpha)} \exp\left\{ - \int_R^{|x|} \sqrt{\frac{r^\beta}{1 + r^\alpha}}\, dr \right\} \exp\left\{ - \int_R^{|x|} v_{2\lambda_0}(r) dr \right\} \\ &\leq C_3 |x|^{-\frac{N-1}{2} - \frac{1}{4}(\beta-\alpha)} \exp\left\{ - \int_R^{|x|} \sqrt{\frac{r^\beta}{1 + r^\alpha}}\, dr \right\}, \end{aligned}$$

since

$$\lim_{|x|\to\infty} \int_R^{|x|} v_\lambda(r)\, dr = \lim_{|x|\to\infty} \sum_{j=1}^k \frac{c_j}{j\xi}(R^{-j\xi} - |x|^{-j\xi}) = \sum_{j=1}^k \frac{c_j}{j\xi} R^{-j\xi}. \tag{9}$$

As regards lower bounds of ψ, we observe that, from (6), we have

$$\Delta f_{\alpha,\beta,0}(x) - \frac{|x|^\beta}{1 + |x|^\alpha} f_{\alpha,\beta,0}(x) = \frac{o(1)}{1 + |x|^\alpha} f_{\alpha,\beta,0}(x) > \frac{\lambda_0}{1 + |x|^\alpha} f_{\alpha,\beta,0}(x)$$

if $|x| \geq R$ for some suitable $R > 0$. Then,

$$\Delta f_{\alpha,\beta,0}(x) > \frac{|x|^\beta}{1 + |x|^\alpha} f_{\alpha,\beta,0}(x) + \frac{\lambda_0}{1 + |x|^\alpha} f_{\alpha,\beta,0}(x)$$

Since $\frac{\lambda_0}{1+|x|^\alpha}\psi = \Delta\psi(x) - \frac{|x|^\beta}{1+|x|^\alpha}\psi$ we have

$$\Delta(f_{\alpha,\beta,0} - \psi) > \frac{|x|^\beta + \lambda_0}{1 + |x|^\alpha}(f_{\alpha,\beta,0} - \psi)\,.$$

We can assume that $|x|^\beta + \lambda_0$ is positive for $|x| \geq R$ and, arguing as above, by the maximum principle and using (9) we get

$$\psi(x) \geq C_1 f_{\alpha,\beta,0}(x) \geq C_1 |x|^{-\frac{N-1}{2}-\frac{1}{4}(\beta-\alpha)} \exp\left\{-\int_R^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}\,dr\right\}$$

for $|x| \geq R$. Since $0 < \psi \in C(\mathbb{R}^N)$, by changing the constants, the above upper and lower estimates remain valid for $1 \leq |x| \leq R$. This ends the proof of the proposition. □

3. Intrinsic ultracontractivity and heat kernel estimates

Let us now introduce on $L^2_\mu := L^2(\mathbb{R}^N, d\mu)$ the bilinear form

$$a_\mu(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \overline{v}\,dx + \int_{\mathbb{R}^N} V u\overline{v}\,d\mu, \quad u,v \in D(a_\mu), \tag{10}$$

where $V(x) = |x|^\beta$, $d\mu(x) = (1+|x|^\alpha)^{-1}dx$ and $D(a_\mu) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_H}$ with H the Hilbert space

$$H = \left\{u \in L^2_\mu : V^{\frac{1}{2}}u \in L^2_\mu,\ \nabla u \in (L^2(\mathbb{R}^N))^N\right\}$$

endowed with the inner product

$$\langle u,v\rangle_H = \int_{\mathbb{R}^N}(1+V)u\overline{v}\,d\mu + \int_{\mathbb{R}^N}\nabla u\cdot\nabla\overline{v}\,dx.$$

Since a_μ is a closed, symmetric and accretive form, to a_μ we associate the self-adjoint operator A_μ defined by

$$\begin{aligned} D(A_\mu) &= \left\{u \in D(a_\mu) : \exists\, g \in L^2_\mu \text{ s.t. } a_\mu(u,v) = -\int_{\mathbb{R}^N} g\overline{v}\,d\mu,\ \forall v \in D(a_\mu)\right\}, \\ A_\mu u &= g, \end{aligned}$$

see e.g., [18, Proposition 1.24]. By general results on positive self-adjoint operators induced by nonnegative quadratic forms in Hilbert spaces (see e.g., [18, Proposition 1.51, Theorems 1.52, 2.6, 2.13]) A_μ generates a positive analytic semigroup $(e^{tA_\mu})_{t\geq 0}$ in L^2_μ.

We need to show that the semigroup e^{tA_μ} coincides with the semigroup $T_p(\cdot)$ generated by A_p in $L^p(\mathbb{R}^N)$ on $L^p(\mathbb{R}^N) \cap L^2_\mu$.

Lemma 3.1. *We have*

$$D(A_\mu) = \left\{u \in D(a_\mu) \cap W^{2,2}_{loc}(\mathbb{R}^N) : (1+|x|^\alpha)\Delta u - V(x)u \in L^2_\mu\right\}$$

and $A_\mu u = (1+|x|^\alpha)\Delta u - V(x)u$ *for* $u \in D(A_\mu)$. *Moreover, if* $\lambda > 0$ *and* $f \in L^p(\mathbb{R}^N) \cap L^2_\mu$, *then*

$$(\lambda - A_\mu)^{-1} f = (\lambda - A_p)^{-1} f.$$

Proof. The inclusion "$\subset$" is obtained, taking $v \in C_c^\infty(\mathbb{R}^N)$ in (10), by local elliptic regularity. As regards the inclusion "$\supset$" we consider $u \in D(a_\mu) \cap W^{2,2}_{loc}(\mathbb{R}^N)$ such that $g := (1+|x|^\alpha)\Delta u - V(x)u \in L^2_\mu$ and consider $v \in C_c^\infty(\mathbb{R}^N)$. Integrating by parts we obtain

$$a_\mu(u,v) = -\int g v \, d\mu. \tag{11}$$

By the density of $C_c^\infty(\mathbb{R}^N)$ in $D(a_\mu)$ we have equation (11) for every $v \in D(a_\mu)$. This implies that $u \in D(A_\mu)$.

To show the coherence of the resolvent, we consider $f \in C_c^\infty(\mathbb{R}^N)$ and let $u = (\lambda - A)^{-1} f$. Since $f \in L^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$, by [3, Theorem 3.7] and [3, Theorem 4.4], it follows that $u \in D_2(A)$. So, we have $\nabla u \in L^2(\mathbb{R}^N)$ and $Vu \in L^2(\mathbb{R}^N)$. Moreover, it is clear that $u \in L^2_\mu$, and

$$\|V^{\frac{1}{2}} u\|^2_{L^2_\mu} \le \int_{\mathbb{R}^N} V(x) u^2 dx \le \int_{B(1)} u^2 dx + \int_{\mathbb{R}^N / B(1)} V^2(x) u^2 dx \le \|u\|_2^2 + \|Vu\|_2^2. \tag{12}$$

This yields $u \in H$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in $D_2(A)$, see [3, Lemma 4.3], we can find a sequence $(u_n) \subset C_c^\infty(\mathbb{R}^N)$ such that u_n converges to u in the operator norm. Then, u_n converges to u in $L^2(\mathbb{R}^N)$ and hence in L^2_μ. By [3, Lemma 4.2] ∇u_n converges to ∇u_n in $L^2(\mathbb{R}^N)$ and hence in L^2_μ. Finally replacing u with $u_n - u$ in (12) we have that $V^{\frac{1}{2}} u_n$ converges to $V^{\frac{1}{2}} u$ in L^2_μ. Thus we have proved that $u \in D(a_\mu)$. Integration by parts we obtain

$$a(u,v) = -(\lambda u - f, v)_{L^2_\mu}.$$

That is $u \in D(A_\mu)$ and $\lambda u - A_\mu u = f$. Therefore, $(\lambda - A_\mu)^{-1} f = (\lambda - A_p)^{-1} f$ for all $f \in C_c^\infty(\mathbb{R}^N)$ and so by density the last statement follows. □

The previous Lemma implies in particular that

$$e^{tA_\mu} f = T_p(t) f = \int_{\mathbb{R}^N} k(t,x,y) f(y)\, dy, \quad f \in L^p(\mathbb{R}^N) \cap L^2_\mu.$$

By density we obtain that the semigroup e^{tA_μ} admits the integral representation $e^{tA_\mu} f(x) = \int_{\mathbb{R}^N} k_\mu(t,x,y) f(y) d\mu(y)$ for all $f \in L^2_\mu$, where

$$k_\mu(t,x,y) = (1+|y|^\alpha) k(t,x,y), \quad t > 0,\ x, y \in \mathbb{R}^N.$$

Let us now give the first application of Proposition 2.2. The proof is similar to the one given in [11, Proposition 3.4] and is based on the semigroup law and the symmetry of $k_\mu(t,\cdot,\cdot)$ for $t>0$.

Proposition 3.2. *If $N>2$, $\alpha\geq 2$ and $\beta>\alpha-2$, then*

$$k(t,x,x)\geq Me^{\lambda_0 t}\left(|x|^{\frac{\alpha-\beta}{4}-\frac{N-1}{2}}e^{-\frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}\right)^2(1+|x|^\alpha)^{-1},\quad t>0,$$

for all $x\in\mathbb{R}^N\setminus B_1$ and some constant $M>0$.

We now give the main result of this section.

Theorem 3.3. *If $N>2$, $\alpha\geq 2$ and $\beta>\alpha-2$ then*

$$k(t,x,y)\leq c_1e^{\lambda_0 t+c_2t^{-b}}|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}\frac{|y|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}}{1+|y|^\alpha}e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}$$

for $t>0$, $x,y\in\mathbb{R}^N\setminus B_1$, where c_1,c_2 are positive constants and $b=\frac{\beta-\alpha+2}{\beta+\alpha-2}$.

Proof. Let us prove first

$$k(t,x,y)\leq c_1e^{c_2t^{-b}}|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}\frac{|y|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}}{1+|y|^\alpha}e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}\quad(13)$$

for $0<t\leq 1$, $x,y\in\mathbb{R}^N\setminus B_1$. By adapting the arguments used in [5, Subsections 4.4 and 4.5], we have only to show the following estimates

$$\int_{\mathbb{R}^N}g|u|^2d\mu\leq C\|g\|_{L_\mu^{\frac{N}{2}}}a_\mu(u,u),\quad u\in D(a_\mu),\ g\in L_\mu^{\frac{N}{2}},\quad(14)$$

and

$$\int_{\mathbb{R}^N}-\log\psi|u|^2d\mu\leq\varepsilon a_\mu(u,u)+(C_1\varepsilon^{-b}+C_2)\|u\|^2_{L^2_\mu},\quad u\in D(a_\mu).\quad(15)$$

To prove (14) we observe that using Hölder and Sobolev inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}g|u|^2d\mu&\leq C\left(\int_{\mathbb{R}^N}|g|^{\frac{N}{2}}d\mu\right)^{\frac{2}{N}}\left(\int_{\mathbb{R}^N}|u|^{\frac{2N}{N-2}}d\mu\right)^{\frac{N-2}{N}}\\&=C\|g\|_{L_\mu^{\frac{N}{2}}}\|u\|^2_{L_\mu^{2^*}}\\&\leq C\|g\|_{L_\mu^{\frac{N}{2}}}\|\nabla u\|_2^2\\&\leq C\|g\|_{L_\mu^{\frac{N}{2}}}a_\mu(u,u),\quad u\in D(a_\mu).\end{aligned}$$

To show (15), we apply the lower estimate of ψ obtained in Proposition 2.2

$$-\log\psi \le -\left(\log C_1 - \frac{2}{\beta-\alpha+2}\right) + \frac{2N-2+\beta-\alpha}{4}\log|x| + \frac{2}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}$$

for $|x|\ge 1$. Hence, there are positive constants C_1, C_2 such that

$$-\log\psi \le C_1|x|^{\frac{\beta-\alpha+2}{2}} + C_2, \quad x\in\mathbb{R}^N.$$

Since $\xi = \frac{\beta-\alpha}{2}+1<\beta$ we have $|x|^\xi \le \varepsilon|x|^\beta + C\varepsilon^{-\frac{\xi}{\beta-\xi}} = \varepsilon V(x) + C\varepsilon^{-b}$ for all $\varepsilon>0$. Thus,

$$-\log\psi \le \varepsilon V + c_1\varepsilon^{-b} + c_2.$$

Taking into account that $\int_{\mathbb{R}^N} V|u|^2 d\mu \le a_\mu(u,u)$ for all $u\in D(a_\mu)$, we obtain (15). This ends the proof of (13).

It remains to prove that

$$k(t,x,y) \le Ce^{\lambda_0 t}|x|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}} \frac{|y|^{-\frac{N-1}{2}-\frac{\beta-\alpha}{4}}}{1+|y|^\alpha} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}$$

for $t>1$, $x,y\in\mathbb{R}^N\setminus B_1$ and some constant $C>0$. To this purpose we use the semigroup law and the symmetry of $k_\mu(t,\cdot,\cdot)$ to infer that

$$k_\mu(t,x,y) = \int_{\mathbb{R}^N} k_\mu\left(t-\frac{1}{2},x,z\right) k_\mu\left(\frac{1}{2},y,z\right) d\mu(z), \quad t>\frac{1}{2},\ x,y\in\mathbb{R}^N.$$

By (13), the function $k_\mu(\frac{1}{2},y,\cdot)$ belongs to L^2_μ. Hence,

$$k_\mu(t,x,y) = \left(e^{(t-\frac{1}{2})A_\mu} k_\mu\left(\frac{1}{2},y,\cdot\right)\right)(x), \quad t>\frac{1}{2},\ x,y\in\mathbb{R}^N.$$

Using again the semigroup law and the symmetry we deduce that

$$\begin{aligned} k_\mu(t,x,x) &= \int_{\mathbb{R}^N}\left|k_\mu\left(\frac{t}{2},x,y\right)\right|^2 d\mu(y) \\ &\le Me^{\lambda_0(t-1)}\left\|k_\mu\left(\frac{1}{2},x,\cdot\right)\right\|^2_{L^2_\mu} \\ &= Me^{\lambda_0(t-1)}k_\mu(1,x,x). \end{aligned}$$

So, by applying (13) to $k_\mu(1,x,x)$ and using the inequality

$$k_\mu(t,x,y) \le (k_\mu(t,x,x))^{\frac{1}{2}}(k_\mu(t,y,y))^{\frac{1}{2}},$$

one obtains (3). □

Remark 3.4. It follows from Proposition 3.2 that the estimates obtained for the heat kernel k in Theorem 3.3 could be sharp in the space variables but certainly not in the time variable as we will prove in Proposition 3.8.

Remark 3.5. In the above proof we use the Sobolev inequality

$$\|u\|^2_{L^{2^*}_\mu} \le C\|\nabla u\|_2^2$$

which holds in $D(a_\mu)$ but not in H (consider for example the case where $\alpha > \beta + N$ and $u = 1$).

As a consequence of Theorem 3.3 we deduce some estimates for the eigenfunctions.

Corollary 3.6. *If the assumptions of Theorem* 3.3 *hold, then all normalized eigenfunctions* ψ_j *of* A_2 *satisfy*

$$|\psi_j(x)| \le C_j |x|^{\frac{\alpha-\beta}{4}-\frac{N-1}{2}} e^{-\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}},$$

for all $x \in \mathbb{R}^N \setminus B_1$, $j \in \mathbb{N}$ *and a constant* $C_j > 0$.

Proof. Let λ_j be an eigenvalue of A_2 and denote by ψ_j any normalized (i.e. $\|\psi_j\|_{L^2(\mathbb{R}^N)} = 1$) eigenfunction associated to λ_j. Then, as in the proof of Theorem 3.3, we have

$$\begin{aligned} e^{\lambda_j t}|\psi_j(x)| &= \left| \int_{\mathbb{R}^N} k_\mu(t,x,y)\psi_j(y)\, d\mu(y) \right| \\ &\le \left(\int_{\mathbb{R}^N} k_\mu(t,x,y)^2 d\mu(y) \right)^{\frac{1}{2}} \|\psi_j\|_{L^2_\mu} \\ &= (k_\mu(2t,x,x))^{\frac{1}{2}}, \end{aligned}$$

for $t > 0$ and $x \in \mathbb{R}^N$. So, the estimates follow from Theorem 3.3. □

Remark 3.7. It is possible to obtain better estimates of the kernels k with respect to the time variable t for small t. In fact if we denote by $S(\cdot)$ the semigroup generated by $(1+|x|^\alpha)\Delta$ in $C_b(\mathbb{R}^N)$, which is given by a kernel p, then by domination we have $0 < k(t,x,y) \le p(t,x,y)$ for $t > 0$ and $x, y \in \mathbb{R}^N$. So, by [14, Theorems 2.6 and 2.14], it follows that

$$\begin{aligned} &k(t,x,y) \le Ct^{-\frac{N}{2}}(1+|x|)^{2-N}(1+|y|)^{2-N-\alpha}, && \alpha > 4, \\ &k(t,x,y) \le Ct^{-\frac{N}{2}}(1+|x|^\alpha)^{\frac{2-N}{4}}(1+|y|^\alpha)^{\frac{2-N}{4}-1}, && 2 < \alpha \le 4 \end{aligned} \tag{16}$$

for $0 < t \le 1$, $x, y \in \mathbb{R}^N$.

Using a domination argument and [14, Proposition 2.10] we can improve the estimate (16).

Proposition 3.8. *If $\alpha \geq 2$, $\beta > \alpha - 2$ and $N > 2$, then the kernel k_μ satisfies*

$$k_\mu(t,x,y) \leq Ct^{-\frac{N}{2}}(1+|x|^\alpha)^{\frac{2-N}{4}}(1+|y|^\alpha)^{\frac{2-N}{4}}$$

for $0 < t \leq 1$ and $x, y \in \mathbb{R}^N$.

Proof. It suffices to consider the case $\alpha > 4$.

By domination one sees that weighted Nash inequalities given in [14, Proposition 2.10] hold for the quadratic form a_μ. Hence, by [1, Corollary 2.8], the results is proved provided that the function $\varphi(x) = (1+|x|^\alpha)^{\frac{2-N}{4}}$ is a Lyapunov function in the sense of [14, Definition 2.1]. A simple computation yields

$$\begin{aligned} A\varphi &= \left[\gamma(N+\alpha-2)|x|^{\alpha-2} + \gamma(\gamma-\alpha)|x|^{\alpha-2}\frac{|x|^\alpha}{1+|x|^\alpha}\right]\varphi(x) \\ &= \left[\gamma\left(\gamma+N-2\right)\frac{|x|^{2\alpha-2}}{1+|x|^\alpha} + \gamma(\alpha-2+N)\frac{|x|^{\alpha-2}}{1+|x|^\alpha} - |x|^\beta\right]\varphi(x) \\ &\leq \left[\gamma\left(\gamma+N-2\right)\frac{|x|^{2\alpha-2}}{1+|x|^\alpha} - |x|^\beta\right]\varphi(x), \end{aligned}$$

where $\gamma := \frac{\alpha(2-N)}{4}$. We note that $\gamma < 2-N$, since $\alpha > 4$. Now, using the fact that $\beta > \alpha - 2$, we deduce that

$$\gamma\left(\gamma+N-2\right)\frac{|x|^{2\alpha-2}}{1+|x|^\alpha} \leq \gamma\left(\gamma+N-2\right)|x|^{\alpha-2} \leq |x|^\beta + \kappa$$

for some $\kappa > 0$. Thus, $A\varphi \leq \kappa\varphi$. Using the same arguments as in [14, Lemma 2.13] we obtain that φ is Lyapunov function for A. □

As in [11] heat kernel estimates can be also obtained for a more general class of elliptic operators.

Let us consider the operator B, defined on smooth functions u by

$$Bu = (1+|x|^\alpha)\sum_{j,k=1}^{N} D_k(a_{kj}D_j u) - Wu,$$

under the following set of assumptions:

Hypotheses 1.

1. the coefficients $a_{kj} = a_{jk}$ belong to $C_b(\mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $j,k = 1,\dots,N$ and there exists a positive constant η such that
$$\eta|\xi|^2 \leq \sum_{j,k=1}^N a_{kj}(x)\xi_k\xi_j, \quad x,\xi \in \mathbb{R}^N;$$
2. $W \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies $W(x) \geq |x|^\beta$ for any $x \in \mathbb{R}^N$ and some $\beta > \alpha - 2$;
3. $\alpha \geq 2$ and $D_j a_{kj}(x) = \mathrm{o}(|x|^{\frac{\beta-\alpha}{2}})$ as $|x| \to \infty$.

On L^2_μ we define the bilinear form
$$b_\mu(u,v) = \sum_{j,k=1}^N \int_{\mathbb{R}^N} a_{kj} D_k u D_j \overline{v}\, dx + \int_{\mathbb{R}^N} W u\overline{v}\, d\mu, \quad u,v \in D(b_\mu),$$
where $D(b_\mu) = \overline{C_c^\infty(\mathbb{R}^N)}^{\|\cdot\|_{\mathcal{H}}}$ with $\mathcal{H}$ the Hilbert space
$$\mathcal{H} = \left\{u \in L^2_\mu : W^{\frac{1}{2}}u \in L^2_\mu,\ \nabla u \in (L^2(\mathbb{R}^N))^N\right\}.$$
Since b_μ is a symmetric, accretive and closable form, we can associate a positive strongly continuous semigroup $S_\mu(\cdot)$ in L^2_μ. The same arguments as in the beginning of this section show that the infinitesimal generator B_μ of this semigroup is the realization in L^2_μ of the operator B with domain $D(B_\mu) = \{u \in D(b_\mu) \cap W^{2,2}_{\mathrm{loc}}(\mathbb{R}^N) : Bu \in L^2_\mu\}$. Let us denotes by p_μ the heat kernel associated to $S_\mu(\cdot)$.

We will also need the bilinear form
$$a_{\mu,\theta}(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \overline{v}\, dx + \theta^2 \int_{\mathbb{R}^N} V u\overline{v} d\mu, \quad u,v \in D(a_{\mu,\theta}) = D(a_\mu).$$
The same arguments as in the proof of Theorem 3.3 can be used to show that the kernel $k_{\mu,\theta}$ of the analytic semigroup associated to the form $a_{\mu,\theta}$ in L^2_μ satisfies
$$0 < k_{\mu,\theta}(t,x,y) \leq K_\theta e^{\lambda_{0,\theta}t} e^{\tilde{c}_\theta t^{-b}} \psi_\theta(x)\psi_\theta(y), \quad t > 0,\ x,y \in \mathbb{R}^N, \qquad (17)$$
where $\tilde{c}_\theta$ and K_θ are positive constants, $\lambda_{0,\theta}$ is the largest (negative) eigenvalue of the minimal realization of operator $A_\theta := (1+|x|^\alpha)\Delta - \theta^2|x|^\beta$ in $L^2(\mathbb{R}^N)$, and ψ_θ is a corresponding positive and bounded eigenfunction. Moreover, there exist $C_{1,\theta}$, $C_{2,\theta} > 0$ such that
$$C_{1,\theta} \leq |x|^{\frac{\beta-\alpha}{4}+\frac{N-1}{2}} e^{\theta\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr}\psi_\theta(x) \leq C_{2,\theta},$$
for any $x \in \mathbb{R}^N \setminus B_1$.

Using Theorem 3.3 and arguing as in [19] and [11, Theorem 3.9] we obtain the following heat kernel estimate.

Theorem 3.9. *Assume that Hypotheses* 1 *are satisfied and let*

$$\Lambda := \sup_{x,\xi\in\mathbb{R}^N\setminus\{0\}} |\xi|^{-2}\sum_{j,k=1}^N a_{kj}(x)\xi_k\xi_j.$$

Then, for any $\theta \in \left(0,\Lambda^{-\frac{1}{2}}\right)$, *we have*

$$p_\mu(t,x,y) \le M_\theta e^{\lambda_{0,\theta}t}e^{c_\theta t^{-b}}(|x||y|)^{\frac{\alpha-\beta}{4}-\frac{N-1}{2}}e^{-\theta\frac{\sqrt{2}}{\beta-\alpha+2}|x|^{\frac{\beta-\alpha+2}{2}}}e^{-\theta\frac{\sqrt{2}}{\beta-\alpha+2}|y|^{\frac{\beta-\alpha+2}{2}}}$$

for any $t>0$ *and* $x,y\in\mathbb{R}^N\setminus B_1$, *where* M_θ, c_θ *are positive constants,* $b=\frac{\beta-\alpha+2}{\beta+\alpha-2}$, *and* $\lambda_{0,\theta}$ *is the largest eigenvalue of the operator* $(1+|x|^\alpha)\Delta-\theta^2|x|^\beta$.

Proof. For the reader's convenience, we give the main ideas of the proof.

Proving the above estimate is equivalent to showing that

$$\phi_\theta(x)^{-1}p_\mu(t,x,y)\phi_\theta(y)^{-1} \le M_\theta e^{\lambda_{0,\theta}t}e^{c_\theta t^{-b}}, \quad t>0,\ x,y\in\mathbb{R}^N, \tag{18}$$

where ϕ_θ is any smooth function satisfying

$$\phi_\theta(x) = |x|^{\frac{\alpha-\beta}{4}-\frac{N-1}{2}}e^{-\theta\int_1^{|x|}\sqrt{\frac{r^\beta}{1+r^\alpha}}dr}, \quad x\in\mathbb{R}^N\setminus B_1.$$

If we denote by $T_{\phi_\theta}: L^2_{\phi_\theta^2\mu}\to L^2_\mu$ the isometry defined by $T_{\phi_\theta}f=\phi_\theta f$, then the left hand side of (18) is the kernel of the semigroup $(T_{\phi_\theta}^{-1}e^{tB_\mu}T_{\phi_\theta})_{t\ge0}$ in $L^2_{\phi_\theta^2\mu}$. It is clear that this semigroup is associated with the form $\tilde{b}_\mu(u,v)=b_\mu(\phi_\theta u,\phi_\theta v)$ for $u,v\in D(\tilde{b}_\mu):=\{u\in L^2_{\phi_\theta^2\mu}:\phi_\theta u\in D(b_\mu)\}$.

As in the proof of Theorem 3.3, it suffices to establish (18) for $t\in(0,1]$. To this purpose one has to prove, as in the proof of [11, Theorem 3.9], the following assertions:

(i) $\min\{u,1\}\in D(\tilde{a}_{\mu,\theta})$ (resp. $D(\tilde{b}_\mu)$) for any nonnegative $u\in D(\tilde{a}_{\mu,\theta})$ (resp. $D(\tilde{b}_\mu)$);

(ii) the semigroup $(T_{\phi_\theta}^{-1}e^{tB_\mu}T_{\phi_\theta})_{t\ge0}$ and the semigroup $(T_{\phi_\theta}^{-1}e^{tA_{\mu,\theta}}T_{\phi_\theta})_{t\ge0}$, associated to the form $\tilde{a}_{\mu,\theta}=a_{\mu,\theta}(\phi_\theta\cdot,\phi_\theta\cdot)$ with domain $D(\tilde{a}_{\mu,\theta})=D(\tilde{b}_\mu)$, are positive, they map $L^\infty(\mathbb{R}^N)$ into itself and satisfy the estimates

$$\|T_{\phi_\theta}^{-1}e^{tB_\mu}T_{\phi_\theta}\|_{L(L^\infty(\mathbb{R}^N))}\le e^{C_1t},\ \ \|T_{\phi_\theta}^{-1}e^{tA_{\mu,\theta}}T_{\phi_\theta}\|_{L(L^\infty(\mathbb{R}^N))}\le e^{C_1t},\ \ t>0,$$

for some positive constant C_1;

(iii) the Log-Sobolev inequality

$$\int_{\mathbb{R}^N}u^2(\log u)\phi_\theta^2d\mu\le\varepsilon\tilde{b}_\mu(u,u)+\|u\|^2_{L^2_{\phi_\theta^2\mu}}\log\|u\|_{L^2_{\phi_\theta^2\mu}}+c_\theta(1+\varepsilon^{-b})\|u\|^2_{L^2_{\phi_\theta^2\mu}} \tag{19}$$

holds true for any nonnegative $u\in D(\tilde{b}_\mu)\cap L^1_{\phi_\theta^2\mu}\cap L^\infty(\mathbb{R}^N)$, where c_θ is the constant in (18).

So, applying (19) and combining [5, Lemma 2.1.2, Corollary 2.2.8 and Example 2.3.4], estimate (18) follows with $t \in (0, 1]$.

The proof of (i)–(iii) is similar to the one in [11, Theorem 3.9]. The proof of (iii) is based on the estimate $\tilde{b}_\mu(u, u) \geq \min\{\mu, \theta^{-1}\}\tilde{a}_{\mu,\theta}(u, u)$ which holds for any $u \in D(\tilde{b}_\mu) \subset D(\tilde{a}_{\mu,\theta})$, and (17). □

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