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Existence of Cylindrically Symmetric Ground States to a Nonlinear Curl-Curl Equation with Non-Constant Coefficients

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Abstract. We consider the nonlinear curl-curl problem $\nabla \times \nabla \times U + V(x)U = f(x, |U|^2)U$ in $\mathbb{R}^3$ related to the nonlinear Maxwell equations with Kerr-type nonlinear material laws. We prove the existence of a symmetric ground-state type solution for a bounded, cylindrically symmetric coefficient V and subcritical cylindrically symmetric nonlinearity f. The new existence result extends the class of problems for which ground-state type solutions are known. It is based on compactness properties of symmetric functions due to Lions [J. Funct. Anal. 41 (1981)(2), 236–275], new rearrangement type inequalities by Brock [Proc. Indian Acad. Sci. Math. Sci. 110 (2000), 157–204] and the recent extension of the Nehari-manifold technique from Szulkin and Weth [Handbook of Nonconvex Analysis and Applications (2010), pp. 597–632].

Keywords. Curl-curl problem, nonlinear elliptic equations, cylindrical symmetry, variational methods

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1. Introduction

We consider the system

$$\nabla \times \nabla \times U + V(x)U = f(x, |U|^2)U \quad \text{in } \mathbb{R}^3 \tag{1}$$

where $V \in L^\infty(\mathbb{R}^3)$ and $f : \mathbb{R}^3 \times [0, \infty) \to [0, \infty)$ is a non-negative Carathéodory function growing at infinity with a power at most $\frac{p-1}{2}$ for $p \in (1, 5)$. The particular feature of (1) is the curl-curl operator. It arises in specific models for standing waves in Maxwell's equations with Kerr-type nonlinear material laws where $f(x, |U|^2)U = \Gamma(x)|U|^2 U$. For a detailed physical motivation of (1) see [2].

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We look for $\mathbb{R}^3$-valued weak solutions U in a cone $K_{4,1}$ of functions with suitable symmetries and $U \in L^2(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$, $\nabla \times U \in L^2(\mathbb{R}^3)$. The condition that 0 lies below the spectrum of $\operatorname{curl}\operatorname{curl} + V(x)$ allows us to find ground-state type critical points of a functional $J(u) = \frac{1}{2}\|u\|^2 - I(u)$, cf. (4), restricted to the so-called Nehari-manifold. The basic idea of applying symmetrizations to minimizing sequences on the Nehari-manifold goes back to Stuart [18] in the context of the stationary nonlinear Schrödinger equation. Compared to [18] the assumptions on the nonlinearity f can be substantially weakened beyond the classical Ambrosetti-Rabinowitz condition. This is based on three important ingredients:

- the recent extension of the Nehari-manifold method by Szulkin and Weth [19],
- the weak sequential continuity of functionals $I(u)$ and $I'(u)[u]$ on $K_{4,1}$ due to compactness properties of symmetric functions by Lions [12, 13],
- new rearrangement inequalities for general nonlinearities due to Brock [7].

Using the combination of these ingredients our main result Theorem 1.1 substantially extends the known results on the existence of ground-state type solutions for (1).

Benci, Fortunato [6] and Azzollini, Benci, D'Aprile, Fortunato [1] were among the first to consider the constant coefficient case of (1) with $V \equiv 0$. Their method was based on cylindrical symmetries of the vector-fields U, cf. [9] for a different class of symmetries. The case where $f(x, |U|^2)U = \Gamma(x)\,|U|^{p-1}\,U$ with periodic coefficients V and Γ has been treated in [2]. In [15] Mederski considered (1) where $f(x, |U|^2)U$ is replaced by, e.g., $\Gamma(x)g(U)$ with $\Gamma > 0$ periodic and bounded, $V \leq 0$, $V \in L^{\frac{p+1}{p-1}}(\mathbb{R}^3) \cap L^{\frac{q+1}{q-1}}(\mathbb{R}^3)$ and $g(U) \sim |U|^{p-1}U$ if $|U| \gg 1$ and $g(U) \sim |U|^{q-1}U$ if $|U| \ll 1$ for $1 < p < 5 < q$. A remarkable feature of Mederski's work is that (1) can be treated without assuming special symmetries of the field U. The nonlinear curl-curl problem on bounded domains with the boundary condition $\nu \times U = 0$ has been elaborated in [3, 4]. For a recent survey on the nonlinear curl-curl problem cf. [5].

An important feature of [1] is the use of cylindrically symmetric ansatz functions for U. Here we make a slightly different ansatz of the form

$$U(x) = u(r, z)\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \quad \text{where } r = \sqrt{x_1^2 + x_2^2},\ z = x_3. \tag{2}$$

Moreover, we assume cylindrically symmetric coefficients, i.e., $V(x) = V(r, z)$ and $f(x, |U|^2) = f(r, z, |U|^2)$. For U of the form (2) we see that $\operatorname{div} U = 0$, and hence (1) reduces to the scalar equation

$$-\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 \frac{\partial u}{\partial r}\right) - \frac{\partial^2 u}{\partial z^2} + V(r, z)u = f(r, z, r^2u^2)u \quad \text{for } (r, z) \in \Omega := (0, \infty) \times \mathbb{R}. \tag{3}$$

It turns out that a suitable space to consider (3) is given by

$$H^1_{\mathrm{cyl}}(r^3 drdz) := \left\{ v \colon (0, \infty) \times \mathbb{R} \to \mathbb{R} : v, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^2_{\mathrm{cyl}}(r^3 drdz) \right\},$$

$$L^2_{\mathrm{cyl}}(r^3drdz) := \left\{ v\colon (0,\infty)\times\mathbb{R}\to\mathbb{R} : \int_\Omega v(r,z)^2 r^3 d(r,z) < \infty \right\},$$

cf. Section 2 for more details on these spaces. Weak solutions of (3) arise as critical points of the functional

$$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla_{r,z}u|^2 + V(r,z)u^2\right) r^3 d(r,z) - \int_\Omega \frac{1}{2r^2} F(r,z,r^2u^2) r^3 d(r,z), \qquad (4)$$

$u \in H^1_{\mathrm{cyl}}(r^3drdz)$, where $F(r,z,t) := \int_0^t f(r,z,s)\,ds$ and $\nabla_{r,z} := \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)$. A ground state u of (3) is defined as a weak solution of (3) in the Nehari-manifold

$$M := \left\{ v \in H^1_{\mathrm{cyl}}(r^3drdz)\setminus\{0\} \,\middle|\, \int_\Omega \left(|\nabla_{r,z}v|^2 + V(r,z)v^2\right) r^3 d(r,z) = \int_\Omega f(r,z,r^2v^2)v^2 r^3 d(r,z) \right\}$$

such that

$$J(u) = \inf_{v\in M} J(v),$$

see the classical papers [16, 17]. We find ground states of (3) under additional assumptions on V and f. To state these assumptions we need the notion of Steiner-symmetrization, cf. [11, Chapter 3]. The Steiner-symmetrization (also called symmetric-decreasing rearrangement) of a cylindrical function $g = g(r,z)$ with respect to z is denoted by $g^\star$. We say that g is *Steiner-symmetric* if g coincides with its Steiner-symmetrization with respect to z, keeping the r-variable fixed. A function $h \in L^\infty(\Omega)$ is *reversed Steiner-symmetric* if $(\operatorname{ess\,sup} h - h)^\star = \operatorname{ess\,sup} h - h$ holds true. In other words h is even and symmetrically increasing.

Now we can state our assumptions on f.

(i) $f : \Omega \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function with $0 \le f(r,z,s) \le c(1 + s^{\frac{p-1}{2}})$ for some $c > 0$ and $p \in (1,5)$,
(ii) $f(r,z,s) = o(1)$ as $s \to 0$ uniformly in $r,z \in [0,\infty)\times\mathbb{R}$,
(iii) $f(r,z,s)$ strictly increasing in $s \in [0,\infty)$,
(iv) $\frac{F(r,z,s)}{s} \to \infty$ as $s \to \infty$ uniformly in $r,z \in [0,\infty)\times\mathbb{R}$,
(v) for all $r \in [0,\infty)$, $s \ge 0$ and $\sigma > 0$ the function

$$\varphi_\sigma(r,z,s) := f(r,z,(s+\sigma)^2)(s+\sigma)^2 - f(r,z,s^2)s^2$$

is symmetrically nonincreasing in z.

Conditions (ii)–(iv) are inspired by the work of Szulkin and Weth [19]. Namely, if we translate (ii)–(iv) into conditions for $\tilde{f}(r,z,s) := f(r,z,r^2s^2)s$ then they become identical to (ii)–(iv) of Theorem 20 from [19]. Condition (v) is used to prove the rearrangement inequality of Lemma 2.10 and it is due to Brock [7].

Next we state our main result.

Theorem 1.1. *Let $V \in L^\infty(\Omega)$ be reversed Steiner-symmetric such that the map*

$$\|\cdot\| : H^1_{\text{cyl}}(r^3 drdz) \to \mathbb{R}; \quad u \mapsto \left(\int_\Omega \left(|\nabla_{r,z} u|^2 + V(r,z) u^2 \right) r^3 d(r,z) \right)^{\frac{1}{2}}$$

is an equivalent norm to $\|\cdot\|_{H^1_{\text{cyl}}(r^3drdz)}$. Additionally, let f satsify the assumptions (i)–(v). *Then* (3) *has a ground state $u \in H^1_{\text{cyl}}(r^3drdz)$ which is symmetric about* $\{z=0\}$.

Remark 1.2. (1) The assumption of norm-equivalence is for instance satisfied if $V \geq 0$ and $\inf_{B_R^c} V > 0$ for some $R > 0$, where $B_R^c := \{(r,z) \in \Omega : r^2 + z^2 > R^2\}$. For the reader's convenience the proof based on Poincaré's inequality is given in the Appendix. Since Poincaré's inequality is applicable for domains bounded in one direction we can weaken $\inf_{B_R^c} V > 0$ to $\inf_{S^c} V > 0$ for strips $S = [0,\infty) \times [0,\rho]$ with $\rho > 0$ or $S = [r_0, r_1] \times [0, \infty)$ with $0 \leq r_0 < r_1 < \infty$.

(2) The conditions on f are satisfied if for instance $f(r,z,s) = \Gamma(r,z)|s|^{\frac{p-1}{2}}$ where $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric, ess $\inf_\Omega \Gamma > 0$ and $p \in (1,5)$. This choice of f corresponds to the equation $\nabla \times \nabla \times U + V(r,z)U = \Gamma(r,z)\,|U|^{p-1}\,U$ in $\mathbb{R}^3$. Another possible choice is $f(r,z,s) = \Gamma(r,z)\log(1+s)$ where again $\Gamma \in L^\infty(\Omega)$ is Steiner-symmetric and ess $\inf_\Omega \Gamma > 0$. This nonlinearity appeared for instance in [14] and it does not satisfy the classical Ambrosetti-Rabinowitz condition. The piecewise defined function

$$f(r,z,s) = \begin{cases} s^{\frac{\tilde{p}-1}{2}}, & 0 \leq s \leq 1 \\ s^{\frac{p(z)-1}{2}}, & s > 1 \end{cases}$$

with $\tilde{p} \in (1,5)$, $1 < \inf_{z\in\mathbb{R}} p(z) \leq \sup_{z\in\mathbb{R}} p(z) < 5$ and p symmetrically decreasing also satisfies the required conditions.

The paper is structured as follows: In Section 2 we give details on the variational formulation of problem (3) and prove pointwise decay estimates of Steiner-symmetric functions in $H^1_{\text{cyl}}(r^3drdz)$. In Section 3 we give the proof of Theorem 1.1, and in the Appendix we show an example for the potential V satisfying the norm-assumption of Theorem 1.1.

2. Variational formulation, decay estimates, rearrangements

Let us consider some properties of the space $H^1_{\text{cyl}}(r^3drdz)$. First, for U of the form (2) we have that $U \in H^1(\mathbb{R}^3)$ if and only if $u \in H^1_{\text{cyl}}(r^3drdz)$. A norm on $H^1_{\text{cyl}}(r^3drdz)$ is given by

$$\|u\|_{H^1_{\text{cyl}}(r^3drdz)} := \left(\int_\Omega \left(|\nabla_{r,z} u|^2 + u^2 \right) r^3 d(r,z) \right)^{\frac{1}{2}}.$$

Notice that the space $H^1_{\mathrm{cyl}}(r^3 drdz)$ behaves like a Sobolev-space in dimension 5. Next we show a useful embedding property. For this we need the following Sobolev and Lebesgue spaces in dimension 3 together with their canonical norms:

$$H^1_{\mathrm{cyl}}(rdrdz) := \left\{ v\colon (0,\infty)\times\mathbb{R}\to\mathbb{R} : v, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial z} \in L^2_{\mathrm{cyl}}(rdrdz)\right\},$$
$$L^q_{\mathrm{cyl}}(rdrdz) := \left\{ v\colon (0,\infty)\times\mathbb{R}\to\mathbb{R} : \int_\Omega |v(r,z)|^q r d(r,z) < \infty\right\} \quad \text{for } q\in[1,\infty).$$

Lemma 2.1. *For $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ Hardy's inequality holds*

$$\int_\Omega \frac{u^2}{r^2} r^3 d(r,z) \le C_H \int_\Omega \left(\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right) r^3 d(r,z). \tag{5}$$

Moreover, if $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$ then $ru \in H^1_{\mathrm{cyl}}(rdrdz)$ and there is a constant $C > 0$ such that for $2 \le q \le 6$

$$\|ru\|_{H^1_{\mathrm{cyl}}(rdrdz)}, \|ru\|_{L^q_{\mathrm{cyl}}(rdrdz)} \le C\,\|u\|_{H^1_{\mathrm{cyl}}(r^3drdz)} \tag{6}$$

Proof. Hardy's inequality (5) is given in [2, Lemma 9 (i)]. For $u \in H^1_{\mathrm{cyl}}(r^3drdz)$ we have ru, $\frac{\partial}{\partial z}(ru)$, $r\frac{\partial u}{\partial r} \in L^2_{\mathrm{cyl}}(rdrdz)$ and by (5) also $u \in L^2_{\mathrm{cyl}}(rdrdz)$. Since $\frac{\partial}{\partial r}(ru) = r\frac{\partial u}{\partial r} + u$ we conclude altogether $ru \in H^1_{\mathrm{cyl}}(rdrdz)$. By the Sobolev embedding in three dimensions this implies $ru \in L^q(rdrdz)$ for $q \in [2,6]$ and (5) yields

$$\begin{aligned}
\|ru\|^2_{H^1_{\mathrm{cyl}}(rdrdz)} &= \int_\Omega \left(|\nabla_{r,z}(ru)|^2 + r^2u^2 \right) rd(r,z) \\
&\le 2\int_\Omega \left(\left(r\frac{\partial u}{\partial z}\right)^2 + \left(r\frac{\partial u}{\partial r}\right)^2 + u^2 + r^2u^2 \right) rd(r,z) \\
&\le \tilde{C}\,\|u\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)}.
\end{aligned} \tag{7}$$

This finishes the proof. □

Next we show that the functional J from the introduction as well as the functional in the defintion of the Nehari-manifold are well-defined.

Lemma 2.2. *There is a constant $C > 0$ such that*

$$\left.\begin{aligned} &\int_\Omega f(r,z,r^2u^2)u^2 r^3 d(r,z) \\ &\int_\Omega \frac{1}{2r^2} F(r,z,r^2u^2) r^3 d(r,z) \end{aligned}\right\} \le C\left(\|u\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)} + \|u\|^{p+1}_{H^1_{\mathrm{cyl}}(r^3drdz)} \right)$$

for all $u \in H^1_{\mathrm{cyl}}(r^3drdz)$.

Proof. Clearly assumption (i) and (ii) show that for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that $0 \le f(r, z, s) \le \epsilon + C_\epsilon s^{\frac{p-1}{2}}$. Hence

$$0 \le f(r, z, r^2u^2)u^2r^3 \le \left(\epsilon r^2u^2 + C_\epsilon |ru|^{p+1}\right) r, \tag{8}$$

$$0 \le \frac{1}{2r^2} F(r, z, r^2u^2)r^3 \le \left(\epsilon r^2u^2 + \tilde{C}_\epsilon |ru|^{p+1}\right) r. \tag{9}$$

Due to (6) this implies the claim. □

In order to find critical points of J we need uniform decay estimates of Steiner-symmetric functions in $H^1_{\text{cyl}}(r^3 drdz)$. These estimates are given in [13] in much more generality but for the sake of completeness we give them here together with the simple proof. We start with a well-known fact concerning radially symmetric functions and afterwards extend the result to cylindrically symmetric functions. Let

$$H^1_{\text{rad}}(\mathbb{R}^n) := \left\{u \in H^1(\mathbb{R}^n) : u \text{ is radially symmetric}\right\}.$$

Lemma 2.3 (see [13])**.** *Let $n \ge 2$. Then there is a constant $C > 0$ such that*

$$|u(x)| \le C \, \|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)} \, \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)} \, |x|^{-\frac{n-1}{2}} \quad \textit{for almost all } x \in \mathbb{R}^n \textit{ and all } u \in H^1_{\text{rad}}(\mathbb{R}^n).$$

Proof. By density it is sufficient to prove the estimate for $u \in H^1_{\text{rad}}(\mathbb{R}^n) \cap C^\infty_c(\mathbb{R}^n)$. Let $r := |x|$. Then

$$\frac{d}{dr}\left(r^{n-1} |u|^2\right) = (n-1)r^{n-2} |u|^2 + r^{n-1} 2u \frac{\partial u}{\partial r} \ge -2 |u| \left|\frac{\partial u}{\partial r}\right| r^{n-1}.$$

Integrating from r to ∞ and expanding the domain of integration to all of $\mathbb{R}^n$ yields

$$r^{n-1} |u(x)|^2 \le C \int_{\mathbb{R}^n} |u| \, |\nabla u| \, dy \le C \, \|\nabla u\|_{L^2(\mathbb{R}^n)} \, \|u\|_{L^2(\mathbb{R}^n)} . \qquad □$$

Now we give an extension of Lemma 2.3 to cylindrically symmetric functions which are Steiner-symmetric in the non-radial component. We make use of the following notation: Let $t \in \mathbb{N}_{\ge 2}$ and $s \in \mathbb{N}$ such that $n = t + s$. We write points in $\mathbb{R}^n$ as (x, y) with $x \in \mathbb{R}^t$ and $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$. Furthermore, let

$$K_{t,s} := \left\{ u \in H^1(\mathbb{R}^n) \,\middle|\, \begin{array}{l} u(\cdot, y) \text{ is a radially symmetric function for every } y \in \mathbb{R}^s \text{ and} \\ u(x, \cdot) \text{ is Steiner-symmetric w.r.t. } y_i, i = 1, \ldots, s, \ \forall\, x \in \mathbb{R}^t \end{array} \right\}$$

In particular, if $u \in K_{t,s}$ then necessarily $u \ge 0$. In this setting we have the following extension of Lemma 2.3.

Lemma 2.4 (see [13])**.** *There is a constant $C > 0$ such that*

$$0 \le u(x, y) \le C \, \|\nabla_x u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)} \, \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)} \, |x|^{-\frac{t-1}{2}} \, |y_1 \cdots y_s|^{-\frac{1}{2}}$$

for almost all $(x, y) \in \mathbb{R}^n$ and all $u \in K_{t,s}$.

Proof. Let $u \in K_{t,s}$ and fix $y \in \mathbb{R}^s$. W.l.o.g. let $y_i > 0$ for all $i = 1, \ldots, s$. We define

$$v(x) := \int_0^{y_1} \cdots \int_0^{y_s} u(x,z)dz \quad \text{for } x \in \mathbb{R}^t.$$

By Hölder's inequality we obtain $v^2(x) \le y_1 \cdots y_s \int_0^{y_1} \cdots \int_0^{y_s} u^2(x,z)dz$, i.e.,

$$\|v\|_{L^2(\mathbb{R}^t)} \le (y_1 \cdots y_s)^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)}. \tag{10}$$

In the same manner we receive

$$\|\nabla v\|_{L^2(\mathbb{R}^t)} \le (y_1 \cdots y_s)^{\frac{1}{2}} \|\nabla_x u\|_{L^2(\mathbb{R}^n)}. \tag{11}$$

Since $v\colon \mathbb{R}^t \to \mathbb{R}$ is radially symmetric we can apply Lemma 2.3 and get from (10) and (11)

$$0 \le v(x) \le C \|\nabla v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^t)} \|v\|^{\frac{1}{2}}_{L^2(\mathbb{R}^t)} |x|^{-\frac{t-1}{2}} \le C(y_1 \cdots y_s)^{\frac{1}{2}} \|\nabla_x u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)} \|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n)} |x|^{-\frac{t-1}{2}}. \tag{12}$$

Due to the monotonicity-property in y-direction we also have $v(x) \ge y_1 \cdots y_s u(x,y)$ and thus (12) gives the desired inequality. □

We prove three additional lemmas which are used in the next section.

Lemma 2.5. *The set $K_{t,s}$ is a weakly closed cone in $H^1(\mathbb{R}^n)$.*

Proof. Take a sequence $(u_k)_{k\in\mathbb{N}} \subset K_{t,s}$ such that $u_k \rightharpoonup u \in H^1(\mathbb{R}^n)$ as $k \to \infty$. By the Sobolev embedding on bounded domains we deduce that a subsequence of u_k converges pointwise almost everywhere on $\mathbb{R}^n$ to u. Since every u_k enjoys the radial symmetry in the first component and the non-increasing property in the second variable, the pointwise convergence implies that also u enjoys these properties, i.e., $u \in K_{t,s}$. □

Lemma 2.6. *The functionals*

$$I(v) = \int_\Omega \frac{1}{2r^2} F(r,z,r^2v^2)r^3\, d(r,z), \quad I'(v)[v] = \int_\Omega f(r,z,r^2v^2)v^2r^3\, d(r,z)$$

are weakly sequentially continuous on the set $K_{4,1} \subset H^1_{\text{cyl}}(r^3drdz)$.

Remark 2.7. In the proof we use twice the following principle: if $S \subset \mathbb{R}^m$ is a set of finite measure and $w_k : S \to \mathbb{R}$ a sequence of measurable functions such that $\|w_k\|_{L^r(S)} \le C$ and $w_k \to w$ pointwise a.e. as $k \to \infty$ then $\|w_k - w\|_{L^q(S)} \to 0$ as $k \to \infty$ for $1 \le q < r$. The proof is as follows: Egorov's theorem allows to choose $\Sigma \subset S$ such that $w_k \to w$ uniformly on Σ and $|S \setminus \Sigma| \le \epsilon$ arbitrary small. By Hölder's inequality the remaining integral is estimated by $\int_{S\setminus\Sigma} |w_k - w|^q\, dx \le \epsilon^{1-\frac{q}{r}} \|w_k - w\|^q_{L^r(S)}$.

Proof. Let us take a weakly convergent sequence $(v_k)_{k\in\mathbb{N}}$ in $K_{4,1}$ such that $v_k \rightharpoonup v$ in $H^1_{\mathrm{cyl}}(r^3 drdz)$ and $v_k \to v$ pointwise a.e. in Ω. By Lemma 2.5 one gets $v \in K_{4,1}$ and using Lemma 2.4 there exists a constant $C > 0$ such that

$$0 \le v_k(r,z), v(r,z) \le Cr^{-\frac{3}{2}}|z|^{-\frac{1}{2}} \quad \text{for all } k \in \mathbb{N} \text{ and almost all } (r,z) \in \Omega. \tag{13}$$

Our goal is now to show at least for a subsequence

$$\int_\Omega \frac{1}{r^2} F(r,z,r^2v_k^2) r^3 d(r,z) \to \int_\Omega \frac{1}{r^2} F(r,z,r^2v^2) r^3 d(r,z) \quad \text{as } k \to \infty \tag{14}$$

and

$$\int_\Omega f(r,z,r^2v_k^2) v_k^2 r^3 d(r,z) \to \int_\Omega f(r,z,r^2v^2) v^2 r^3 d(r,z) \quad \text{as } k \to \infty. \tag{15}$$

By (9) we find $\frac{1}{r^2}\Big|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)\Big| r^3 \le \epsilon r^2(v_k^2+v^2)r + C_\epsilon\Big(|rv_k|^{p+1} + |rv|^{p+1}\Big)r$ and hence

$$\Big(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2+v^2)\Big)^+ r \le C_\epsilon \Big(|rv_k|^{p+1} + |rv|^{p+1}\Big) r. \tag{16}$$

Inspired by [12, 13] the idea is to show

$$rv_k \to rv \text{ in } L^{p+1}(rdrdz) \quad \text{as } k \to \infty. \tag{17}$$

Once (17) is established we obtain a majorant $|rv_k|, |rv| \le w \in L^{p+1}(r\,drdz)$ (cf. [20, Lemma A.1]). Together with (16) this majorant allows to apply Lebesgue's dominated convergence theorem and yields

$$\lim_{k\to\infty} \int_\Omega \Big(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2+v^2)\Big)^+ r\,drdz = 2\epsilon\|v\|^2_{L^2(r^3drdz)}. \tag{18}$$

If we set

$$a_k := \int_\Omega \Big|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)\Big| r\,drdz$$

and

$$b_k := \epsilon\|r^2(v_k^2+v^2)\|_{L^1(rdrdz)} = \epsilon(\|v_k\|^2_{L^2(r^3drdz)} + \|v\|^2_{L^2(r^3drdz)}) \le C\epsilon$$

then

$$\begin{aligned}
\limsup_{k\in\mathbb{N}} a_k &\le \limsup_{k\in\mathbb{N}} b_k + \limsup_{k\in\mathbb{N}} (a_k - b_k)^+ \\
&\le C\epsilon + \limsup_{k\in\mathbb{N}} \left(\int_\Omega \Big(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2+v^2)\Big) rdrdz\right)^+ \\
&\le C\epsilon + \limsup_{k\in\mathbb{N}} \int_\Omega \Big(|F(r,z,r^2v_k^2) - F(r,z,r^2v^2)| - \epsilon r^2(v_k^2+v^2)\Big)^+ rdrdz \\
&\le \epsilon(C + 2\|v\|^2_{L^2(r^3drdz)}) \quad \text{by (18).}
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary this shows that $\lim_{k\to\infty} a_k = 0$ and therefore (14) holds. The proof of (15) is similar since $\left(f(r,z,r^2v_k^2)r^2v_k^2 - f(r,z,r^2v^2)r^2v^2 - \epsilon r^2(v_k^2+v^2)\right)^+ r$ satisfies an estimate just like (16) if we use (8) instead of (9).

It remains to prove (17). For this, we split Ω into four parts $\Omega_1,\dots,\Omega_4$ and show (17) on each of these parts separately. The definitions of $\Omega_1,\dots,\Omega_4$ are as follows: For $R > 0$ let

$$\Omega_1 := \{(r,z)\in\Omega : r<R, |z|<R\},\quad \Omega_2 := \{(r,z)\in\Omega : r\geq R, |z|\geq R\},$$
$$\Omega_3 := \{(r,z)\in\Omega : r<R, |z|\geq R\},\quad \Omega_4 := \{(r,z)\in\Omega : r\geq R, |z|< R\}.$$

Convergence on Ω_1: Follows from $rv_k \to rv$ in $L^q(K; r\,drdz)$ for every compact subset $K\subset[0,\infty)\times\mathbb{R}$ and every $q\in[1,6)$. This step works independently of the choice of $R>0$.

Convergence on Ω_2: Let $\varepsilon > 0$. With the help of (13) we calculate

$$\begin{aligned}
\int_{\Omega_2} |rv_k - rv|^{p+1}\, rd(r,z) &\leq 2^{p+1}\int_{\Omega_2} r^{p+1}\left(|v_k|^{p+1}+|v|^{p+1}\right) rd(r,z)\\
&\leq 2^{p+1}C^{p-1}\int_{\Omega_2} r^{-\frac{p-1}{2}}\,|z|^{-\frac{p-1}{2}}\left(|v_k(r,z)|^2+|v(r,z)|^2\right) r^3 d(r,z)\\
&\leq C_1\left(\|v_k\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)} + \|v\|^2_{H^1_{\mathrm{cyl}}(r^3drdz)}\right) R^{-(p-1)}\\
&\leq C_2 R^{-(p-1)}
\end{aligned}$$

which is less or equal ε if we choose $R>0$ large enough.

Convergence on Ω_3: Due to symmetry in z-direction it is enough to focus on $\tilde{\Omega}_3 := \{(r,z)\in\Omega : r<R, z\geq R\}$. Let $\alpha>0$ be arbitrary. Again by (13) we obtain

$$\{(r,z)\in\tilde{\Omega}_3 : v_k(r,z)>\alpha\}\subset\{(r,z)\in\tilde{\Omega}_3 : r\,z^{\frac{1}{3}}\leq C_\alpha\} =: S_\alpha,$$

where $C_\alpha = \left(\frac{C}{\alpha}\right)^{\frac{2}{3}}$ and C is the constant from (13). The set S_α has finite measure since

$$|S_\alpha| \leq \int_R^\infty\int_0^{C_\alpha z^{-\frac{1}{3}}} r^3 dr\, dz = \frac{C_\alpha^4}{4}\int_R^\infty z^{-\frac{4}{3}}dz = \frac{3}{4}C_\alpha^4 R^{-\frac{1}{3}} < \infty.$$

By the convergence principle from Remark 2.7 and since by (7) $\|rv_k\|_{L^6(rdrdz)} \leq \|v_k\|_{H^1_{\mathrm{cyl}}(r^3drdz)}$ is bounded we obtain $\int_{S_\alpha} r^{p-1}|v_k - v|^{p+1}r^3 d(r,z) \to 0$ as $k\to\infty$ for $1\leq p<5$. It remains to prove the convergence on $\tilde{\Omega}_3\backslash S_\alpha$. For allmost all $(r,z)\in\tilde{\Omega}_3\backslash S_\alpha$ we have that $v(r,z) = \lim_{k\to\infty} v_k(r,z)\leq\alpha$. Hence,

$$\int_{\tilde{\Omega}_3\backslash S_\alpha} r^{p-1}|v_k - v|^{p+1}r^3 d(r,z) \leq R^{p-1}(2\alpha)^{p-1}\int_\Omega |v_k - v|^2 r^3 d(r,z) \leq C\alpha^{p-1}.$$

In summary, since $\alpha > 0$ is arbitrary this shows (17) on Ω_3.

Convergence on Ω_4: Again it is enough to focus on $\tilde{\Omega}_4 := \{(r,z) \in \Omega : r \geq R, 0 \leq z < R\}$. Fix $z \in (0,R)$. Let us first show that

$$\int_{\{r\geq R\}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr \to 0 \quad \text{as } k \to \infty. \tag{19}$$

Since $v_k(r,\cdot)$ is nonincreasing in its last component we deduce

$$\int_0^\infty r^q v_k^q(r,z) r\, dr \leq \frac{1}{z}\int_0^z \int_0^\infty r^q v_k^q(r,\zeta) r\, dr d\zeta \leq \frac{1}{z}\int_\Omega r^q v_k^q(r,\zeta) r d(r,\zeta) \leq \frac{C}{z} \tag{20}$$

for all $q \in [2,6]$ by (7). Thus for $q \in [2,6]$ the sequence $\|\cdot v_k(\cdot,z)\|_{L^q((0,\infty),rdr)}$ is uniformly bounded in $k \in \mathbb{N}$. Moreover, (13) implies $v_k(r,z) \leq C(z) r^{-\frac{3}{2}}$ uniformly in $k \in \mathbb{N}$. Hence for $\tilde{R} > R$

$$\begin{aligned}\int_{\tilde{R}}^\infty r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr &\leq (2C(z))^{p-1}\int_{\tilde{R}}^\infty r^{-\frac{p-1}{2}}|v_k(r,z) - v(r,z)|^2 r^3 dr\\ &\leq (2C(z))^{p-1}\tilde{R}^{\frac{1-p}{2}}\frac{C}{z} \quad \text{by (20).}\end{aligned}$$

The last term can be made arbitrarily small provided $\tilde{R}$ is chosen big enough. To finish the proof of (19) it remains to prove $\int_R^{\tilde{R}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr \to 0$ as $k \to \infty$. Since for almost all $z \in (0,R)$ we have $v_k(\cdot,z) \to v(\cdot,z)$ pointwise almost everywhere on $(R,\tilde{R})$ as well as the boundedness of $\|\cdot v_k(\cdot,z)\|_{L^6((0,\infty),rdr)}$ by (20) we can apply the convergence principle from the remark above and deduce

$$\int_R^{\tilde{R}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr \to 0 \quad \text{as } k \to \infty.$$

Hence (19) is accomplished for almost all $z \in (0,R)$.

Defining $\varphi_k(z) := \int_{\{r\geq R\}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr$ we have $\varphi_k \to 0$ as $k \to \infty$ pointwise a.e. in $[0,R)$. The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $L^1([0,R),dz)$ since by (6)

$$\int_0^R \int_{\{r\geq R\}} r^{p-1}|v_k(r,z) - v(r,z)|^{p+1} r^3 dr dz \leq C\int_\Omega r^{p-1}\left(|v_k|^{p+1} + |v|^{p+1}\right) r^3 d(r,z) \leq \tilde{C}.$$

Moreover, for $p \in (1,3]$, the sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,1}([0,R),dz)$ since

$$\begin{aligned}\left\|\frac{\partial \varphi_k}{\partial z}\right\|^2_{L^1([0,R],dz)} &\leq \left(\int_0^R \int_R^\infty (p+1) r^{p-1}|v_k - v|^p \left|\frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z}\right| r^3 dr dz\right)^2\\ &\leq \left(\int_\Omega (p+1) r^{p-1}|v_k - v|^p \left|\frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z}\right| r^3 d(r,z)\right)^2\end{aligned}$$

and consequently $\left\|\frac{\partial \varphi_k}{\partial z}\right\|^2_{L^1([0,R],dz)} \leq C \int_\Omega r^{2p-2}|v_k - v|^{2p} r^3 d(r,z) \int_\Omega \left|\frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z}\right|^2 r^3 d(r,z) = C\|r(v_k - v)\|^{2p}_{L^{2p}(rdrdz)} \int_\Omega \left|\frac{\partial v_k}{\partial z} - \frac{\partial v}{\partial z}\right|^2 r^3 d(r,z) \leq C$. Hence, by the compact embedding $W^{1,1}([0,R),dz) \hookrightarrow L^1([0,R),dz)$ we conclude that at least a subsequence of $(\varphi_k)_{k\in\mathbb{N}}$ is converging in $L^1([0,R),dz)$ to a limit function, which must be 0 since we have already asserted the pointwise a.e. convergence to 0 on $[0,R)$. This shows (17) on Ω_4 for $p \in (1,3]$. For $p \in (3,5)$ we make use of Hölder's interpolation, namely,

$$\begin{aligned}\|rv_k - rv\|^{p+1}_{L^{p+1}_{\text{cyl}}(\Omega_4,rdrdz)} &\leq \|rv_k - rv\|^{4\theta}_{L^4_{\text{cyl}}(\Omega_4,rdrdz)} \|rv_k - rv\|^{6(1-\theta)}_{L^6_{\text{cyl}}(\Omega_4,rdrdz)} \\ &\leq \tilde{C}\,\|rv_k - rv\|^{4\theta}_{L^4_{\text{cyl}}(\Omega_4,rdrdz)} \to 0\end{aligned}$$

as $k \to \infty$, where $\theta \in (0,1)$ is chosen such that $p + 1 = 4\theta + 6(1-\theta)$, i.e., $\theta = \frac{5-p}{2}$.

The combination of convergences on $\Omega_1, \ldots, \Omega_4$ finally proves (17). □

For our last lemma we need the notion of cylindrical C_c^∞-functions which we introduce now.

Definition 2.8. A function $u = u(r,z)$ belongs to $C_c^\infty([0,\infty) \times \mathbb{R})$ if and only if $u \in C^\infty([0,\infty) \times \mathbb{R})$, $\operatorname{supp} u$ is compact in $[0,\infty) \times \mathbb{R}$ and $\frac{\partial^j u}{\partial r^j}(0,z) = 0$ for all odd integers $j \in 2\mathbb{N} - 1$.

Remark 2.9. Since $u \in C_c^\infty([0,\infty) \times \mathbb{R})$ is equivalent to $\tilde{u} \in C_c^\infty(\mathbb{R}^5)$ with $\tilde{u}(x) := u(|(x_1,\ldots,x_4)|, x_5)$ we see that $C_c^\infty([0,\infty) \times \mathbb{R})$ is dense in $H^1_{\text{cyl}}(r^3 drdz)$.

Lemma 2.10. *For $u \in H^1_{\text{cyl}}(r^3 drdz)$ we have $\|u^\star\| \leq \|u\|$ where $\star$ denotes Steiner-symmetrization with respect to z and $\|\cdot\|$ is the equivalent norm from Theorem 1.1. Moreover*

$$I(u) \leq I(u^\star) \quad \textit{and} \quad I'(u)[u] \leq I'(u^\star)[u^\star].$$

Proof. We begin by recalling several classical rearrangement inequalities from [10, 11]. Recall first the Pólya-Szegö inequality

$$\int_{\mathbb{R}^n} |\nabla f^\circledast|^2 dx \leq \int_{\mathbb{R}^n} |\nabla f|^2 dx \tag{21}$$

for $f \in H^1(\mathbb{R}^n)$ and $\circledast$ denoting Schwarz-symmetrization (also called symmetrically decreasing rearrangement). Furthermore we have for $0 \leq f, g \in L^2(\mathbb{R}^n)$ the classical rearrangement inequality

$$\int_{\mathbb{R}} fg dx \leq \int_{\mathbb{R}} f^\circledast g^\circledast dx \tag{22}$$

and the nonexpansivity of rearrangement

$$\int_{\mathbb{R}^n} |f^\circledast - g^\circledast|^2 dx \leq \int_{\mathbb{R}^n} |f - g|^2\, dx. \tag{23}$$

From (21) we immediately receive for $u \in H^1_{\text{cyl}}(r^3 drdz)$ that

$$\int_{\mathbb{R}} |\nabla_z u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_z u|^2 dz. \tag{24}$$

Next we want to establish a similar inequality for $\nabla_r u$. We do this first for $u \in C_c^\infty([0,\infty) \times \mathbb{R})$. With the help of (23) we find that

$$\int_{\mathbb{R}} \left| \frac{u^\star(r+t,z) - u^\star(r,z)}{t} \right|^2 dz \le \int_{\mathbb{R}} \left| \frac{u(r+t,z) - u(r,z)}{t} \right|^2 dz$$

for almost all $r, t \in [0,\infty)$. Sending $t \to 0$ and using Fatou's lemma on the left side of the inequality yields

$$\int_{\mathbb{R}} |\nabla_r u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_r u|^2 dz \tag{25}$$

for $u \in C_c^\infty([0,\infty) \times \mathbb{R})$ and almost all $r \in [0,\infty)$. Since Steiner Symmetrization is continuous in H^1 (see [8, Theorem 1]) we obtain by approximation that (25) is indeed valid for all $u \in H^1_{cyl}(r^3 drdz)$. Together with (24) we obtain $\int_{\mathbb{R}} |\nabla_{r,z} u^\star|^2 dz \le \int_{\mathbb{R}} |\nabla_{r,z} u|^2 dz$ for almost all $r \ge 0$ and integration leads to

$$\int_{\mathbb{R}} \int_0^\infty |\nabla_{r,z} u^\star|^2 r^3 drdz \le \int_{\mathbb{R}} \int_0^\infty |\nabla_{r,z} u|^2 r^3 drdz. \tag{26}$$

Fixing $r \in [0,\infty)$ and applying (22) to $f(\cdot) = \operatorname{ess\,sup} V - V(r,\cdot)$ and $g(\cdot) = u^2(r,\cdot)$ gives

$$\begin{aligned}\int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))\, u^2(r,\cdot) dz &\le \int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))^\star \, (u^2)^\star(r,\cdot) dz \\ &= \int_{\mathbb{R}} (\operatorname{ess\,sup} V - V(r,\cdot))\, (u^\star)^2\, (r,\cdot) dz.\end{aligned}$$

Using $\|u(r,\cdot)\|_{L^2(\mathbb{R})} = \|u^\star(r,\cdot)\|_{L^2(\mathbb{R})}$ this results in

$$\int_{\mathbb{R}} \int_0^\infty V(r,z)\, (u^\star)^2\, r^3 drdz \le \int_{\mathbb{R}} \int_0^\infty V(r,z) u^2 r^3 drdz. \tag{27}$$

The combination of (26) and (27) yields the claimed inequality $\|u^\star\|^2 \le \|u\|^2$.

Assumption (v) on f allows to apply [7, Theorem 5.1] and to deduce

$$I'(u)[u] = \int_\Omega f(r,z,r^2u^2) u^2 r^3 d(r,z) \le \int_\Omega f(r,z,r^2 u^{\star 2}) u^{\star 2} r^3 d(r,z) = I'(u^\star)[u^\star].$$

Moroever, using (v) with $s = 0$ shows that for all $r \in [0, \infty)$, $\sigma \geq 0$ the function $z \mapsto f(r, z, \sigma^2)$ is symmetrically nonincreasing in z and hence

$$\Phi_\sigma(r, z, s) := F(r, z, r^2(s+\sigma)^2) - F(r, z, r^2 s^2) = \int_{r^2 s^2}^{r^2(s+\sigma)^2} f(r, z, t)\, dt$$

is symmetrically nonincreasing in z. Applying once more [7, Theorem 5.1] yields

$$I(u) = \int_\Omega \frac{1}{2r^2} F(r, z, r^2 u^2) r^3 d(r, z) \leq \int_\Omega \frac{1}{2r^2} F(r, z, r^2 u^{\star 2}) r^3 d(r, z) = I(u^\star).$$

This finishes the proof of the lemma. □

3. Proof of Theorem 1.1

Recall from Lemma 2.6 the definition $I(u) := \int_\Omega \frac{1}{2r^2} F(r, z, r^2 u^2) r^3 d(r, z)$ for $u \in H^1_{\mathrm{cyl}}(r^3 drdz)$. We show that the assumptions (i)–(iii) of [19, Theorem 12] are satisfied. Let $\varepsilon > 0$. The growth assumptions (i) and (ii) on f imply that for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that the global estimate $0 \leq f(r, z, s) \leq \epsilon + C_\epsilon |s|^{\frac{p-1}{2}}$ holds. Together with (6) we obtain

$$\begin{aligned}
|I'(u)[v]| &= \left| \int_\Omega f(r, z, r^2 u^2) u v r^3 d(r, z) \right| \\
&\leq \varepsilon \int_\Omega |ru||rv| r d(r, z) + C_\epsilon \int_\Omega |ru|^p |rv| r d(r, z) \\
&\leq \varepsilon C \, \|u\|_{H^1_{cyl}(r^3 drdz)} \, \|v\|_{H^1_{cyl}(r^3 drdz)} + \tilde{C}_\epsilon \, \|u\|^p_{H^1_{cyl}(r^3 drdz)} \, \|v\|_{H^1_{cyl}(r^3 drdz)}
\end{aligned}$$

Taking the supremum over all $v \in H^1_{cyl}(r^3 drdz)$ with $\|v\|_{H^1_{cyl}(r^3 drdz)} = 1$ we see that

$$I'(u) = o(\|u\|) \quad \text{as } u \to 0. \tag{28}$$

Moreover, due to assumption (iii) on f the map

$$s \mapsto \frac{I'(su)[u]}{s} = \int_\Omega f(r, z, s^2 r^2 u^2) u^2 r^3 d(r, z) \tag{29}$$

is strictly increasing for all $u \neq 0$ and $s > 0$. Next we claim that

$$\frac{I(su)}{s^2} \to \infty \quad \text{as } s \to \infty \begin{cases} \text{uniformly for } u \text{ on weakly compact} \\ \text{subsets } W \text{ of } H^1_{cyl}(r^3 drdz) \setminus \{0\}. \end{cases} \tag{30}$$

Suppose not. Then there are $(u_k)_{k\in\mathbb{N}} \subset W$ and $s_k \to \infty$ as $k \to \infty$ such that $\frac{I(s_k u_k)}{s_k^2}$ is bounded as $k \to \infty$. But along a subsequence we have $u_k \rightharpoonup u \neq 0$ and $u_k(x) \to u(x)$

pointwise almost everywhere. Let $\Omega^\sharp := \{(r,z) \in \Omega : u(r,z) \neq 0\}$. Then $|\Omega^\sharp| > 0$ and on $\Omega^\sharp$ we have $|s_k u_k(r,z)| \to \infty$ as $k \to \infty$. Fatou's lemma and assumption (iv) on F imply

$$\frac{I(s_k u_k)}{s_k^2} = \int_\Omega \frac{F(r,z,s_k^2 r^2 u_k^2)}{2 s_k^2 r^2} r^3 d(r,z) \geq \int_{\Omega^\sharp} \frac{F(r,z,s_k^2 r^2 u_k^2)}{2 s_k^2 r^2 u_k^2} u_k^2 r^3 d(r,z) \to \infty \quad \text{as } k \to \infty,$$

a contradiction. In summary, (28)–(30) imply that (i)–(iii) of [19, Theorem 12] are satisfied.

Now we take a sequence $(u_k)_{k\in\mathbb{N}} \subset M$ such that $J(u_k) \to \inf_M J$ as $k \to \infty$. Since $\|\nabla_{r,z}|u_k|\,\|_{L^2} = \|\nabla_{r,z} u_k\|_{L^2}$ we can assume that $u_k \geq 0$ for all $k \in \mathbb{N}$. Then [19, Theorem 12] guarantees that for every k there is a unique $t_k > 0$ such that $v_k := t_k u_k^\star \in M$. We show next that $t_k \leq 1$ for all $k \in \mathbb{N}$. Assume $t_k > 1$. Then

$$\begin{aligned}
\int_\Omega f(r,z,r^2 u_k^{\star 2}) u_k^{\star 2} r^3 d(r,z) &< \int_\Omega f(r,z,t_k^2 r^2 u_k^{\star 2}) u_k^{\star 2} r^3 d(r,z) && \text{by assumption (iii)} \\
&= \|u_k^\star\|^2 && \text{since } t_k u_k^\star \in M \\
&\leq \|u_k\|^2 && \text{by Lemma 2.10} \\
&= \int_\Omega f(r,z,r^2 u_k^2) u_k^2 r^3 d(r,z) && \text{since } u_k \in M.
\end{aligned}$$

This contradicts the inequality $I'(u_k)[u_k] \leq I'(u_k^\star)[u_k^\star]$ from Lemma 2.10 and thus $t_k \leq 1$ for all $k \in \mathbb{N}$.

Next notice that for fixed $(r,z,s) \in [0,\infty) \times \mathbb{R} \times [0,\infty)$ and $t \in (0,1]$ one has

$$\frac{d}{dt}\left(t^2 f(r,z,s^2)s^2 - F(r,z,t^2 s^2)\right) = 2ts^2\left(f(r,z,s^2) - f(r,z,t^2 s^2)\right) > 0$$

since f is strictly increasing in its last variable by assumption (iii). This shows that the map $t \mapsto t^2 f(r,z,s^2)s^2 - F(r,z,t^2 s^2)$ is strictly increasing for $t \in [0,1]$. From this monotonicity and the inequality $I(t_k u_k) \leq I(t_k u_k^\star)$ from Lemma 2.10 we conclude

$$\begin{aligned}
2J(v_k) &= \int_\Omega \left(t_k^2 |\nabla_{r,z} u_k^\star|^2 + V(r,z) t_k^2 u_k^{\star 2} - \frac{1}{r^2} F(r,z,r^2 t_k^2 u_k^{\star 2})\right) r^3 d(r,z) \\
&\leq \int_\Omega \left(t_k^2 |\nabla_{r,z} u_k|^2 + V(r,z) t_k^2 u_k^2 - \frac{1}{r^2} F(r,z,r^2 t_k^2 u_k^2)\right) r^3 d(r,z) \\
&= \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2 u_k^2) t_k^2 r^2 u_k^2 - F(r,z,r^2 t_k^2 u_k^2)\right) r^3 d(r,z) \\
&\leq \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2 u_k^2) r^2 u_k^2 - F(r,z,r^2 u_k^2)\right) r^3 d(r,z) \\
&= 2J(u_k).
\end{aligned} \tag{31}$$

So $(v_k)_{k\in\mathbb{N}} \subset M$ is also a minimizing sequence for J which belongs to $K_{4,1}$. The boundedness of $(v_k)_{k\in\mathbb{N}}$ is established in Proposition 14 in [19]. Hence, we find

$v_\infty \in H^1_{\mathrm{cyl}}(r^3 drdz)$ such that $v_k \rightharpoonup v_\infty$ in $H^1_{\mathrm{cyl}}(r^3 drdz)$ along a subsequence as $k \to \infty$. In addition, $v_\infty \in K_{4,1}$ due to Lemma 2.5 and $v_\infty \neq 0$ by [19, Proposition 14] where instead of the weak sequential continuity of I on all of $H^1_{\mathrm{cyl}}(r^3 drdz)$ we use it only on $K_{4,1}$ as stated in Lemma 2.6.

Let us show that $v_\infty \in M$. Since $v_\infty \neq 0$ we can choose $t_\infty > 0$ such that $t_\infty v_\infty \in M$. In the same manner as before for the sequence t_k we can show that $t_\infty \leq 1$. Assume $t_\infty < 1$. Then as in (31) and using the weak sequential continuity on $K_{4,1}$ as shown in Lemma 2.6 we find

$$\begin{aligned} 2J(t_\infty v_\infty) &< \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2v_\infty^2)r^2v_\infty^2 - F(r,z,r^2v_\infty^2)\right) r^3 d(r,z) \\ &= \lim_{k\to\infty} \int_\Omega \frac{1}{r^2}\left(f(r,z,r^2v_k^2)r^2v_k^2 - F(r,z,r^2v_k^2)\right) r^3 d(r,z) \\ &= 2\inf_M J \\ &\leq 2J(t_\infty v_\infty) \end{aligned}$$

which is a contradiction. So $t_\infty = 1$ and thus $v_\infty \in M$. Then by the weak lower semi-continuity of $\|\cdot\|$ and once again the weak sequential continuity of I we conclude

$$J(v_\infty) \leq \liminf_{k\to\infty} J(v_k) = \inf_M J \leq J(v_\infty).$$

Hence, $v_\infty \in K_{4,1}$ is a minimizer of J on M, i.e., a ground state of (3) which is Steiner symmetric in z with respect to $\{z = 0\}$.

Appendix

Here we prove that the condition $V \geq 0$ and $\inf_{B_R^c} V > 0$ for some $R > 0$ implies that on $H^1_{\mathrm{cyl}}(r^3 drdz)$ the expression $\left(\int_\Omega \left(|\nabla_{r,z}u|^2 + V(r,z)u^2\right) r^3 d(r,z)\right)^{\frac{1}{2}}$ is an equivalent norm. Suppose not. Then there is a sequence $(u_k)_{k\in\mathbb{N}}$ such that $\|u_k\|_{L^2(r^3drdz)} = 1$ and $\int_\Omega \left(|\nabla_{r,z}u_k|^2 + V(r,z)u_k^2\right) r^3 d(r,z) \to 0$ as $k \to \infty$. In particular,

$$\int_\Omega |\nabla_{r,z}u_k|^2 r^3 d(r,z) \to 0 \quad \text{and} \quad \int_{B_R^c} u_k^2 r^3 d(r,z) \to 0 \quad \text{as } k \to \infty. \tag{32}$$

Let χ denote a smooth cut-off function such that $\chi(r,z) = 1$ for $0 \leq \sqrt{r^2+z^2} < R$ and $\chi(r,z) = 0$ for $\sqrt{r^2+z^2} \geq R+1$. Then $v_k := \chi u_k \in H^1_{0,\mathrm{cyl}}(B_{R+1}, r^3 drdz)$ and

$$|\nabla_{r,z}v_k|^2 = \chi^2|\nabla_{r,z}u_k|^2 + |\nabla_{r,z}\chi|^2u_k^2 + 2u_k\chi\nabla_{r,z}u_k \cdot \nabla_{r,z}\chi.$$

Hence, by (32)

$$\begin{aligned}\int_\Omega |\nabla_{r,z} v_k|^2 r^3 d(r,z) &\le 2\int_\Omega \chi^2 |\nabla_{r,z} u_k|^2 r^3 d(r,z) + 2\int_\Omega u_k^2 |\nabla_{r,z}\chi|^2 r^3 d(r,z) \\ &\le 2\int_\Omega |\nabla_{r,z} u_k|^2 r^3 d(r,z) + 2\|\nabla_{r,z}\chi\|_\infty^2 \int_{B_{R+1}\setminus B_R} u_k^2 r^3 d(r,z) \to 0\end{aligned} \tag{33}$$

as $k \to \infty$. In particular, $\int_{B_{R+1}} |\nabla_{r,z} v_k|^2 r^3 d(r,z) \to 0$. By Poincaré's inequality, (32) and $\|u_k\|_{L^2(r^3 drdz)} = 1$ we see

$$C_P \int_{B_{R+1}} |\nabla_{r,z} v_k|^2 r^3 d(r,z) \ge \int_{B_{R+1}} v_k^2 r^3 d(r,z) \ge \int_{B_R} u_k^2 r^3 d(r,z) = 1 - o(1),$$

contradicting (33). □

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