

Solvability of Hammerstein Integral Equations with Applications to Boundary Value Problems

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Abstract. In this paper we present some new results regarding the solvability of nonlinear Hammerstein integral equations in a special cone of continuous functions. The proofs are based on a certain fixed point theorem of Leggett and Williams type. We give an application of the abstract result to prove the existence of nontrivial solutions of a periodic boundary value problem. We also investigate, via a version of Krasnosel'skii's theorem for the sum of two operators, the solvability of perturbed Hammerstein integral equations in the space of continuous functions of bounded variation in the sense of Jordan. As an application of these results, we study the solvability of a boundary value problem subject to integral boundary conditions of Riemann–Stieltjes type. Some examples are presented in order to illustrate the obtained results.

Keywords. Boundary value problem, cone, Hammerstein integral equation, functions of bounded variation

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1. Introduction

Numerous problems of various branches of science lead to the necessity of investigating the solvability of nonlinear Hammerstein integral equations. It is for this reason the theory of nonlinear integral equations has become an important part of nonlinear functional analysis and has attracted the interest of many mathematicians.

In the first part of the paper we study the existence of eigenvalues of Hammerstein integral equations of the form

$$\lambda x(t) = \int_{\bar{\Omega}} k(t, s) f(s, x(s)) ds, \quad t \in \bar{\Omega}, \quad (1.1)$$

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where Ω is an open and bounded subset of \mathbb{R}^n , k is allowed to change sign and f is non-negative.

A number of tools have been utilized to study the solvability of (1.1); for example, variational methods have been employed in the case of *symmetric* kernels by Faraci [11] and Faraci and Moroz [12], topological methods have been used by Lan [24–26], Lan and Webb [27] and Ma [31] and a combination of topological and iterative techniques has been utilized by Franco and co-authors [13]. In particular, Lan [24] proved the existence of a positive eigenfunction under non-negativity assumptions on the kernel k . The results of [24] were complemented by Infante [17], who proved, under weaker assumptions regarding the sign of the kernel k , the existence of eigenfunctions within a cone of functions that are allowed to *change sign*, namely

$$K = \left\{ x \in C(\overline{\Omega}) : \min_{t \in \Omega_0} x(t) \geq c \|x\|_\infty \right\}, \quad (1.2)$$

where $\Omega_0 \subseteq \overline{\Omega}$ is a closed set of positive Lebesgue measure. This type of cone has been introduced by Infante and Webb in [19]. Let us note that the functions in (1.2) are positive on the subset Ω_0 but are allowed to change sign elsewhere.

A key assumption in [17] is the positivity of the kernel k on $\Omega_0 \times \overline{\Omega}$. Here we drop this condition and utilize instead the cone

$$C = \left\{ x \in C(\overline{\Omega}) : \int_{\Omega_0} x(t) dt \geq c \|x\|_\infty \right\}. \quad (1.3)$$

Let us note that we do not require the functions in C to be *positive* in Ω_0 , but only to have a *positive average* in this subset. In Section 3 we use a recent Leggett–Williams type theorem due to Bugajewski and Kasprzak [8] in order to prove the existence of eigenfunctions for (1.1) in the cone (1.3). Let us note that we do not require symmetry for our kernels.

Motivated by the works of Graef, Kong and Wang [14] and Webb [41], we apply our results to study the periodic boundary value problem (BVP)

$$x''(t) + \omega^2 x(t) = \lambda f(t, x(t)), \quad t \in [0, 1], \quad (1.4)$$

$$x(0) = x(1), \quad x'(0) = x'(1). \quad (1.5)$$

Our approach, in comparison with the ones used in the papers [14, 41], has the advantage of working for all admissible values of the parameter ω , that is, for all $\omega > 0$ such that $\omega \neq 2n\pi$, $n \in \mathbb{N}$ (in other words, we are able to prove the existence of solutions to the periodic BVP (1.4), (1.5) even in the case when the corresponding Green's function takes negative values – see Theorem 3.5). It also enables us to pinpoint the localization of the solution by means of the supremum and the integral norm. Let us add that if $\omega \in (0, \pi]$ our result ensures the existence of positive solutions to (1.4), (1.5) for some $\lambda > 0$.

Note that, although we impose strictly weaker conditions on the function f than the authors of [14, 41], our result – Theorem 3.5 – is in a sense “incomparable” with the existence results obtained in [14, 41] (for the discussion of the relation of Theorem 3.5 and the theorems from [14, 41] we refer the reader to Remark 3.6 in Section 3.1).

In the second part of the paper (see Section 4), by means of some version of Krasnosel’skiĭ’s theorem for the sum of two operators, we study the existence of solutions of the following perturbed nonlinear Hammerstein integral equation

$$x(t) = \alpha[x]v(t) + \beta[x]w(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \quad t \in [0, 1], \quad (1.6)$$

in the space $CBV[0, 1]$ consisting of continuous functions of bounded variation in the sense of Jordan; here $\lambda \in \mathbb{R}$, $\alpha, \beta \in CBV^*[0, 1]$ and $v, w \in CBV[0, 1]$. Let us add that the approach via Krasnosel’skiĭ’s fixed point theorem relies heavily on the ideas and techniques concerning the compactness of nonlinear integral operators in the space of functions of bounded variation developed in [7]. (It may seem surprising, but sufficient conditions guaranteeing the compactness of Hammerstein integral operators in the space $BV[0, 1]$ of functions of bounded variation, were not described until very recently – for more details see [7].)

The motivation to seek solutions of bounded variation is connected with their numerous applications (see e.g. [1, 35, 40]). Such functions can be used to describe some real world phenomena; for example, functions of bounded variation appear in mathematical biology or economics (see [4, 15]). Furthermore, it turns out that by a suitable choice of the space of functions of bounded variation it is possible, for example, to obtain solutions to certain nonlinear Hammerstein integral equations which are constant on each interval of continuity (for more details see [5]).

The necessity of studying the integral equation (1.6) in the space $CBV[0, 1]$ comes, *for example*, from the fact that it naturally arises, when dealing with BVPs

$$x''(t) = -\lambda f(t, x(t)), \quad t \in [0, 1], \quad (1.7)$$

with non-local boundary conditions (BCs) of the form

$$x(0) = \int_0^1 A(s)dx(s) \quad \text{and} \quad x(1) = \int_0^1 B(s)dx(s), \quad (1.8)$$

or

$$x(0) = \int_0^1 x(s)dA(s) \quad \text{and} \quad x(1) = \int_0^1 x(s)dB(s); \quad (1.9)$$

let us add that the integrals occurring in (1.8) and (1.9) are understood in the Riemann–Stieltjes sense.

In the context of ODEs the study of non-local BCs can be traced back to Picone [38], who investigated multi-point BCs. For an introduction to non-local problems we refer to the reviews [10,32,36,39,44] as well as the papers [9,20–22].

The existence of positive solutions (and their multiplicity) to the BVP (1.7), (1.9) was studied, for example, in [18,20,42,43]. In this paper (see Section 4.1), using a different approach from the one used in the above-mentioned articles, we establish existence type results for the BVPs with non-local integral conditions (1.7), (1.8) and (1.7), (1.9). Furthermore, we provide some examples of multi-point BVPs to which our results apply (see Examples 4.18 and 4.19).

2. Preliminaries

The aim of this Section is to fix the notation and to recall some basic definitions and facts which will be used in the sequel.

Notation. The closed ball in a normed space X with center at x and radius $r \in (0, +\infty)$ will be denoted by $B_X(x, r)$. For simplicity, instead of $B_{\mathbb{R}}(x, r)$ we will simply write $[x - r, x + r]$. The symbol θ will stand for the zero element of the normed space X .

If Ω is an open and bounded subset of \mathbb{R}^n , then by $C(\overline{\Omega})$ we will denote the Banach space of all continuous real-valued functions defined on $\overline{\Omega}$, endowed with the supremum norm $\|\cdot\|_{\infty}$. In the particular case when Ω is an open interval, that is $\Omega = (a, b)$, we will write $C[a, b]$ instead of $C(\overline{(a, b)})$ or $C([a, b])$. Moreover, by $BV[a, b]$ we will denote the Banach space of all real-valued functions defined on $[a, b]$ of bounded variation (in the sense of Jordan), endowed with the norm $\|f\|_{BV} := |f(a)| + \text{var}_{[a,b]} f$, and by $CBV[a, b]$ its closed subspace consisting of continuous functions; here the symbol $\text{var}_{[a,b]} f$ denotes the (Jordan) variation of the function $f: [a, b] \rightarrow \mathbb{R}$, that is,

$$\text{var}_{[a,b]} f = \sup_{\pi} \sum_{i=1}^n |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $\pi: a = t_0 < t_1 < \dots < t_n = b$ of the interval $[a, b]$. If no confusion concerning the interval over which we compute the variation can arise, instead of $\text{var}_{[a,b]} f$ we will simply write $\text{var} f$. Let us also recall that BV -functions are bounded and $\|f\|_{\infty} \leq \|f\|_{BV}$ for every $f \in BV[0, 1]$. As usual, by $CBV^*[a, b]$ we will denote the dual space of $CBV[a, b]$, that is, the space of all continuous linear functionals $\alpha: CBV[a, b] \rightarrow \mathbb{R}$. For a thorough treatment of functions of bounded variation of various kinds we refer the reader to [2].

Although throughout the paper we will use the same symbol “ f ” to denote both the Lebesgue and the Riemann–Stieltjes integral, it should always be clear from the context which integral we use. The Lebesgue measure in \mathbb{R}^n will be denoted by μ .

2.1. Partially ordered structures. Now, we are going to recall some definitions concerning partially ordered structures.

Definition 2.1 (cf. [29, p. 249]). Let X be a real normed space. A non-empty closed and convex set $C_X \subseteq X$ is called a (positive) cone if the following conditions are satisfied:

- (a) if $x \in C_X$ and $\lambda \geq 0$, then $\lambda x \in C_X$;
- (b) if $x \in C_X$ and $-x \in C_X$, then $x = \theta$.

Notation. If C_X is a cone in a normed space X , then by $C_X(\theta, r)$ we will denote the intersection of C_X and $B_X(\theta, r)$, that is, $C_X(\theta, r) := C_X \cap B_X(\theta, r)$.

Remark 2.2. A cone C_X in a normed space X induces a partial order \preceq given by the following formula

$$x \preceq y \quad \text{if and only if} \quad y - x \in C_X. \quad (2.1)$$

Let us note that the relation \preceq is compatible with the linear structure of the normed space X , that is, if $x \preceq y$, then $x + z \preceq y + z$ and $\lambda x \preceq \lambda y$ for all $x, y, z \in X$ and all $\lambda \geq 0$.

In the sequel, considering an ordered normed space with a cone, we will always assume that the partial order is defined by the formula (2.1).

Definition 2.3 (cf. [29, p. 249]). A real Banach space endowed with the partial order induced by a cone is called an *ordered Banach space*.

2.2. Fixed point theorems. Let us begin with recalling the following extension of the Leggett–Williams theorem.

Theorem 2.4 ([8, Theorem 4]). *Let $(X, \|\cdot\|)$ be an ordered Banach space with a cone C_X and suppose $F: C_X(\theta, r) \rightarrow C_X$, where $r > 0$, is a compact mapping. Moreover, assume that a continuous seminorm $|\cdot|: X \rightarrow [0, +\infty)$, together with positive numbers m, M and δ satisfy the conditions:*

- (i) $\|x\| \leq M |x|$ if $x \in C_X$;
- (ii) $m \leq rM^{-1}$ and $|x| = m$ for some $x \in C_X(\theta, r)$;
- (iii) $|F(x)| \geq \delta$ if $x \in C_X(\theta, r)$ and $|x| = m$.

Then there exist $\lambda_0 > 0$ and $x_0 \in C_X(\theta, r) \setminus \{\theta\}$ such that $F(x_0) = \lambda_0 x_0$ and $|x_0| = m$.

In the sequel, we will also use the following version of the well-known Krasnosel'skiĭ's fixed-point theorem for the sum of two operators. In comparison with the original result, the nonlinear contraction has been replaced by a bounded linear operator with spectral radius less than one (for a quite general

version of Krasnosel'skiĭ's fixed-point theorem encompassing almost 30 previously known generalizations of that theorem we refer the reader to [37]). For completeness, let us recall that the spectral radius of a bounded linear operator A is given by the formula $r(A) := \lim_{n \rightarrow \infty} \|A^n\|^{\frac{1}{n}}$ (see, for example, [28, p. 109]).

Theorem 2.5. *Let M be a non-empty closed and convex subset of a Banach space X , and let $F_1: X \rightarrow X$, $F_2: M \rightarrow X$ be two mappings such that:*

- (i) F_1 is a bounded linear operator such that $r(F_1) < 1$;
- (ii) F_2 is compact, that is, F_2 is continuous and $F_2(M)$ is contained in a compact subset of X ;
- (iii) if $x = F_1(x) + F_2(y)$ for some $y \in M$, then $x \in M$.

Then $F_1 + F_2$ has a fixed point in M .

The proof of Theorem 2.5 is standard, and hence it will be omitted.

Remark 2.6. The condition (iii) of Theorem 2.5, which guarantees that a certain nonlinear operator maps the set M into itself, is often the most difficult condition among the three one has to verify. However, if $M := B_X(\theta, r)$ and the assumption (ii) of Theorem 2.5 is strengthened to: " F_2 is continuous and $F_2(M)$ is contained in a compact subset of $B_X(\theta, \frac{r}{\|(I - F_1)^{-1}\|})$," where $\|(I - F_1)^{-1}\|$ denotes the operator norm of the continuous inverse of $I - F_1$ which exists due to (i), then the condition (iii) of Theorem 2.5 holds. Indeed, if $x = F_1(x) + F_2(y)$, then $(I - F_1)(x) = F_2(y)$, and thus $\|x\| = \|(I - F_1)^{-1} \circ F_2(y)\| \leq \|(I - F_1)^{-1}\| \cdot \|F_2(y)\| \leq r$. Hence, $x \in M$.

Remark 2.7. Let us also add that a Krasnosel'skiĭ–Schaefer type result similar to Theorem 2.5 can be found in [3], where the mapping F_2 , defined on the whole Banach space X , is required to be completely continuous (that is, F_2 is continuous and maps bounded sets into relatively compact ones) rather than compact (see [3, Theorem 4.2]).

3. Existence results via a Leggett-Williams type theorem

In this Section we are going to study the existence of continuous solutions to Hammerstein integral equations with kernels which may change sign. Let Ω be an open and bounded subset of \mathbb{R}^n and let us consider the following Hammerstein integral equation

$$\lambda x(t) = \int_{\Omega} k(t, s) f(s, x(s)) ds, \quad t \in \bar{\Omega}, \quad (3.1)$$

where $k: \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}$, $f: \bar{\Omega} \times \mathbb{R} \rightarrow [0, +\infty)$ and $\lambda \neq 0$.

Before we proceed to the main part of this Section, let us recall a Leggett–Williams type theorem for Hammerstein integral equations with non-negative kernels.

Theorem 3.1 (see [8, Theorem 12] and [29, Theorem 2]). *Let Ω be an open and bounded subset of \mathbb{R}^n , and let the functions $k: \bar{\Omega} \times \bar{\Omega} \rightarrow [0, +\infty)$ and $f: \bar{\Omega} \times [0, r] \rightarrow [0, +\infty)$ be continuous for some $r > 0$. Moreover, suppose that there exist positive numbers δ_1, δ_2, m , and a closed set $\Omega_0 \subseteq \bar{\Omega}$ of positive Lebesgue measure such that:*

- (i) $\int_{\Omega_0} k(t, s) dt \geq \delta_1$ for each $s \in \Omega_0$;
- (ii) $\int_{\Omega_0} k(t, s) dt \geq \delta_2 k(u, s)$ for each $(u, s) \in \bar{\Omega} \times \bar{\Omega}$;
- (iii) $f(t, x) > 0$ if $m\mu(\Omega_0)^{-\frac{1}{p}} \leq x \leq r$ and $t \in \Omega_0$, where $p \in [1, +\infty)$;
- (iv) $0 < m \leq r\delta_2\mu(\Omega_0)^{-\frac{1}{q}}$, where $q \in (1, +\infty]$ is such that $p^{-1} + q^{-1} = 1$.

Then there exists a positive parameter $\lambda > 0$ such that the Hammerstein integral equation (3.1) admits a continuous and positive solution $x: \bar{\Omega} \rightarrow [0, r]$ such that

$$\left(\int_{\Omega_0} [x(t)]^p dt \right)^{\frac{1}{p}} = m. \quad (3.2)$$

Remark 3.2. Assume that $k: \bar{\Omega} \times \bar{\Omega} \rightarrow [0, +\infty)$ is continuous. Let us observe that if we define the mapping $\Phi: \bar{\Omega} \rightarrow [0, +\infty)$ by the formula

$$\Phi(s) = \delta_2^{-1} \int_{\Omega_0} k(t, s) dt, \quad s \in \bar{\Omega}, \quad (3.3)$$

then the assumptions (i) and (ii) of Theorem 3.1 may be restated as follows:

- (i) $\Phi(s) \geq \delta_1 \delta_2^{-1}$ for each $s \in \Omega_0$;
- (ii) $k(u, s) \leq \Phi(s)$ for each $(u, s) \in \bar{\Omega} \times \bar{\Omega}$.

On the other hand, if there exists a mapping $\Phi: \bar{\Omega} \rightarrow [0, +\infty)$ which satisfies the above properties (i), (ii) (with the equality sign in (3.3) replaced by the “less than or equal to” sign), then

$$\delta_2 k(u, s) \leq \delta_2 \Phi(s) \leq \int_{\Omega_0} k(t, s) dt \quad \text{for } (u, s) \in \bar{\Omega} \times \bar{\Omega}$$

and

$$\int_{\Omega_0} k(t, s) dt \geq \delta_2 \Phi(s) \geq \delta_1 \quad \text{for } s \in \Omega_0.$$

This shows that in this case the kernel k satisfies the assumptions (i) and (ii) of Theorem 3.1. We will use the above observation in the main result of this Section.

Now, we will prove an extension of Theorem 3.1 for kernels that may change sign. Clearly, since the kernels are allowed to take negative values, one cannot expect, in general, to prove the existence of positive solutions. However, we

will still be able to pinpoint the localization of the solutions by means of the supremum norm and the integral semi-norm (cf. formula (3.2)).

Let us assume that for some $r > 0$ there exists a set $\Omega_0 \subseteq \bar{\Omega}$ with positive Lebesgue measure, together with constants $\vartheta \in (0, 1)$, $\eta_1, \eta_2, m, c \in (0, +\infty)$ and $p \in [1, +\infty)$ such that the functions $f: \bar{\Omega} \times [-r, r] \rightarrow [0, +\infty)$ and $k: \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}$ satisfy the following conditions:

- (A₁) f satisfies the Carathéodory conditions, that is,
 - (i) for every $u \in [-r, r]$ the function $t \mapsto f(t, u)$ is Lebesgue measurable;
 - (ii) for a.e. $t \in \bar{\Omega}$ the function $u \mapsto f(t, u)$ is continuous;
 - (iii) there exists a Lebesgue measurable function $g_r: \bar{\Omega} \rightarrow [0, +\infty)$ such that for a.e. $t \in \bar{\Omega}$ we have $f(t, u) \leq g_r(t)$ for all $u \in [-r, r]$;
- (A₂) for a.e. $t \in \Omega_0$ we have $\inf\{f(t, u) : \vartheta m \mu(\Omega_0)^{-\frac{1}{p}} \leq |u| \leq r\} \geq \eta_1$;
- (A₃) k is Lebesgue measurable on $\bar{\Omega} \times \bar{\Omega}$, and for every $\tau \in \bar{\Omega}$ the function $s \mapsto k(\tau, s)$ is Lebesgue measurable¹ on $\bar{\Omega}$ and

$$\lim_{t \rightarrow \tau} \int_{\bar{\Omega}} |k(t, s) - k(\tau, s)| g_r(s) ds = 0;$$

- (A₄) there exists a Lebesgue measurable function $\Phi: \bar{\Omega} \rightarrow [0, +\infty)$ such that
 - (i) $\Phi(t) \geq \eta_2$ for a.e. $t \in \Omega_0$;
 - (ii) for every $t \in \bar{\Omega}$ we have $|k(t, s)| \leq \Phi(s)$ for a.e. $s \in \bar{\Omega}$;
 - (iii) $c\Phi(s) \leq \int_{\Omega_0} k(t, s) dt$ for a.e. $s \in \bar{\Omega}$;
- (A₅) $\int_{\bar{\Omega}} \Phi(s) g_r(s) ds < +\infty$;
- (A₆) $0 < m \leq rc \mu(\Omega_0)^{-\frac{1}{q}}$, where $q \in (1, +\infty]$ is such that $p^{-1} + q^{-1} = 1$.

Theorem 3.3. *Suppose that the above assumptions hold. Then there exists a parameter $\lambda > 0$ such that the Hammerstein integral equation (3.1) admits a continuous solution $x: \bar{\Omega} \rightarrow [-r, r]$ such that*

$$\left(\int_{\Omega_0} |x(t)|^p dt \right)^{\frac{1}{p}} = m.$$

Proof. Let us set

$$C := \left\{ x \in C(\bar{\Omega}) : \int_{\Omega_0} x(t) dt \geq c \|x\|_\infty \right\}$$

and

$$|x| = \left(\int_{\Omega_0} |x(t)|^p dt \right)^{\frac{1}{p}} \quad \text{for } x \in C(\bar{\Omega}).$$

¹The fact that $s \mapsto k(\tau, s)$ is Lebesgue measurable for every $\tau \in \bar{\Omega}$, in general, does not follow from the Lebesgue measurability of k on $\bar{\Omega} \times \bar{\Omega}$; the latter condition implies only that *almost all* vertical sections are Lebesgue measurable (cf. [30, Section 6.3]).

Clearly, C is a cone in $C(\bar{\Omega})$, and moreover, it is easy to check that the assumption (i) of Theorem 2.4 holds with $M = c^{-1}\mu(\Omega_0)^{\frac{1}{q}}$ (cf. [16, Theorem 13.17]).

Let F be the mapping defined on $C(\theta, r)$ by

$$F(x)(t) = \int_{\bar{\Omega}} k(t, s)f(s, x(s))ds, \quad t \in \bar{\Omega}. \quad (3.4)$$

It can be shown that F is a compact mapping from $C(\theta, r)$ into $C(\bar{\Omega})$ (cf. [34, Proposition 3.1, p. 164]). Furthermore, if $x \in C(\theta, r)$, then given any $u \in \bar{\Omega}$ we have

$$\begin{aligned} c|F(x)(u)| &\leq \int_{\bar{\Omega}} c|k(u, s)|f(s, x(s))ds \\ &\leq \int_{\bar{\Omega}} c\Phi(s)f(s, x(s))ds \\ &\leq \int_{\bar{\Omega}} \left(\int_{\Omega_0} k(t, s)dt \right) f(s, x(s))ds \\ &= \int_{\Omega_0} \left(\int_{\bar{\Omega}} k(t, s)f(s, x(s))ds \right) dt \\ &= \int_{\Omega_0} F(x)(t)dt, \end{aligned}$$

which shows that $F(C(\theta, r)) \subseteq C$.

Observe that from the assumptions imposed on k it follows that $c \leq \mu(\Omega_0)$, and hence if we define $x(t) = m\mu(\Omega_0)^{-\frac{1}{p}}$ for $t \in \bar{\Omega}$, we see that

$$|x(t)| = m\mu(\Omega_0)^{-\frac{1}{p}} \leq cr\mu(\Omega_0)^{-1} \leq r \quad \text{and} \quad c\|x\|_{\infty} = cm\mu(\Omega_0)^{-\frac{1}{p}} \leq \int_{\Omega_0} x(t)dt.$$

Thus, we have $x \in C(\theta, r)$ and $|x| = m$.

Finally, we shall show that the assumption (iii) of Theorem 2.4 holds. Assume that $x \in C(\theta, r)$ is such that $|x| = m$. Let us set

$$\Omega_1 = \{t \in \Omega_0 : \vartheta m\mu(\Omega_0)^{-\frac{1}{p}} \leq |x(t)| \leq r\}.$$

Then we have $m^p = \int_{\Omega_0} |x(t)|^p dt = \int_{\Omega_1} |x(t)|^p dt + \int_{\Omega_0 \setminus \Omega_1} |x(t)|^p dt \leq r^p \mu(\Omega_1) + \frac{\vartheta^p m^p}{\mu(\Omega_0)} \cdot \mu(\Omega_0 \setminus \Omega_1) \leq r^p \mu(\Omega_1) + \vartheta^p m^p$. Thus, we obtain $\mu(\Omega_1) \geq m^p(1 - \vartheta^p)r^{-p} > 0$.

Hence, we have

$$\begin{aligned}
 |F(x)| &= \left(\int_{\Omega_0} |F(x)(t)|^p dt \right)^{\frac{1}{p}} \\
 &\geq \mu(\Omega_0)^{-\frac{1}{q}} \int_{\Omega_0} |F(x)(t)| dt \\
 &\geq \mu(\Omega_0)^{-\frac{1}{q}} \int_{\Omega_0} F(x)(t) dt \\
 &= \mu(\Omega_0)^{-\frac{1}{q}} \int_{\bar{\Omega}} \left(\int_{\Omega_0} k(t, s) dt \right) f(s, x(s)) ds \\
 &\geq c\mu(\Omega_0)^{-\frac{1}{q}} \int_{\Omega_0} \Phi(s) f(s, x(s)) ds \\
 &\geq c\eta_1\mu(\Omega_0)^{-\frac{1}{q}} \int_{\Omega_1} \Phi(s) ds \\
 &\geq c\eta_1\eta_2\mu(\Omega_0)^{-\frac{1}{q}} m^p(1 - \vartheta^p)r^{-p} =: \delta > 0.
 \end{aligned}$$

To end the proof it suffices to apply Theorem 2.4. □

Remark 3.4. Let us note that in the simplest, yet very common, case when the functions $f: \bar{\Omega} \times [-r, r] \rightarrow [0, +\infty)$ and $k: \bar{\Omega} \times \bar{\Omega} \rightarrow \mathbb{R}$ are continuous, the assumptions (A₁)–(A₆) are implied by the following set of conditions:

“There exists a closed set $\Omega_0 \subseteq \bar{\Omega}$ with positive Lebesgue measure, together with constants $m \in (0, +\infty)$ and $p \in [1, +\infty)$ such that:

- (B₁) $f(t, u) > 0$ for $t \in \Omega_0$ and $m\mu(\Omega_0)^{-\frac{1}{p}} \leq |u| \leq r$;
- (B₂) $\min_{s \in \bar{\Omega}} \int_{\Omega_0} k(t, s) dt > 0$;
- (B₃) $0 < m \leq r\mu(\Omega_0)^{-\frac{1}{q}} \|k\|_{\infty}^{-1} \min_{s \in \bar{\Omega}} \int_{\Omega_0} k(t, s) dt.$ ”

Indeed, (A₁), (A₃) and (A₆) trivially hold; (A₁)(iii) and (A₃) with $g_r(t) = \|f\|_{\infty}$ for $t \in \bar{\Omega}$, whereas (A₆) with $c := \|k\|_{\infty}^{-1} \min_{s \in \bar{\Omega}} \int_{\Omega_0} k(t, s) dt$. To see that the assumptions (A₄), (A₅) also hold it suffices to set

$$\Phi(s) = \frac{\|k\|_{\infty} \int_{\Omega_0} k(t, s) dt}{\min_{s \in \bar{\Omega}} \int_{\Omega_0} k(t, s) dt} \quad \text{for } s \in \bar{\Omega}.$$

3.1. Periodic BVPs. In this short Section, we would like to show how to apply Leggett–Williams type theorems to proving the existence of solutions to BVPs. Therefore, let us consider the following periodic BVP

$$x''(t) + \omega^2 x(t) = \lambda f(t, x(t)), \quad t \in [0, 1], \tag{3.5}$$

$$x(0) = x(1), \quad x'(0) = x'(1), \tag{3.6}$$

where $\lambda \neq 0$ and ω is a positive constant such that $\omega \neq 2n\pi$ for $n \geq 1$. For simplicity, we assume that $f: [0, 1] \times \mathbb{R} \rightarrow [0, +\infty)$ is continuous. It can be

checked that each continuous solution of the following nonlinear Hammerstein integral equation

$$x(t) = \lambda \int_0^1 k(t, s) f(s, x(s)) ds, \quad t \in [0, 1], \quad (3.7)$$

where

$$k(t, s) = \begin{cases} \frac{\cos[\omega(\frac{1}{2} - t + s)]}{2\omega \sin(\frac{1}{2}\omega)}, & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{\cos[\omega(\frac{1}{2} - s + t)]}{2\omega \sin(\frac{1}{2}\omega)}, & \text{if } 0 \leq t < s \leq 1, \end{cases}$$

is a solution² to the BVP (3.5), (3.6).

Furthermore, let us note that if, for example, $\omega = \frac{3}{2}\pi$, then $k(t, t) = -\frac{1}{3\pi} < 0$ for $t \in [0, 1]$. This means that to study the existence of solutions of the BVP (3.5), (3.6) with $\omega = \frac{3}{2}\pi$ we cannot apply results for mappings in cones which require k to be non-negative on a rectangle of the form $[a, b] \times [0, 1]$ such as, for example, Theorem 3.1 or the results in [41].

Theorem 3.5. *Let ω be a positive constant such that $\omega \neq 2n\pi$ for $n \geq 1$. Moreover, let $r \geq 1$ and let the continuous function $f: [0, 1] \times [-r, r] \rightarrow [0, +\infty)$ be such that $f(t, u) > 0$ if $0 \leq t \leq 1$ and $2\omega^{-1}|\sin(\frac{1}{2}\omega)| \leq |u| \leq r$. Then for every $p \in [1, +\infty)$ there exists a positive parameter $\lambda := \lambda(\omega, p)$ that the BVP (3.5), (3.6) has a solution $x: [0, 1] \rightarrow [-r, r]$ with $(\int_0^1 |x(t)|^p dt)^{\frac{1}{p}} = 2\omega^{-1}|\sin(\frac{1}{2}\omega)|$.*

Proof. Since $\int_0^1 k(t, s) dt = \omega^{-2}$ for every $s \in [0, 1]$, the proof of Theorem 3.5 is a direct consequence of Theorem 3.3 if one sets: $\bar{\Omega} = \Omega_0 = [0, 1]$ and $m = 2\omega^{-1}|\sin(\frac{1}{2}\omega)|$ (cf. Remark 3.4). \square

Remark 3.6. The existence of solutions to the BVP (3.5), (3.6) was also studied in, for example, [14, 41] (more general periodic BVPs were investigated in [33]), where, under some growth conditions on the function f , it was shown that the BVP (3.5), (3.6) admits a positive solution, provided $\lambda = 1$ and $\omega \in (0, \pi]$.

Let us note that for such values of ω also our result ensures the existence of positive solutions to the BVP (3.5), (3.6) (this follows from the fact that $k(t, s) \geq 0$ for $(t, s) \in [0, 1] \times [0, 1]$, if $\omega \in (0, \pi]$).

However, our result and the results obtained in [14, 41] are in a sense “incomparable” (even in the case $\omega \in (0, \pi]$), since on the one hand our assumptions imposed on the function f are less restrictive than those introduced in the above-mentioned articles (for example, we do not require f to satisfy certain

²Let us recall that by a solution to the BVP (3.5), (3.6) we understand a real-valued twice continuously differentiable function defined on $[0, 1]$ which satisfies both the equation (3.5) and the periodic BCs (3.6).

growth conditions at zero and at infinity), but on the other hand we are able to prove the existence of solutions to the BVP (3.5), (3.6) for some positive parameter λ , whereas in [14, 41] $\lambda = 1$.

What is worth mentioning is that the fact that Theorem 3.5 ensures the existence of non-zero solutions to the BVP (3.5), (3.6) only for some $\lambda > 0$ is something that cannot be avoided and is not a consequence of the approach, but is, one might say, “forced” by the additional constraints imposed on the sought solution and the problem itself. To better understand this phenomenon let us consider the BVP (3.5), (3.6) with $f(t, u) = u^2$, $\omega = \pi$ and $\lambda = 1$. It is easy to see that the integral operator F corresponding to the considered BVP in the integral form (3.7) (cf. formula (3.4)) is a contraction with respect to the supremum norm (with Lipschitz constant $\frac{r}{\pi}$) on the closed ball $B_{C[0,1]}(\theta, r)$, where $r \in (0, \pi)$, and therefore it has at most one fixed point in this ball. However, we know that $F(\theta) = \theta$, which means that there are no solutions to the BVP in question with the supremum norm not exceeding r other than the zero solution.

4. Existence results via Krasnosel’skiĭ’s theorem

The aim of this Section is to prove the existence of CBV -solutions to the following perturbed nonlinear Hammerstein integral equation

$$x(t) = \alpha[x]v(t) + \beta[x]w(t) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \quad t \in [0, 1], \quad (4.1)$$

where $\lambda \in \mathbb{R}$, using Krasnosel’skiĭ’s fixed-point theorem. As in Section 3, we are also going to provide applications of our result to some BVPs.

Before we proceed further, let us make the following assumptions on the functionals $\alpha, \beta \in CBV^*[0, 1]$ and the functions $v, w \in CBV[0, 1]$:

(A₇) $\alpha[e] = \beta[e] = 0$; here e denotes the constant function given by $e(t) = 1$ for $t \in [0, 1]$;

(A₈) $|\alpha[v] - \beta[v]| < 1$;

(A₉) $v(t) + w(t) = 1$ for every $t \in [0, 1]$.

Furthermore, let us assume that the nonlinearity $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ and the kernel $k: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ satisfy the following conditions:

(A₁₀) f satisfies the Carathéodory conditions, that is,

(i) for every $u \in \mathbb{R}$ the function $t \mapsto f(t, u)$ is Lebesgue measurable;

(ii) for a.e. $t \in [0, 1]$ the function $u \mapsto f(t, u)$ is continuous;

(iii) there exist a non-decreasing function $\psi: [0, +\infty) \rightarrow [0, +\infty)$ and an L^p -function $\phi: [0, 1] \rightarrow [0, +\infty)$ with $p \in (1, +\infty]$ such that $|f(t, u)| \leq \phi(t)\psi(|u|)$ for $t \in [0, 1]$ and $u \in \mathbb{R}$;

(A₁₁) $\lim_{r \rightarrow +\infty} \frac{\psi(r)}{r} = 0$;

- (A₁₂) for every $t \in [0, 1]$ the function $s \mapsto k(t, s)$ is an L^q -function; here $q \in [1, +\infty)$ is such that $p^{-1} + q^{-1} = 1$;
- (A₁₃) there exists an L^q -function $m: [0, 1] \rightarrow [0, +\infty)$ such that $\text{var}_{[0,1]} k(\cdot, s) \leq m(s)$ for a.e. $s \in [0, 1]$;
- (A₁₄) for every $\tau \in [0, 1]$ we have $\lim_{t \rightarrow \tau} \int_0^1 |k(t, s) - k(\tau, s)| \phi(s) ds = 0$.

Remark 4.1. If the kernel k satisfies (A₁₃), then in (A₁₂) we do not need to assume that the function $s \mapsto k(t, s)$ is an L^q -function for every $t \in [0, 1]$; it is enough to assume that all the vertical sections are Lebesgue measurable and only one of them is integrable with q th power (cf. [7, Remark 7] or [6, Remark 7.3]).

Remark 4.2. Let us note that the assumptions (A₁₂)–(A₁₄) hold if, for example, the kernel $k: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is continuous and satisfies the following condition:

- (B₄) there is an L^q -function $m: [0, 1] \rightarrow [0, +\infty)$ such that $|k(t, s) - k(\tau, s)| \leq m(s)|t - \tau|$ for all $s, t, \tau \in [0, 1]$.

We set

$$F_1(x)(t) = \alpha[x]v(t) + \beta[x]w(t), \quad t \in [0, 1] \quad (4.2)$$

and

$$F_2(x)(t) = \int_0^1 k(t, s)f(s, x(s))ds, \quad t \in [0, 1], \quad (4.3)$$

so that (4.1) with $\lambda = 1$ takes the following operator form

$$x = F_1(x) + F_2(x).$$

In order to show that the set of fixed points of $F_1 + F_2$ is non-empty (which would obviously imply that the Hammerstein integral equation (4.1) with $\lambda = 1$ has a solution) we are going to apply Theorem 2.5, and therefore we begin with proving that the operators F_1 and F_2 have the required properties.

Lemma 4.3. *Suppose that the assumptions (A₇) and (A₉) hold. Then the map $F_1: CBV[0, 1] \rightarrow CBV[0, 1]$ given by (4.2) is a bounded linear operator with*

$$\|F_1\| \leq \|\alpha\| + \|\alpha - \beta\| \cdot \|w\|_{BV}$$

and

$$\|F_1^{n+2}\| \leq \|\alpha - \beta\| \cdot |\alpha[v] - \beta[v]|^n \cdot \|\alpha[v]v + \beta[v]w\|_{BV} \quad \text{for } n \geq 0.^3 \quad (4.4)$$

In particular, $r(F_1) \leq |\alpha[v] - \beta[v]|$.

Proof. The proof of Lemma 4.3 is straightforward (although tedious) and hence it will be omitted. \square

³If $n = 0$, then, by definition, we take $|\alpha[v] - \beta[v]|^n = 1$, even if $|\alpha[v] - \beta[v]| = 0$.

Remark 4.4. Let us note that in order to prove the estimate for $\|F_1\|$ one does not have to use the assumption (A_7) . This assumption is used in showing (4.4).

Lemma 4.5. *If the assumptions (A_{10}) and (A_{12}) – (A_{14}) hold, then the mapping $F_2: BV[0, 1] \rightarrow CBV[0, 1]$ given by (4.3) is completely continuous.*

Proof. First, we shall show that for every $x \in BV[0, 1]$ the function $F_2(x)$ is well-defined. So let us fix $x \in BV[0, 1]$. Then for every $s \in [0, 1]$ we have $|f(s, x(s))| \leq \phi(s)\psi(\|x\|_{BV})$. Therefore, the function $s \mapsto k(t, s)f(s, x(s))$ is Lebesgue integrable for every $t \in [0, 1]$; Lebesgue measurability of the above function follows from the Carathéodory conditions imposed on f and the fact that x is Lebesgue measurable (cf. [2, Theorem 1.5]).

Now, we are going to prove that F_2 maps the space $BV[0, 1]$ into $CBV[0, 1]$. Fix $x \in BV[0, 1]$. If $0 = t_0 < \dots < t_n = 1$ is an arbitrary finite partition of the interval $[0, 1]$, then, in view of the assumption (A_{13}) , we infer that

$$\sum_{i=1}^n |k(t_i, s) - k(t_{i-1}, s)| \leq m(s) \quad \text{for a.e. } s \in [0, 1], \tag{4.5}$$

and so

$$\begin{aligned} \sum_{i=1}^n |F_2(x)(t_i) - F_2(x)(t_{i-1})| &\leq \int_0^1 \sum_{i=1}^n |k(t_i, s) - k(t_{i-1}, s)| \phi(s)\psi(\|x\|_{BV}) ds \\ &\leq \psi(\|x\|_{BV}) \int_0^1 m(s)\phi(s) ds. \end{aligned}$$

Thus, $\text{var}_{[0,1]} F_2(x) \leq \psi(\|x\|_{BV}) \int_0^1 m(s)\phi(s) ds$. This shows that $F_2(x)$ belongs to $BV[0, 1]$. Continuity of the function $F_2(x)$ is a simple consequence of the assumption (A_{14}) . Hence, $F_2(x) \in CBV[0, 1]$, which proves our claim.

Finally, we will show that F_2 is completely continuous. Suppose that $(x_n)_{n \in \mathbb{N}}$ is an arbitrary sequence of BV -functions which is convergent to some $x \in BV[0, 1]$ with respect to the BV -norm⁴. In particular, $(x_n)_{n \in \mathbb{N}}$ is bounded, which means that $\sup_{n \in \mathbb{N}} \|x_n\|_{BV} \leq R_1$ and $\|x\|_{BV} \leq R_1$ for some $R_1 > 0$. Using the fact that the estimate (4.5) is true for an arbitrary finite partition of the interval $[0, 1]$, we see that

$$\|F_2(x_n) - F_2(x)\|_{BV} \leq \int_0^1 (m(s) + |k(0, s)|) |f(s, x_n(s)) - f(s, x(s))| ds$$

for $n \in \mathbb{N}$. Since all integrands on the right-hand side of the above formula can be majorized by the Lebesgue integrable function $s \mapsto 2(m(s) + |k(0, s)|)\phi(s)\psi(R_1)$,

⁴Let us recall that if a sequence of BV -functions is convergent to a function $x \in BV[0, 1]$ with respect to the BV -norm, then it is uniformly convergent to x on the whole interval $[0, 1]$.

which, clearly, does not depend on n , by the Lebesgue dominated convergence theorem, we infer that $\|F_2(x_n) - F_2(x)\|_{BV} \rightarrow 0$ as $n \rightarrow +\infty$. This shows that F_2 is continuous.

To prove that F_2 maps bounded subsets of $BV[0, 1]$ into relatively compact subsets of $CBV[0, 1]$, we will use similar techniques to those used in [7]. Fix $R_2 > 0$ and let $(x_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of BV -functions such that $\|x_n\|_{BV} \leq R_2$ for $n \in \mathbb{N}$. By Helly's selection theorem there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ and a function $x \in BV[0, 1]$ such that $x_{n_k} \rightarrow x$ pointwise on $[0, 1]$ and $\|x\|_{BV} \leq R_2$ (see [2, Theorem 1.11] or [30, Theorem 1.4.5]). From the first part of the proof it follows that $y := F_2(x) \in CBV[0, 1]$. Reasoning similar to that used in the proof of the continuity of F_2 shows that $F_2(x_{n_k}) \rightarrow y$ in $CBV[0, 1]$ as $k \rightarrow +\infty$, which means that the sequence $(F_2(x_n))_{n \in \mathbb{N}}$ has a convergent subsequence. Therefore, F_2 is completely continuous. \square

Now, we are ready to prove the first main result of this Section.

Theorem 4.6. *Under the assumptions (A₇)–(A₁₄), the perturbed nonlinear Hammerstein integral equation (4.1) with $\lambda = 1$ has a CBV -solution.*

Proof. First, let us observe that by the assumption (A₁₁) it is possible to find a positive number r such that

$$\psi(r) \leq rc^{-1} \left(1 + \int_0^1 (m(s) + |k(0, s)|) \phi(s) ds \right)^{-1},$$

where

$$c := 1 + \|\alpha\| + \|\alpha - \beta\| \cdot \|w\|_{BV} + \frac{\|\alpha - \beta\| \cdot \|\alpha[v]v + \beta[v]w\|_{BV}}{1 - |\alpha[v] - \beta[v]|}. \tag{4.6}$$

Set $M := B_{CBV}(\theta, r)$ and consider the mappings $F_1: CBV[0, 1] \rightarrow CBV[0, 1]$ and $F_2: M \rightarrow CBV[0, 1]$ given by the formulas (4.2) and (4.3), respectively. From Lemmas 4.3 and 4.5 it follows that F_1 and F_2 satisfy the assumptions (ii) and (i) of Theorem 2.5. Moreover, for every $x \in M$, we have

$$\|F_2(x)\|_{BV} \leq \psi(r) \cdot \int_0^1 (m(s) + |k(0, s)|) \phi(s) ds$$

(cf. the proof of Lemma 4.5). This, together with the fact that

$$\|(I - F_1)^{-1}\| = \left\| \sum_{n=0}^{\infty} F_1^n \right\| \leq \sum_{n=0}^{\infty} \|F_1^n\| = c$$

(cf. Lemma 4.3), yields

$$\|F_2(x)\|_{BV} \leq \frac{r}{\|(I - F_1)^{-1}\|} \quad \text{for } x \in M.$$

To end the proof it suffices to apply Remark 2.6 and Theorem 2.5. \square

Remark 4.7. From the proof of Theorem 4.6 it follows that the condition (A_{11}) can be replaced by a weaker (but at the same time more technical) condition:

(B_5) there exists $r > 0$ such that $\psi(r) \leq rc^{-1} \left(1 + \int_0^1 (m(s) + |k(0, s)|) \phi(s) ds\right)^{-1}$, where c is given by (4.6).

Corollary 4.8. *Suppose that the assumptions (A_7) – (A_{10}) and (A_{12}) – (A_{14}) hold. Then there exists $\lambda_0 > 0$ such that the perturbed nonlinear Hammerstein integral equation (4.1) has a CBV-solution for every $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \lambda_0$.*

Proof. Let

$$\lambda_0 := \frac{1}{c(\psi(1) + 1)} \left(1 + \int_0^1 (m(s) + |k(0, s)|) \phi(s) ds\right)^{-1},$$

where the number c is given by (4.6). The claim follows from Remark 4.7 and Theorem 4.6 if we replace f and ψ with λf and $|\lambda|\psi$. □

It turns out that if we strengthen the condition (A_8) , then (A_7) is not required. Indeed, by a similar argument to the proof of Theorem 4.6, in view of the classical Krasnosel'skiĭ's theorem (see, for example, [23, 37]) and Lemma 4.3 (cf. also Remark 4.4), one can establish the following result.

Theorem 4.9. *Suppose that the assumptions (A_9) – (A_{14}) hold. If, additionally, the functionals $\alpha, \beta \in CBV^*[0, 1]$ satisfy the following condition:*

(B_6) $\|\alpha\| + \|\alpha - \beta\| \cdot \|w\|_{BV} < 1$,

then the perturbed nonlinear Hammerstein integral equation (4.1) with $\lambda = 1$ has a CBV-solution.

Corollary 4.10. *Suppose that the conditions (A_9) , (A_{10}) , (A_{12}) – (A_{14}) and (B_6) hold. Then there exists $\lambda_0 > 0$ such that the perturbed nonlinear Hammerstein integral equation (4.1) has a CBV-solution for every $\lambda \in \mathbb{R}$ such that $|\lambda| \leq \lambda_0$.*

4.1. BVPs with non-local BCs. In this Section, as an application of the abstract results for the perturbed nonlinear Hammerstein integral equation, we are going to study the existence of solutions to the following second-order differential equation

$$x''(t) = -\lambda f(t, x(t)), \quad t \in [0, 1], \tag{4.7}$$

with the non-local BCs of the form

$$x(0) = \int_0^1 A(s) dx(s) \quad \text{and} \quad x(1) = \int_0^1 B(s) dx(s). \tag{4.8}$$

For simplicity, as in Section 3.1, we assume that the function $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Before we proceed to the main theorems of this Section, we will discuss a class of functions for which the BCs (4.8) are well-posed and we will prove a result concerning continuous linear functionals on $CBV[0, 1]$.

Let us introduce the following notation. Given a number $\varepsilon > 0$, we say that a bounded function $A: [a, b] \rightarrow \mathbb{R}$, where $-\infty < a < b < +\infty$, belongs to the family $\Omega_\varepsilon[a, b]$, if there exists $\delta > 0$ such that $\text{osc}_{[t,s]} A \leq \varepsilon$, whenever $t, s \in [a, b]$ are such that $0 \leq s - t \leq \delta$; here the symbol $\text{osc}_{[t,s]} A$ denotes the *oscillation* of the function A over the interval $[t, s]$, that is, $\text{osc}_{[t,s]} A := \sup_{t \leq \tau \leq \sigma \leq s} |A(\sigma) - A(\tau)|$. Let us denote by $\Omega[0, 1]$ the set of all bounded functions $A: [0, 1] \rightarrow \mathbb{R}$ such that for every $\varepsilon > 0$ there is $a \in (0, 1)$ such that $A|_{[0,a]} \in \Omega_\varepsilon[0, a]$ and $A|_{[a,1]} \in BV[a, 1]$. For simplicity, let us also set $\widehat{\Omega}[0, 1] := \Omega[0, 1] \cup C[0, 1] \cup BV[0, 1]$.

Example 4.11. An example of a bounded function $A: [0, 1] \rightarrow \mathbb{R}$ which belongs to $\widehat{\Omega}[0, 1]$ but is neither continuous nor of bounded variation on the interval $[0, 1]$ is given by the following formula

$$A(t) = \begin{cases} \frac{1}{n}, & \text{if } t \in (\frac{1}{n+1}, \frac{1}{n}), n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.12. *If $x \in CBV[0, 1]$ and $A \in \widehat{\Omega}[0, 1]$, then the Riemann–Stieltjes integral $\int_0^1 A(s)dx(s)$ exists.*

Proof. In view of [30, Theorems 1.5.5 and 1.5.6] it is clear that we may assume that $A \in \Omega[0, 1]$. We may also assume that $\|A\|_\infty > 0$ and $\text{var}_{[0,1]} x > 0$. Then, given $\varepsilon > 0$ there exists $a \in (0, 1)$ such that $A|_{[0,a]} \in \Omega_\eta[0, a]$ and $A|_{[a,1]} \in BV[a, 1]$, where $\eta := \varepsilon(3 \text{var}_{[0,1]} x)^{-1}$. Let $\delta = \min\{\delta_1, \delta_2\}$, where $\delta_1 \in (0, 1)$ is chosen according to the definition of the family $\Omega_\eta[0, a]$ and $\delta_2 \in (0, 1)$ is such a number that

$$\text{var}_{[t,s]} x \leq \min \left\{ \frac{\varepsilon}{6 \|A\|_\infty}, \frac{\varepsilon}{3(1 + \text{var}_{[a,1]} A)} \right\},$$

whenever $t, s \in [0, 1]$ are such that $0 \leq s - t \leq \delta_2$; the number δ_2 exists by the continuity of the function $t \mapsto \text{var}_{[0,t]} x$ (cf. [30, Theorem 1.3.4]).

If $0 = t_0 < t_1 < \dots < t_n = 1$ is an arbitrary finite partition of the interval $[0, 1]$ such that $\max_{1 \leq i \leq n} |t_i - t_{i-1}| \leq \delta$, then⁵

$$\begin{aligned} & \sum_{i=1}^n \text{osc}_{[t_{i-1}, t_i]} A \cdot \text{var}_{[t_{i-1}, t_i]} x \\ &= \sum_{i=1}^j \text{osc}_{[t_{i-1}, t_i]} A \cdot \text{var}_{[t_{i-1}, t_i]} x + \text{osc}_{[t_j, t_{j+1}]} A \cdot \text{var}_{[t_j, t_{j+1}]} x + \sum_{i=j+2}^n \text{osc}_{[t_{i-1}, t_i]} A \cdot \text{var}_{[t_{i-1}, t_i]} x, \end{aligned}$$

⁵If the upper summation limit is smaller than the lower one, then, by definition, the sum is equal to zero.

where the index $j \in \{0, \dots, n - 1\}$ is chosen in such a way that $t_j \leq a < t_{j+1}$. Thus

$$\sum_{i=1}^n \operatorname{osc}_{[t_{i-1}, t_i]} A \cdot \operatorname{var}_{[t_{i-1}, t_i]} x \leq \frac{\varepsilon \operatorname{var}_{[0, a]} x}{3 \operatorname{var}_{[0, 1]} x} + \frac{2\varepsilon \|A\|_\infty}{6 \|A\|_\infty} + \frac{\varepsilon \operatorname{var}_{[a, 1]} A}{3(1 + \operatorname{var}_{[a, 1]} A)} \leq \varepsilon,$$

since $\operatorname{osc}_{[t_{i-1}, t_i]} A \leq \operatorname{var}_{[t_{i-1}, t_i]} A$ (see [30, Formula (1.3.4)]). This shows that the condition (B) of [30, Theorem 1.5.2] is satisfied, and therefore the Riemann–Stieltjes integral $\int_0^1 A(s)dx(s)$ exists. \square

From Lemma 4.12 and the properties of the Riemann–Stieltjes integral we have the following straightforward consequence.

Corollary 4.13. *If $A \in \widehat{\Omega}[0, 1]$, then the formula $x \mapsto \int_0^1 A(s)dx(s)$ defines a continuous linear functional on $CBV[0, 1]$.*

Remark 4.14. Clearly, Lemma 4.12 and Corollary 4.13 would also be true, if in the definition of the class $\Omega[0, 1]$ we assumed that $A|_{[a, 1]} \in \Omega_\varepsilon[a, 1]$ and $A|_{[0, a]} \in BV[0, a]$.

Now, let us return to the BVP (4.7), (4.8). It can be shown that if $A, B \in \widehat{\Omega}[0, 1]$ and the function $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then the BVP (4.7), (4.8) is equivalent to the following perturbed nonlinear Hammerstein integral equation

$$x(t) = \int_0^1 (1-t)A(s)dx(s) + \int_0^1 tB(s)dx(s) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \tag{4.9}$$

$t \in [0, 1]$, where the kernel k has the following form

$$k(t, s) = \begin{cases} s(1-t), & \text{if } 0 \leq s \leq t \leq 1, \\ t(1-s), & \text{if } 0 \leq t < s \leq 1, \end{cases} \tag{4.10}$$

that is, each twice continuously differentiable function $x: [0, 1] \rightarrow \mathbb{R}$ which satisfies (4.7), (4.8) is a CBV -solution to (4.9), and vice-versa.

Remark 4.15. It is easy to see that the kernel k given by (4.10) is continuous and satisfies the condition (B₄) with the function $m: [0, 1] \rightarrow [0, +\infty)$ given by $m(s) = 1$.

Theorem 4.16. *Let $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Moreover, let the functions $A, B \in \widehat{\Omega}[0, 1]$ be such that*

$$\left| \int_0^1 [A(s) - B(s)]ds \right| < 1.$$

Then there exists a number $\lambda_0 > 0$ such that for any $\lambda \in \mathbb{R}$ satisfying $|\lambda| \leq \lambda_0$ the BVP (4.7), (4.8) has a solution.

Proof. Since f is continuous (and thus locally bounded), $\psi: [0, +\infty) \rightarrow [0, +\infty)$ given by

$$\psi(r) := \sup\{|f(t, u)| : t \in [0, 1] \text{ and } u \in [-r, r]\} \quad \text{for } r \geq 0$$

is well-defined and non-decreasing. Moreover, k given by (4.10) is continuous and satisfies the condition (B_4) with $m(s) = 1$ (see Remark 4.15). Therefore, the proof of Theorem 4.16 is a direct consequence of Corollary 4.8, Corollary 4.13 and Remark 4.2 if one sets $\phi(t) = 1$, $v(t) = 1 - t$ and $w(t) = t$ for $t \in [0, 1]$ as well as

$$\alpha[x] = \int_0^1 A(s)dx(s) \quad \text{and} \quad \beta[x] = \int_0^1 B(s)dx(s) \quad \text{for } x \in CBV[0, 1]. \quad \square$$

Similar approach to the above one can also be used to study the existence of solutions to BVPs with non-local BCs slightly different from (4.8), namely to BVPs of the following form

$$\begin{aligned} x''(t) &= -\lambda f(t, x(t)), \quad t \in [0, 1], \\ x(0) &= \int_0^1 x(s)dA(s), \quad x(1) = \int_0^1 x(s)dB(s), \end{aligned} \quad (4.11)$$

where $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $A, B \in BV[0, 1]$. Let us note that since we are interested in classical twice continuously differentiable solutions of the BVP (4.11), the integral BCs are well-posed.

Let us also add that such problems have been investigated via the fixed point index theory approach by, for example, Webb and Infante in [43].

The BVP (4.11) is clearly equivalent with the perturbed nonlinear Hammerstein integral equation

$$x(t) = \int_0^1 (1-t)x(s)dA(s) + \int_0^1 tx(s)dB(s) + \lambda \int_0^1 k(t, s)f(s, x(s))ds, \quad (4.12)$$

$t \in [0, 1]$, where $k: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ is given by (4.10), which means, as before, that each twice continuously differentiable function $x: [0, 1] \rightarrow \mathbb{R}$ which satisfies the BVP (4.11) is a CBV -solution to (4.12), and vice-versa.

In the case of the BVP (4.11) we have the following existence result.

Theorem 4.17. *Let $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Moreover, let the functions $A, B \in BV[0, 1]$ be such that $\text{var}_{[0,1]} A + \text{var}_{[0,1]}(A - B) < 1$. Then there exists a number $\lambda_0 > 0$ such that for any $\lambda \in \mathbb{R}$ satisfying $|\lambda| \leq \lambda_0$ the BVP (4.11) has a solution.*

Proof. Let the continuous functionals α and β on the space $CBV[0, 1]$ be given by

$$\alpha[x] = \int_0^1 x(s)dA(s) \quad \text{and} \quad \beta[x] = \int_0^1 x(s)dB(s) \quad \text{for } x \in CBV[0, 1].$$

Then, in view of [30, Theorem 1.6.1], we have

$$\|\alpha\| = \sup_{\|x\|_{BV}=1} |\alpha[x]| \leq \operatorname{var}_{[0,1]} A \cdot \sup_{\|x\|_{BV}=1} \|x\|_\infty \leq \operatorname{var}_{[0,1]} A$$

and $\|\alpha - \beta\| \leq \operatorname{var}_{[0,1]}(A - B)$.

To end the proof it suffices to apply Corollary 4.10 with $v(t) = 1 - t$ and $w(t) = t$ as well as $m(s) = 1$, $\phi(t) = 1$ and

$$\psi(r) := \sup\{|f(t, u)| : t \in [0, 1] \text{ and } u \in [-r, r]\} \quad \text{for } r \geq 0$$

(cf. the proof of Theorem 4.16). □

Finally, we will devote the last part of this Section to illustrating the above existence results by two examples.

Example 4.18. Let us consider the BVP

$$x''(t) = -\lambda f(t, x(t)), \quad t \in [0, 1], \tag{4.13}$$

$$x(0) = \frac{1}{5}x(a) + \frac{1}{5}x(c), \quad x(1) = \frac{1}{5}x(b) + \frac{1}{5}x(c), \tag{4.14}$$

where $0 < a < b < c < 1$ and $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. It is easy to see that the BVP (4.13), (4.14) is equivalent with the BVP (4.11) with $A = \frac{1}{5}\chi_{[a,1]} + \frac{1}{5}\chi_{[c,1]}$ and $B = \frac{1}{5}\chi_{[b,1]} + \frac{1}{5}\chi_{[c,1]}$.

Since $\operatorname{var}_{[0,1]} A + \operatorname{var}_{[0,1]}(A - B) = \frac{4}{5} < 1$, from Theorem 4.17 it follows that there is $\lambda_0 > 0$ such that for every $\lambda \in \mathbb{R}$ with $|\lambda| \leq \lambda_0$ the BVP (4.13), (4.14) admits at least one solution. Of course, if the function $t \mapsto f(t, 0)$ is not identically equal to zero on $[0, 1]$, the solution is non-zero.

On the other hand, transforming the BVP (4.13), (4.14) into an equivalent BVP with integral conditions of the form

$$x(0) = \int_0^1 \widehat{A}(s) dx(s) \quad \text{and} \quad x(1) = \int_0^1 \widehat{B}(s) dx(s) \tag{4.15}$$

with some functions $\widehat{A}, \widehat{B} \in \widehat{\Omega}[0, 1]$ (and then applying Theorem 4.16), in general, seems to be a more difficult task, since the functionals $\alpha[x] = \frac{1}{5}x(a) + \frac{1}{5}x(c)$ and $\beta[x] = \frac{1}{5}x(b) + \frac{1}{5}x(c)$ and the functionals generated by the right-hand sides of the BCs (4.15) have different properties; for example, $\alpha[e] = \beta[e] = \frac{2}{5} > 0$, whereas⁶

$$\int_0^1 \widehat{A}(s) de(s) = \int_0^1 \widehat{B}(s) de(s) = 0.$$

⁶Let us recall that by e we denote the constant function given by $e(t) = 1$ for $t \in [0, 1]$.

However, in some cases it can be done. For example, it can be checked that the BVP (4.13), (4.14) where the function f is given by $f(t, u) = 0$ for $(t, u) \in [0, 1] \times \mathbb{R}$ and the following BVP

$$\begin{aligned} x''(t) &= 0, \quad t \in [0, 1], \\ x(0) &= \int_0^1 dx(s), \quad x(1) = \int_0^1 dx(s), \end{aligned}$$

have only zero solution. This, in particular, means that those BVPs are equivalent.

Now, let us pass to the second example.

Example 4.19. Let us consider the BVP

$$x''(t) = -\lambda f(t, x(t)), \quad t \in [0, 1], \quad (4.16)$$

$$x(0) = 2x(a) - 2x(c), \quad x(1) = 2x(b) - 2x(c), \quad (4.17)$$

where $0 < a < b < c < 1$ and $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. The BVP (4.16), (4.17) is equivalent with (4.7), (4.8), where $A = 2\chi_{[c,1]} - 2\chi_{[a,1]}$ and $B = 2\chi_{[c,1]} - 2\chi_{[b,1]}$ (cf. Example 4.18 and [30, Theorem 1.6.7]). Of course, $A, B \in \widehat{\Omega}[0, 1]$, and moreover

$$\left| \int_0^1 [A(s) - B(s)] ds \right| = 2|a - b|.$$

So if, for example, $a = \frac{1}{5}$, $b = \frac{3}{5}$ and $c = \frac{4}{5}$, then $2|a - b| = \frac{4}{5} < 1$, and hence from Theorem 4.16 it follows that there exists $\lambda_0 > 0$ such that for every $\lambda \in \mathbb{R}$ with $|\lambda| \leq \lambda_0$ the BVP (4.16), (4.17) has at least one solution.

Now, we would like to show that in general it is not possible to transform the BVP (4.16), (4.17) into an equivalent BVP of the form (4.11) to which one could apply Theorem 4.17.

Suppose that the function f is given by $f(t, u) = 2$ for $(t, u) \in [0, 1] \times \mathbb{R}$ and that the BVP (4.16), (4.17) can be equivalently rewritten as the following BVP with non-local integral BCs

$$x''(t) = -2\lambda, \quad t \in [0, 1], \quad (4.18)$$

$$x(0) = \int_0^1 x(s) d\widehat{A}(s), \quad x(1) = \int_0^1 x(s) d\widehat{B}(s), \quad (4.19)$$

for some $\widehat{A}, \widehat{B} \in BV[0, 1]$. Let us note that one of such equivalent reformulations of the BVP (4.16), (4.17) can be obtained using the functions $\widehat{A} = 2\chi_{[a,1]} - 2\chi_{[c,1]}$ and $\widehat{B} = 2\chi_{[b,1]} - 2\chi_{[c,1]}$.

It can be checked that for every $\lambda \in \mathbb{R}$ the BVP (4.16), (4.17) with the above-defined function f and $a = \frac{1}{5}$, $b = \frac{3}{5}$, $c = \frac{4}{5}$ has a unique solution $x_\lambda: [0, 1] \rightarrow \mathbb{R}$ given by $x_\lambda(t) = -\lambda t^2 + \frac{9}{5}\lambda t - \frac{24}{25}\lambda$. Since the BVPs (4.16), (4.17) and (4.18), (4.19) are equivalent, we infer that for every $\lambda \in \mathbb{R}$ we have

$$\frac{24}{25}|\lambda| = |x_\lambda(0)| = \left| \int_0^1 x_\lambda(s) d\widehat{A}(s) \right| \leq \|x_\lambda\|_\infty \cdot \operatorname{var}_{[0,1]} \widehat{A} = \frac{24}{25}|\lambda| \cdot \operatorname{var}_{[0,1]} \widehat{A}.$$

In particular, $\operatorname{var}_{[0,1]} \widehat{A} \geq 1$, which shows that $\operatorname{var}_{[0,1]} \widehat{A} + \operatorname{var}_{[0,1]}(\widehat{A} - \widehat{B}) \geq 1$ and means that the assumptions of Theorem 4.17 cannot be satisfied.

Remark 4.20. In connection with Example 4.19 a natural question arises whether instead of transforming the BVP (4.16), (4.17) into an equivalent BVP of the form (4.11) and trying to apply Theorem 4.17, it would not be better to use directly Corollary 4.10 with the functionals $\alpha, \beta \in CBV^*[0, 1]$ given by $\alpha[x] = 2x(a) - 2x(c)$ and $\beta[x] = 2x(b) - 2x(c)$.

It turns out that (in general) it would be not, since, as we will show below, the norm of the functional α is at least 2, and thus the condition (B_6) cannot be satisfied.

Let $x \in CBV[0, 1]$ be a continuous piecewise linear function whose graph is a polygonal line spanned by the points $(0, 0)$, $(a, 0)$, $(c, 1)$ and $(1, 1)$. Then $\|x\|_{BV} = 1$ and $|\alpha[x]| = 2$, which proves that $\|\alpha\| \geq 2$.

Remark 4.21. The existence of (positive) solutions (and their multiplicity) to BVPs with non-local BCs of the form (4.11) was also studied, for example, in [18, 20, 42, 43]. The main tool used in those papers was the fixed point index theory in the space of continuous functions, and hence additional assumptions on the nonlinearity f , such as sub- or superlinearity, were required.

For completeness, let us also add that if we use the functions $\widehat{A} = 2\chi_{[a,1]} - 2\chi_{[c,1]}$ and $\widehat{B} = 2\chi_{[b,1]} - 2\chi_{[c,1]}$ to transform the BVP (4.16), (4.17) from Example 4.19 into an equivalent BVP with non-local integral conditions, and additionally we assume that $a = \frac{1}{5}$, $b = \frac{3}{5}$, $c = \frac{4}{5}$, then we cannot apply the existence result from [43], since the assumption (C_6) from the aforementioned article is not satisfied.

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References

- [1] Ambrosio, L., Fusco, N. and Pallara, D., *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Math. Monogr. New York: Oxford Univ. Press 2000.
- [2] Appell, J., Banaś, J. and Merentes, N., *Bounded Variation and Around*. De Gruyter Ser. Nonlinear Anal. Appl. 17. Berlin: De Gruyter 2014.
- [3] Borkowski, M., Bugajewski, D. and Zima, M., On some fixed-point theorems for generalized contractions and their perturbations. *J. Math. Anal. Appl.* 367 (2010)(2), 464 – 475.
- [4] Brauer, F., Constant rate harvesting of populations governed by Volterra integral equations. *J. Math. Anal. Appl.* 56 (1976) 18 – 27.
- [5] Bugajewska, D., Bugajewski, D. and Lewicki, L., On nonlinear integral equations in the space of functions of bounded generalized ϕ -variation. *J. Integral Equ. Appl.* 21 (2009)(1), 1 – 20.
- [6] Bugajewski, D., On BV-solutions of some nonlinear integral equations. *Integral Equ. Oper. Theory* 46 (2003), 387 – 398.
- [7] Bugajewski, D., Gulowski, J. and Kasprzak, P., On continuity and compactness of some nonlinear operators in the spaces of functions of bounded variation. *Ann. Mat. Pura Appl. (4)* 195 (2016)(5), 1513 – 1530.
- [8] Bugajewski, D. and Kasprzak, P., Leggett–Williams type theorems with applications to nonlinear differential and integral equations. *Nonlinear Anal.* 114 (2015), 116 – 132.
- [9] Byszewski, L., Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* 162 (1991)(2), 494 – 505.
- [10] Conti, R., Recent trends in the theory of boundary value problems for ordinary differential equations. *Boll. Un. Mat. Ital. (3)* 22 (1967), 135 – 178.
- [11] Faraci, F., Existence and multiplicity results for a non linear Hammerstein integral equation. In: *Variational Analysis and Applications* (Proceedings Erice (Italy) 2003; eds.: F. Giannessi et al.). Nonconvex Optim. Appl. 79. New York: Springer 2005, pp. 359 – 371.
- [12] Faraci, F. and Moroz, V., Solutions of Hammerstein integral equations via a variational principle. *J. Integral Equ. Appl.* 15 (2003)(4), 385 – 402.
- [13] Franco, D., Infante, G. and Perán, J., A new criterion for the existence of multiple solutions in cones. *Proc. Royal Soc. Edinburgh Sect. A* 142 (2012)(5), 1043 – 1050.
- [14] Graef, J. R., Kong, L. and Wang, H., A periodic boundary value problem with vanishing Green's function. *Appl. Math. Lett.* 21 (2008)(2), 176 – 180.,
- [15] Harris, H. and Laibson, D., Dynamic choices of hyperbolic consumers. *Econometrica* 68 (2001)(2), 935 – 957.

- [16] Hewitt, E. and Stromberg, K., *Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable*. Graduate Texts Math. 25. New York: Springer 1975.
- [17] Infante, G., Eigenvalues of some non-local boundary-value problems. *Proc. Edinb. Math. Soc. (2)* 46 (2003)(1), 75 – 86.
- [18] Infante, G., Positive solutions of nonlocal boundary value problems with singularities. *Discrete Contin. Dyn. Syst. (Suppl.)* (2009), 377 – 384.
- [19] Infante, G. and Webb, J. R. L., Three-point boundary value problems with solutions that change sign. *J. Integral Equ. Appl.* 15 (2003)(1), 37 – 57.
- [20] Infante, G. and Webb, J. R. L., Nonlinear non-local boundary-value problems and perturbed Hammerstein integral equations. *Proc. Edinb. Math. Soc. (2)* 49 (2006)(3), 637 – 656.
- [21] Karakostas, G. L. and Tsamatos, P. Ch., Existence of multiple positive solutions for a nonlocal boundary value problem. *Topol. Methods Nonlinear Anal.* 19 (2002)(1), 109 – 121.
- [22] Karakostas, G. L. and Tsamatos, P. Ch., Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems. *Electron. J. Diff. Equ.* 30 (2002), 17 pp.
- [23] Krasnosel'skiĭ, M. A., Some problems of nonlinear analysis (in Russian). *Uspehi Mat. Nauk (N.S.)* 9 (1954), 57 – 114; Engl.-transl: *Amer. Math. Soc. Transl. Ser. 2* 10 (1958), 345 – 409.
- [24] Lan, K. Q., Eigenvalues of second order differential equations with singularities. *Discrete Contin. Dyn. Syst. (Added Volume)* (2001), 241 – 247.
- [25] Lan, K. Q., Multiple eigenvalues for singular Hammerstein integral equations with applications to boundary value problems. *J. Comput. Appl. Math.* 189 (2006)(1–2), 109 – 119.
- [26] Lan, K. Q., Eigenvalues of semi-positone Hammerstein integral equations and applications to boundary value problems. *Nonlinear Anal.* 71 (2009)(12), 5979 – 5993.
- [27] Lan, K. Q. and Webb, J. R. L., Positive solutions of semilinear differential equations with singularities. *J. Diff. Equ.* 148 (1998)(2), 407 – 421.
- [28] Lebedev, V. I., *An Introduction to Functional Analysis in Computational Mathematics*. Boston: Birkhäuser 1997.
- [29] Leggett, R. W. and Williams, L. R., An extension of Jentzsch's theorem to nonlinear Hammerstein operators. *J. Math. Anal. Appl.* 60 (1977), 248 – 254.
- [30] Łojasiewicz, S., *An Introduction to the Theory of Real Functions*. Chichester: Wiley 1988.
- [31] Ma, R., Multiplicity of positive solutions for second-order three-point boundary value problems. *Comput. Math. Appl.* 40 (2000)(2–3), 193 – 204.
- [32] Ma, R., A survey on nonlocal boundary value problems. *Appl. Math. E-Notes* 7 (2007), 257 – 279.

- [33] Ma, R., Nonlinear periodic boundary value problems with sign-changing Green's function. *Nonlinear Anal.* 74 (2011)(5), 1714 – 1720.
- [34] Martin, R. H., Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*. Pure Appl. Math. New York: Wiley 1976.
- [35] Maz'ya, V., *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*. Grundlehren Math. Wiss. 342. Heidelberg: Springer 2011.
- [36] Ntouyas, S. K., Nonlocal initial and boundary value problems: a survey. In: *Handbook of Differential Equations: Ordinary Differential Equations. Vol. II* (eds.: A Cañada et al.). Amsterdam: Elsevier 2005, pp. 461 – 557.
- [37] Park, S., Generalizations of the Krasnoselskii fixed point theorem. *Nonlinear Anal.* 67 (2007), 3401 – 3410.
- [38] Picone, M., Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine (in Italian). *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 10 (1908), 95.
- [39] Štikonas, A., A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions. *Nonlinear Anal. Model. Control* 19 (2014)(3), 301 – 334.
- [40] Waterman, D., On convergence of Fourier series of functions of generalized bounded variation. *Studia Math.* 44 (1972), 107 – 117.
- [41] Webb, J. R. L., Boundary value problems with vanishing Green's function. *Comm. Appl. Anal.* 13 (2009)(4), 587 – 595.
- [42] Webb, J. R. L., Positive solutions of a boundary value problem with integral boundary conditions. *Electron. J. Diff. Equ.* 55 (2011), 10 pp.
- [43] Webb, J. R. L. and Infante, G., Positive solutions of nonlocal boundary value problems: a unified approach. *J. London Math. Soc.* 74 (2006)(2), 673 – 693.
- [44] Whyburn, W. M., Differential equations with general boundary conditions. *Bull. Amer. Math. Soc.* 48 (1942), 692 – 704.

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