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Uniform Asymptotic Expansions for the Fundamental Solution of Infinite Harmonic Chains

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Abstract. We study the dispersive behavior of waves in linear oscillator chains. We show that for general general dispersions it is possible to construct an expansion such that the remainder can be estimated by $\frac{1}{t}$ uniformly in space. In particlar we give precise asymptotics for the cross-over from the $t^{-\frac{1}{2}}$ decay of nondegenerate wave numbers to the degenerate $t^{-\frac{1}{3}}$ decay of degenerate wave numbers. This involves a careful description of the oscillatory integral involving the Airy function.

Keywords. Asymptotic analysis, method of stationary phase, dispersive decay, oscillatory integrals, Airy function, Fermi–Pasta–Ulam chain

Mathematics Subject Classification (2010). 37K60, 41A60, 42B20

1. Introduction

In this work we study the dispersive behavior of waves in linear oscillator chains. While there is a large body of the analysis in certain regimes of the dispersion relation there seems to be no general theory providing a uniform estimates. The main problem derives from the fact that the large-time asymptotics of the solutions can be estimated along the rays given by the group velocity by the presentation via oscillatory integrals. The difficulty to obtain uniform estimates for the remainders of an asymptotic expansion stems from the fact that the dispersion relation $\theta \mapsto \omega(\theta)$ necessarily contains degenerate points (i.e. where $\omega''(\hat{\theta}) = 0$). While for nondegenerate points the solutions decay like $t^{-\frac{1}{2}}$, the decay at degenerate points the decay is only of order $t^{-\frac{1}{k}}$ with $k \geq 3$. Such separate estimates for dispersive partial differential equations or discrete lattices

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are classical (cf. [4, 5, 8, 10, 12, 14–16] and the references there), but our aim is to find a uniform expansion providing also a sharp estimate in the transition regions, i.e. for nearly degenerate wave numbers. Moreover, for nondegenerate wave numbers the asymptotic profiles are given in terms of simple trigonometric functions, the degenerate case with $k = 3$ (i.e. $\omega'''(\hat{\theta}) \neq 0$) leads to fronts with a profile given in terms of the Airy function, see Figure 1.1.

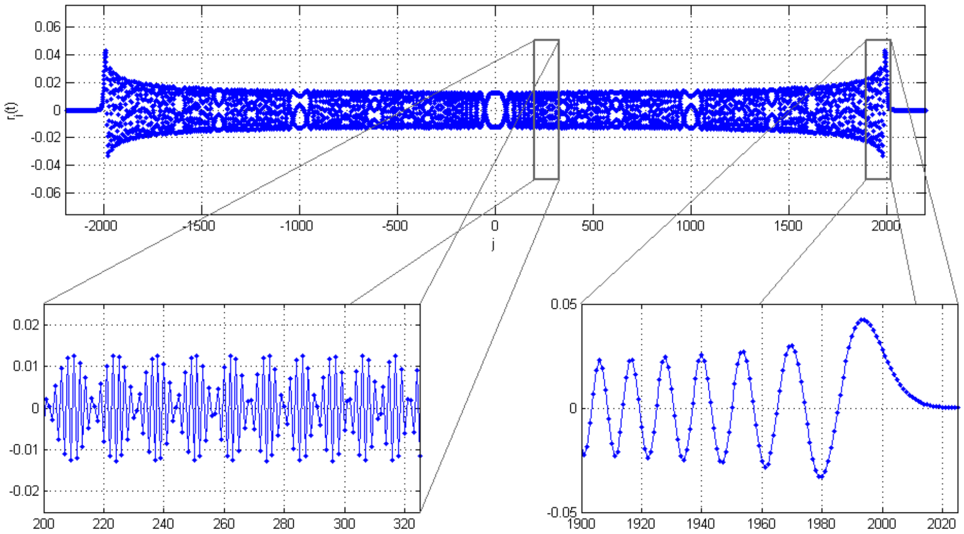

Figure 1.1: Green's function $G_{11}(t, j)$ at $t = 2000$ for the linear FPU chain. Lower left: periodic wave trains for nondegenerate group velocities. Lower right: Airy-type behavior at the degenerate front.

To be more precise we study the general linear oscillator chain

$$\ddot{x}_j = -a_0 x_j + \sum_{k=1}^{K} a_k \big(x_{j+k} - 2x_j + x_{j-k} \big), \quad j \in \mathbb{Z}, \tag{1.1}$$

where $a_0 \geq 0$ is due a stabilizing background potential and $a_1, \dots, a_K \in \mathbb{R}$ give the interaction coefficients. With $a_0 = 0$, $a_1 = 1$, and $K = 1$ we obtain the linearized Fermi-Pasta-Ulam (FPU) system (cf. [3]). The dispersion relation is given via

$$\omega^2 = \Lambda(\theta) := a_0 + \sum_{k=1}^{K} 2a_k \left[1 - \cos(k\theta) \right] \quad \text{for } \theta \in \mathbb{S}^1 := \mathbb{R}/_{(2\pi\mathbb{Z})},$$

where we always assume stability in the form $\Lambda(\theta) \geq 0$. Thus, we define the positive branch of the dispersion relation and the group velocity via

$$\omega(\theta) := \sqrt{\Lambda(\theta)} \geq 0 \quad \text{and} \quad c(\theta) := \omega'(\theta).$$

The oscillatory integrals to be estimated have the form

$$\int_{\mathbb{S}^1} A(\theta)\mathrm{e}^{\mathrm{i}(\omega(\theta)t+\theta j)}\,\mathrm{d}\theta = \int_{\mathbb{S}^1} A(\theta)\mathrm{e}^{\mathrm{i}(\omega(\theta)+c\theta)t}\,\mathrm{d}\theta \;=:\; g_A(t,c),$$

where we always use the relation $j = ct \in \mathbb{Z}$, which is important to keep the periodicity of the integrand.

Two of the analytical difficulties in the analysis can be explained at this point. First, if $\Lambda(\theta) = \gamma^2(\theta-\theta_1)^{2n} + O(|\theta-\theta_1|^{2n+1})$ the dispersion relation $\omega(\theta)$ may be nonanalytic. We treat this case by assuming $a_0 = 0$, which gives $\omega(\theta) = \gamma|\theta| + O(\theta^2)$ where we assume $\gamma = \big(\sum_{k=1}^{K} k^2 a_k\big)^{\frac{1}{2}} > 0$. The second difficulty arises from degeneracies of the dispersion relation. The group velocity is given by the relation

$$c = c_{\mathrm{gr}}(\theta) = \omega'(\theta).$$

Fixing a θ with $\omega''(\theta) \neq 0$ and thus fixing $c = \omega'(\theta)$ and assuming that the support of the A in the definition of g_A is contained in a sufficiently small neighborhood of θ, we have the expansion

$$g_A(t,c) = c_A \cos\big[(\omega(\theta){+}c\theta)t + \mathrm{sign}(\omega''(\theta))\frac{\pi}{4}\big]\, t^{\frac{1}{2}} \; + R_A^{\mathrm{non}}(t,c)$$

with $R_A^{\mathrm{non}}(t,c) = O(t^{-1})$, see Subsection 4.2. However, for θ near a degenerate case with $\omega''(\hat\theta) = 0$ but $\omega'''(\theta) \neq 0$, we obtain the expansion

$$g_A(t,\hat c) = \mathrm{e}^{itb(c)}\big[c_A \mathrm{Ai}(a(c)t^{\frac{2}{3}})t^{-\frac{1}{3}} + d_A \mathrm{Ai}'(a(c)t^{\frac{2}{3}})t^{-\frac{2}{3}}\big] + R_A^{\mathrm{deg}}(t,c)$$

with $R_A^{\mathrm{deg}}(t,c) = O(t^{-1})$ and suitable functions $a(c)$ and $b(c)$, see [8] or Subsection 4.3. To obtain a uniform error estimate we give quantitative estimates for the two error terms $R_A^{\mathrm{non}}(t,c)$ and $R_A^{\mathrm{deg}}(t,c)$ and show that the degenerate expansion based on the Airy function coincides up to an error $O(t^{-1})$ with the harmonic expansion in an overlapping region.

In summary our main result reads as follows.

Theorem 1.1. *Assume that* (1.1) *satisfies* $a_0 = 0$ *and the dispersion relation has the form* $\omega(\theta) = \mathrm{sign}(\theta)\tilde\omega(\theta)$*, where* $\tilde\omega$ *is smooth and satisfies the*

$$\textit{nondegeneracy condition:}\quad \tilde\omega''(\theta) = 0 \quad\Longrightarrow\quad \tilde\omega'''(\theta) \neq 0.$$

Then there exists a constant $C(\tilde\omega)$ *such that for all* $t > 0$ *the Green's function matrix* $G_j(t) \in \mathbb{R}^{2\times 2}$ *for* (1.1) *written for the vectors* $\mathbf{r} = (x_{j+1}{-}x_j)_{j\in\mathbb{Z}}$ *and* $\mathbf{p} = (\dot x_j)_{j\in\mathbb{Z}}$ *satisfies the estimate*

$$\left|G_j(t) - \mathcal{G}^{\mathrm{expan}}\left(t, \frac{j}{t}\right)\right| \leq \frac{C(\tilde\omega)}{t} \quad \textit{for all } j \in \mathbb{Z} \textit{ and all } t > 0,$$

where the function $\mathcal{G}^{\mathrm{expan}}(t,c)$ *is given in* (3.6).

Using the classical decay estimates for the Green's functions for group velocities outside the range of $\tilde{\omega}'$ (see Proposition 2.3), we easily obtain bounds in ℓ^p spaces, namely for each $p > 1$ there exists $C_p > 0$ such that

$$\|G_{(\cdot)}(t) - \mathcal{G}^{\mathrm{expan}}(t, \tfrac{\cdot}{t})\|_{\ell^p} \leq C_p \, t^{-\frac{p-1}{p}} \quad \text{for } t > 0.$$

In particular, this result implies that the dispersive decay estimates for the Green's function given in [12] for $p \in (2,4) \cup (4,\infty)$

$$\|G_{(\cdot)}(t)\|_{\ell^p} \leq C_p^{\mathrm{upper}} t^{-\alpha_p} \text{ for } t > 0 \quad \text{with } \alpha_p = \min\{\tfrac{p-2}{2p}\,,\,\tfrac{p-1}{3p}\}$$

are sharp. Our hope is that using the specific form of $G_j^{\mathrm{expan}}(t)$ one can improve the decay results for nonlinear systems as well, see [6, 12, 14].

Our analysis was stimulated by the work in [5], which analyzed synchronization effects occurring for $c = 0$, which corresponds to the wave number $\theta = \pm\pi$. The analysis is done for fixed j and can be made uniform on parabolic regions $j^2 \leq Ct$. The dispersion of the energy was analyzed in [7, 11] via the Husimi and Wigner transform, even in multidimensional cases. The usage of dispersion in the error control for discretized PDEs is discussed in [9, 10].

Another interesting question relates to the two or three-dimensional case, where only few optimal decay rates are available, see e.g. [2] for an ℓ^∞ estimate of $G_{(\cdot)}$ for the two-dimensional square lattice.

Notations. Bold face letters $\mathbf{r}, \mathbf{p}, \ldots$ denote elements in $\ell^p(\mathbb{R})$ or, with the common abuse of notation, smooth functions $\mathbf{r}(\cdot), \mathbf{p}(\cdot), \ldots : \mathbb{R} \to \ell^p(\mathbb{R})$. Capital letters normally refer to linear operators, e.g. $G_j(t) : \mathbb{R}^2 \to \mathbb{R}^2$ and $\mathbf{G}(t) : \ell^p(\mathbb{R}^2) \to \ell^p(\mathbb{R}^2)$ or, again, to smooth functions mapping into these spaces.

To simplify the notations we denote bounds in general by C and carry out the distinction via indices $C_1, C_2, \ldots$ only if it is necessary. To highlight the dependency on parameters we write for instance $C = C(\omega, \delta)$. For $\delta \in \mathbb{R}$ the notation is obvious. For ω being a sufficiently smooth function this refers to a dependency on $\|\omega\|_{W^{n,p}}$ for suitable $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$.

2. Dispersion in the generalized linear FPU

2.1. The generalized linear FPU. We consider an infinite number of equal particles with unit mass interacting with a finite number K of neighbors via linear forces. According to Newton's law, the equations of motion are

$$\ddot{x}_j = \sum_{1 \leq k \leq K} \big[a_k(x_{j+k} - x_j) - a_k(x_j - x_{j-k})\big], \quad j \in \mathbb{Z}. \tag{2.1}$$

Here $x_j \in \mathbb{R}$ denote the displacements. We write $\mathbf{x} := (x_j)_{j \in \mathbb{Z}}$. The system (2.1) is Hamiltonian, i.e. $(\dot{\mathbf{x}}, \dot{\mathbf{p}})^T = \mathcal{J}_{\mathrm{can}} \, \mathrm{d}\,\mathcal{H}_{\mathrm{x}}(\mathbf{x}, \mathbf{p})^T$ with momentum

$\mathbf{p} := \dot{\mathbf{x}}$, Hamiltonian function $\mathcal{H}_{\mathrm{x}}(\mathbf{x},\mathbf{p}) = \sum_{j\in\mathbb{Z}} \big(\frac{1}{2}p_j^2 + \sum_{1\le k\le K} \frac{a_k}{2}(x_{j+k}-x_j)^2\big)$ and $\mathcal{J}_{\mathrm{can}}$ the canonical Poisson operator defined by $\langle(\mathbf{x},\mathbf{p})^T, \mathcal{J}_{\mathrm{can}}(\tilde{\mathbf{x}},\tilde{\mathbf{p}})^T\rangle_{\ell^2\oplus\ell^2} = \langle \mathbf{x},\tilde{\mathbf{p}}\rangle_{\ell^2} - \langle\tilde{\mathbf{x}},\mathbf{p}\rangle_{\ell^2}$.

The system (2.1) exhibits plane waves solution of the form $x_j(t) = \mathrm{e}^{\mathrm{i}(\theta j+\hat{\omega}t)}$ if and only if the dispersion relation

$$\hat{\omega}^2 = \Lambda(\theta) := \sum_{1\le k\le K} 2a_k\big[1-\cos(k\theta)\big] \tag{2.2}$$

is satisfied. By periodicity, it suffices to take $\theta\in(-\pi,\pi]$. We have $\Lambda(0)=0$ which is a consequence of Galilean invariance, i.e. for all $\xi, c\in\mathbb{R}$ the transformation $(x_j,p_j)\mapsto(x_j{+}\xi{+}ct, p_j{+}c)$ leaves (2.1) invariant. Throughout, we assume the stability condition

$$\forall\,\theta\in(-\pi,\pi]\setminus\{0\}: \quad \Lambda(\theta)>0 \tag{2.3}$$

holds. This certainly holds if all a_k are positive, however more general cases are possible. Thus we are able to define the relevant branch $\hat{\omega}=\omega(\theta)$ of the dispersion relation via

$$\omega(\theta) := \sqrt{\Lambda(\theta)} \ge 0. \tag{2.4}$$

With a slight abuse of notation we simply call ω the dispersion relation.

Due to the Galilean invariance it is convenient to use distances instead of the displacements $\mathbf{r} := (\partial_1{-}\mathbf{1})\mathbf{x} = (x_{j+1}{-}x_j)_{j\in\mathbb{Z}}$ as new variables. Then the Hamiltonian function turns into $\mathcal{H}_{\mathrm{r}}(\mathbf{r},\mathbf{p}) = \frac{1}{2}\sum_{j\in\mathbb{Z}}\big(p_j^2 + \sum_{1\le k\le K} a_k|\sum_{0\le l<k} r_{j+l}|^2\big)$. The transformed Hamiltonian system reads as

$$(\dot{\mathbf{r}},\dot{\mathbf{p}})^T = \mathcal{J}_{\mathrm{r}}\,\mathrm{d}\mathcal{H}_{\mathrm{r}}(\mathbf{r},\mathbf{p})^T =: \mathcal{L}(\mathbf{r},\mathbf{p})^T \tag{2.5a}$$

with

$$\mathcal{J}_{\mathrm{r}} := \begin{pmatrix} \mathbf{0} & \partial_1{-}\mathbf{1} \\ \mathbf{1}{-}\partial_{-1} & \mathbf{0}\end{pmatrix}, \quad \mathrm{d}\mathcal{H}_{\mathrm{r}} = \begin{pmatrix} \sum_{|l|<K}\sum_{|l|<k\le K}(k{-}|l|)a_k\partial_l & \mathbf{0} \\ \mathbf{0} & \mathbf{1}\end{pmatrix}, \tag{2.5b}$$

and $(\partial_l\mathbf{z})_j = z_{j+l}$. The operator $\mathcal{J}_{\mathrm{r}}$ is a non-canonical Poisson structure arising from the push-forward of the Poisson tensor $\mathcal{J}_{\mathrm{can}}$, that is $\mathcal{J}_{\mathrm{r}} = \mathcal{T}\mathcal{J}_{\mathrm{can}}\mathcal{T}^*$ where $\mathcal{T}$ is the linear map defined by $(\mathbf{r},\mathbf{p})^T = \mathcal{T}(\mathbf{x},\mathbf{p})^T$.

Using the Fourier transform $\mathcal{F}:\ell^2(\mathbb{Z},\mathbb{R}^2)\to L^2(\mathbb{S}^1,\mathbb{R}^2)$ defined by $\hat{z}(\theta) = \sum_{j\in\mathbb{Z}} z_j\mathrm{e}^{-\mathrm{i}j\theta}$, it is possible to solve (2.5) explicitly. Applying $\mathcal{F}$ leads to

$$\begin{pmatrix}\dot{\hat{\mathbf{r}}}\\ \dot{\hat{\mathbf{p}}}\end{pmatrix} = \begin{pmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta}-1 \\ 1-\mathrm{e}^{-\mathrm{i}\theta} & 0\end{pmatrix}\begin{pmatrix}\omega_{\mathrm{r}}^2(\theta) & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\mathbf{r}}\\ \hat{\mathbf{p}}\end{pmatrix}, \tag{2.6}$$

where

$$\omega_{\mathrm{r}}^2(\theta) = \sum_{0<k\le K} ka_k + 2\sum_{0<l\le K-1}\Big(\sum_{l<k\le K}(k-l)a_k\Big)\cos(l\cdot\theta). \tag{2.7}$$

Now, solving the linear system (2.6) we obtain the fundamental matrix $\hat{\mathbf{G}}_{\mathrm{r}}(t,\theta)$ and Green's function of our original problem is given by inverse Fourier transform, $\mathbf{G}(t) = \mathcal{F}^{-1}\{\hat{\mathbf{G}}_{\mathrm{r}}(t,\theta)\} = \frac{1}{2\pi}\int_{S^1}\hat{\mathbf{G}}_{\mathrm{r}}(t,\theta)\,\mathrm{d}\theta$ with

$$G_j(t) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\begin{pmatrix} \cos(\omega(\theta)t) & \frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\omega(\theta)}\sin(\omega(\theta)t) \\ \frac{-\omega(\theta)}{\mathrm{e}^{-\mathrm{i}\theta}-1}\sin(\omega(\theta)t) & \cos(\omega(\theta)t)\end{pmatrix}\mathrm{e}^{\mathrm{i}j\,\theta}\,\mathrm{d}\theta, \quad j\in\mathbb{Z} \tag{2.8}$$

Thus the long time behavior of solutions is determined by oscillatory integrals. Altogether we proved the following lemma.

Proposition 2.1 (Explicit solution)**.** *Given some initial conditions* $(\mathbf{r}^0,\mathbf{p}^0)^T \in \ell^2(\mathbb{Z},\mathbb{R}^2)$*, the unique solution of* $(\dot{\mathbf{r}},\dot{\mathbf{p}})^T = \mathcal{L}(\mathbf{r},\mathbf{p})^T$ *defined in* (2.5) *is determined by*

$$(\mathbf{r}(t),\mathbf{p}(t))^T = \mathrm{e}^{\mathcal{L}\,t}(\mathbf{r}^0,\mathbf{p}^0)^T \tag{2.9a}$$

where $(\mathrm{e}^{\mathcal{L}\,t})_{t\in\mathbb{R}}$ *is a differentiable group of bounded operators on* $\ell^2(\mathbb{Z},\mathbb{R}^2)$ *defined by*

$$\left(\mathrm{e}^{\mathcal{L}\,t}(\mathbf{r},\mathbf{p})^T\right)_j = \sum_{k\in\mathbb{Z}} G_k(t)\cdot(r_{j-k},p_{j-k})^T \quad \textit{for } j\in\mathbb{Z} \tag{2.9b}$$

with $G_j(t)$ *defined in* (2.8)*.*

To characterize the dispersion relation, from now on, we will assume that, additionally to the stability condition, the following non-degeneracy condition is satisfied,

$$\omega'(0)>0 \quad \text{and} \quad \forall\,\hat{\theta}\in\mathbb{S}^1:\ \omega''(\hat{\theta})=0 \quad\Longrightarrow\quad \omega'''(\hat{\theta})\neq 0 \tag{2.10}$$

These conditions as will be discussed below in connection with (2.15). In fact, these are not fundamental for the upcoming discussions, but a violation would lead to different decay rates and necessitate case-by-case analysis. Now we may highlight important properties of the dispersion relation ω in the following lemma.

Lemma 2.2. *For the dispersion relation defined by* (2.2) *and* (2.4) *holds*

$$\omega(\theta) = 2\left|\sin\left(\frac{\theta}{2}\right)\right|\omega_{\mathrm{r}}(\theta) \tag{2.11}$$

with ω_{r} *given by* (2.7)*. Furthermore, for* ω_{r} *holds*

$$\forall\,\theta\in\mathbb{S}^1:\ \omega_{\mathrm{r}}(\theta)=\omega_{\mathrm{r}}(-\theta)=\omega_{\mathrm{r}}(\theta+2\pi) \tag{2.12a}$$

and, if additionally the stability and non-degeneracy conditions (2.3) *and* (2.10) *are satisfied, then*

$$\exists\,c_{\mathrm{r}}>0\ \forall\,\theta\in\mathbb{S}^1:\ \omega_{\mathrm{r}}(\theta)\geq c_{\mathrm{r}}. \tag{2.12b}$$

Proof. Since the linear equation (2.1) is invariant under the transformation $(\partial_1 - \mathbf{1})$, the first statement follows from $\ddot{\hat{\mathbf{r}}} = -\Lambda(\theta)\hat{\mathbf{r}}$ and the fact that (2.6) implies $\ddot{\hat{\mathbf{r}}} = 2(\cos\theta - 1)\omega_r(\theta)^2\hat{\mathbf{r}}$. The symmetry and periodicity of ω_{r} is obvious from the explicit formula (2.7). To construct $c_{\mathrm{r}} > 0$ note first that, in view of (2.3), it is sufficient to check $\omega_{\mathrm{r}}(0) \neq 0$. But this follows from $\omega'(0) = \omega_{\mathrm{r}}(0)$ and (2.10). □

The first factor of ω is due to the transformation of the Poisson structure and arises independently of the actual interactions between the particles. This is why we carried out the transformation in detail.

Finally, we state a second equivalent representation of Green's function which will be used below. It is obtained by rewriting the dispersion relation,

$$\tilde{\omega}(\theta) = 2\sin\left(\frac{\theta}{2}\right)\omega_{\mathrm{r}}(\theta) \tag{2.13}$$

and using the symmetry of ω_{r}, namely

$$G(t,c) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\begin{pmatrix} \mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} & \pm\frac{\mathrm{e}^{\mathrm{i}\theta}}{\omega_{\mathrm{r}}(2\theta)}\mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} \\ \pm\frac{\omega_{\mathrm{r}}(2\theta)}{\mathrm{e}^{\mathrm{i}\theta}}\mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} & \mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} \end{pmatrix}\mathrm{d}\theta$$
$$\text{where}\quad \phi_\pm(\theta,c) = \theta c \pm \tilde{\omega}(\theta) \quad\text{and}\quad c = \frac{j}{t}. \tag{2.14}$$

The proof as well as further useful representations of G are given in Appendix A. Note that $\tilde{\omega}$ now is 4π-periodic. The new variable $c \in \mathbb{R}$ characterizes the rays $j = ct$ and refers to the the group velocity $c_{\mathrm{gr}}(\theta) = \pm\omega'(\theta)$. In view of the fact that $\theta \mapsto c_{\mathrm{gr}}(\theta)$ is not injective, it is useful to define

$$\boldsymbol{\Theta}(c) := \{\theta \in \mathbb{S}^1 \,|\, \omega'(\theta) = c\}.$$

Although we often will consider c as a continuous variable, it is to be evaluated in $\frac{j}{t}$. This is crucial in view of the fact that $\theta \mapsto \mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)}$ remains 2π-periodic for that choice.

2.2. Critical wave numbers, dispersive decay and finite sonic velocity. The asymptotic behavior of solutions of (2.5) is dominated by dispersion occurring in such lattices systems. A typical consequence in this context is the decay of solutions to localized initial conditions: Consider the group $(\mathrm{e}^{\mathcal{L}t})_{t\in\mathbb{R}}$ in (2.9) and assume that the dispersion relation ω satisfies the stability condition (2.3) and the non-degeneracy condition (2.10). Then, for $p \in [2,4) \cup (4,\infty]$ there exists C_p such that, for all $t \geq 0$, we have

$$\|\mathrm{e}^{\mathcal{L}t}\|_{\ell^1,\ell^p} \leq \frac{C_p}{(1+t)^{\alpha_p}}, \quad\text{where}\quad \alpha_p = \begin{cases} \dfrac{p-2}{2p} & \text{for } p \in [2,4), \\ \dfrac{p-1}{3p} & \text{for } p \in (4,\infty]. \end{cases}$$

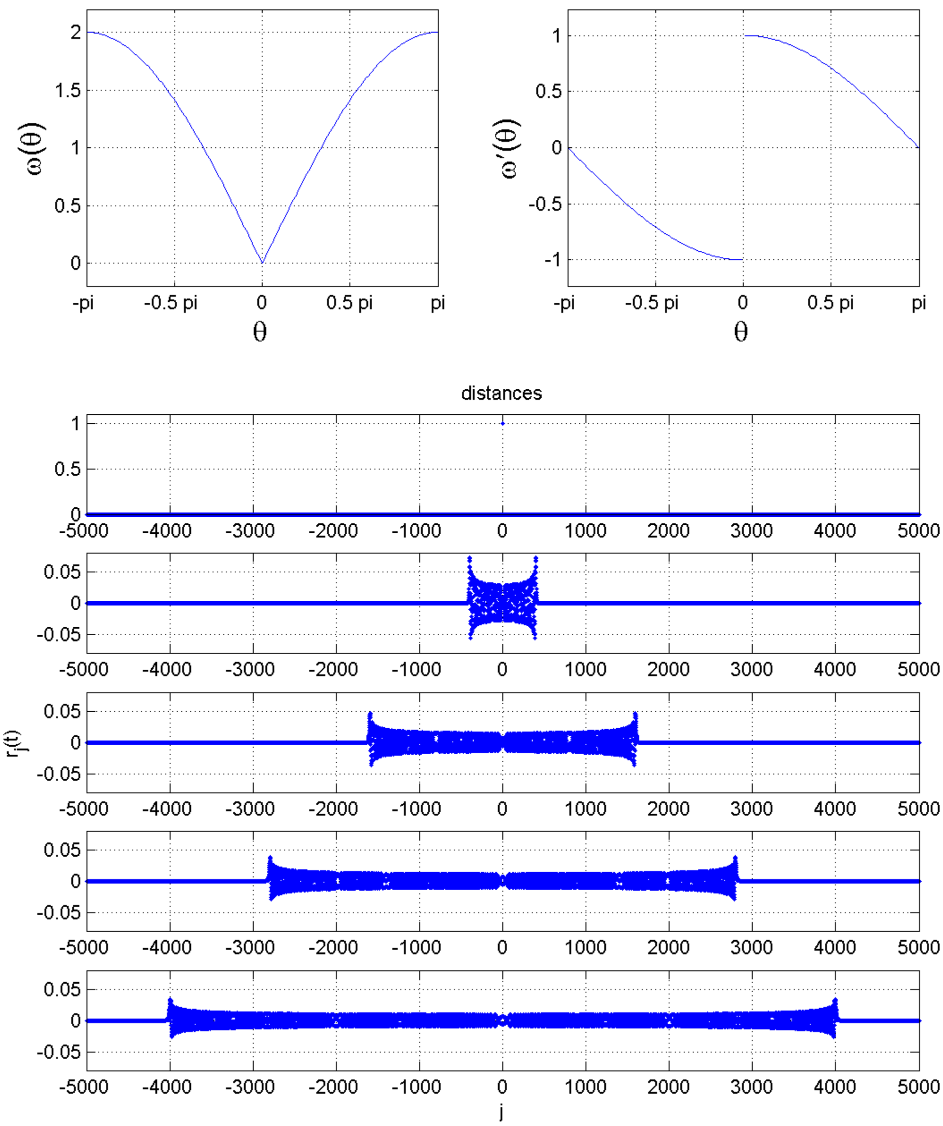

Figure 2.1: Dispersion relation and solutions for different times for the linearized FPU, i.e. $\omega(\theta) = 2|\sin(\frac{\theta}{2})|$.

Here we skipped the case $p = 4$ where an additional logarithmic correction term occurs, see [12] for details and the proof. Like in PDE theory, this decay estimate carries over to nonlinear equations if the nonlinearity is weak and the initial conditions are sufficiently small, see again [12] and also [6, 14].

To obtain these decay rates α_p, which are better than those one gets by standard Riesz Thorin interpolation based on $\alpha_2 = 0$ and $\alpha_\infty = \frac{1}{3}$, the local decay of $(e^{\mathcal{L}t})_{t\in\mathbb{R}}$ needs to be estimated carefully. We will from now on focus on

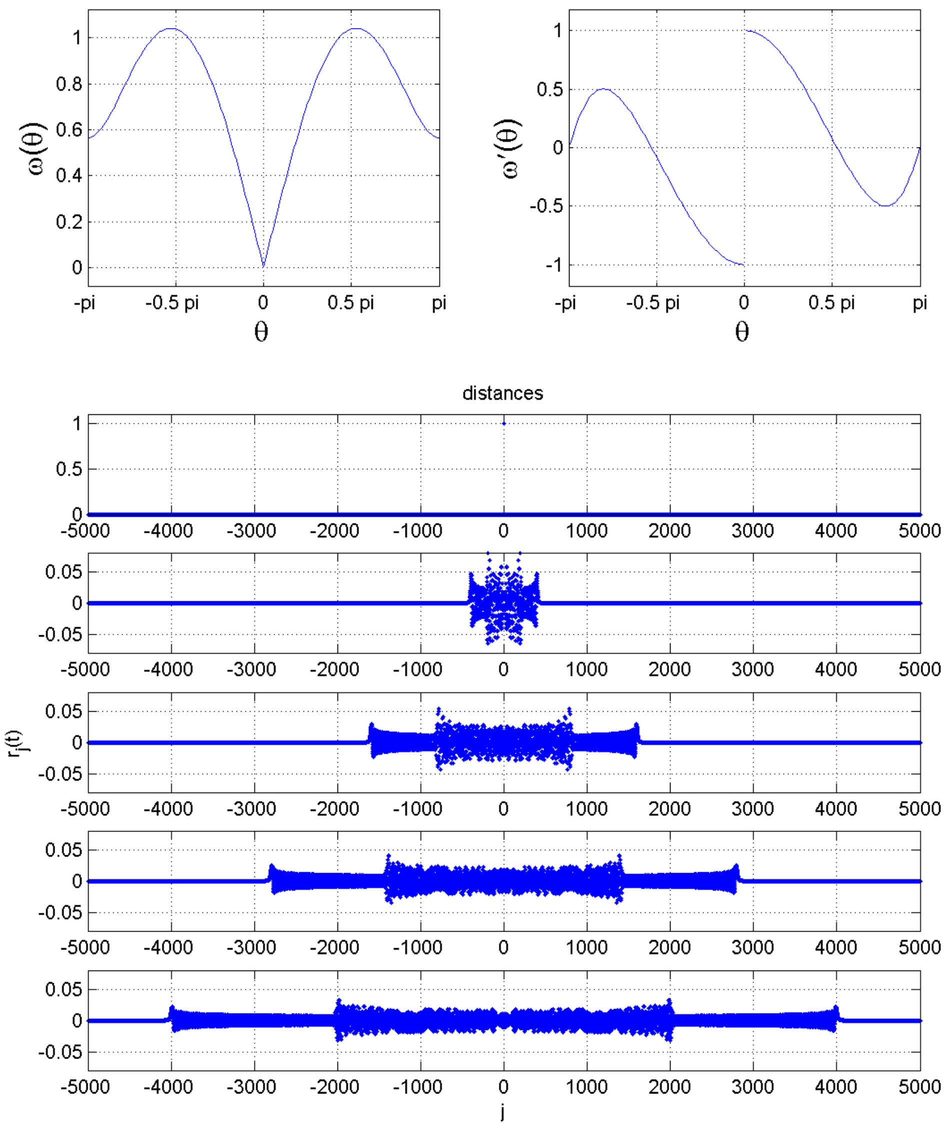

Figure 2.2: Dispersion relation and solutions for different times for a system with second nearest-neighbor interaction with $a_1 = 0.08$ and $a_2 = 0.23$. The coefficients are chosen such that the critical group velocities are approximately 1 and $\frac{1}{2}$.

this local behavior. In the remaining section we summarize important results concerning upper bounds of solutions. In Section 3 and 4, respectively, we will derive uniform asymptotic expansions for the solutions. In this context we will see that these results are optimal - not only in term of the decay rates, but also in terms of $c = \frac{j}{t}$.

The dispersive decay relies on the fact that the behavior of solutions to (2.5) is determined by oscillatory integrals of the form

$$g(t,c) = \int_{\mathbb{S}^1} A(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta \tag{2.15}$$

with $A(\theta)$ standing for 1, $\frac{1}{\omega_{\mathrm{r}}(\theta)}$ or $\omega_{\mathrm{r}}(\theta)$. According to Lemma 2.2, A is real-analytic in any case. Thus, oscillations with wave numbers θ travel along rays $j = c_{\mathrm{gr}}(\theta)t$, where the group velocity is defined by the relation $\partial_\theta \phi(\theta, c_{\mathrm{gr}}(\theta)) = 0$, i.e. $c_{\mathrm{gr}}(\theta) = \pm\tilde{\omega}'(\theta)$. The method of stationary phase, see e.g. [15–17] indicates that the decay along these rays is like $t^{-\frac{1}{2}}$ if $\partial_\theta^2\phi(\theta,c) = \tilde{\omega}''(\theta) \neq 0$ and like $t^{-\frac{1}{3}}$ for $\tilde{\omega}''(\theta) = 0$. Weaker decay rates, for instance $t^{-\frac{1}{4}}$, are excluded by the non-degeneracy condition (2.10). We define the set of critical wave numbers $\mathbf{\Theta}_{\mathrm{cr}}$ and the maximal wave speed $c_{\max}$ via

$$\mathbf{\Theta}_{\mathrm{cr}} := \left\{ \hat{\theta} \in \mathbb{S}^1 \mid \tilde{\omega}''(\hat{\theta}) = 0 \right\} \quad \text{and} \quad c_{\max} := \max\left\{ |\tilde{\omega}'(\theta)| \,\middle|\, \theta \in \mathbb{S}^1 \right\}.$$

Note that $0 \in \mathbf{\Theta}_{\mathrm{cr}}$ and, since K in (2.2) is finite, the set $\mathbf{\Theta}_{\mathrm{cr}}$ is discrete. In this paper we in particular aim to understand the cross-over between the two different decay rates, i.e. the behavior for $c \in \mathcal{C}_{\mathrm{cr}}(\varepsilon)$ with

$$\mathcal{C}_{\mathrm{cr}}(\varepsilon) := \bigcup_{\hat{\theta}\in\mathbf{\Theta}_{\mathrm{cr}}} \left[c_{\mathrm{gr}}(\hat{\theta}){-}\varepsilon\,,\, c_{\mathrm{gr}}(\hat{\theta}){+}\varepsilon \right] \subset \mathbb{R}.$$

In the upcoming sections two other sets will be relevant. For the sake of completeness we define these already here, where $\mathcal{U}_\delta(\hat{\theta}) = \left\{ \theta \in \mathbb{S}^1 \,\middle|\, |\theta{-}\hat{\theta}| < \delta \right\}$:

$$\mathbf{\Theta}_{\mathrm{cr}}(\delta) := \bigcup_{\hat{\theta}\in\mathbf{\Theta}_{\mathrm{cr}}} \overline{\mathcal{U}_\delta(\hat{\theta})} \subset \mathbb{S}^1, \quad \mathbf{\Theta}_{\mathrm{cr}}(c,\varepsilon) := \left\{ \hat{\theta} \in \mathbf{\Theta}_{\mathrm{cr}} \,\middle|\, \left|c_{\mathrm{gr}}(\hat{\theta}) - c\right| \leq \varepsilon \right\}$$

The first is analogous to $\mathcal{C}_{\mathrm{cr}}(\varepsilon)$, but in terms of the wave numbers and the $\mathbf{\Theta}_{\mathrm{cr}}(c,\varepsilon)$ characterizes critical wave numbers corresponding to group velocities in a neighborhood of a given group velocity c.

In Figures 2.1 and 2.2 we plot two dispersion relations and associated solutions $r_j(t)$ for different times to display the influence of the critical wave numbers $\hat{\theta}_j \in \mathbf{\Theta}_{\mathrm{cr}}$. As a consequence of the Galilean invariance the wave number $\theta = 0$ is outstanding in two respects (cf. (2.11)). First, as already mentioned above, $0 \in \mathbf{\Theta}_{\mathrm{cr}}$ such that there are always wave fronts traveling with speed $c = \pm\tilde{\omega}'(0)$ and second, since $\tilde{\omega}(0) = 0$, these wave front are monotone. We will prove this monotonicity of the front in Section 4. The latter holds for all $t > 0$ (not only in the limit $t \to \infty$) if $\tilde{\omega}'(0) = \max_{\theta\in\mathbb{S}^1} \tilde{\omega}'(\theta) := c_{\max}$, i.e. the sonic velocity, which is not satisfied in general for $K > 1$.

The tool to derive upper bound on (2.15) is van der Corput's lemma, see e.g. [15]. It states that if $\left|\phi^{(k)}(\theta)\right| \geq \lambda > 0$ for $\theta \in (\underline{\theta}, \overline{\theta})$ where either $k \geq 2$, or $k = 1$ and ϕ' is monotonic, then

$$\left| \int_{\underline{\theta}}^{\overline{\theta}} A(\theta) \mathrm{e}^{\mathrm{i}t\phi(\theta)} \, \mathrm{d}\theta \right| \; \leq \; C(k, A) \, (\lambda t)^{-\frac{1}{k}}$$

with $C(k, A) = (5{\cdot}2^{k-1}{-}2) \left(\max_{\theta \in [\underline{\theta}, \overline{\theta}]} |A(\theta)| + \int_{\underline{\theta}}^{\overline{\theta}} |A'(\theta)| \mathrm{d}\theta \right)$. Based on this a global bound on $g(t, c)$ is straightforward, cf. [12, Lemma 3.5]: There exists a constant $C(\tilde{\omega}, A) > 0$ depending on the dispersion relation $\tilde{\omega}$ and A such that

$$\forall\, t \geq 0,\; c \in \mathbb{R}: \quad |g(t, c)| \leq \frac{C(\tilde{\omega}, A)}{(1 + t)^{\frac{1}{3}}}.$$

Similarly, it is possible to obtain an upper bound of order $t^{-\frac{1}{2}}$ for c bounded away from group velocities corresponding to critical wave numbers, i.e. for $c \notin \mathcal{C}_{\mathrm{cr}}(\varepsilon)$ with $\varepsilon > 0$. But a careful application of van der Corput's lemma gives an upper bound for the singularity occurring as $c \to \tilde{\omega}'(\hat{\theta})$, $\hat{\theta} \in \mathbf{\Theta}_{\mathrm{cr}}$. Namely, according to [12, Lemma 3.6] we can choose $\varepsilon = t^{-\frac{2}{3}}$, and there exists again a constant $\tilde{C}(\tilde{\omega}, A) > 0$ such that

$$\forall\, t {\geq}\, 0 \;\forall\, c {\in} \mathbb{R} \backslash \mathcal{C}_{\mathrm{cr}}(t^{-\frac{2}{3}}): \quad \big|g(t, c)\big| {\leq} \frac{\tilde{C}(\tilde{\omega}, A)}{(1{+}\,t)^{\frac{1}{2}}} \left(1 {+} \sum_{\hat{\theta} \in \mathbf{\Theta}_{\mathrm{cr}}} \frac{1}{|\tilde{\omega}'(\hat{\theta})^2 {-} c^2|^{\frac{1}{4}}} \right). \tag{2.16}$$

Combining both upper bounds suggests that the wave fronts with decay $t^{-\frac{1}{3}}$ expand like $t^{\frac{1}{3}}$ (in terms of $r_j(t)$). In Section 3 we will see that this indeed holds and furthermore that (2.16) is sharp.

The third important decay estimate concerns the exponential decay of $g(t, c)$ for $|c| > c_{\max}$. Thus, ignoring exponentially small tails, this is equivalent to a finite sonic velocity. Although we suppose that the result is known we give the statement and the full proof since we were not able to find a citable reference. Only in [5] there is a similar statement for the case of pure nearest-neighbor interaction (with a different proof), but unfortunately this work is unpublished. For this result we choose $\gamma > 0$ such that ω_{r} can be holomorphically extended on the comples stripe $\{\theta \in \mathbb{C} \,|\, 0 \leq \mathfrak{Im}\, \theta \leq \gamma\}$ such that $\omega_{\mathrm{r}}(\theta) > 0$ still holds.

Proposition 2.3. *Consider the Green's function defined in* (2.9) *with dispersion relation* ω *satisfying the stability and non-degeneracy condition* (2.3) *and* (2.10), *respectively. Then, for all* $t > 0$ *and all* $c = \frac{j}{t} \in \mathbb{R} \setminus [-c_{\max}, c_{\max}]$ *holds*

$$|G(t, c)| \leq \begin{pmatrix} 1 & \frac{1}{\underline{\omega}_*(c)}\, \mathrm{e}^{-\frac{\mathrm{sign}(c)\alpha_*(c)}{2}} \\ \overline{\omega}_*(c)\, \mathrm{e}^{\frac{\mathrm{sign}(c)\alpha_*(c)}{2}} & 1 \end{pmatrix} \mathrm{e}^{-\kappa(c)t} \tag{2.17a}$$

$$\textit{with} \quad \kappa(c) := \max_{\alpha\in[0,\gamma]} \Big(\alpha|c| - \max_{\theta\in[-\pi,\pi]} \mathfrak{Im}\,\tilde\omega(2\theta+\mathrm{i}\alpha) \Big) > 0, \qquad (2.17\mathrm{b})$$

$\underline{\omega}_*(c) := \min_{\theta\in[-\pi,\pi]} |\omega_\mathrm{r}(2\theta+\mathrm{i}\alpha_*(c))|$, *and* $\overline{\omega}_*(c) := \max_{\theta\in[-\pi,\pi]} |\omega_\mathrm{r}(2\theta+\mathrm{i}\alpha_*(c))|$, *where* $\alpha_*(c)$ *maximizes the term defining* $\kappa(c)$.

Proof. According to (2.14) the components of $G_j(t)$ are given by

$$g_\pm(t,c) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\theta)\mathrm{e}^{\mathrm{i}t(2\theta c\pm\tilde\omega(2\theta))}\,\mathrm{d}\theta \quad \text{with} \quad c = \frac{j}{t},$$

in which $A(\theta) = 1$, $\frac{1}{\omega_\mathrm{r}(2\theta)}\mathrm{e}^{\mathrm{i}\theta}$, or $\omega_\mathrm{r}(2\theta)\mathrm{e}^{-\mathrm{i}\theta}$. In all cases $A(\theta)$ is analytic and 2π-periodic.

To prove the statement for $c > c_{\max}$ we consider $g_+(t,c)$ and continue $\tilde\omega$ and A analytically to $[-\pi,\pi]+\mathrm{i}\,[0,\gamma]$. Applying Cauchy's integral theorem we get for $\alpha > 0$

$$\int_{-\pi}^{\pi} A(\theta)\mathrm{e}^{\mathrm{i}t(2\theta c+\tilde\omega(2\theta))}\,\mathrm{d}\theta = \Big(\int_{-\pi}^{-\pi+\mathrm{i}\alpha} + \int_{-\pi+\mathrm{i}\alpha}^{\pi+\mathrm{i}\alpha} + \int_{\pi+\mathrm{i}\alpha}^{\pi} \Big) A(\theta)\mathrm{e}^{\mathrm{i}t(2\theta c+\tilde\omega(2\theta))}\,\mathrm{d}\theta.$$

Exploiting the symmetry of $A(\theta)$ we find

$$\begin{aligned}
&\int_{-\pi}^{-\pi+\mathrm{i}\alpha} A(\theta)\mathrm{e}^{\mathrm{i}t(2\theta c+\tilde\omega(2\theta))}\,\mathrm{d}\theta + \int_{\pi+\mathrm{i}\alpha}^{\pi} A(\theta)\mathrm{e}^{\mathrm{i}t(2\theta c+\tilde\omega(2\theta))}\,\mathrm{d}\theta \\
&= 2\int_0^{\alpha} A(\pi+\mathrm{i}\alpha)\sin(2\pi ct)\mathrm{e}^{(-2yc+\mathrm{i}\tilde\omega(2\pi+2\mathrm{i}y))t}\,\mathrm{d}y \\
&= 0,
\end{aligned}$$

since $c = \frac{j}{t}$. For the remaining integral holds

$$\begin{aligned}
\Big| \int_{-\pi+\mathrm{i}\alpha}^{\pi+\mathrm{i}\alpha} A(\theta)\mathrm{e}^{\mathrm{i}t(2\theta c+\tilde\omega(2\theta))}\,\mathrm{d}\theta \Big| &= \mathrm{e}^{-2\alpha ct} \Big| \int_{-\pi}^{\pi} A(\theta+\mathrm{i}\alpha)\mathrm{e}^{\mathrm{i}t(2\theta c+\tilde\omega(2\theta+2\mathrm{i}\alpha))}\,\mathrm{d}\theta \Big| \\
&\le 2\pi\mathrm{e}^{-2\alpha ct} \max_{\theta\in[-\pi,\pi]} |A(\theta+\mathrm{i}\alpha)|\,\mathrm{e}^{t\max_{\theta\in[-\pi,\pi]}\mathfrak{Im}\,\tilde\omega(2\theta+2\mathrm{i}\alpha)}.
\end{aligned}$$

Now we choose α such that the overall exponent is minimized and set $\alpha_* = 2\alpha$ to obtain

$$|g_+(t,c)| \le \max_{\theta\in[-\pi,\pi]} \Big| A\Big(\theta+\mathrm{i}\frac{\alpha_*}{2}\Big) \Big|\ \mathrm{e}^{-\kappa(c)t}$$

with $\kappa(c)$ defined in (2.17b) which proves (2.17a). To see $\kappa(c) > 0$ for $c > c_{\max}$ note that $\max_{\theta\in[-\pi,\pi]} \mathfrak{Im}\,\tilde\omega(2\theta+\mathrm{i}\alpha) = c_{\max}\alpha + \mathcal{O}(\alpha^3)$ as $\alpha \to 0$.

To prove the statements for $c < -c_{\max}$ we consider the representation

$$g_-(t,c) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A(\theta)\mathrm{e}^{\mathrm{i}t(2\theta(-c)+\tilde\omega(2\theta))}\,\mathrm{d}\theta$$

with $A(\theta) = 1$, $\frac{1}{\omega_\mathrm{r}(2\theta)}\mathrm{e}^{-\mathrm{i}\theta}$, or $\omega_\mathrm{r}(2\theta)\mathrm{e}^{\mathrm{i}\theta}$. Thus, the proof follows by the same arguments. □

Solving condition (2.17b) locally, the decay of solutions in terms of $c - c_{\max}$ near the wave fronts becomes evident: Consider $0 \le c - c_{\max} \le \varepsilon$ and $0 < \alpha \le \alpha_0$ for $\varepsilon, \alpha_0 > 0$ sufficiently small. Then there exists $B = B(\tilde{\omega}, \alpha_0) > 0$ such that

$$\begin{aligned}\kappa(c) &\ge c\alpha - c_{\max}\alpha - B(\tilde{\omega}, \alpha_0)\alpha^3 \\ &= (1-\xi)(c-c_{\max})\alpha + (1-\xi)(c-c_{\max})\alpha - B(\tilde{\omega}, \alpha_0)\alpha^3\end{aligned}$$

with $0 < \xi < 1$. Now we claim $(1-\xi)(c-c_{\max})\alpha - B(\tilde{\omega}, \alpha_0)\alpha^3 = 0$. This implies $\alpha = \left(\frac{\varepsilon(c-c_{\max})}{B(\tilde{\omega},\alpha_0)}\right)^{\frac{1}{2}}$. Thus, for $0 \le c - c_{\max} \le \varepsilon$ with $\varepsilon > 0$ sufficiently small, there exists $\tilde{\kappa}$ such that

$$\forall\, t \ge 0: \quad |G(t,c)| \le C_{\omega,A}\mathrm{e}^{-\tilde{\kappa}(c-c_{\max})^{-\frac{3}{2}}\, t}. \tag{2.18}$$

3. Uniform asymptotic behavior

In this section we first state the main results. Namely, the uniform asymptotic expansions of solutions of (2.5) near wave fronts where the decay rate is $\sim t^{-\frac{1}{3}}$ and in the inner regions where we have a decay in time $\sim t^{-\frac{1}{2}}$. Here the leading order behavior as well as the Airy-like wave fronts are well known. Our contribution is to make the dependency on $c - c_{\mathrm{gr}}(\hat{\theta})$ for $\hat{\theta} \in \boldsymbol{\Theta}_{\mathrm{cr}}$ more explicit. Afterwards we discuss the results, in particular the cross-over between the different scales and give illustrating examples. The actual proofs are shifted out to Section 4.

3.1. Asymptotic expansions. Now we are in place to state the main results. The first concerns the nondegenerate case, namely $c \in [-c_{\max}, c_{\max}] \setminus \mathcal{C}_{\mathrm{cr}}(\varepsilon)$ for some $\varepsilon > 0$. We emphasize that the obtain we obtain will blow up, since the expansion becomes singular as $c \to c_{\mathrm{gr}}(\hat{\theta})$ for $\hat{\theta} \in \boldsymbol{\Theta}_{\mathrm{cr}}$. It would be interesting to keep track of the blowup behavior to optimize the overall error. However we are content with some, not necessarily optimal bound.

Theorem 3.1 (Asymptotic behavior in the nondegenerate case)**.** *Consider Green's function of the Hamiltonian system* (2.5) *and assume that the corresponding dispersion relation* ω *defined in* (2.4) *satisfies the non-degeneracy condition* (2.10). *Then, for all* $\varepsilon > 0$ *there exists a constant* $C_{\mathrm{nd}}(\varepsilon, \omega) > 0$ *such that for all* $c \in [-c_{\max}, c_{\max}] \setminus \mathcal{C}_{\mathrm{cr}}(\varepsilon)$ *and* $t > 0$ *we have the estimate*

$$|G(t,c) - \mathcal{G}_{\mathrm{non}}(t,c)| \le C_{\mathrm{nd}}(\varepsilon, \omega)\, t^{-1} \quad \textit{with} \tag{3.1a}$$

$$\mathcal{G}_{\mathrm{non}}(t,c) = \sum_{\theta\in\theta(c)} \frac{1}{\sqrt{8\pi|\omega''(\theta)|}} \begin{pmatrix} 1 & \frac{1}{\omega_{\mathrm{r}}(\theta)} \\ \omega_{\mathrm{r}}(\theta) & 1 \end{pmatrix} \cos\!\left[(\omega(\theta)+c\theta)t + \tfrac{\pi}{4}\operatorname{sign}\omega''(\theta)\right] t^{-\frac{1}{2}}. \tag{3.1b}$$

Indeed, for the constant in (3.1a) we will show the upper bound

$$C_{\rm in}(\varepsilon,\omega) = C_1(\omega) + C_2(\omega)\varepsilon^{-\frac{5}{2}}$$

as $\varepsilon \to 0$. Note furthermore that the coefficient in (3.1b) becomes unbounded in that case. According to Lemma 4.1 holds $\sqrt{|\omega''(\theta)|} \sim \varepsilon^{\frac{1}{4}}$ as $\theta \to \hat\theta \in \mathbf{\Theta}_{\rm cr}$.

Concerning the second result, the challenge is that the dispersion relation degenerates. Thus, in the general case the implicit function theorem is not sufficient. Here a suitable version of Weierstraß' preparation theorem is necessary. In this context a classical statement is the following.

Theorem 3.2 (Method of stationary phase, cf. [8, Theorem 7.7.18]). *Let ϕ be a real-valued C^∞ function of $(\theta, y) \in \mathbb{R}^{1+n}$ near 0 such that*

$$\phi(0,0) = \partial_\theta\phi(0,0) = \partial_\theta^2\phi(0,0) = 0 \quad \textit{and} \quad \partial_\theta^3\phi(0,0) \neq 0.$$

Then there exist C^∞ real-valued functions $a(y)$ and $b(y)$ near 0 such that $a(0) = b(0) = 0$ and

$$\begin{aligned}&\int A(\theta,y)\mathrm{e}^{\mathrm{i}t\phi(\theta,y)}\,\mathrm{d}\theta\\ &\sim \mathrm{e}^{\mathrm{i}tb(y)}\Big(\mathrm{Ai}\big(a(y)t^{\frac{2}{3}}\big)t^{-\frac{1}{3}}\sum_{0\le j\le\infty}u_{0,j}(y)t^{-j} + \mathrm{Ai}'\big(a(y)t^{\frac{2}{3}}\big)t^{-\frac{2}{3}}\sum_{0\le j\le\infty}vu_{1,j}(y)t^{-j}\Big),\end{aligned} \tag{3.2}$$

provided that $A \in \mathrm{C}_0^\infty$ and supp A *is sufficiently close to* 0. *Here $u_{0,j}$, $u_{1,j} \in \mathrm{C}_0^\infty$.*

Here $\mathrm{Ai}(\cdot)$ denotes the Airy function defined by $\mathrm{Ai}(z) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{e}^{i(\frac{1}{3}u^3+zu)}\mathrm{d}z$, see for instance [13] for basic properties. Figure 3.1 show wave fronts together with the correspondingly scale Airy function.

There remain two questions. First, the result does neither state the functions a, b, $u_{0,j}$ and $u_{1,j}$ nor the error bounds in terms explicitly. Second, these functions are not determined uniquely, i.e. the theorem does not state that the asymptotic series (3.2) is unique. We will make these functions and the error estimates for this expansion more explicit in Subsection 4.3. For this we give a short summary on a suitable version of Weierstraß' preparation theorem and explain that the functions a and b can be made more explicit. This allows us to estimate the remainder terms explicitly.

For the following result we recall that that for a degenerate wave number $\theta_{\rm cr}$ there may be other nondegenerate wave numbers θ having the same group velocity, namely $c_{\rm gr}(\theta_{\rm cr}) = c_{\rm gr}(\theta)$. Thus, an Airy-type degenerate behavior may be superimposed by a nondegenerate harmonic wave train. This is expressed in the following result by the additive structure $\mathcal{G}_{\rm dg} + \mathcal{G}_0$ and can be see in the example displayed in Figure 2.2.

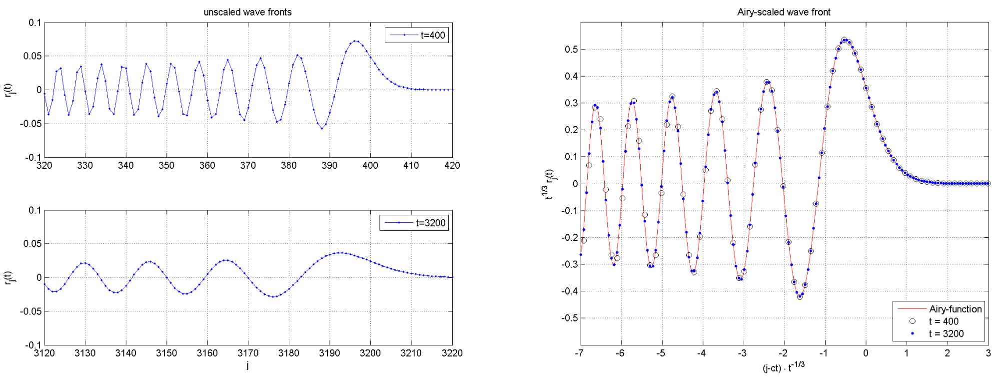

Figure 3.1: Wave fronts for the linearized FPU chain in comparison with the corresponding Airy function approximation.

Theorem 3.3 (Asymptotic behavior near degenerate points)**.** *Consider Green's function of the Hamiltonian system* (2.5) *and assume that the corresponding dispersion relation* ω *defined in* (2.4) *satisfies the non-degeneracy condition* (2.10)*. Then, there exists* $\varepsilon_0 = \varepsilon_0(\tilde{\omega}) > 0$, $\delta_0 = \delta_0(\tilde{\omega}) > 0$ *and* $C_{\rm dg}(\tilde{\omega}) > 0$ *such that for all* $c \in \mathcal{C}_{\rm cr}(\varepsilon_0)$

$$\left|G(t,c) - \mathcal{G}_{\rm dg}(t,c) - \mathcal{G}_0(t,c)\right| \le C_{\rm dg}(\omega)t^{-1} \quad \textit{with} \tag{3.3a}$$

$$\mathcal{G}_{\rm dg}(t,c) = \sum_{\hat{\theta}\in\mathbf{\Theta}_{\rm cr}(c,\varepsilon_0)} \Big(\mathbb{A}_{\hat{\theta}}(c)\cos\big([\omega(\hat{\theta}){-}\hat{c}\hat{\theta}{-}b_{\hat{\theta}}(c)]t\big)\,\mathrm{Ai}\left(a_{\hat{\theta}}(c)t^{\frac{2}{3}}\right)t^{-\frac{1}{3}} \\ - \mathbb{B}_{\hat{\theta}}(c)\sin\big([\omega(\hat{\theta}){-}\hat{c}\hat{\theta}{-}b_{\hat{\theta}}(c)]t\big)\,\mathrm{Ai}'\left(a_{\hat{\theta}}(c)t^{\frac{2}{3}}\right)t^{-\frac{2}{3}}\Big), \tag{3.3b}$$

where $\hat{c} = \omega'(\hat{\theta})$, *and*

$$\mathcal{G}_0(t,c) \\ = \sum_{\substack{\theta\in\mathbf{\Theta}(c)\ \text{and}\\ \mathrm{dist}(\theta,\mathbf{\Theta}_{\rm cr}(c,\varepsilon_0))>\delta_0}} \frac{1}{\sqrt{8\pi|\omega''(\theta)|}}\begin{pmatrix} 1 & \frac{1}{\omega_{\rm r}(\theta)} \\ \omega_{\rm r}(\theta) & 1\end{pmatrix}\cos\Big[(\omega(\theta){-}c\theta)t{+}\tfrac{\pi}{4}\,\mathrm{sign}\,\omega''(\theta)\Big]t^{-\frac{1}{2}}. \tag{3.3c}$$

The scalar-valued functions $a_{\hat{\theta}}$, $b_{\hat{\theta}}$ *as well as the* $\mathbb{R}^{2\times 2}$*-valued functions* $\mathbb{A}_{\hat{\theta}}$ *and* $\mathbb{B}_{\hat{\theta}}$ *are real-analytic on* $[\hat{c}{-}\varepsilon_0, \hat{c}{+}\varepsilon_0]$*. Furthermore, for* $(c{-}\omega'(\hat{\theta}))\omega'''(\hat{\theta}) > 0$ *we have*

$$a_{\hat{\theta}}(c) = \left(\tfrac{3}{4}[\varsigma_-(c){-}\varsigma_+(c)]\right)^{\frac{2}{3}} = -\sqrt[3]{\frac{2}{\tilde{\omega}'''(\hat{\theta})}}\,(c{-}\hat{c}) + \mathcal{O}\left((c{-}\hat{c})^2\right) \tag{3.4}$$

$$b_{\hat{\theta}}(c) = \tilde{\omega}(\hat{\theta}) - \hat{c}\hat{\theta} - \tfrac{1}{2}[\varsigma_-(c) + \varsigma_+(c)] = 2\hat{\theta}(c{-}\hat{c}) + \mathcal{O}\left((c{-}\hat{c})^2\right), \tag{3.5}$$

where $\varsigma_\pm(c) := \tilde{\omega}(\theta_\pm(c)) - c\theta_\pm(c)$ *and* $\theta_+(c), \theta_-(c) \in \mathcal{U}_{\delta_0}(\hat{\theta})$ *are the two wave numbers such that* $c = \tilde{\omega}'(\theta_\pm(c))$ *and* $\theta_-(c) < \theta_+(c)$*. Moreover, we have*

$$\mathbb{A}_{\hat{\theta}}(c) = \frac{1}{\sqrt[3]{4|\tilde{\omega}'''(\hat{\theta})|}}\begin{pmatrix} 1 & \frac{1}{\omega_{\rm r}(\hat{\theta})} \\ \omega_{\rm r}(\hat{\theta}) & 1\end{pmatrix} + \mathcal{O}(c{-}\hat{c}).$$

Note that the error bound $C_{\rm dg}(\omega)$ depends on ω only. If c is near the sonic wave speed $c_{\max}$ the nondegenerate part $\mathcal{G}_0$ vanishes and the sum in (3.3b) reduces to one term (or in the degenerated case of several wave numbers traveling with the same sonic wave speed to just as many terms). Furthermore in case $\hat\theta = 0 \bmod \pi$ we have $b_{\hat\theta} \equiv 0$.

3.2. Full approximation and crossover. We are now in the position to define the full expansion that gives rise to the uniform approximation result stated in Theorem 1.1. We simply choose the ε_0 according to Theorem 3.3 and define $\mathcal{G}^{\rm expan}$ via

$$\mathcal{G}^{\rm expan}(t,c) = \begin{cases} \mathcal{G}_{\rm non}(t,c) & \text{for } c \in [-c_{\max}, c_{\max}] \setminus \mathcal{C}_{\rm cr}(\varepsilon_0), \\ \mathcal{G}_{\rm deg}(t,c) + \mathcal{G}_0(t,c) & \text{for } c \in \mathcal{C}_{\rm cr}(\varepsilon_0), \\ 0 & \text{for } |c| > c_{\max} + \varepsilon_0. \end{cases} \tag{3.6}$$

Combining Proposition 2.3 and the Theorems 3.1 and 3.3 we have established the main Theorem 1.1 on the uniform approximation of the exact Green's function by $\mathcal{G}^{\rm expan}$ up to a uniform error of order $\mathcal{O}(\frac{1}{t})$.

Now we discuss the cross-over between the three different regions of definition for the function $\mathcal{G}^{\rm expan}$. We first look at the expansions for $|c| > c_{\max}$. For all such c we have exponential decay for $t \to \infty$, by Proposition 2.3. Thus, replacing this function by 0 will certainly keep the error order $\mathcal{O}(\frac{1}{t})$.

Second we look at the overlap between the degenerate and the nondegenerate case inside $(-c_{\max}, c_{\max})$. Certainly we can make ε_0 for the degenerate as large as possible and make ε for the nondegenerate much smaller. Then we have an overlap of the two domains where the expansions are valid. Since both approximations decay slower that $\frac{1}{t}$, the theory can only be consistent if the error between the two approximations is at most $\mathcal{O}(\frac{1}{t})$. To understand the cross-over between these to regimes explicitly, we consider $\hat\theta \in \boldsymbol{\Theta}_{\rm cr}$ with $\tilde\omega'''(\hat\theta) > 0$, i.e. a wave front traveling to $j \to +\infty$ as $t \to \infty$. The leading order term of (3.3b) is

$$g_{\rm deg}(t,c) = \mathbb{A}_{\hat\theta}(c) \cdot \cos\Big(\big[\omega(\hat\theta) - \hat c\hat\theta - b_{\hat\theta}(c)\big]t\Big) \cdot \mathrm{Ai}\left(a_{\hat\theta}(c)t^{\frac{2}{3}}\right) t^{-\frac{1}{3}}.$$

We fix $c \in \tilde\omega'(S^1)$ such that $-\varepsilon_0 < c - \hat c < 0$ and aim to determine the behavior as $t \to \infty$. Recall the asymptotic behavior of Airy's function,

$$\mathrm{Ai}(z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \left(\cos\left(\tfrac{2}{3}z^{\frac{3}{2}} - \tfrac{\pi}{4}\right) + \mathcal{O}\left(z^{-\frac{3}{2}}\right)\right) \quad \text{as} \quad z \to -\infty.$$

Now, $z = a_{\hat\theta}(c)t^{\frac{2}{3}}$ and (3.4) imply $\frac{2}{3}z^{\frac{3}{2}} = \frac{1}{2}\big[\varsigma_-(c) - \varsigma_+(c)\big]$. With (3.5) we find

$$\cos\big([\omega(\hat\theta) - \hat c\hat\theta - b_{\hat\theta}(c)]t\big) \cos\left(\tfrac{2}{3}z^{\frac{3}{2}} - \tfrac{\pi}{4}\right) = \tfrac{1}{2}\cos\left(\varsigma_+(c)t - \tfrac{\pi}{4}\right) + \tfrac{1}{2}\cos\left(\varsigma_-(c)t + \tfrac{\pi}{4}\right).$$

Since the saddle point of order two in $\hat{\theta}$ splits up into the two saddle points $\theta_{\pm}(c)$ of order one, we have two oscillating terms; each corresponding to one term in (3.1b).

Concerning the amplitude we expand $\tilde{\omega}''(\theta)$ and $c = \tilde{\omega}'(\theta)$ in $\hat{\theta}$, which implies $\tilde{\omega}''(\theta) = 2\tilde{\omega}'''(\hat{\theta})(\hat{c} - c) + \mathcal{O}\left((\hat{c} - c)^{\frac{2}{3}}\right)$. Thus, using the second representation of $a_{\hat{\theta}}$ from (3.4) yields

$$\frac{1}{\sqrt{\pi}}\mathbb{A}_{\hat{\theta}}(c)z^{-\frac{1}{4}}t^{-\frac{1}{3}} = \frac{1}{\sqrt{2\pi|\tilde{\omega}''(\theta)|}}\begin{pmatrix} 1 & \frac{1}{\omega_{\mathrm{r}}(\hat{\theta})} \\ \omega_{\mathrm{r}}(\hat{\theta}) & 1 \end{pmatrix} t^{-\frac{1}{2}} + \mathcal{O}(c-\hat{c}).$$

Combining this with the oscillating terms we obtain two terms of (3.1b).

4. Asymptotic expansions - proofs

This section is dedicated to the proofs of the asymptotic expansions stated in the last section, namely to Theorem 3.3 and 3.1. Before starting the actual work, we outline the general strategy followed in Subsections 4.2 and 4.3 to derive the asymptotic expansions.

4.1. General strategy for the proofs. Consider an oscillatory integral of the form

$$g(t, c) = \int_{\mathcal{S}^1} A(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta \tag{4.1}$$

We aim to derive an asymptotic expansion which holds (locally) uniformly with respect to c, for instance for $c \in [c_*-\varepsilon, c_*+\varepsilon]$. The procedure splits into the following four steps.

Localization principle. As mentioned in Subsection 2.2, the asymptotic behavior of (4.1) for $t \to \infty$ is dominated by the wave numbers $\{\theta_1, \ldots, \theta_K\} = \mathbf{\Theta}(c_*)$ which fulfill the condition $\partial_\theta\phi(\theta_k, c_*) = 0$. This is due to fact to which [15, Chap. VIII] refers as the first principle of oscillatory integrals, namely the localization principle. Here the underlying idea is that, if $|\partial_\theta\phi(\theta_k, c)|$ is uniformly bounded from below on $[\underline{\theta}, \overline{\theta}]$, we may apply partial integration to obtain

$$\left|\int_{\underline{\theta}}^{\overline{\theta}} A(\theta)\,\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right| = \left|\frac{A(\theta)}{\mathrm{i}t\partial_\theta\phi(\theta,c)}\right|_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}}\partial_\theta\left(\frac{A(\theta)}{\mathrm{i}t\partial_\theta\phi(\theta,c)}\right)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right| \le \frac{C}{t}.$$

If the boundary terms cancel due to periodicity or compact support we may iterate the argument to obtain bounds $\sim t^{-N}$. To utilize this fact for (4.1) we use a partition of unity $\{\psi_1, \ldots, \psi_K, 1-\sum_k \psi_k\}$ on $\mathcal{S}^1$ with $\psi_k \in \mathrm{C}_0^k(\mathcal{U}_\delta(\theta_k))$ for some $\delta_k > 0$ which in general might depend on k. Then asymptotic behavior is dominated by terms of the form

$$I_k(t,c) = \int_{\theta_k-\delta}^{\theta_k+\delta} A(\theta)\psi_k(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta. \tag{4.2}$$

For the localization error holds

$$\big|R_{\mathrm{loc}}(t,c)\big| = \left|\int_{S^1} A(\theta)\Big[1-\sum_k \psi_k(\theta)\Big]\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta\right| \le C\Big(\omega,\varepsilon,\min_k \delta_k\Big)t^{-N}$$

for some $N \in \mathbb{N}$. The bound $C(\omega,\varepsilon,\min_k \delta_k)$ depends on δ_k via $\|\psi_k\|_{W^{n,p}}$.

Local coordinate transformation. The next step consists in rewriting (4.2) using a suitable coordinate transform $u = U(\theta,c)$. In the simplest case with $|\partial_\theta^2\phi(\theta_k,c)| = |\omega''(\theta_k)| \neq 0$, we may use the implicit function theorem to see that locally we can achieve $\phi(\theta,c) = \pm(u^2+\phi(\theta_k,c))$. This applies in case of the inner regions, cf. Subsection 4.2. At the wave fronts the dispersion relation degenerates: For $\hat\theta \in \mathbf{\Theta}_{\mathrm{cr}}$ and $\hat c = c_{\mathrm{gr}}(\hat\theta)$ holds $|\partial_\theta^2\phi(\hat\theta,\hat c)| = |\omega''(\hat\theta)| = 0$ but $|\partial_\theta^2\phi(\theta,c)| = |\omega''(\theta)| \neq 0$ for arbitrary c (in a sufficiently small neighborhood) if $\theta \neq \hat\theta$. In that case we may apply a suitable version of Weierstraß' preparation theorem which is derived in Appendix B. For instance, if $|\partial_\theta^3\phi(\theta_k,c)| = |\omega'''(\theta_k)| \neq 0$ we obtain $\phi(\theta,c) = \frac{\sigma}{3}u^3 + a(c,\theta_k)u + b(c,\theta_k)$. The asymptotic behavior of (4.1) in this case is discussed in detail in Subsection 4.3. Since the preparation theorems are quite general the theory also applies for higher order of degeneracy.

In any case we may substitute $\theta = \Theta(u,c) := U^{-1}(u,c)$ in (4.2) such that

$$I_k(t,c) = \int_{-B}^{B} f(u,c)\,\mathrm{e}^{\mathrm{i}tp(u,c,\theta_k)}\,\mathrm{d}u$$

with $f(u) = A\circ\Theta(u,c)\cdot\psi_k\circ\Theta(u,c)\cdot\partial_u\Theta(u,c)$ and $p(\cdot,c,\theta_k)$ is a polynomial in a suitable normal form.

Actual decay rate. In this step the actual leading order term of the asymptotic expansion is derived by tracing back $I_k(t,c)$ to special functions. In the non-degenerated case we obtain integral of Fresnel-type, in the degenerated case we obtain Airy's function. In any case we basically obtain

$$I_k(t,c) = \mathcal{G}(t,c)\,t^{-\frac{1}{n}} + R_k(t,c) \quad \text{with} \quad n = 2 \text{ or } 3.$$

Estimation of the error terms. It remains to estimate the error term. We will prove that

$$|R_k(t,c)| \le C(\omega,\varepsilon)\, t^{-1}.$$

In general, this might not be optimal with respect to the decay in t. But we focus on the dependency which actually is more complicated. In fact, in the non-degenerated case C, as well as $\mathcal{G}$, becomes singular as $\varepsilon \to 0$. In this step we finally fix $\delta_k = \delta_k(\varepsilon)$. Thus, combining all estimates we end up with

$$\left| g(t,c) - \mathcal{G}(t,c)\, t^{-\frac{1}{n}} \right| \le C(\omega,\varepsilon)\, t^{-1}.$$

4.2. Asymptotic expansion in the nondegenerate case. In this section we proof Theorem 3.1. We follow the strategy outlined in Subsection 4.1. The proof splits into two parts. First we apply standard method of stationary state, cf. for instance [15, 17] to derive the leading order term if the asymptotic expansion. Second we derive a uniform upper bound on the error term when the expansion becomes singular as $\varepsilon \to 0$.

We consider the Green's function represented by (2.14) and choose a parametrization of $\mathbb{S}^1$ on $(2\pi, 2\pi]$, i.e. the components of $G(t,c)$ are given by

$$g(t,c) = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} A(\theta)\, \mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\, \mathrm{d}\theta \quad \text{with} \quad \phi(\theta,c) = c\theta - \tilde{\omega}(\theta), \quad c = \frac{j}{t},$$

and $A(\theta) = 1$, $\frac{1}{\omega_{\mathrm{r}}(\theta)}\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}$, or $\omega_{\mathrm{r}}(\theta)\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}$.

Assume $\varepsilon > 0$ and $c \in \omega'(\mathbb{S}^1)\backslash\mathcal{C}_{\mathrm{cr}}(\varepsilon)$. We implicitly exclude the degenerated case $\omega'(\mathbb{S}^1)\backslash\mathcal{C}_{\mathrm{cr}}(\varepsilon) = \emptyset$ where the statement becomes meaningless, i.e. we assume $\varepsilon < 2c_{\max}$. Assume $\Theta(c) = \{\theta_1, \ldots, \theta_K\}$.

Localization. For localization to the relevant wave numbers consider $\delta > 0$ such that

$$\forall\, k_1 \neq k_2 \;:\quad \mathcal{U}_\delta(\theta_{k_1}) \cap \mathcal{U}_\delta(\theta_{k_2}) = \emptyset \tag{4.3}$$

Without loss of generality we may assume $\mathcal{U}_\delta(\theta_k) \subset (-2\pi, 2\pi]$ for all $k = 1, \ldots, K$. Otherwise we simply may shift the parametrization of S^1. We choose a smooth partition of unity $\{\psi_1, \ldots, \psi_K, 1 - \sum \psi_k\}$ on $(-2\pi, 2\pi]$ with $\psi_k \in \mathrm{C}_0^2(\mathcal{U}_\delta(\theta_k))$ and $\psi_k\big|_{\mathcal{U}_{\delta/2}(\theta_k)} = 1$. For later use we record that there exist $C_1, C_2 > 0$ such that

$$\|\psi_k\|_\infty = 1, \quad \|\psi_k'\|_\infty = \frac{C_1}{\delta}, \quad \|\psi_k''\|_\infty = \frac{C_2}{\delta^2}. \tag{4.4}$$

Now, the components of Green's function read as

$$g(t,c) = \frac{1}{4\pi}\sum_k \int_{\theta_k-\delta}^{\theta_k+\delta} \psi_k(\theta) A(\theta)\, \mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\, \mathrm{d}\theta + R_{\mathrm{loc}}(t,c)$$

with

$$R_{\rm loc}(t,c) = \frac{1}{4\pi}\int_{-2\pi}^{2\pi}\Big[1-\sum_k \psi_k(\theta)\Big]A(\theta)\,\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta. \qquad (4.5)$$

Obviously, the localization error is $\mathcal{O}(t^{-N})$ for all $N\in\mathbb{N}$. But here we want to point out the dependency of the bound on ε. We postpone the discussion. First we derive the leading order asymptotic behavior. To do so we consider

$$I(t,\theta_k) = \int_{\theta_k}^{\theta_k+\delta} \psi_k(\theta)A(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}. \qquad (4.6)$$

The discussion for $\int_{\theta_k-\delta}^{\theta_k}$ works analogous. Since from now on we will focus on on single integral of the this form, we omit the index k in ψ_k to simplify notation.

Local coordinate transform. We introduce a new variable u by

$$\phi(\theta,c) = -\operatorname{sign}\tilde\omega''(\theta_k)u^2 + \phi(\theta_k,c)$$

which defines the coordinate transform

$$U(\theta) = \sqrt{-\operatorname{sign}\tilde\omega''(\theta_k)\big(\phi(\theta,c)-\phi(\theta_k,c)\big)}.$$

The function $U(\theta)$ is smooth and monotone on $[\theta_k,\theta_{k+1})$ For later use we introduce

$$\begin{aligned} h(\theta,\theta_k) &= \frac{-2\operatorname{sign}\tilde\omega''(\theta_k)\big(\phi(\theta,c)-\phi(\theta_k,c)\big)}{(\theta-\theta_k)^2} - |\tilde\omega''(\theta_k)| \\ &= 2\operatorname{sign}\tilde\omega''(\theta_k)\sum_{n=1}^{\infty}\frac{\tilde\omega^{(n+2)}(\theta_k)}{(n+2)!}(\theta-\theta_k)^n \end{aligned}$$

such that we may write

$$U(\theta) = \tfrac{1}{\sqrt{2}}(\theta-\theta_k)\sqrt{|\tilde\omega''(\theta_k)|+h(\theta,\theta_k)}. \qquad (4.7)$$

Here, the analyticity of $\tilde\omega$ carries over to h. Moreover, note that $h(\theta,\theta_k) = \mathcal{O}(\theta-\theta_k)$ as $\theta\to\theta_k$ and $U'(\theta_k) = \sqrt{\frac{|\tilde\omega''(\theta_k)|}{2}}$.

Using the implicit function theorem we invert the coordinate transform U, i.e. $\theta = \Theta(u) := U^{-1}(u)$ and substitute the integration variable in (4.6). Since ψ is compactly supported we may replace the upper bound $U(\theta_k+\delta)$ by ∞ such that

$$I(t,\theta_k) = \mathrm{e}^{\mathrm{i}t\operatorname{sign}\tilde\omega''(\theta_k)\phi(c,\theta_k)}\int_0^\infty f(u)\mathrm{e}^{\mathrm{i}tu^2}\,\mathrm{d}u \qquad (4.8)$$

with $f(u) = A\circ\Theta(u)\cdot\psi\circ\Theta(u)\cdot\Theta'(u)$. Note that, via Θ and ψ, f depends on θ_k.

Actual decay rate. To derive the actual asymptotic expansion we use arguments based on partial integration. We basically follow the procedure presented in [17, Section II.3]. We define

$$K_0(u) = \mathrm{e}^{\mathrm{i}tu^2} \quad \text{and} \quad K_{n+1}(u) = -\int_u^{u+\infty \mathrm{e}^{\mathrm{i}\pi/4}} K_n(\tilde u)\,\mathrm{d}\tilde u.$$

Thus, applying partial integration to (4.8) leads to

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}t\,\mathrm{sign}\,\tilde\omega''(\theta_k)\phi(c,\theta_k)} I(t,\theta_k) &= f(u)K_1(u)\Big|_{u=0}^{\infty} - \int_0^\infty f'(u)K_1(u)\,\mathrm{d}u\\ &= -f(0)K_1(0) - \int_0^\infty f'(u)K_1(u)\,\mathrm{d}u.\end{aligned}$$

Using $f(0) = \psi(\theta_k)A(\theta_k)\Theta'(0) = A(\theta_k)\sqrt{\frac{|\tilde\omega''(\theta_k)|}{2}}$ and

$$K_1(0) = -\int_0^{\infty \mathrm{e}^{\mathrm{i}\pi/4}} \mathrm{e}^{\mathrm{i}tu^2}\,\mathrm{d}u = -\int_0^\infty \mathrm{e}^{-\frac{1}{2}u^2}\,\mathrm{d}u \cdot \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{2t}} = -\frac{\sqrt{\pi}}{2}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\,t^{-\frac{1}{2}}$$

we obtain the first order term of the asymptotic expansion,

$$I(t,\theta_k) = \mathrm{e}^{\mathrm{i}t\,\mathrm{sign}\,\tilde\omega''(\theta_k)\phi(c,\theta_k)} A(\theta_k)\sqrt{\frac{2\pi}{|\tilde\omega''(\theta_k)|}}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\,t^{-\frac{1}{2}} + R(t,\theta_k).$$

in which, after a second partial integration we find

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}t\,\mathrm{sign}\,\tilde\omega''(\theta_k)\phi(c,\theta_k)} R(t,\theta_k) &= -\int_0^\infty f'(u)K_1(u)\,\mathrm{d}u\\ &= -f'(0)K_2(0) + \int_0^\infty f''(u)K_2(u)\,\mathrm{d}u\end{aligned}$$

for the error term. Thus we get

$$|R(t,\theta_k)| \le \frac{1}{2}\left(|f'(0)| + \int_0^\infty |f''(u)|\,\mathrm{d}u\right) t^{-1}. \tag{4.9}$$

Here the right hand side depends on ψ and via A and the coordinate transform Θ on the dispersion relation $\tilde\omega$ on $[\theta_k, \theta_k{+}\delta]$.

It remains to determine how the right-hand side depends on ε, which will take the larger part of the proof. Preliminary for that consider first the simplest example, namely the case of pure nearest neighbor interaction. In that case we have $\tilde\omega(\theta) = \frac{1}{2}\sin(\frac{\theta}{2})$ and $\hat c = \pm 1$ which implies $|\tilde\omega''(\theta)| = \frac{1}{2}\sqrt{\hat c^2 - c^2} \sim \sqrt{\varepsilon}$. The next basic lemma states that this holds in general.

Lemma 4.1. *Consider the dispersion relation* $\tilde{\omega}$ *defined in* (2.13) *and assume that the non-degeneracy condition* (2.10) *is satisfied. Then there exists* $\delta_0 > 0$ *and constants* $\underline{C}(\tilde{\omega})$, $\overline{C}(\tilde{\omega}) > 0$ *such that with* $c = \tilde{\omega}'(\theta)$ *we have*

$$\forall\, \theta \in \bigcup_{\hat{\theta}\in\mathbf{\Theta}_{\mathrm{cr}}} \mathcal{U}_{\delta_0}(\hat{\theta}): \quad \underline{C}(\tilde{\omega})|c-\hat{c}|^{\frac{1}{2}} \le |\tilde{\omega}''(\theta)| \le \overline{C}(\tilde{\omega})|c-\hat{c}|^{\frac{1}{2}}. \tag{4.10}$$

Proof. According to the non-degeneracy condition we have $\tilde{\omega}'''(\hat{\theta}) \neq 0$ for all $\hat{\theta} \in \mathbf{\Theta}_{\mathrm{cr}}$. We define $\underline{\alpha} := \frac{1}{2}\min_{\hat{\theta}\in\mathbf{\Theta}_{\mathrm{cr}}} |\tilde{\omega}'''(\hat{\theta})|$ and $\overline{\alpha} := \max_{\theta\in\mathbb{S}^1} |\tilde{\omega}'''(\theta)|$. Since $\mathbf{\Theta}_{\mathrm{cr}}$ is finite, there exists $\delta_0 > 0$ such that

$$\forall\, \theta \in \bigcup_{\hat{\theta}\in\mathbf{\Theta}_{\mathrm{cr}}} \mathcal{U}_{\delta_0}(\hat{\theta}): \quad \underline{\alpha} \le |\tilde{\omega}'''(\theta)| \le \overline{\alpha}.$$

As from now we assume $\theta \in \mathcal{U}_{\delta_0}(\hat{\theta})$ for some $\hat{\theta} \in \mathbf{\Theta}_{\mathrm{cr}}$. Due to $\tilde{\omega}''(\hat{\theta}) = 0$ it is straight forward that

$$\underline{\alpha}|\theta-\hat{\theta}| \le |\tilde{\omega}''(\theta)| \le \overline{\alpha}|\theta-\hat{\theta}|.$$

Now we consider $c - \hat{c} = \tilde{\omega}'(\theta) - \tilde{\omega}'(\hat{\theta})$ and find $|c-\hat{c}| \le |\tilde{\omega}''(\theta)| \cdot |\theta-\hat{\theta}|$. Thus we obtain $|c-\hat{c}|^{\frac{1}{2}} \le \frac{1}{\sqrt{\underline{\alpha}}}|\tilde{\omega}''(\theta)|$ which proves the first estimate of (4.10) with $\underline{C}(\tilde{\omega}) = \sqrt{\underline{\alpha}}$. To prove the second estimate note that, possibly after decreasing δ_0, $|c-\hat{c}| \ge \frac{1}{2}\underline{\alpha}|\theta-\hat{\theta}|^2$ again holds for all $\theta \in \mathcal{U}_{\delta_0}(\hat{\theta})$. Thus we get $|\tilde{\omega}''(\theta)| \le \frac{\sqrt{2}\overline{\alpha}}{\sqrt{\underline{\alpha}}}|c-\hat{c}|^{\frac{1}{2}}$ which proves the second estimate in (4.10) with $\overline{C}(\tilde{\omega}) = \frac{\sqrt{2}\overline{\alpha}}{\sqrt{\underline{\alpha}}}$. □

Estimation of the error term $|R(t,\theta_k)|$**.** Now we are able to prove the second statement of Theorem 3.1. At first we determine a uniform bound on $|R(t,\theta_k)|$ for ε small where the asymptotic expansion becomes singular. To do so we first provide some formulas to express $f'(u)$ and $f''(u)$ in (4.9), respectively, in terms of the dispersion relation $\tilde{\omega}$. Obviously we have

$$\begin{aligned} f' &= \psi A \Theta'' + (\psi_\theta A + \psi A_\theta)\,\Theta'^2, \\ f'' &= \psi A \Theta''' + 3\,(\psi_\theta A + \psi A_\theta)\,\Theta'\Theta'' + (\psi_{\theta\theta} A + \psi A_{\theta\theta} + 2\psi_\theta A_\theta)\,\Theta'^3, \end{aligned} \tag{4.11}$$

and, with $\theta = \Theta(u)$, for the coordinate transform holds

$$\Theta'(u) = \frac{1}{U'(\theta)}, \quad \Theta''(u) = -\frac{U''(\theta)}{U'(\theta)^3}, \quad \Theta'''(u) = \frac{3U''(\theta)^2 - U'(\theta)U'''(\theta)}{U'(\theta)^5}. \tag{4.12}$$

Finally we express the derivatives of U in terms of $\tilde\omega$ and h. According to (4.7) we find

$$\begin{aligned}
U' &= \frac{2(|\tilde\omega''(\theta_k)|+h_k) + (\theta-\theta_k)h_k'}{2\sqrt{2}(|\tilde\omega''(\theta_k)|+h_k)^{\frac12}}\,,\\
U'' &= \frac{\big(4h_k' + 2h_k''(\theta-\theta_k)\big)(|\tilde\omega''(\theta_k)|+h_k) - {h_k'}^2(\theta-\theta_k)}{4\sqrt{2}(|\tilde\omega''(\theta_k)|+h_k)^{\frac32}}\,,\\
U''' &= \frac{\big(12h_k''+4h_k'''(\theta-\theta_k)\big)(|\tilde\omega''(\theta_k)|+h_k)^2}{8\sqrt{2}(|\tilde\omega''(\theta_k)|+h_k)^{\frac52}}\\
&\quad + \frac{-\big(6{h_k'}^2+8h_k'h_k''(\theta-\theta_k)\big)(|\tilde\omega''(\theta_k)|+h_k) + 3{h_k'}^3(\theta-\theta_k)}{8\sqrt{2}(|\tilde\omega''(\theta_k)|+h_k)^{\frac52}}.
\end{aligned}\tag{4.13}$$

Now we consider the first term on the right hand side of (4.9). Using (4.12), (4.13), (4.7) and $\psi'(\Theta(0)) = \psi'(\theta_k) = 0$ we find $\psi_\theta A{\Theta'}^2\big|_{u=0} = 0$, $\psi A_\theta{\Theta'}^2\big|_{u=0} = \frac{2A'(\theta_k)}{|\tilde\omega''(\theta_k)|}$ and $\psi A\Theta''\big|_{u=0} = \frac{2\,\mathrm{sign}\,\tilde\omega''(\theta_k)\,\tilde\omega'''(\theta_k)}{3|\tilde\omega''(\theta_k)|^2}$. Thus we have

$$|f'(0)| \le \left|\frac{2A'(\theta_k)}{|\tilde\omega''(\theta_k)|} + \frac{2\,\mathrm{sign}\,\tilde\omega''(\theta_k)\,\tilde\omega'''(\theta_k)}{3|\tilde\omega''(\theta_k)|^2}\right| =: B_1(\theta_k). \tag{4.14}$$

Concerning the second term on the right hand side of (4.9), due to the compact support of ψ and $\int_0^{U(\theta_k+\delta)} A(\theta(u))\Theta'(u)\,\mathrm{d}u = \int_{\theta_k}^{\theta_k+\delta} A(\theta)\,\mathrm{d}\theta$, (4.11) leads to

$$\begin{aligned}
\left|\int_0^\infty f''(u)\,\mathrm{d}u\right| \le \|A\|_{W^{2,1}(\theta_k,\theta_k+\delta)}\Bigg(&\left\|\frac{\Theta'''}{\Theta'}\right\|_{L^\infty(0,U(\theta_k+\delta))}\\
&+ 3\Big(1+\|\psi'\|_{L^\infty(\theta_k,\theta_k+\delta)}\Big)\|\Theta''\|_{L^\infty(0,U(\theta_k+\delta))}\\
&+ \Big(1+2\|\psi'\|_{L^\infty(\theta_k,\theta_k+\delta)}+\|\psi''\|_{L^\infty(\theta_k,\theta_k+\delta)}\Big)\|{\Theta'}^2\|_{L^\infty(0,U(\theta_k+\delta))}\Bigg).
\end{aligned}$$

Using (4.4) and (4.12) this implies

$$|R(t,\theta_k)| \le \frac12\left(B_1(\theta_k) + B_2(\theta_k,\delta) + \frac{B_3(\theta_k,\delta)}{\delta} + \frac{B_4(\theta_k,\delta)}{\delta^2}\right)t^{-1} \tag{4.15}$$

with

$$\begin{aligned}
B_2(\theta_k,\delta) &= \|A\|_{W^{2,1}(S^1)}\left\|\frac{3{U''}^2 - U'U'''}{{U'}^4}\right\|_{L^\infty(\theta_k,\theta_k+\delta)},\\
B_3(\theta_k,\delta) &= C\|A\|_{W^{2,1}(S^1)}\left\|\frac{U''}{{U'}^3}\right\|_{L^\infty(\theta_k,\theta_k+\delta)},\\
B_4(\theta_k,\delta) &= C\|A\|_{W^{2,1}(S^1)}\left\|\frac{1}{{U'}^2}\right\|_{L^\infty(\theta_k,\theta_k+\delta)},
\end{aligned}\tag{4.16}$$

and B_1 defined in (4.14).

Now recall the assumptions of the first part of the proof: $\varepsilon > 0$, $c \in \tilde{\omega}'(\mathbb{S}^1) \setminus \mathcal{C}_{\rm cr}(\varepsilon)$, $\theta_k \in \{\theta_1, \ldots, \theta_K\} = \Theta(c)$ and $\delta > 0$ such that $\mathcal{U}_\delta(\theta_{k_1}) \cap \mathcal{U}_\delta(\theta_{k_2}) \neq \emptyset$ if $k_1 \neq k_2$ (cf. (4.3)) is fulfilled. We aim to prove that the bound on $R(t, \theta_k)$ is uniform with respect to $c \in \tilde{\omega}'(\mathbb{S}^1) \setminus \mathcal{C}_{\rm cr}(\varepsilon)$. To do so we express the dependency on c in terms of θ_k. Since $\tilde{\omega}'$ is continuous and $\boldsymbol{\Theta}_{\rm cr}$ finite, there exists $\delta_\varepsilon > 0$ such that $\tilde{\omega}'(\boldsymbol{\Theta}_{\rm cr}(\delta_\varepsilon)) \subset \mathcal{C}_{\rm cr}(\varepsilon)$. For later use we choose δ_ε maximal in the sense that there exists $\hat{\theta} \in \boldsymbol{\Theta}_{\rm cr}$ such that $|\tilde{\omega}'(\hat{\theta}+\delta_\varepsilon) - \tilde{\omega}'(\hat{\theta})| = \varepsilon$ or $|\tilde{\omega}'(\hat{\theta}-\delta_\varepsilon) - \tilde{\omega}'(\hat{\theta})| = \varepsilon$. Thus, proving the assertion for $\theta_k \in \boldsymbol{\Theta}^c_{\rm cr}(\delta_\varepsilon) = S^1 \setminus \boldsymbol{\Theta}_{\rm cr}(\delta_\varepsilon)$ is sufficient since, due to $\tilde{\omega}'(\mathbb{S}^1) \setminus \mathcal{C}_{\rm cr}(\varepsilon) \subset \tilde{\omega}'(\boldsymbol{\Theta}^c_{\rm cr}(\delta_\varepsilon))$, this includes all demanded cases.

According to Lemma 4.1 there exists $\delta_0 > 0$ such that (4.10) holds. By eventually decreasing δ_0 we may ensure furthermore that $\boldsymbol{\Theta}_{\rm cr}(\delta_0) = \overline{\bigcup_{\hat{\theta} \in \boldsymbol{\Theta}_{\rm cr}} \mathcal{U}_\delta(\hat{\theta})}$ is a disjoint union. Note that this choice of δ_0 only depends on $\tilde{\omega}$. Now, to provide an upper bound on the right hand side of (4.15), we distinguish two cases.

First we consider $\theta_k \in \boldsymbol{\Theta}^c_{\rm cr}(\delta_\varepsilon) \setminus \boldsymbol{\Theta}_{\rm cr}(\delta_0)$. An upper bound on B_1 defined in (4.14) is provided by $B_1(\theta_k) \leq \max\Big\{ B_1(\theta) \big| \theta \in \overline{S^1 \setminus \boldsymbol{\Theta}_{\rm cr}(\delta_0)} \Big\} := \bar{B}_1$ which only depends on $\tilde{\omega}$. To provide an upper bound on B_2, B_3 and B_4 we fix $\delta < \delta_0$, e.g. $\delta := \frac{\delta_0}{2}$. This is consistent with condition (4.3) since $\operatorname{dist}(\theta_k, \boldsymbol{\Theta}_{\rm cr}) \geq \delta_0$ and the fact that there always exists $\hat{\theta} \in \boldsymbol{\Theta}_{\rm cr}$ such that $\theta_k < \hat{\theta} < \theta_{k+1}$. We again have $B_n(\theta_k, \delta) \leq \max\Big\{ B_n\big(\theta, \frac{\delta_0}{2}\big) \big| \theta \in \overline{S^1 \setminus \boldsymbol{\Theta}_{\rm cr}(\delta_0)} \Big\} := \bar{B}_n$. Thus, in case we proved the statement with a bound independently of ε,

$$\forall\, \theta_k \in \boldsymbol{\Theta}^c_{\rm cr}(\delta_\varepsilon) \setminus \boldsymbol{\Theta}_{\rm cr}(\delta_0): \quad |R(t, \theta_k)| \leq C(\tilde{\omega})\, t^{-1}.$$

In the second case, $\theta_k \in \boldsymbol{\Theta}^c_{\rm cr}(\delta_\varepsilon) \cap \boldsymbol{\Theta}_{\rm cr}(\delta_0)$, the singularity as $\varepsilon \to 0$ comes into play. The choice of δ_0 implies $\frac{1}{|\tilde{\omega}''(\theta_k)|} \leq \frac{1}{\underline{C}(\tilde{\omega}) \varepsilon^{\frac{1}{2}}}$. Thus we obtain

$$B_1(\theta_k) \leq \frac{2|A'(\theta_k)|}{\underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}}} + \frac{2|\tilde{\omega}'''(\theta_k)|}{3\, \underline{C}^2(\tilde{\omega})\, \varepsilon} \leq \frac{C(\tilde{\omega})}{\varepsilon}.$$

For B_2, B_3 and B_4 we have to provide lower bounds on the denominators on the right hand side of (4.16). To do so note that according to Lemma 4.1 holds $|\tilde{\omega}''(\theta_k)| \geq \underline{C}(\tilde{\omega}) \varepsilon^{\frac{1}{2}}$. We introduce $v(\theta, \theta_k) := h(\theta, \theta_k) + \frac{1}{2} \partial_1 h(\theta, \theta_k)(\theta - \theta_k)$. Claiming that

$$\begin{aligned} &\forall\, \theta_k \in \boldsymbol{\Theta}^c_{\rm cr}(\delta_\varepsilon) \cap \boldsymbol{\Theta}_{\rm cr}(\delta_0)\ \forall\, \theta \in [\theta_k, \theta_k+\delta]: \\ &|h(\theta, \theta_k)| \leq \tfrac{1}{2}\, \underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}} \quad \text{and} \quad |v(\theta, \theta_k)| \leq \tfrac{1}{2}\, \underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}} \end{aligned} \tag{4.17}$$

we obtain

$$\begin{aligned} &|\tilde{\omega}''(\theta_k)| + h(\theta, \theta_k) \geq \tfrac{1}{2}\, \underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}} \quad \text{and} \\ &2\big(|\tilde{\omega}''(\theta_k)| + h(\theta, \theta_k)\big) + \partial_1 h(\theta, \theta_k)(\theta - \theta_k) \geq \underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}}. \end{aligned}$$

To see that it is possible to choose $\delta > 0$ such that (4.17) holds consider $\tilde{h}(\Delta\theta, \theta_k) := h(\theta_k + \Delta\theta, \theta_k)$ on $[0, \delta_\varepsilon] \times \overline{\mathbf{\Theta}^c_{\rm cr}(\delta_\varepsilon) \cap \mathbf{\Theta}_{\rm cr}(\delta_0)}$. Since the domain is compact, $\tilde{h}$ continuous and $\tilde{h}(0, \theta_k) = 0$ holds for all θ_k, there exists $\delta_1 \in (0, \delta_\varepsilon]$ such that

$$\forall\, \Delta\theta \in [0, \delta_1]: \quad \max_{\theta_k} |\tilde{h}(\Delta\theta, \theta_k)| \leq \tfrac{1}{2}\, \underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}}.$$

We choose δ_1 maximal in the sense that $\max_{\theta_k} |\tilde{h}(\Delta\theta, \theta_k)| \geq \frac{1}{2}\, \underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}}$ for $\Delta\theta \in [\delta_1, \tilde{\delta}_1)$ with some $\tilde{\delta}_1 > 0$ or $\delta_1 = \delta_\varepsilon$. Repeating the arguments for $\tilde{v}(\Delta\theta, \theta_k) := v(\theta, \theta_k)$ leads to $\delta_2 > 0$. Choosing $\delta := \min\{\delta_1, \delta_2\}$ provides (4.17). Now, using (4.13) and $\varepsilon \leq 2 \max_{\theta \in \mathbb{S}^1} |\tilde{\omega}'(\theta)|$ we obtain

$$\left\| \frac{3U''(\theta)^2 - U'(\theta)U'''(\theta)}{U'(\theta)^4} \right\|_{L^\infty(\theta_k, \theta_k+\delta)} \leq \frac{C(\tilde{\omega})}{\varepsilon^{\frac{5}{2}}},$$

$$\left\| \frac{U''(\theta)}{U'(\theta)^3} \right\|_{L^\infty(\theta_k, \theta_k+\delta)} \leq \frac{C(\tilde{\omega})}{\varepsilon^{\frac{3}{2}}}, \quad \text{and} \quad \left\| \frac{1}{U'(\theta)} \right\|^2_{L^\infty(\theta_k, \theta_k+\delta)} \leq \frac{C(\tilde{\omega})}{\varepsilon^{\frac{1}{2}}}$$

with constants $C(\tilde{\omega}) > 0$ depending only on $\tilde{\omega}$.

It remains to express δ in terms of ε. To do so we distinguish three cases, namely $\delta = \delta_1 < \delta_\varepsilon$, $\delta = \delta_2 < \delta_\varepsilon$ and $\delta = \delta_\varepsilon$. In the first case we consider $\bar{\theta}_k := \operatorname{argmax} |\tilde{h}(\delta, \theta_k)|$. In view of (4.7) and the fact that $\tilde{\omega}$ is real-analytic $\tilde{h}$ in fact is well defined and smooth on a compact domain which is a superset of $[0, \delta_\varepsilon] \times \overline{\mathbf{\Theta}^c_{\rm cr}(\delta_\varepsilon) \cap \mathbf{\Theta}_{\rm cr}(\delta_0)}$ and independent of ε, for instance $[0, \pi] \times [-\pi, \pi]$. On this set there exists a global Lipschitz constant $L_{\tilde{h}} = L_{\tilde{h}}(\tilde{\omega})$ only depending on $\tilde{\omega}$. We obtain $\frac{1}{2}\, \underline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}} = |\tilde{h}(\delta, \theta_k)| \leq L_{\tilde{h}}(\tilde{\omega})\, \delta$, i.e. $\delta \geq \frac{\underline{C}(\tilde{\omega})}{2L_{\tilde{h}}(\tilde{\omega})} \varepsilon^{\frac{1}{2}} = \frac{1}{C(\tilde{\omega})}\, \varepsilon^{\frac{1}{2}}$. In the second case we again repeat the arguments for $\delta = \delta_2$ and $\tilde{v}(\Delta\theta, \theta_k)$. For the third case recall that there exists $\hat{\theta} \in \mathbf{\Theta}_{\rm cr}$ such that $|\tilde{\omega}(\hat{\theta} \pm \delta_\varepsilon) - \tilde{\omega}(\hat{\theta})| = \varepsilon$. According to Lemma 4.1 we have $\varepsilon = |\tilde{\omega}(\hat{\theta} \pm \delta_\varepsilon) - \tilde{\omega}(\hat{\theta})| \leq \max_{\theta \in [\hat{\theta}, \hat{\theta}+\delta_\varepsilon]} |\tilde{\omega}''(\theta)|\, \delta_\varepsilon \leq \overline{C}(\tilde{\omega})\, \varepsilon^{\frac{1}{2}}\, \delta_\varepsilon$, which leads to the same order of decay as in the first two cases, i.e. we have

$$\delta \geq \frac{\varepsilon^{\frac{1}{2}}}{C(\tilde{\omega})}. \tag{4.18}$$

To summarize, the terms on the right-hand side of (4.15) are bounded as follows, $B_1(\theta_k) \leq C(\tilde{\omega})\varepsilon^{-\frac{1}{2}}$, $B_2(\theta_k) \leq C(\tilde{\omega})\varepsilon^{-\frac{5}{2}}$, $\frac{B_3(\theta_k)}{\delta} \leq C(\tilde{\omega})\varepsilon^{-2}$ and $\frac{B_4(\theta_k)}{\delta^2} \leq C(\tilde{\omega})\varepsilon^{-\frac{3}{2}}$. Thus we proved

$$\forall\, \theta_k \in \mathbf{\Theta}^c_{\rm cr}(\delta_\varepsilon) \cap \mathbf{\Theta}_{\rm cr}(\delta_0): \quad |R(t, \theta_k)| \leq C(\tilde{\omega})\varepsilon^{-\frac{5}{2}}\, t^{-1}$$

Combining all bounds we obtain

$$|R(t, \theta_k)| \leq \frac{1}{2} \left(C_1(\tilde{\omega}) + \frac{C_2(\tilde{\omega})}{\varepsilon^{\frac{5}{2}}} \right) t^{-1}. \tag{4.19}$$

Estimation of the error term $|R_{\text{loc}}(t,c)|$. Finally we consider the localization error given in (4.5). Using the notation $\theta_k < \hat{\theta}_k < \theta_{k+1}$ the right-hand side of (4.5) decomposes into terms of the form $\int_{\theta_k+\frac{\delta}{2}}^{\hat{\theta}_k} \big[1-\psi_k(\theta)\big] A(\theta)\, \mathrm{e}^{it\phi(\theta,c)}\, \mathrm{d}\theta$. Note that $|\phi'(\theta,c)|$ strictly monotone on $[\theta_k, \hat{\theta}_k]$. Using Lemma 4.1 and (4.18) we find $|\phi'(\theta,c)| \geq \big|\phi'\big(\theta_k{+}\frac{\delta}{2}, c\big)\big| \geq \min_{\theta\in\mathbb{S}^1\setminus\mathbf{\Theta}_{\text{cr}}(\frac{\delta_\varepsilon}{2})} |\tilde{\omega}''(\theta)|\,\frac{\delta}{2} \geq \min\big\{C_1(\tilde{\omega}), C_2(\tilde{\omega})\,\varepsilon\big\}$. Thus, applying partial integration we find

$$\left|\int_{\theta_k+\frac{\delta}{2}}^{\hat{\theta}_k} \big[1-\psi_k(\theta)\big] A(\theta)\, \mathrm{e}^{it\phi(\theta,c)}\, \mathrm{d}\theta\right| \leq \left(C_1(\tilde{\omega}) + \frac{C_2(\tilde{\omega})}{\varepsilon}\right) t^{-1}.$$

This decay rate is already covered by (4.19), which completes the proof of (3.1).

4.3. Asymptotic expansion at the wave fronts. This section is dedicated to the proof of Theorem 3.3. Arguments presented in [1, 8, 17] are underlying the following considerations.

Consider again Green's function represented by (2.14). We choose a parametrization of $\mathbb{S}^1$ on $(-2\pi, 2\pi]$, i.e. the components of $G(t,c)$ are given by

$$g(t,c) = \frac{1}{4\pi}\int_{-2\pi}^{2\pi} A(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta \quad \text{with} \quad \phi(\theta,c) = c\theta - \tilde{\omega}(\theta), \quad c = \frac{j}{t}$$

and $A(\theta) = 1$, $\frac{1}{\omega_{\mathrm{r}}(\theta)}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}$, or $\omega_{\mathrm{r}}(\theta)\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}$.

Localization. We aim to determine the asymptotic behavior of $g(t,c)$ as $t \to \infty$ for group velocities c near caustics, i.e. $|c-\hat{c}| = \varepsilon$ with $\hat{c} = \tilde{\omega}'(\hat{\theta})$, $\hat{\theta} \in \mathbf{\Theta}_{\text{cr}}$. Here, in general, the asymptotic behavior splits up into two components. One contribution $\sim t^{-\frac{1}{3}}$ is made by a neighborhood of the actual critical wave number $\hat{\theta}$ and one $\sim t^{-\frac{1}{2}}$ by further wave numbers $\theta \in \mathbf{\Theta}(c)$ which are bounded away from that. The latter vanishes if we are close to the sonic wave speeds $\pm c_{\max} = \pm \max_{\theta\in\mathbf{\Theta}_{\text{cr}}} \tilde{\omega}'(\hat{\theta})$. To formalize this define

$$\delta_0 := \tfrac{1}{4} \min_{\hat{\theta}_1\neq\hat{\theta}_2\in\mathbf{\Theta}_{\text{cr}}} |\hat{\theta}_1 - \hat{\theta}_2| \quad \text{and} \quad \varepsilon_0 := \min_{\hat{\theta}\in\mathbf{\Theta}_{\text{cr}}} \left\{\tfrac{1}{2}\left|\tilde{\omega}'(\hat{\theta}) - \tilde{\omega}'(\hat{\theta}{\pm}\delta_0)\right|\right\}.$$

Note that both are uniquely determined by $\tilde{\omega}$, i.e. bounds depending beside $\tilde{\omega}$ on δ_0 or ε_0 are according to our convention denoted by $C(\tilde{\omega})$.

Now consider $c \in \mathcal{C}_{\text{cr}}(\varepsilon_0)$ and $\{\hat{\theta}_1, \ldots, \hat{\theta}_K\} = \mathbf{\Theta}_{\text{cr}}(c, \varepsilon_0)$. To simplify notation we assume $\mathcal{U}_{2\delta_0}(\hat{\theta}) \subset (-2\pi, 2\pi]$ for all k. For localization we use $\psi_k \in C_0^\infty\big(\mathcal{U}_{2\delta_0}(\hat{\theta}_k)\big)$ with $\psi|_{[\hat{\theta}_k-\delta_0, \hat{\theta}_k+\delta_0]} \equiv 1$ and obtain

$$g(t,c) = \frac{1}{4\pi}\sum_k \int_{\hat{\theta}_k-\delta_0}^{\hat{\theta}_k+\delta_0} A(\theta)\, \mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\, \mathrm{d}\theta + R_0(t,c) + R_{\text{loc}}(t,c) \tag{4.20}$$

with

$$R_0(t,c) = \frac{1}{4\pi}\sum_k \left(\int_{\hat\theta_k-2\delta_0}^{\hat\theta_k-\delta_0} + \int_{\hat\theta_k+\delta_0}^{\hat\theta_k+2\delta_0}\right)\psi_k(\theta)A(\theta)\,\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta \quad \text{and}$$

$$R_{\mathrm{loc}}(t,c) = \frac{1}{4\pi}\left(\int_{-\pi}^{\hat\theta_1-\delta_0} + \sum_{k=1}^{K-1}\int_{\hat\theta_k+\delta_0}^{\theta_{k+1}-\delta_0} + \int_{\hat\theta_K+\delta_0}^{\pi}\right)\Big[1-\sum_k \psi_k(\theta)\Big]A(\theta)\,\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta.$$

First we treat the error term R_0. In view of our choice of δ_0, $|\hat c - \tilde\omega'(\theta)|$ is monotonically increasing on $[\hat\theta_k, \hat\theta_k{+}2\delta_0]$. Utilizing the definition of ε_0 we find that $|\partial_\theta\phi(\theta,c)| \geq -|\hat c - c| + |\hat c - \tilde\omega'(\theta)| \geq \varepsilon_0$ for $\theta \in [\hat\theta_k{+}\delta_0, \hat\theta_k{+}2\delta_0]$. Since the same estimate holds on $[\hat\theta_k{-}2\delta_0, \hat\theta_k]$, integration by parts leads to

$$|R_0(t,c)| \leq C(\tilde\omega)\,t^{-1}.$$

Introducing $R_0(t,c)$ would actually not have been necessary to determine the asymptotic expansion since the upcoming steps also do the job if it is merged with the remaining integral on the right hand side of (4.20), cf. [8]. But by doing so we may treat the remaining integral in an analytic setting which makes it more easy to determine the leading order error terms explicitly. Here the pay-off are error terms $\sim t^{-1}$ at the boundary $\hat\theta{\pm}\delta_0$. But this is what we get anyway from R_{loc}. We also could have used a simple cut-off, but we introduced ψ_k to highlight the origin of the contribution.

Concerning the localization error R_{loc} we may distinguish two cases. First, assume there exists $\theta \in \mathbf{\Theta}(c) \setminus \{\theta \mid \mathrm{dist}(\mathbf{\Theta}_{\mathrm{cr}}(c,\varepsilon_0),\theta) \leq \delta_0\}$. Since we are again uniformly bounded away from other critical wave numbers, we may use the theory presented in Subsection 4.2 to prove

$$\left|R_{\mathrm{loc}}(t,c) - \mathcal{G}_{\mathrm{non}}(t,c)t^{-\frac{1}{2}}\right| \leq C(\tilde\omega)\,t^{-1}.$$

Here $\mathcal{G}_{\mathrm{non}}(t,c)$ is defined in (3.1b), where in this case we only sum over $\mathbf{\Theta}(c)\setminus\mathbf{\Theta}_{\mathrm{cr}}(\delta_0)$. In the second case, if $\theta \in \mathbf{\Theta}(c)\setminus\{\theta \mid \mathrm{dist}(\mathbf{\Theta}_{\mathrm{cr}}(c,\varepsilon_0),\theta) \leq \delta_0\} = \emptyset$, we are obviously near the sonic wave speed. We again have $|\partial_\theta\phi| \geq \varepsilon_0$ but this time iterative application of partial integration leads to

$$|R_{\mathrm{loc}}(t,c)| \leq C_N(\tilde\omega)\,t^{-N} \quad \text{for all } N \in \mathbb{N}$$

since the boundary terms cancel due to periodicity and the compact support of ψ_k, respectively.

Local coordinate transform. Since from now on we will consider only one single integral $\int_{\hat\theta_k-\delta_0}^{\hat\theta_k+\delta_0} A(\theta)\,\mathrm{e}^{\mathrm{i}t\phi(\theta,c)}\,\mathrm{d}\theta$ we skip the indices k and $\hat\theta$. Using the nondegeneracy condition $\omega'''(\hat\theta) \neq 0$ we can use the following proposition summarizing all the necessary results concerning the local coordinate transform, cf. [1, 17] for a similar procedure.

Proposition 4.2 (Coordinate change for wave front)**.** *Assume that* $\theta \mapsto \omega(\theta)$ *is real-analytic near* $\hat\theta$ *and that*

$$\hat c := \omega'(\hat\theta), \quad \omega''(\hat\theta) = 0, \quad \textit{and} \quad \omega'''(\hat\theta) \neq 0.$$

Then, for the function $\phi(\theta, \hat c) = c\theta - \omega(\theta)$ *there exists a unique local, analytical coordinate change* $u = U(\theta, c)$ *near* $(\hat\theta, \hat c)$ *with inverse* $\theta = \Theta(u, c)$ *satisfying* $U(\hat\theta, \hat c) = 0$, $\partial_\theta U(\hat\theta, \hat c) > 0$ *and*

$$\begin{aligned}&\phi(\theta, c) = \phi(\Theta(u, c), c) = -\frac{\sigma}{3}u^3 - a(c)u + b(c) \quad \textit{with}\\ &\sigma := \operatorname{sign}\big(\omega'''(\hat\theta)\big) \in \{-1, 1\}, \quad a(\hat c) = 0, \quad \sigma a'(\hat c) > 0, \quad b(\hat c) = \phi(\hat\theta, \hat c).\end{aligned}$$

The functions a *and* b *can explicitly be constructed as follows. For* $0 < \sigma(c-\hat c) \ll 1$ *the equation* $0 = \partial_\theta\phi(\theta, c) = c - \omega'(\theta)$ *has exactly two solutions near* $\hat\theta$, *namely* $\theta_-(c) < \theta_+(c)$. *Now, for* $0 < \sigma(c-\hat c) \ll 1$ *the functions* a *and* b *are given by*

$$a(c) = \sigma\bigg(\frac{3\sigma}{4}\big[\phi(\theta_+(c), c) - \phi(\theta_-(c), c)\big]\bigg)^{\frac{2}{3}}, \quad b(c) = \frac{1}{2}\big[\phi(\theta_+(c), c) + \phi(\theta_-(c), c)\big].$$

In particular, if ϕ *is odd in* θ *(i.e.* $\phi(-\theta, c) = -\phi(\theta, c)$*), then* $b \equiv 0$.

Remark 4.3. Before we give the proof of this result, we show how the result looks like for the FPU chain with $\omega(\theta) = 2\sin(\frac{\theta}{2})$ with $\hat\theta = 0$, where $\hat c = 1$ and $\omega'''(0) = -\frac{1}{4}$. We find $\theta_\pm(c) = \pm 2\arccos(c)$ for $0 < c < \hat c = 1$. By symmetry we have $b \equiv 0$ and find $a(c) = (3[\sqrt{1-c^2} - c\arccos c])^{\frac{2}{3}}$. This function can be extended analytically into a neighborhood of $\hat c$ as follows. With $\alpha = \arccos c$ we have

$$a(c) = (3[\sin\alpha - \alpha\cos\alpha])^{\frac{2}{3}} = \alpha^2 g(\alpha^2)^{\frac{2}{3}},$$

where g is analytic around 0 with $g(0) = 1$ by using that $3[\sin\alpha - \alpha\cos\alpha] = \alpha^3(1 + \sum_{k=1}^\infty c_k\alpha^{2k})$. Now defining T as the inverse of $C(r) = \sum_{k=0}^\infty \frac{(-r)^k}{(2k)!}$ (i.e. $C(r^2) = \cos r$ and $C(-r^2) = \cosh r$) we find $a(c) = T(c)g(T(c))^{\frac{2}{3}}$, which is real analytic for $c \in (-1, c_*)$ with $c_* \approx 44.7$, see Figure 4.1.

Proof of Proposition 4.2. Without loss of generality we carry out the details of the proof for the case that $\tilde\omega'$ takes a local maximum at $\hat\theta$, i.e. $\tilde\omega'''(\hat\theta) < 0$. We introduce $\varepsilon := \hat c - c$. Due to $c = \tilde\omega'(\theta)$ we have $\varepsilon \geq 0$ in the considered neighborhood of $\hat\theta$. For

$$\hat\phi(\theta, \varepsilon) := \phi(\theta, \hat c-\varepsilon) - \phi(\hat\theta, \hat c) = -\theta\varepsilon - \tilde\omega(\theta) + \tilde\omega(\hat\theta) + \tilde\omega'(\hat\theta)(\theta - \hat\theta)$$

holds

$$\hat\phi(\hat\theta, 0) = \partial_\theta\hat\phi(\hat\theta, 0) = \partial_\theta^2\hat\phi(\hat\theta, 0) = 0 \quad \text{and} \quad \partial_\theta^3\hat\phi(\hat\theta, 0) > 0.$$

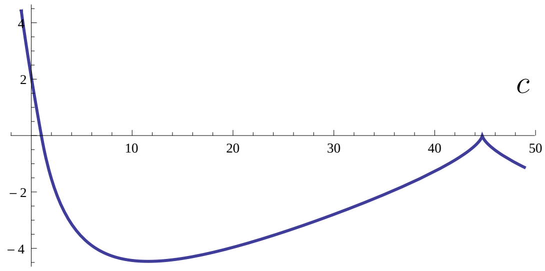

Figure 4.1: Function $a(c)$ for FPU dispersion relation $\omega(\theta) = 2\sin(\frac{\theta}{2})$ at $\hat{\theta} = 0$.

Thus we are able to apply the suitable version of Weierstraß' preparation theorem yielding a normal form for $\hat{\phi}$. According to Theorem B.5, in a neighborhood of the critical point $\hat{\theta}$, there exists a coordinate transform

$$\begin{aligned} &u = U(\theta,\varepsilon) \text{ or } \theta = \Theta(u,\varepsilon) \quad \text{such that} \\ &\hat{\phi}(\Theta(u,\varepsilon),\varepsilon) = \Phi(u,\varepsilon) := \tfrac{1}{3}u^3 - a(\varepsilon)u + b(\varepsilon). \end{aligned} \tag{4.21}$$

The transform $U(\theta,\varepsilon)$ as well as $a(\varepsilon)$ and $b(\varepsilon)$ are real-analytic. Furthermore we have $a(0) = b(0) = 0$.

In case $\hat{\phi}$ is symmetric in $\hat{\theta}$, we have $b(\varepsilon) \equiv 0$ such that the oscillatory contribution depends on $\hat{\theta}$ only.

To make a and b explicit we use that, for $\theta \neq \hat{\theta}$, the saddle point of order 2 splits up into two saddle points of order 1. Thus, for $\varepsilon > 0$ there exist $\theta_\pm(\varepsilon)$ such that $\hat{c} - \tilde{\omega}'\big(\hat{\theta}_\pm(\varepsilon)\big) = \varepsilon$, which means $\partial_\theta\hat{\phi}(\theta_\pm(\varepsilon),\varepsilon) = 0$ for all ε. We now set $u_\pm(\varepsilon) := U(\theta_\pm(\varepsilon),\varepsilon)$ and observe

$$\big(u_\pm(\varepsilon)\big)^2 - a(\varepsilon) = \partial_u\Phi(u_\pm(\varepsilon),\varepsilon) = \partial_\theta\hat{\phi}(\theta_\pm(\varepsilon),\varepsilon)\,\partial_u\Theta(u_\pm(\varepsilon),\varepsilon) = 0.$$

Hence, we find $u = \pm\sqrt{a(\varepsilon)}$, and using the definition of Φ in (4.21) gives

$$a(\varepsilon) = \left(\frac{3}{4}\big[\hat{\phi}\big(\theta_+(\varepsilon),\varepsilon\big) - \hat{\phi}(\theta_-(\varepsilon),\varepsilon)\big]\right)^{\frac{2}{3}},\ b(\varepsilon) = \frac{1}{2}\big[\hat{\phi}\big(\theta_+(\varepsilon),\varepsilon\big) + \hat{\phi}\big(\theta_-(\varepsilon),\varepsilon\big)\big].$$

Expanding $\theta_\pm(\varepsilon)$ in powers of $\varepsilon^{\frac{1}{2}}$ we obtain

$$\theta_\pm(\varepsilon) = \hat{\theta} \pm \sqrt{\frac{2}{-\tilde{\omega}'''(\hat{\theta})}}\,\varepsilon^{\frac{1}{2}} + \frac{\tilde{\omega}^{(4)}(\hat{\theta})}{3\tilde{\omega}'''(\hat{\theta})^2}\,\varepsilon + \mathcal{O}(\varepsilon^{\frac{3}{2}}). \tag{4.22}$$

Inserting this into $\hat{\phi}(\theta_\pm(\varepsilon),\varepsilon)$ leads to the expansions

$$a(\varepsilon) = -\sqrt[3]{\tfrac{2}{\tilde{\omega}'''(\hat{\theta})}}\,\varepsilon + \mathcal{O}(\varepsilon^2), \quad b(\varepsilon) = 2\hat{\theta}\,\varepsilon + \mathcal{O}(\varepsilon^2).$$

We derived these identities for $\varepsilon \in [0, \varepsilon_0]$. But since a and b are real-analytic this actually holds at least for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Concerning the symmetric case where we have $b \equiv 0$ note that $\hat{\phi}(\hat{\theta}-\Delta\theta, c) = -\hat{\phi}(\hat{\theta}+\Delta\theta, c)$ implies $2\hat{\theta}c = \tilde{\omega}(\hat{\theta}+\Delta\theta) + \tilde{\omega}(\hat{\theta}-\Delta\theta)$. Keeping in mind that $c = \frac{j}{t}$ and that fact that we actually consider $\mathrm{e}^{\mathrm{i}t\phi}$ we find that the last identity needs to be valid $\bmod \frac{2\pi}{t}$. I.e. since c is arbitrary, we have $b \equiv 0$ if $\hat{\theta} = 0 \bmod \pi$.

Thus, Proposition 4.2 is established. □

Actual decay rate. Obviously, $\overline{\mathcal{U}_{\delta_0}(\hat{\theta})} \times [-\varepsilon_0, \varepsilon_0]$ is included in the neighborhood where the statement holds. Introducing the new variable u we may rewrite the integral on the right hand side of (4.20) as follows,

$$\begin{aligned} I(t,\varepsilon) &= \int_{\hat{\theta}-\delta_0}^{\hat{\theta}+\delta_0} A(\theta)\,\mathrm{e}^{\mathrm{i}t\phi(\theta,\hat{c}-\varepsilon)}\,\mathrm{d}\theta \\ &= \mathrm{e}^{\mathrm{i}t[b(\varepsilon)-\phi(\hat{\theta},\hat{c})]} \int_{U(\hat{\theta}-\delta_0,\varepsilon)}^{U(\hat{\theta}+\delta_0,\varepsilon)} A\big(\Theta(u,\varepsilon)\big)\,\partial_u\Theta(u,\varepsilon)\,\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\,\mathrm{d}u. \end{aligned} \tag{4.23}$$

To calculate the actual decay rate we again apply a version of Weierstraß' division theorem. Regarding $f(u,\varepsilon) := A\big(\Theta(u,\varepsilon)\big)\,\partial_u\Theta(u,\varepsilon)$, Theorem B.2 yields

$$f(u,\varepsilon) = q(u,\varepsilon)\,\big[u^2 - a(\varepsilon)\big] + \beta(\varepsilon)u + \alpha(\varepsilon) \tag{4.24}$$

with real-analytic functions q, β and α. Thus, (4.23) takes the form

$$I(t,\varepsilon) = \mathrm{e}^{\mathrm{i}t[b(\varepsilon)-\phi(\hat{\theta},\hat{c})]} \int_{U(\hat{\theta}-\delta_0,\varepsilon)}^{U(\hat{\theta}+\delta_0,\varepsilon)} \big[\beta(\varepsilon)u+\alpha(\varepsilon)\big]\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\,\mathrm{d}u + R_1(t,\varepsilon) \tag{4.25}$$

with

$$\begin{aligned} R_1(t,\varepsilon) &= \mathrm{e}^{\mathrm{i}t[b(\varepsilon)-\phi(\hat{\theta},\hat{c})]} \int_{U(\hat{\theta}-\delta_0,\varepsilon)}^{U(\hat{\theta}+\delta_0,\varepsilon)} q(u,\varepsilon)\,\big[u^2 - a(\varepsilon)\big]\,\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\,\mathrm{d}u \\ &= \frac{-\mathrm{i}}{t}\,\mathrm{e}^{\mathrm{i}t[b(\varepsilon)-\phi(\hat{\theta},\hat{c})]} \\ &\quad\times\bigg(q(u,\varepsilon)\,\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\Big|_{U(\hat{\theta}-\delta_0,\varepsilon)}^{U(\hat{\theta}+\delta_0,\varepsilon)} - \int_{U(\hat{\theta}-\delta_0,\varepsilon)}^{U(\hat{\theta}+\delta_0,\varepsilon)} \partial_u q(u,\varepsilon)\,\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\,\mathrm{d}u\bigg). \end{aligned} \tag{4.26}$$

The first two terms on the right hand side of (4.25) bear the leading order asymptotic behavior. We have

$$\int_{U(\hat\theta-\delta_0,\varepsilon)}^{U(\hat\theta+\delta_0,\varepsilon)} \mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\,\mathrm{d}u = 2\pi\,\mathrm{Ai}\big(-a(\varepsilon)t^{\frac{2}{3}}\big)\,t^{-\frac{1}{3}} + R_2(t,\varepsilon) \quad \text{and}$$

$$\int_{U(\hat\theta-\delta_0,\varepsilon)}^{U(\hat\theta+\delta_0,\varepsilon)} u\,\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\,\mathrm{d}u = 2\pi\mathrm{i}\,\mathrm{Ai}'\big(-a(\varepsilon)t^{\frac{2}{3}}\big)\,t^{-\frac{2}{3}} + R_3(t,\varepsilon),$$

$$\text{where}\quad R_j(t,\varepsilon) = \int_{\mathbb{R}\setminus[U(\hat\theta-\delta_0,\varepsilon),U(\hat\theta+\delta_0,\varepsilon)]} u^{j-2}\,\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\mathrm{d}u \quad \text{for } j=2,3.$$

Here $\mathrm{Ai}(\cdot)$ denotes Airy's function $\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i(\frac{1}{3}u^3+zu)}\mathrm{d}u$.

Before estimating the error terms we want to make the last two functions on the right hand side of (4.24) explicit to determine the leading order terms. To do so note that

$$f\big(\pm\sqrt{a(\varepsilon)},\varepsilon\big) = \pm\beta(\varepsilon)\sqrt{a(\varepsilon)} + \alpha(\varepsilon)$$

which implies

$$\alpha(\varepsilon) = \frac{1}{2}\Big[f\big(\sqrt{a(\varepsilon)},\varepsilon\big) + f\big(-\sqrt{a(\varepsilon)},\varepsilon\big)\Big],$$
$$\beta(\varepsilon) = \frac{1}{2\sqrt{a(\varepsilon)}}\Big[f\big(\sqrt{a(\varepsilon)},\varepsilon\big) - f\big(-\sqrt{a(\varepsilon)},\varepsilon\big)\Big],$$

and $q(u,\varepsilon) = \frac{f(u,\varepsilon)-\beta(\varepsilon)u-\alpha(\varepsilon)}{u^2-a(\varepsilon)}$. The first factor of $f(\pm\sqrt{a(\varepsilon)},\varepsilon)$ we have at hand explicitly. We either have $A\equiv 1$ or, in view of (4.22), $A\big(\Theta(\pm\sqrt{a(\varepsilon)},\varepsilon)\big) = A\big(\hat\theta+\Delta\theta_\pm\big) = A(\hat\theta) \pm \tilde A_1\varepsilon^{\frac{1}{2}} + \tilde A_2\varepsilon \pm \tilde A_3\varepsilon^{\frac{3}{2}} + \mathcal{O}(\varepsilon^2)$ with $A(\hat\theta)\neq 0$. For the second multiplier we start from identity (4.21) and apply l'Hopital's rule to $\partial_u\Theta = \frac{u^2-a(\varepsilon)}{\partial_\theta\hat\phi}$. This leads to

$$\Big(\partial_u\Theta(\pm\sqrt{a(\varepsilon)},\varepsilon)\Big)^2 = \frac{2u}{\partial_\theta^2\hat\phi}\bigg|_{u=\pm\sqrt{a(\varepsilon)}} = \frac{\pm 2\sqrt{a(\varepsilon)}}{-\tilde\omega''(\hat\theta+\Delta\theta_\pm)} = \left(\frac{2}{\tilde\omega'''(\hat\theta)}\right)^{\frac{2}{3}} + \mathcal{O}(\varepsilon^{\frac{1}{2}}).$$

Thus, we obtain

$$\alpha(\varepsilon) = A(\hat\theta)\sqrt[3]{\tfrac{2}{|\tilde\omega'''(\hat\theta)|}} + \mathcal{O}(\varepsilon),$$

which holds for $\varepsilon\in[-\varepsilon_0,\varepsilon_0]$. Note that the leading order term is always nonzero.

Concerning β we know by Theorem B.2 that the singularity in $\varepsilon = 0$ is removable. Indeed, we have $\beta = \partial_u f(0,0)$. Here we do not determine the leading order term explicitly. But note that in general this is not nonzero. For instance, in case $\hat\phi$ and with it U are symmetric (cf. special case of Theorem B.5) we have $\partial_u^2\Theta(0,0) = 0$. Thus for the first component of G (i.e. $A\equiv 1$) holds $\partial_u f(0,0) = A'(\hat\theta)(\partial_u\Theta(0,0))^2 + A(\hat\theta)\partial_u^2\Theta(0,0) = 0$.

Estimation of the error terms. Now we turn to estimate the error terms. We will prove that R_1 as well as R_2 and R_3 are at least $\mathcal{O}(t^{-1})$. To shorten the notion we introduce $u_\pm := U(\hat\theta\pm\delta_0,\varepsilon)$.

For estimating R_2 again partial integration does the trick:

$$\int_{\mathbb{R}\setminus[u_-,u_+]} \mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\mathrm{d}u = \frac{1}{t\big[u^2-a(\varepsilon)\big]}\Big|_{u_+}^{u_-} + \int_{\mathbb{R}\setminus[u_-,u_+]} \frac{2u}{t\big[u^2-a(\varepsilon)\big]^2}\,\mathrm{e}^{\mathrm{i}t(\frac{1}{3}u^3-a(\varepsilon)u)}\mathrm{d}u.$$

Recall that $\big|\partial_\theta\hat\phi(\hat\theta\pm\delta_0,\varepsilon)\big| \geq \varepsilon_0$. Since $\partial_\theta\hat\phi\,\partial_u\Theta = u^2 - a(\varepsilon)$ and the fact that $|\partial_u\Theta|$ is uniformly bounded from below on $[\hat\theta-\delta,\hat\theta+\delta]\times[0,\varepsilon_0]$ we conclude $\big|u_\pm^2 - a(\varepsilon)\big| \geq C(\tilde\omega)\varepsilon_0$. Obviously, the remaining integral on the right hand side also is uniformly bounded. Since the same arguments apply to $R_3(t,\varepsilon)$ we obtain

$$|R_2(t,\varepsilon)| \leq C(\tilde\omega)\,t^{-1} \quad \text{and} \quad |R_3(t,\varepsilon)| \leq C(\tilde\omega)\,t^{-1}$$

with $C(\tilde\omega)$ only depending on $\tilde\omega$.

It remains to determine an upper bound on R_1. Using (4.26) we easily find

$$|R_1(t,\varepsilon)| \leq \big(|q(u_+,\varepsilon)| + |q(u_-,\varepsilon)| + \|\partial_u q(\cdot,\varepsilon)\|_{L^1(u_-,u_+)}\big)\,t^{-1} \leq C(\tilde\omega)\,t^{-1}$$

which completes the proof.

We comment the proof with two final remarks. First, one cannot expect that the error bounds going to 0 as $\varepsilon\to 0$. The reason is that R_2 and R_3, which represent the deviation from Airy's function due to the bounded domain of integration, as well as R_1, which represents the higher order terms of the asymptotic expansion, remain in case of a pointwise expansion for $\varepsilon = 0$. Second, the bounds $\sim t^{-1}$ of the error terms are again due to the cut-off. Using a smooth partition of unity R_2 and R_3 would behave like t^{-N} for all $N\in\mathbb{N}$ (cf. R_{loc}) and R_1 like the next term of the asymptotic expansion, namely $\sim t^{-\frac{4}{3}}$.

A. Representations of Green's function

Recall the formulas of the dispersion relation,

$$\omega(\theta) := 2\left|\sin\left(\frac{\theta}{2}\right)\right|\omega_{\mathrm{r}}(\theta) \quad \text{and} \quad \tilde\omega(\theta) := 2\sin\left(\frac{\theta}{2}\right)\omega_{\mathrm{r}}(\theta)$$

with $\omega_{\mathrm{r}}\in\mathrm{C}^\omega(\mathbb{S}^1)$ and (2.12), i.e.

$$\exists\, c_{\mathrm{r}} > 0\ \forall\,\theta\in\mathbb{S}^1: \quad c_{\mathrm{r}} \leq \omega_{\mathrm{r}}(\theta) = \omega_{\mathrm{r}}(-\theta) = \omega_{\mathrm{r}}(\theta+2\pi).$$

The following five representations of the Green's function (2.8) are equivalent:

$$G_j(t) = \frac{1}{2\pi}\int_{\mathbb{S}^1}\begin{pmatrix} \cos(\omega(\theta)t) & \frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\omega(\theta)}\sin(\omega(\theta)t) \\ \frac{-\omega(\theta)}{\mathrm{e}^{\mathrm{i}\theta}-1}\sin(\omega(\theta)t) & \cos(\omega(\theta)t)\end{pmatrix}\mathrm{e}^{\mathrm{i}j\,\theta}\,\mathrm{d}\theta; \tag{A.1}$$

$$G_j(t) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \begin{pmatrix} \cos(\tilde\omega(\theta)t) & \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{\omega_{\mathrm{r}}(\theta)}\sin(\tilde\omega(\theta)t) \\ \mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}\omega_{\mathrm{r}}(\theta)\sin(\tilde\omega(\theta)t) & \cos(\tilde\omega(\theta)t) \end{pmatrix} \mathrm{e}^{\mathrm{i}j\theta}\,\mathrm{d}\theta; \tag{A.2}$$

$$G_j(t) = \frac{1}{2\pi}\int_0^{\pi} \begin{pmatrix} h_+(\theta,t,\frac{j}{t}) & \frac{1}{\omega_{\mathrm{r}}(\theta)}h_-\left(\theta,t,\frac{j+\frac12}{t}\right) \\ \omega_{\mathrm{r}}(\theta)h_-\left(\theta,t,\frac{j-\frac12}{t}\right) & h_+(\theta,t,\frac{j}{t}) \end{pmatrix}\mathrm{d}\theta, \tag{A.3}$$

where $h_\pm(\theta,t,c) = \cos(t(\omega(\theta)+\theta c)) \pm \cos(t(\omega(\theta)-\theta c))$;

$$G_j(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \begin{pmatrix} \cos(\omega(\theta)t\pm\theta j) & \frac{\pm\hat s_\theta}{\omega_{\mathrm{r}}(\theta)}\cos(\omega(\theta)t\pm\theta(j+\frac12)) \\ \pm\omega_{\mathrm{r}}(\theta)\hat s_\theta\cos(\omega(\theta)t\pm\theta(j-\frac12)) & \cos(\omega_{\mathrm{r}}t\pm\theta j) \end{pmatrix}\mathrm{d}\theta; \tag{A.4}$$

$$G_j(t) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \begin{pmatrix} \mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} & \pm\frac{\mathrm{e}^{\mathrm{i}\theta}}{\omega_{\mathrm{r}}(2\theta)}\mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} \\ \pm\frac{\omega_{\mathrm{r}}(2\theta)}{\mathrm{e}^{\mathrm{i}\theta}}\mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} & \mathrm{e}^{\mathrm{i}t\phi_\pm(2\theta,c)} \end{pmatrix}\mathrm{d}\theta; \tag{A.5}$$

where $\hat s_\theta = \mathrm{sign}(\theta)$ and $\phi_\pm(\theta,c) = \theta c \pm \tilde\omega(\theta)$.

Proof. We only give the equivalence proof for the two components $G_j^{1,1}(t)$ and $G_j^{1,2}(t)$. The proof for $G_j^{2,1}(t)$ is analogous to that of $G_j^{1,2}(t)$.

(A.1) ⇔ (A.2). The result for the first component is obvious by $\cos(\omega(\theta)t) = \cos(\tilde\omega(\theta)t)$. For the second component we use $\sin(\omega(\theta)t) = \mathrm{sign}\,\theta\sin(\tilde\omega(\theta)t)$, and

$$2\sin\left(\frac{\theta}{2}\right) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}(\mathrm{e}^{\mathrm{i}\theta}-1), \quad \text{i.e.} \quad \frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\omega(\theta)} = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{\omega_{\mathrm{r}}(\theta)}\,\mathrm{sign}\,\theta.$$

(A.1) ⇔ (A.3). For the first component we use

$$\int_0^{\pi}\cos(\omega t)\mathrm{e}^{\mathrm{i}\theta j}\,\mathrm{d}\theta = \frac12\int_0^{\pi}\left[\mathrm{e}^{\mathrm{i}(\omega t+\theta j)} + \mathrm{e}^{-\mathrm{i}(\omega t-\theta j)}\right]\mathrm{d}\theta,$$

$$\int_{-\pi}^{0}\cos(\omega t)\mathrm{e}^{\mathrm{i}\theta j}\,\mathrm{d}\theta = \frac12\int_0^{\pi}\left[\mathrm{e}^{\mathrm{i}(\omega t-\theta j)} + \mathrm{e}^{-\mathrm{i}(\omega t+\theta j)}\right]\mathrm{d}\theta,$$

and for the second we use

$$\begin{aligned}\int_0^{\pi}\frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\omega}\sin(\omega t)\mathrm{e}^{\mathrm{i}\theta j}\,\mathrm{d}\theta &= \int_0^{\pi}\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{\omega_{\mathrm{r}}}\cdot\frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}(\omega t+\theta j)} - \mathrm{e}^{-\mathrm{i}(\omega t-\theta j)}\right]\mathrm{d}\theta \\ &= \frac12\int_0^{\pi}\frac{1}{\omega_{\mathrm{r}}}\left[\mathrm{e}^{\mathrm{i}(\omega t+\theta(j+\frac12))} - \mathrm{e}^{-\mathrm{i}(\omega t-\theta(j+\frac12))}\right]\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}\int_{-\pi}^{0}\frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\omega}\sin(\omega t)\mathrm{e}^{\mathrm{i}\theta j}\,\mathrm{d}\theta &= \int_{-\pi}^{0}\frac{-\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{\omega_{\mathrm{r}}}\cdot\frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}(\omega t+\theta j)} - \mathrm{e}^{-\mathrm{i}(\omega t-\theta j)}\right]\mathrm{d}\theta \\ &= \frac12\int_0^{\pi}\frac{1}{\omega_{\mathrm{r}}}\left[\mathrm{e}^{-\mathrm{i}(\omega t+\theta(j+\frac12))} - \mathrm{e}^{\mathrm{i}(\omega t-\theta(j+\frac12))}\right]\mathrm{d}\theta.\end{aligned}$$

(A.3) ⇔ (A.4). The proof for the first component is again straight forward using $\int_0^\pi \cos(\omega t \mp \theta j)\,\mathrm{d}\theta = \int_{-\pi}^0 \cos(\omega t \pm \theta j)\,\mathrm{d}\theta$. With $\int_0^\pi \frac{1}{\omega_\mathrm{r}} \cos(\omega t \mp \theta(j+\frac12))\,\mathrm{d}\theta = \int_{-\pi}^0 \frac{1}{\omega_\mathrm{r}} \cos(\omega t \pm \theta(j+\frac12))\,\mathrm{d}\theta$ for the second component, we conclude

$$\int_0^\pi \frac{1}{\omega_\mathrm{r}}\Big[\cos(\omega(\theta)t+\theta(j+\tfrac12)) - \cos(\omega(\theta)t-\theta(j+\tfrac12))\Big]\,\mathrm{d}\theta$$
$$= \int_{-\pi}^\pi \frac{\mathrm{sign}(\theta)}{\omega_\mathrm{r}} \cos(\omega(\theta)t+\theta(j+\tfrac12))\,\mathrm{d}\theta = -\int_{-\pi}^\pi \frac{\mathrm{sign}(\theta)}{\omega_\mathrm{r}} \cos(\omega(\theta)t-\theta(j+\tfrac12))\,\mathrm{d}\theta.$$

(A.2) ⇔ (A.5). Because of $\tilde\omega(2\pi+\theta) = -\tilde\omega(\theta)$ we have $\int_{-\pi}^\pi \mathrm{e}^{\mathrm{i}(\mp\tilde\omega t+\theta j)}\,\mathrm{d}\theta = \int_{-3\pi}^{-\pi} \mathrm{e}^{\mathrm{i}(\pm\tilde\omega t+\theta j)}\,\mathrm{d}\theta$. Thus, using the 4π-periodicity of $\tilde\omega$ we get

$$\int_{-\pi}^\pi \cos(\tilde\omega t)\mathrm{e}^{\mathrm{i}\theta j}\,\mathrm{d}\theta = \frac12 \int_{-\pi}^\pi \big[\mathrm{e}^{\mathrm{i}(\tilde\omega t+\theta j)} + \mathrm{e}^{\mathrm{i}(-\tilde\omega t+\theta j)}\big]\mathrm{d}\theta$$
$$= \frac12 \int_{-3\pi}^\pi \mathrm{e}^{\mathrm{i}(\pm\tilde\omega t+\theta j)}\,\mathrm{d}\theta = \int_{-\pi}^\pi \mathrm{e}^{\mathrm{i}(\pm\tilde\omega(2\theta)t+2\theta j)}\,\mathrm{d}\theta,$$

which proves the result for the first component. For the second component we use $\int_{-\pi}^\pi \frac{1}{\omega_\mathrm{r}} \mathrm{e}^{\mathrm{i}(\mp\tilde\omega t+\theta(j+\frac12))}\,\mathrm{d}\theta = -\int_{-3\pi}^{-\pi} \frac{1}{\omega_\mathrm{r}} \mathrm{e}^{\mathrm{i}(\pm\tilde\omega t+\theta(j+\frac12))}\,\mathrm{d}\theta$ such that

$$\int_{-\pi}^\pi \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{\omega_\mathrm{r}(\theta)} \sin(\tilde\omega(\theta)t)\mathrm{e}^{\mathrm{i}\theta j}\,\mathrm{d}\theta = \frac12 \int_{-\pi}^\pi \frac{1}{\omega_\mathrm{r}}\Big[\mathrm{e}^{\mathrm{i}(\tilde\omega t+\theta(j+\frac12))} - \mathrm{e}^{\mathrm{i}(-\tilde\omega t+\theta(j+\frac12))}\Big]\,\mathrm{d}\theta$$
$$= \pm\frac12 \int_{-3\pi}^\pi \frac{1}{\omega_\mathrm{r}} \mathrm{e}^{\mathrm{i}(\pm\tilde\omega t+\theta(j+\frac12))}\,\mathrm{d}\theta$$
$$= \pm \int_{-\pi}^\pi \frac{1}{\omega_{\mathrm{r}(2\theta)}} \mathrm{e}^{\mathrm{i}(\pm\tilde\omega(2\theta)t+2\theta(j+\frac12))}\,\mathrm{d}\theta.$$

Finally, since $t\phi_\pm(\theta+2\pi, c) = t\phi_\pm(\theta, c) + 4\pi k$, the integrand is well defined on $\mathbb{S}^1$ and we actually have $\int_{\mathbb{S}^1} \cdots \mathrm{d}\theta$ instead of $\int_{-\pi}^\pi \cdots \mathrm{d}\theta$. □

Remark A.1. We make the following observations.

- In case of $\omega_\mathrm{r} \equiv 1$ the components of $G_j(t)$ are Bessel functions (cf. [5]).
- The function ω is 2π-periodic, but $\tilde\omega$ is only 4π-periodic.
- For all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we have the relations $\omega^{(2n)}(0) = 0$, $\omega^{(2n+1)}(\pm\pi) = 0$, and $\tilde\omega^{(2n)}(2k\pi) = 0$.
- The first term of h in (A.3) corresponds to wave numbers $c \le 0$, while the second corrensponds to $c \ge 0$.
- The kernel of (A.4) is only 4π-periodic, i.e. it is not $\in \mathrm{C}^\omega(\mathbb{S}^1)$ anymore and we really have $\int_{-\pi}^\pi \cdots \mathrm{d}\theta$ instead of $\int_{\mathbb{S}^1} \cdots \mathrm{d}\theta$. In contrast to that, the kernel of (A.5) is $\mathrm{C}^\omega(\mathbb{S}^1)$.
- In passing to the representation (A.5) the number of critical wave numbers doubles: While the sin and cos terms for each wavenumber θ always involve two group velocities $\pm\omega'(\theta)$, the exponential terms necessitates two different wave numbers.

B. Preparation theorems

In Section 4 we derive uniform asymptotic expansions of integrals

$$I(t,y)=\int_{-\delta}^{\delta}A(\theta)\mathrm{e}^{\mathrm{i}t\phi(\theta,y)}\,\mathrm{d}\theta. \tag{B.1}$$

To do so we are faced with two challenges. First, it is necessary to rewrite the phase function ϕ in a suitable normal form, which is not straight forward if ϕ is degenerated. For instance in Subsection 4.3 we have $\phi(\theta,y)$ analytic with

$$\phi(0,0)=\partial_\theta\phi(0,0)=\partial_\theta^2\phi(0,0)=0,\quad \partial_\theta^3\phi(0,0)>0 \tag{B.2}$$

but $\partial_\theta^2\phi(\theta,y)\neq 0$ for $(\theta,y)\neq 0$ in a neighborhood on $(0,0)$. Second, at one point we want to rewrite A in a suitable factorized form to be able to separate leading order terms, cf. (4.24) and (4.25). In both cases the key ingredients are the preparation theorems of Weierstraß and Malgrange, respectively.

To avoid confusion due to the non-uniform labeling we first recall the different versions of the preparation theorems according to [8]. Then we state and proof a special version to be applied in Subsection 4.3.

Preparation theorems of Weierstraß and Malgrange. The first two theorems are the classical version in the analytic setting.

Theorem B.1 (Weierstraß' preparation theorem, [8, Theorem 7.5.1])**.** *Let g be an analytic function of $(\theta,z)\in\mathbb{C}^{1+n}$ in a neighborhood of $(0,0)$ such that*

$$g(0,0)=\partial_\theta g(0,0)=\cdots=\partial_\theta^{k-1}g(0,0)=0,\quad \partial_\theta^k g(0,0)\neq 0 \tag{B.3}$$

Then there exists a unique factorization

$$g(\theta,z)=h(\theta,z)\left(\theta^k+a_{k-1}(z)\theta^{k-1}+\cdots+a_0(z)\right)$$

where a_j and h are analytic in a neighborhood of 0 and $(0,0)$ respectively, $h(0,0)\neq 0$ and $a_j(0)=0$.

Sometimes the following division theorem is also referred to as Weierstraß' preparation theorem. It is a generalization of the last result and also known as Weierstraß' formula.

Theorem B.2 (Weierstraß' division theorem, [8, Theorem 7.5.2])**.** *Let g and f be analytic functions of $(\theta,z)\in\mathbb{C}^{1+n}$ in a neighborhood of $(0,0)$ and g satisfy* (B.3)*. Then, we have the decomposition*

$$f(\theta,z)=q(\theta,z)g(\theta,z)+r_{k-1}(z)\theta^{k-1}+\cdots+r_0(z),$$

where r_j and q are uniquely determined and analytic in a neighborhood of 0 and $(0,0)$.

Theorem B.1 is a special case of Theorem B.2 with $f(\theta, z) = \theta^k$, where $q(\theta, z) = \frac{1}{h(\theta,z)}$ and $r_j(z) = -a_j(z)$.

The C^∞ counterparts of the last two results are due to Malgrange. The analog of Theorem B.1, namely the Malgrange preparation theorem is again stated and proven in [8]. Here we only state the second result which is sometimes also referred to as Malgrange's preparation theorem or Mather's division theorem.

Theorem B.3 (Malgrange division theorem, [8, Theorem 7.5.6])**.** *Let g and f be C^∞ functions of $(\theta, x) \in \mathbb{R}^{1+n}$ in a neighborhood of $(0,0)$ and g satisfy (B.3). Then, we have the representation*

$$f(\theta, x) = q(\theta, w)g(\theta, x) + r_{k-1}(x)\theta^{k-1} + \cdots + r_0(x),$$

where r_j and q are C^∞ functions in a neighborhood of 0 and $(0,0)$.

The proof can be deduced from the Weierstraß' preparation theorem by decomposing a smooth function as a sum of analytic functions. But note that waiving the analyticity leads to a loss of uniqueness.

Essential in deriving a suitable normal for ϕ to rewrite (B.1) is that the normal form can also be achieved by a change of variables instead of a multiplication.

Theorem B.4 ([8, Theorem 7.5.13])**.** *Let ϕ be a C^∞ function of $(\theta, y) \in \mathbb{R}^{1+n}$ in a neighborhood of $(0,0)$ that satisfies*

$$\phi(0,0) = \partial_\theta\phi(0,0) = \cdots = \partial_\theta^{k-1}\phi(0,0) = 0 \quad \text{and} \quad \partial_\theta^k\phi(0,0) > 0.$$

Then one can find a real-valued C^∞ function $U = U(\theta, w)$ with $U(0,0) = 0$, $\partial_\theta U(0,0) > 0$ and C^∞ functions $a_j(y)$ with $a_j(0) = 0$ such that for $u = U(\theta, y)$ holds

$$\phi(\theta, y) = \tfrac{1}{k}u^k + a_{k-2}(y)u^{k-2} + \cdots + a_0(y).$$

Real-analytic coordinate transform. In view of (B.2) we may apply Theorem B.4 to introduce a new coordinate $u = U(\theta, y)$ such that

$$\phi(\theta, y) = \tfrac{1}{3}u^3 + a(y)u + b(y).$$

The functions U, a and b are C^∞ and $U(0,0) = a(0) = b(0) = 0$. But this result is insufficient in two ways. First it does not yield a real-analytic coordinate change. Second we claim to have $b(y) \equiv 0$ in a special case. Concerning this note that the 3rd power mixes even and odd powers of θ such that $U(-\theta, y) = -U(\theta, y)$, which would imply $b(y) \equiv 0$, it is not obvious. Here we give a modified version of Theorem B.4 which accounts for these two aspects.

Theorem B.5. *Let ϕ be real-analytic a function of $(\theta, y) \in \mathbb{R}^2$ in a neighborhood of $(0,0)$ which satisfies*

$$\phi(0,0) = \partial_\theta \phi(0,0) = \cdots = \partial_\theta^{k-1}\phi(0,0) = 0, \quad \partial_\theta^k \phi(0,0) > 0. \tag{B.4}$$

Then one can find a real-analytic coordinate transform $u = U(\theta, y)$ and real-analytic functions $a_j(y)$ such that

$$\phi(\theta, y) = \tfrac{1}{k}u^k + a_{k-2}(y)u^{k-2} + \cdots + a_1(y)u + a_0(y)$$

Here $U(0,0) = 0$, $\partial_\theta U(0,0) > 0$ and $a_j(0) = 0$.

Furthermore if ϕ is odd with respect to θ, i.e. $\phi(-\theta, y) = -\phi(\theta, y)$, then so is U and $a_0(y) \equiv a_2(y) \equiv \cdots \equiv a_{k-3}(y) \equiv 0$.

Proof. For the general case the proof can be copied from that of Theorem B.4, see [8, Theorem 7.5.13], using Weierstraß' instead of Malgrange's division theorem. Here we only give the proof for the specialization to the case where ϕ is odd.

Due to (B.4) there exists a function ϕ_0 such that $\phi(\theta, 0) = \frac{1}{k}\theta^k \phi_0(\theta)$ with $\phi_0(\theta) > 0$ in a sufficiently small neighborhood of 0. Now we may introduce the new coordinate $\zeta := \theta\phi_0^k(\theta)$, i.e. $\theta = \Theta(\zeta)$, such that $\phi(\Theta(\zeta), 0) = \frac{\zeta^k}{k}$ holds. Note that since ϕ odd with respect to θ, $k = 2J{+}1$ with $J \in \mathbb{N}$. We will use the notation $\tilde{\phi}(\zeta, y) := \phi(\Theta(\zeta), y)$. Here the symmetry of ϕ carries over to that of Θ and $\tilde{\phi}$ with respect to ζ.

We define

$$F(\zeta, y, \alpha) := \tilde{\phi}(\zeta, y) + \sum_{j=0}^{J-1} \alpha_j \zeta^{2j+1}.$$

Then we have in particular $F(\zeta, y, 0) = \phi(\Theta(\zeta), y)$. Now we want to let $\zeta = \zeta(y)$ and $\alpha = \alpha(y)$ vary such that $F(\zeta, y, \alpha)$ remains constant. Then $\frac{\mathrm{d}}{\mathrm{d}y}F = 0$ would lead to

$$\left(\frac{\partial\tilde{\phi}}{\partial\zeta} + \sum_{j=0}^{J-1}(2j+1)\alpha_j\zeta^{2j}\right)\frac{\mathrm{d}\zeta}{\mathrm{d}y} + \frac{\partial\tilde{\phi}}{\partial y} + \sum_{j=0}^{J-1}\frac{\mathrm{d}\alpha_j}{\mathrm{d}y}\zeta^{2j+1} = 0. \tag{B.5}$$

Now we apply Weierstraß' preparation theorem to justify the last equation. To do so we utilize the the symmetry of $\tilde{\phi}$. Note first that the function $\frac{\partial F}{\partial \zeta}$ is even with respect to ζ. Thus, the function defined via

$$\Lambda(\xi, y, \alpha) := \left.\frac{\partial F}{\partial\zeta}\right|_{\zeta^2=\xi} = \left.\frac{\partial\tilde{\phi}}{\partial\zeta}\right|_{\zeta^2=\xi} + \sum_{j=0}^{J-1}(2j{+}1)\alpha_j\xi^j$$

is real-analytic and satisfies (B.3) with $k = J$ and $w = (y, \alpha)$. For the same reason $\frac{\partial\tilde{\phi}}{\partial y}$ is odd with respect to ζ. Thus $f(\theta, y) := \left.\frac{1}{\zeta}\frac{\partial\tilde{\phi}}{\partial y}\right|_{\zeta^2=\xi}$ is again real-analytic. Applying Theorem B.2 to Λ and f leads to

$$\frac{1}{\zeta}\frac{\partial\tilde{\phi}}{\partial y} = q(\zeta^2, y, \alpha)\Big(\frac{\partial\tilde{\phi}}{\partial\zeta} + \sum_{j=0}^{J-1}(2j{+}1)\alpha_j\zeta^{2j}\Big) + \sum_{j=0}^{J-1} r_j(y, \alpha)\zeta^{2j}.$$

In view of this relation (B.5) holds, if the relations

$$\frac{\mathrm{d}\zeta}{\mathrm{d}y} = -\zeta q(\zeta^2, y, \alpha) \quad \text{and} \quad \frac{\mathrm{d}\alpha_j}{\mathrm{d}y} = -r_j(y, \alpha) \quad \text{for } j = 0, \ldots, J-1$$

do as well. Solving these ODE's with initial conditions $\zeta(0) = u$ and $\alpha_j(0) = a_j$ with u and a_j sufficiently close to 0 leads to $\zeta = \zeta(u, y, a)$ and $\alpha_j = \alpha_j(y, a_j)$. These functions are again real-analytic in a neighborhood of 0.

Since $\frac{\mathrm{d}}{\mathrm{d}y} F = 0$ in a neighborhood of 0, we conclude

$$\begin{aligned} F(\zeta, y, \alpha) &= \phi\big(\Theta(\zeta(u, y, a)), y\big) + \sum_{j=0}^{J-1} \alpha_j(y, a_j)[\zeta(u, y, a)]^{2j+1} \\ &= \phi\big(\Theta(\zeta(u, 0, a)), 0\big) + \sum_{j=0}^{J-1} \alpha_j(0, a_j)[\zeta(u, 0, a)]^{2j+1} \\ &= \frac{1}{k} u^k + \sum_{j=0}^{J-1} a_j u^{2j+1}. \end{aligned}$$

Because of $\frac{\partial}{\partial a_j} \alpha_j(0, a_j) = 1$ and $\frac{\partial}{\partial u} \zeta(u, 0, a) = 1$ we can locally invert the functions $\alpha_j = \alpha_j(y, a_j)$ and $\zeta = \zeta(u, y, a)$. Hence $a_j = a_j(y, \alpha_j)$ and $u = u(\zeta, y, \alpha)$, where the dependence is again real-analytic. Thus we get

$$F(\zeta, y, \alpha) = \frac{1}{k}[u(\zeta, y, \alpha)]^k + \sum_{j=0}^{J-1} a_j(y, \alpha_j)[u(\zeta, y, \alpha)]^{2j+1}.$$

Using the definition of F and setting $\alpha = 0$ leads to the desired normal form and $U(\theta, y) := u(\Theta^{-1}(\theta), y, 0)$ is the corresponding coordinate transformation. Finally, the conditions on U and a_j in 0 as well as the symmetry of U with respect to θ are easily checked. Thus, Theorem B.5 is established. □

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