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Multivariate Wave-Packet Transforms

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Abstract. This paper presents a study for square-integrability of classical multivariate wave-packets in $L^2(\mathbb{R}^d)$ via group representation theory. The abstract notions of multivariate wave-packet groups and multivariate wave-packet representations will be introduced and as the main result, we prove an admissibility condition on closed subgroups of $\mathrm{GL}(d,\mathbb{R})$, which guarantees the square integrability of classical multivariate wave-packet representations on $L^2(\mathbb{R}^d)$. Finally, we present application of our results in the case of different admissible subgroups.

Keywords. Multivariate wavelet (Gabor) transforms, multivariate wave-packet representations, multivariate wave-packet groups, multivariate wave-packet transforms
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1. Introduction

The mathematical theory of covariant and coherent states transforms is one of the main building blocks of theoretical physics, modern high frequency approximation techniques and time-frequency (resp. time-scale) analysis [4, 30, 31, 34, 35]. Over the last decades, abstract and computational aspects of covariant and coherent states transforms have achieved significant popularity in mathematical and theoretical physics, scientific computing, and computational engineering, see [6] and references therein. In a nutshell, coherent state transforms are obtained by a given coherent function systems. Then admissibility conditions on the coherent system imply analyzing of functions with respect to the system by the inner product evaluation [22]. Such coherent structures are classically originated from representation theory of locally compact groups, see [18,22,29,32] and references therein. Commonly used coherent states transforms in theoretical physics, are wavelet transform [11, 30], Gabor transform [20, 21], wave-packet transform [13, 16, 17].

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The mathematical theory of Gabor analysis is based on the coherent state generated by modulations and translations of a given window function. Wavelet analysis is a time-scale analysis which is based on the continuous affine group as the group of dilations and translations. Wave packet analysis which is also well-known as Gabor-wavelet analysis is a shrewd extensions of the two most prominent coherent states analysis, namely Gabor and wavelet analysis [28, 39, 40]. The mathematical theory of wave-packet analysis on the real line is originated from classical dilations, translations, and modulations of a given window function. The structure of discrete wave-packet systems over the real line has been studied for higher dimensions by several authors, see [7].

The following paper consists of nature of multivariate wave-packet transforms over $L^2(\mathbb{R}^d)$. We aim to introduce the notion of multivariate wave-packet transform over the Hilbert function space $L^2(\mathbb{R}^d)$ associated to closed subgroups of the general linear group $\mathrm{GL}(d,\mathbb{R})$. We shall address analytic aspects of multivariate wave-packet transforms over $L^2(\mathbb{R}^d)$ using classical tools in coherent state analysis. This article contains 6 sections. Section 2 is devoted to fix notations and a summary of classical Fourier analysis on $\mathbb{R}^d$ and harmonic analysis on projective representations and square integrable representations over locally compact groups. In section 3 we present a brief study of harmonic analysis over the matrix Lie group $\mathrm{GL}(d,\mathbb{R})$. Then we introduce the abstract notion of multivariate wave-packet groups associated to closed subgroups of $\mathrm{GL}(d,\mathbb{R})$. We shall also show that the group structure of multivariate wave-packet groups canonically determines an irreducible projective (unitary) group representation of the group, which is called multivariate wave-packet representation. We then present an admissibility criterion on closed subgroups of $\mathrm{GL}(d,\mathbb{R})$ to guarantee the square integrablity of the associated multivariate wave-packet representation on $L^2(\mathbb{R}^d)$. As an application of our results we study analytic aspects of multivariate wave-packet transforms associated to closed subgroups of $\mathrm{GL}(d,\mathbb{R})$. It is also shown that, if $\mathbb{H}$ is a compact subgroup of $\mathrm{GL}(d,\mathbb{R})$, for all non-zero window functions we can continuously reconstruct any L^2-function from multivariate wave-packet coefficients. Finally, we will illustrate application of these techniques in the case of well-known compact subgroups of $\mathrm{GL}(d,\mathbb{R})$.

2. Preliminaries and notations

Let G be a locally compact group and $\mathcal{H}$ be a Hilbert space. Let $\mathcal{U}(\mathcal{H})$ be the multiplicative group of all unitary operators on $\mathcal{H}$. A projective group representation of G on $\mathcal{H}$ is a mapping $\Gamma : G \to \mathcal{U}(\mathcal{H})$ which satisfies

$$\Gamma(gg') = z(g,g')\Gamma(g)\Gamma(g') \quad \text{for all } g,g' \in G$$

where $z(g,g')$ are unimodular numbers. The projective group representation Γ is called irreducible on $\mathcal{H}$, if $\{0\}$ and $\mathcal{H}$ are the only closed Γ-invariant subspaces of $\mathcal{H}$.

A projective group representation $(\Gamma, \mathcal{H})$ is called left square integrable if there exists a non-zero vector $\zeta \in \mathcal{H}$ such that

$$\int_G |\langle \zeta, \Gamma(g)\zeta\rangle|^2 \mathrm{d}m_G(g) < \infty,$$

for some left Haar measure m_G of G. Similarly, it is called right square integrable if there exists a non-zero vector $\zeta \in \mathcal{H}$ such that

$$\int_G |\langle \zeta, \Gamma(g)\zeta\rangle|^2 \mathrm{d}n_G(g) < \infty,$$

for some right Haar measure n_G of G.

Since $\mathbb{R}^d$ is an LCA (locally compact Abelian) group, according to the Schur's Lemma, all irreducible representations of $\mathbb{R}^d$ are one-dimensional. Thus any irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of $\mathbb{R}^d$ satisfies $\mathcal{H}_\pi = \mathbb{C}$ and hence there exists a continuous homomorphism ω of $\mathbb{R}^d$ into the circle group $\mathbb{T}$, such that for each $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $z \in \mathbb{C}$ we have $\pi(x)(z) = \omega(x)z$. Such homomorphisms are called characters of $\mathbb{R}^d$ and the set of all such characters of $\mathbb{R}^d$ is denoted by $\widehat{\mathbb{R}^d}$. If $\widehat{\mathbb{R}^d}$ equipped with the topology of compact convergence on $\mathbb{R}^d$ which coincides with the w^*-topology that $\widehat{\mathbb{R}^d}$ inherits as a subset of $L^\infty(\mathbb{R}^d)$, then $\widehat{\mathbb{R}^d}$ with respect to the product of characters is an LCA group which is called the dual (character) group of $\mathbb{R}^d$. The character group $\widehat{\mathbb{R}^d}$, that is the multiplicative group of all continuous additive homomorphisms of $\mathbb{R}^d$ into the circle group $\mathbb{T}$, can be parametrizes by $\mathbb{R}^d$ via the following duality notation $\widehat{\mathbb{R}^d}$ with $\mathbb{R}^d$ via

$$\omega(x) = \langle x, \omega\rangle = e^{2\pi i \omega^T \cdot x}$$

for each $\omega \in \widehat{\mathbb{R}^d}$. The linear map $\mathcal{F}_{\mathbb{R}^d} : L^1(\mathbb{R}^d) \to \mathcal{C}(\widehat{\mathbb{R}^d})$ defined by $f \mapsto \mathcal{F}_{\mathbb{R}^d}(f) = \widehat{f}$ via

$$\mathcal{F}_{\mathbb{R}^d}(f)(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} f(s)\overline{\omega(s)} \mathrm{d}m_{\mathbb{R}^d}(s),$$

is called the Fourier transform on $\mathbb{R}^d$. It is a norm-decreasing $*$-homomorphism from $L^1(\mathbb{R}^d)$ into $\mathcal{C}_0(\widehat{\mathbb{R}^d})$ with a uniformly dense range in $\mathcal{C}_0(\widehat{\mathbb{R}^d})$. If a Haar measure $m_{\mathbb{R}^d}$ on $\mathbb{R}^d$ is given and fixed then there is a Haar measure $m_{\widehat{\mathbb{R}^d}}$ on $\widehat{\mathbb{R}^d}$, which is called the normalized Plancherel measure associated to $m_{\mathbb{R}^d}$, such that the Fourier transform (2) is an isometric transform on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and hence it can be extended uniquely to a unitary isomorphism from $L^2(\mathbb{R}^d)$ onto

$L^2(\widehat{\mathbb{R}^d})$, see [10, 25]. Then each $f \in L^1(\mathbb{R}^d)$ with $\widehat{f} \in L^1(\widehat{\mathbb{R}^d})$ satisfies the following Fourier inversion formula

$$f(s) = \int_{\widehat{\mathbb{R}^d}} \widehat{f}(\omega)\omega(s)\mathrm{d}m_{\widehat{\mathbb{R}^d}}(\omega) \quad \text{for a.e. } s \in \mathbb{R}^d.$$

For $x \in \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$, the translation of f by x is defined by $T_x f(y) = f(y-x)$ for $y \in \mathbb{R}^d$. The translation $T_x : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a unitary operator. For $\omega \in \widehat{\mathbb{R}^d}$ and $f \in L^2(\mathbb{R}^d)$, the modulation of f by ω is defined by $M_\omega f(y) = \overline{\omega(y)} f(y)$ for $s \in \mathbb{R}^d$. The modulation operator $M_\omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is unitary as well. The modulation and translation operators are connected via the Fourier transform by

$$\widehat{M_\omega f} = T_{-\omega}\widehat{f}, \quad \widehat{T_k f} = M_k \widehat{f},$$

for all $f \in L^2(\mathbb{R}^d)$, $\omega \in \widehat{\mathbb{R}^d}$, and $k \in \mathbb{R}^d$, see [10, 24, 37].

From now on and in this article, for a fixed Haar (Lebesgue) measure $m_{\mathbb{R}^d}$ on $\mathbb{R}^d$, by $\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}$ or $\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}$ we mean the induced product measure on $\mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathbb{R}^d \times \widehat{\mathbb{R}^d}$, that is $\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(x, \omega) = \mathrm{d}m_{\mathbb{R}^d}(x)\mathrm{d}m_{\widehat{\mathbb{R}^d}}(\omega)$, where $m_{\widehat{\mathbb{R}^d}}$ is the normalized Plancherel measure associated to $m_{\mathbb{R}^d}$.

For $\lambda = (x, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathbb{R}^d \times \widehat{\mathbb{R}^d}$, the time-frequency shift operator $\pi(\lambda) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is defined by $\pi(\lambda) = M_\omega T_x$. Then, it is well-known as the Moyal's formula, that

$$\int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} |\langle f, \pi(\lambda)g\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) = \|f\|^2_{L^2(\mathbb{R}^d)} \|g\|^2_{L^2(\mathbb{R}^d)}, \tag{1}$$

for all $f, g \in L^2(\mathbb{R}^d)$, see [11, 23] and classical references therein.

3. Harmonic analysis over general linear groups

Throughout this section we briefly present basics of harmonic analysis over the multiplicative matrix group $\mathrm{GL}(d, \mathbb{R})$, for a complete picture of this matrix group we referee the readers to [26, 27, 33] and the comprehensive list of references references therein.

For $d \geq 1$, the real general linear group $\mathrm{GL}(d, \mathbb{R})$, is the multiplicative group consists of all $d \times d$ invertible matrices with real entries, that is

$$\mathrm{GL}(d, \mathbb{R}) := \{A \in M_{d\times d}(\mathbb{R}) : \det(A) \neq 0\}.$$

It is a d^2-dimensional real Lie group. It is non-compact but unimodular. A Haar integral (measure) of $\mathrm{GL}(d, \mathbb{R})$ is given by

$$\int_{\mathrm{GL}(d,\mathbb{R})} \phi(A)\mathrm{d}\sigma_{\mathrm{GL}(d,\mathbb{R})}(A) = \int_{M_{d\times d}(\mathbb{R})} \phi(A)|\det(A)|^{-d}\mathrm{d}A,$$

for all $\phi \in \mathcal{C}_c(\mathrm{GL}(d,\mathbb{R}))$, where $\mathrm{d}A$ is the Lebesgue measure over the linear vector space of all $d \times d$ matrices with real entries.

Proposition 3.1. *Let $d \geq 1$ and $m_{\mathbb{R}^d}$ be the Lebesgue measure on $\mathbb{R}^d$. Let $m_{\widehat{\mathbb{R}^d}}$ be the normalized Plancherel measure associated to $m_{\mathbb{R}^d}$ and $A \in \mathrm{GL}(d,\mathbb{R})$. Then*

1. $\mathrm{d}m_{\mathbb{R}^d}(Ax) = |\det(A)|\mathrm{d}m_{\mathbb{R}^d}(x)$.
2. $\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(A\cdot\lambda) = \mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$, *where* $A\cdot\lambda := (Ax, A^{-1}\omega)$ *for* $\lambda = (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}^d}$.

Proof. 1. It is a straightforward consequence of the structure of the Lebesgue measure.

2. Based on our notations, and using 1., we can write

$$\begin{aligned}
\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(A\cdot\lambda) &= \mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(Ax, A^{-1}\omega)\\
&= \mathrm{d}m_{\mathbb{R}^d}(Ax)\mathrm{d}m_{\widehat{\mathbb{R}^d}}(A^{-1}\omega)\\
&= |\det(A)|\cdot|\det(A^{-1})|\mathrm{d}m_{\mathbb{R}^d}(x)\mathrm{d}m_{\widehat{\mathbb{R}^d}}(\omega)\\
&= \mathrm{d}m_{\mathbb{R}^d}(x)\mathrm{d}m_{\widehat{\mathbb{R}^d}}(\omega)\\
&= \mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda).
\end{aligned}$$ □

For $A \in \mathrm{GL}(d,\mathbb{R})$, the dilation operator $D_A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is given by

$$D_A f(t) := |\det A|^{-\frac{1}{2}} f(A^{-1}\cdot t),$$

for all $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$.

The following observations state basic properties of dilation operators.

Proposition 3.2. *Let $d \geq 1$, $A, B \in \mathrm{GL}(d,\mathbb{R})$, and $f \in L^2(\mathbb{R}^d)$. Then*

1. $D_A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is a unitary linear operator.*
2. $D_{AB} = D_A D_B$.
3. $\widehat{D_A f} = D_{A^{-1}}\widehat{f}$.
4. $A \mapsto D_A$ *is a unitary representation of* $\mathrm{GL}(d,\mathbb{R})$ *on the Hilbert function space* $L^2(\mathbb{R}^d)$.

Next proposition summarizes commuting relations of basic operators in multivariate wave packet analysis.

Proposition 3.3. *Let $d \geq 1$ and $\mathbb{H}$ be a subgroup of the general linear group $\mathrm{GL}(d,\mathbb{R})$. Then,*

1. *For* $(A,x) \in \mathbb{H}\times\mathbb{R}^d$ *we have* $D_A T_x = T_{Ax} D_A$.
2. *For* $(A,\omega) \in \mathbb{H}\times\widehat{\mathbb{R}^d}$ *we have* $D_A M_\omega = M_{A^{-1}\omega} D_A$.
3. *For* $(x,\omega) \in \mathbb{R}^d\times\widehat{\mathbb{R}^d}$ *we have* $T_x M_\omega = \omega(x) M_\omega T_x$.

4. Multivariate wave-packet representations

In this section we present the abstract structure of multivariate wave-packet groups associated to closed subgroups of $\mathrm{GL}(d,\mathbb{R})$. Then we introduce the associated multivariate multivariate wave-packet representation. We shall also study classical properties of these representations.

For a closed subgroup $\mathbb{H}$ of the general linear group $\mathrm{GL}(d,\mathbb{R})$, the underlying manifold

$$\mathbb{W}(\mathbb{H}) := \mathbb{H} \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathbb{H} \times \mathbb{R}^d \times \mathbb{R}^d,$$

equipped with operations given by

$$(A, x, \omega) \rtimes (A', x', \omega') = (AA', A'^{-1}x + x', A'\omega + \omega'),$$

and

$$(A, x, \omega)^{-1} := (A^{-1}, -Ax, -A^{-1}\omega),$$

is a group with the identity element $(\mathbf{1}, 0, 0)$.

We call the group $\mathbb{W}(\mathbb{H})$ as multivariate wave-packet group associated to the subgroup $\mathbb{H}$ over $\mathbb{R}^d$.

Remark 4.1. (i) The groups $\mathbb{H}$ and $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ can be considered as closed subgroups of $\mathbb{W}(\mathbb{H})$.

(ii) Let $\mathbb{H}$ be a closed subgroup of $\mathrm{GL}(d,\mathbb{R})$ and K be a closed subgroup of $\mathbb{H}$. Then $\mathbb{W}(K)$ is a closed subgroup of $\mathbb{W}(\mathbb{H})$.

Then we present the following theorem concerning basic properties of the group $\mathbb{W}(\mathbb{H})$.

Theorem 4.2. *Let $\mathbb{H}$ be a closed subgroup of the general linear group $\mathrm{GL}(d,\mathbb{R})$ with the modular function $\Delta_{\mathbb{H}}$ and $m_{\mathbb{H}}$ (resp. $n_{\mathbb{H}}$) be a left (resp. right) Haar measure for $\mathbb{H}$. Then*

1. $\mathbb{W}(\mathbb{H})$ *is a locally compact group with a left Haar measure given by*

$$\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A, \lambda) := \mathrm{d}m_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda),$$

and a right Haar measure given by

$$\mathrm{d}n_{\mathbb{W}(\mathbb{H})}(A, \lambda) := \mathrm{d}n_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda).$$

2. *The modular function $\Delta_{\mathbb{W}(\mathbb{H})} : \mathbb{W}(\mathbb{H}) \to (0, \infty)$ is given by $\Delta_{\mathbb{W}(\mathbb{H})}(A, \lambda) := \Delta_{\mathbb{H}}(A)$. In particular, the multivariate wave-packet group $\mathbb{W}(\mathbb{H})$ is unimodular if and only if $\mathbb{H}$ is unimodular.*
3. *The closed subgroup $\mathbb{H}$ is normal in $\mathbb{W}(\mathbb{H})$ if and only if $\mathbb{H} = \{\mathbf{I}\}$.*
4. *The closed subgroup $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is a normal Abelian subgroup of $\mathbb{W}(\mathbb{H})$.*

Proof. 1. It is easy to see that the mapping $\tau : \mathbb{H} \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} \to \mathbb{R}^d \times \widehat{\mathbb{R}^d}$ given by $(A, \lambda) \to A \cdot \lambda$ is continuous. This automatically implies that the multivariate wave-packet group $\mathbb{W}(\mathbb{H})$ is a locally compact group. Let $F \in \mathcal{C}_c(\mathbb{W}(\mathbb{H}))$ and $\mathbf{g} = (A, \lambda) \in \mathbb{W}(\mathbb{H})$. Since the Lebesgue measure $\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}$ is translation invariant and also $m_{\mathbb{H}}$ is a left Haar measure on $\mathbb{H}$, we have

$$\begin{aligned}
&\int_{\mathbb{W}(\mathbb{H})} F(\mathbf{g}\cdot\mathbf{g}')\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(\mathbf{g}') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((A,\lambda)\rtimes(A',\lambda'))\mathrm{d}m_{\mathbb{H}}(A')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((AA', A'^{-1}\cdot\lambda+\lambda'))\mathrm{d}m_{\mathbb{H}}(A')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((AA', A'^{-1}\cdot\lambda+\lambda'))\mathrm{d}m_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}m_{\mathbb{H}}(A') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(AA', \lambda')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}m_{\mathbb{H}}(A') \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\left(\int_{\mathbb{H}} F(AA', \lambda')\mathrm{d}m_{\mathbb{H}}(A')\right)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}\left(\int_{\mathbb{H}} F(A', \lambda')\mathrm{d}m_{\mathbb{H}}(A')\right)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(A', \lambda')\mathrm{d}m_{\mathbb{H}}(A')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{W}(\mathbb{H})} F(\mathbf{g}')\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(\mathbf{g}'),
\end{aligned}$$

which implies that $\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A,\lambda) := \mathrm{d}m_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$ is a left Haar measure for $\mathbb{W}(\mathbb{H})$. Similarly, using Proposition 3.1, Fubini's theorem and also since the Lebesgue measure $\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}$ is translation invariant, we get

$$\begin{aligned}
&\int_{\mathbb{W}(\mathbb{H})} F(\mathbf{g}'\cdot\mathbf{g})\mathrm{d}n_{\mathbb{W}(\mathbb{H})}(\mathbf{g}') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F((A',\lambda')\rtimes(A,\lambda))\mathrm{d}n_{\mathbb{H}}(A')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(A'A, A^{-1}\cdot\lambda'+\lambda)\mathrm{d}n_{\mathbb{H}}(A')\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(A'A, A^{-1}\cdot\lambda'+\lambda)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}n_{\mathbb{H}}(A') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(A'A, \lambda'+\lambda)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(A\cdot\lambda')\right)\mathrm{d}n_{\mathbb{H}}(A') \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} F(A'A, \lambda'+\lambda)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda')\right)\mathrm{d}n_{\mathbb{H}}(A')
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{H}} \left(\int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(A'A, \lambda') \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \right) \mathrm{d}n_{\mathbb{H}}(A') \\
&= \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \left(\int_{\mathbb{H}} F(A'A, \lambda') \mathrm{d}n_{\mathbb{H}}(A') \right) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \left(\int_{\mathbb{H}} F(A', \lambda') \mathrm{d}n_{\mathbb{H}}(A') \right) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(A', \lambda') \mathrm{d}n_{\mathbb{H}}(A') \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{W}(\mathbb{H})} F(\mathbf{g}') \mathrm{d}n_{\mathbb{W}(\mathbb{H})}(\mathbf{g}'),
\end{aligned}
$$

implying that $\mathrm{d}n_{\mathbb{W}(\mathbb{H})}(A, \lambda) := \mathrm{d}n_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda)$ is a right Haar measure for $\mathbb{W}(\mathbb{H})$.

2. Let $F \in \mathcal{C}_c(\mathbb{W}(\mathbb{H}))$ be a non-zero and positive function. Also, let $(A, \lambda) \in \mathbb{W}(\mathbb{H})$. Then we can write

$$
\begin{aligned}
&\Delta_{\mathbb{W}(\mathbb{H})}(A, \lambda)^{-1} \cdot \int_{\mathbb{W}(\mathbb{H})} F(A', \lambda') \mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A', \lambda') \\
&= \int_{\mathbb{W}(\mathbb{H})} F((A', \lambda') \rtimes (A, \lambda)) \mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A', \lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(A', \lambda') \rtimes (A, \lambda) \mathrm{d}m_{\mathbb{H}}(A') \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(A'A, A^{-1} \cdot \lambda' + \lambda) \mathrm{d}m_{\mathbb{H}}(A') \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(A'A, \lambda' + \lambda) \mathrm{d}m_{\mathbb{H}}(A') \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(A \cdot \lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(A'A, \lambda + \lambda') \mathrm{d}m_{\mathbb{H}}(A') \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{H}} \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} F(A'A, \lambda') \mathrm{d}m_{\mathbb{H}}(A') \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \left(\int_{\mathbb{H}} F(A'A, \lambda') \mathrm{d}m_{\mathbb{H}}(A') \right) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \Delta_{\mathbb{H}}(A)^{-1} \cdot \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} \left(\int_{\mathbb{H}} F(A', \lambda') \mathrm{d}m_{\mathbb{H}}(A') \right) \mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda') \\
&= \Delta_{\mathbb{H}}(A)^{-1} \cdot \int_{\mathbb{W}(\mathbb{H})} F(A', \lambda') \mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A', \lambda'),
\end{aligned}
$$

implying that $\Delta_{\mathbb{W}(\mathbb{H})}(A, \lambda) = \Delta_{\mathbb{H}}(A)$ for all $(A, \lambda) \in \mathbb{W}(\mathbb{H})$.

3. and 4. are straightforward from structure of the group $\mathbb{W}(\mathbb{H})$. □

Remark 4.3. From now on, once the left (resp. right) Haar measure $m_{\mathbb{H}}$ (resp. $n_{\mathbb{H}}$) over $\mathbb{H}$ is fixed, we call the associated left (resp. right) Haar measure on $\mathbb{W}(\mathbb{H})$, which is constructed via Theorem 4.2, as left (resp. right) Haar measure induced by $m_{\mathbb{H}}$ (resp. $n_{\mathbb{H}}$).

For $(A,\lambda) = (A,x,\omega) \in \mathbb{W}(\mathbb{H})$, define the linear operator

$$\Gamma_{\mathbb{H}}(A,\lambda) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \quad \text{by} \quad \Gamma_{\mathbb{H}}(A,\lambda) := D_A\pi(\lambda) = D_A T_x M_\omega. \tag{2}$$

Thus for $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$ we get

$$\begin{aligned}[\Gamma_{\mathbb{H}}(A,x,\omega)f](t) &= D_A T_x M_\omega f(t)\\ &= |\det A|^{-\frac{1}{2}} T_x M_\omega f(A^{-1}t)\\ &= |\det A|^{-\frac{1}{2}} M_\omega f(A^{-1}t - x)\\ &= |\det A|^{-\frac{1}{2}} \omega(x)\overline{\omega(A^{-1}t)} f(A^{-1}t - x).\end{aligned}$$

Remark 4.4. Let $\mathbb{H}$ be a closed subgroup of the general linear group $\mathrm{GL}(d,\mathbb{R})$. The restriction of $\Gamma_{\mathbb{H}}$ to the closed subgroup $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ is unitarily equivalent to the projective Schrödinger representation of the group $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ on $L^2(\mathbb{R}^d)$ (see [23] and references therein) and similarly restriction of $\Gamma_{\mathbb{H}}$ to the closed subgroup $\mathbb{H} \times \mathbb{R}^d$ is unitarily equivalent to the quasi-regular representation of the group $\mathbb{H} \rtimes \mathbb{R}^d$ on $L^2(\mathbb{R}^d)$, see [2] and references therein.

The following theorem shows that $(A,\lambda) \mapsto \Gamma_{\mathbb{H}}(A,\lambda)$ given by (2), defines an irreducible projective group representation of the multivariate wave-packet group $\mathbb{W}(\mathbb{H})$ on the Hilbert space $L^2(\mathbb{R}^d)$.

Theorem 4.5. *Let $\mathbb{H}$ be a closed subgroup of the general linear group $\mathrm{GL}(d,\mathbb{R})$ and $\mathbb{W}(\mathbb{H})$ be the multivariate wave-packet group associated to $\mathbb{H}$. Then $\Gamma_{\mathbb{H}} : \mathbb{W}(\mathbb{H}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ given by $(A,\lambda) \mapsto \Gamma_{\mathbb{H}}(A,\lambda)$ is an irreducible projective group representation of the locally compact group $\mathbb{W}(\mathbb{H})$ on the Hilbert space $L^2(\mathbb{R}^d)$.*

Proof. It is evident to check that $\Gamma_{\mathbb{H}}(1,0,0) = I$. Then the operator $\Gamma_{\mathbb{H}}(A,x,\omega)$ is a unitary operator on $L^2(\mathbb{R}^d)$ for all $(A,x,\omega) \in \mathbb{W}(\mathbb{H})$, because it is the composition of three unitary operators, namely D_A, T_x and M_ω. Now let $(A,x,\omega),(A',x',\omega') \in \mathbb{W}(\mathbb{H})$. Then we have

$$\begin{aligned}D_{AA'}T_{A'^{-1}x+x'}M_{A'\omega+\omega'} &= D_A(D_{A'}T_{A'^{-1}x})T_{x'}M_{A'\omega}M_{\omega'}\\ &= D_A(T_x D_{A'})T_{x'}M_{A'\omega}M_{\omega'}\\ &= D_A T_k D_{A'}(T_{x'}M_{A'\omega})M_{\omega'}\\ &= \omega(A'x')D_A T_x D_{A'}(M_{A'\omega}T_{x'})M_{\omega'}\\ &= \omega(A'x')D_A T_x(D_{A'}M_{A'\omega})T_{x'}M_{\omega'}\\ &= \omega(A'x')D_A T_x(M_\omega D_{A'})T_{x'}M_{\omega'}\\ &= \omega(A'x')(D_A T_x M_\omega)(D_{A'}T_{x'}M_{\omega'}).\end{aligned}$$

Thus invoking the group law of the wave packet group $\mathbb{W}(\mathbb{H})$, we get

$$\begin{aligned}\Gamma_{\mathbb{H}}\left((A,x,\omega)\rtimes(A',x',\omega')\right) &= \Gamma_{\mathbb{H}}(AA',A'^{-1}x+x',A'\omega+\omega')\\ &= D_{AA'}T_{A'^{-1}x+x'}M_{A'\omega+\omega'}\\ &= \omega(A'x')(D_AT_xM_\omega)(D_{A'}T_{x'}M_{\omega'})\\ &= \omega(A'x')\Gamma_{\mathbb{H}}(A,x,\omega)\Gamma_{\mathbb{H}}(A',x',\omega'),\end{aligned}$$

which implies that $\Gamma_{\mathbb{H}}\colon \mathbb{W}(\mathbb{H})\to\mathcal{U}(L^2(\mathbb{R}^d))$ is a unitary projective group representation of the locally compact group $\mathbb{W}(\mathbb{H})$ on the Hilbert space $L^2(\mathbb{R}^d)$. Using Remark 4.4 and since the projective Schrödinger representation of $\mathbb{R}^d\times\widehat{\mathbb{R}^d}$ is irreducible on $L^2(\mathbb{R}^d)$, we deduce that $\Gamma_{\mathbb{H}}$ is a unitary irreducible projective group representation of the locally compact group $\mathbb{W}(\mathbb{H})$ on the Hilbert space $L^2(\mathbb{R}^d)$ as well. □

5. Multivariate wave-packet transforms

Throughout this section, we still assume that $\mathbb{H}$ is a closed subgroup of the multiplicative matrix group $\mathrm{GL}(d,\mathbb{R})$.

Let $\psi\in L^2(\mathbb{R}^d)$ be a window function. The multivariate wave-packet transform of $f\in L^2(\mathbb{R}^d)$ with respect to the window function ψ is given by the voice transform associated to the multivariate wave-packet representation, that is

$$\mathcal{V}_\psi f(A,x,\omega) := \langle f,\Gamma_{\mathbb{H}}(A,x,\omega)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle f, D_AT_xM_\omega\psi\rangle_{L^2(\mathbb{R}^d)}, \tag{3}$$

for $(A,x,\omega)\in\mathbb{H}\times\mathbb{R}^d\times\widehat{\mathbb{R}^d}$.

Remark 5.1. (i) The restriction of the multivariate wave-packet transform to the closed subgroup $\mathbb{R}^d\times\widehat{\mathbb{R}^d}$ is the continuous Gabor (short-time Fourier) transform over $L^2(\mathbb{R}^d)$, see [21] and references therein.

(ii) Let $\mathbb{H}$ be a closed subgroup of the general linear group $\mathrm{GL}(d,\mathbb{R})$. Then the restriction of the multivariate wave-packet transform to the closed subgroup $\mathbb{H}\times\mathbb{R}^d$ is the wavelet transform induced by the action of the multiplicative group $\mathbb{H}$, see [2].

The following theorem can be considered as a constructive criterion on the subgroup $\mathbb{H}$, which guarantees the square integrability of the associated multivariate wave-packet representation $\Gamma_{\mathbb{H}}$ on $L^2(\mathbb{R}^d)$.

Theorem 5.2. *Let $\mathbb{H}$ be a closed subgroup of the multiplicative matrix group $\mathrm{GL}(d,\mathbb{R})$ and $\mathbb{W}(\mathbb{H})$ be the associated multivariate wave-packet group. Then, the multivariate wave-packet representation $\Gamma_{\mathbb{H}}$ is left (resp. right) square integrable over $\mathbb{W}(\mathbb{H})$ if and only if $\mathbb{H}$ is compact. In this case, all non-zero functions in $L^2(\mathbb{R}^d)$ are square integrable over $\mathbb{W}(\mathbb{H})$ with respect to $\Gamma_{\mathbb{H}}$.*

Proof. Let $m_{\mathbb{H}}$ be a left Haar measure for $\mathbb{H}$. Then by Theorem 4.2, the positive Radon measure $m_{\mathbb{W}(\mathbb{H})}$ given by $\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A,\lambda) = \mathrm{d}m_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$ is a left Haar measure for $\mathbb{W}(\mathbb{H})$. Now, suppose that the multivariate wave-packet representation $\Gamma_{\mathbb{H}}$ be left square integrable over $\mathbb{W}(\mathbb{H})$. Then there exists a non-zero function $\psi \in L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{W}(\mathbb{H})} |\langle \psi, \Gamma_{\mathbb{H}}(\mathbf{g})\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}m_{\mathbb{W}(\mathbb{H})}(\mathbf{g}) < \infty.$$

Using Fubini's theorem and also the Moyal's formula (1), we get

$$\begin{aligned}
&\int_{\mathbb{W}(\mathbb{H})} |\langle \psi, \Gamma_{\mathbb{H}}(\mathbf{g})\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}m_{\mathbb{W}(\mathbb{H})}(\mathbf{g}) \\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |\langle \psi, \Gamma_{\mathbb{H}}(A,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}m_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda) \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |\langle \psi, \Gamma_{\mathbb{H}}(A,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\right)\mathrm{d}m_{\mathbb{H}}(A) \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |\langle \psi, D_A\pi(\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\right)\mathrm{d}m_{\mathbb{H}}(A) \\
&= \int_{\mathbb{H}}\left(\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} |\langle D_A^*\psi, \pi(\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\right)\mathrm{d}m_{\mathbb{H}}(A) \\
&= \int_{\mathbb{H}}\left(\|D_A^*\psi\|^2_{L^2(\mathbb{R}^d)}\|\psi\|^2_{L^2(\mathbb{R}^d)}\right)\mathrm{d}m_{\mathbb{H}}(A) \\
&= \|\psi\|^2_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\|D_A^*\psi\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}m_{\mathbb{H}}(A)\right).
\end{aligned}$$

Since dilation operators are unitary on $L^2(\mathbb{R}^d)$, we deduce that

$$\begin{aligned}
\|\psi\|^4_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\mathrm{d}m_{\mathbb{H}}\right) &= \|\psi\|^2_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\|\psi\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}m_{\mathbb{H}}(A)\right) \\
&= \|\psi\|^2_{L^2(\mathbb{R}^d)}\left(\int_{\mathbb{H}}\|D_A^*\psi\|^2_{L^2(\mathbb{R}^d)}\mathrm{d}m_{\mathbb{H}}(A)\right) \\
&= \int_{\mathbb{W}(\mathbb{H})} |\langle \psi, \Gamma_{\mathbb{H}}(\mathbf{g})\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}m_{\mathbb{W}(\mathbb{H})}(\mathbf{g}) < \infty.
\end{aligned}$$

Thus $m_{\mathbb{H}}(\mathbb{H}) < \infty$ and hence $\mathbb{H}$ is compact. Conversely, let $\mathbb{H}$ be a compact subgroup of $\mathrm{GL}(d,\mathbb{R})$ with the normalized Haar measure $\sigma_{\mathbb{H}}$, that is the unique positive Radon measure $\sigma_{\mathbb{H}}$ which is both left and right Haar measure of $\mathbb{H}$ with $\sigma_{\mathbb{H}}(\mathbb{H}) = 1$. Then, each non-zero function $\psi \in L^2(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{W}(\mathbb{H})} |\langle \psi, \Gamma_{\mathbb{H}}(A,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}|^2 \mathrm{d}\sigma_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^4_{L^2(\mathbb{R}^d)}, \tag{4}$$

which implies the square integrabilty of the multivariate wave-packet representation $\Gamma_{\mathbb{H}}$ over $\mathbb{W}(\mathbb{H})$. □

As a consequence of Theorem 5.2, we deduce the following orthogonality relation concerning the multivariate wave-packet transforms.

Corollary 5.3. *Let $\mathbb{H}$ be a compact subgroup of the multiplicative matrix group* $\mathrm{GL}(d,\mathbb{R})$ *with the normalized (probability) Haar measure $\sigma_{\mathbb{H}}$ and $\mathbb{W}(\mathbb{H})$ be the multivariate wave-packet group associated to $\mathbb{H}$ with the induced Haar measure $m_{\mathbb{W}(\mathbb{H})}$ by $\sigma_{\mathbb{H}}$. Also, let $\psi,\varphi \in L^2(\mathbb{R}^d)$ be non-zero window functions and $f,g \in L^2(\mathbb{R}^d)$. Then*

$$\langle \mathcal{V}_\psi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{W}(\mathbb{H}),m_{\mathbb{W}(\mathbb{H})})} = \langle \varphi,\psi\rangle_{L^2(\mathbb{R}^d)}\langle f,g\rangle_{L^2(\mathbb{R}^d)}. \tag{5}$$

Proof. The same argument used in Theorem 5.2 implies that

$$\|\mathcal{V}_\psi f\|^2_{L^2(\mathbb{W}(\mathbb{H}),m_{\mathbb{W}(\mathbb{H})})} = \|\psi\|^2_{L^2(\mathbb{R}^d)}\|f\|^2_{L^2(\mathbb{R}^d)}. \tag{6}$$

Then (6) and also twice applying the polarization identity guarantees (5). □

Next result is an inversion (reconstruction) formula for the multivariate wave-packet transform defined by (3).

Theorem 5.4. *Let $\mathbb{H}$ be a compact subgroup of the multiplicative matrix group* $\mathrm{GL}(d,\mathbb{R})$ *with the normalized Haar measure $\sigma_{\mathbb{H}}$ and $\mathbb{W}(\mathbb{H})$ be the multivariate wave-packet group associated to $\mathbb{H}$ with the induced Haar measure $m_{\mathbb{W}(\mathbb{H})}$ by $\sigma_{\mathbb{H}}$. Also, let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. Then each $f \in L^2(\mathbb{R}^d)$ can be recovered continuously in the weak sense of the Hilbert space $L^2(\mathbb{R}^d)$, from multivariate wave-packet coefficients generated by ψ, via*

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A,\lambda)\Gamma_{\mathbb{H}}(A,\lambda)\psi \ \mathrm{d}\sigma_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda). \tag{7}$$

Proof. Let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. For $f \in L^2(\mathbb{R}^d)$, define

$$f_{(\psi)} := \int_{\mathbb{H}}\int_{\mathbb{R}^d}\int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A,\lambda)\Gamma_{\mathbb{H}}(A,\lambda)\psi \ \mathrm{d}\sigma_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda),$$

in the weak sense of the Hilbert space $L^2(\mathbb{R}^d)$. Using (5), for all $g \in L^2(\mathbb{R}^d)$

we have

$$\begin{aligned}
\langle f_{(\psi)}, g\rangle_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A,\lambda)\langle\Gamma_{\mathbb{H}}(A,\lambda)\psi, g\rangle_{L^2(\mathbb{R}^d)}\ \mathrm{d}\sigma_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A,\lambda)\overline{\langle g, \Gamma_{\mathbb{H}}(A,\lambda)\psi\rangle_{L^2(\mathbb{R}^d)}}\ \mathrm{d}\sigma_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\\
&= \int_{\mathbb{H}}\int_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A,\lambda)\overline{\mathcal{V}_\psi g(A,\lambda)}\ \mathrm{d}\sigma_{\mathbb{H}}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)\\
&= \langle \mathcal{V}_\psi f, \mathcal{V}_\psi g\rangle_{L^2(\mathbb{W}(\mathbb{H}), m_{\mathbb{W}(\mathbb{H})})}\\
&= \|\psi\|^2_{L^2(\mathbb{R}^d)}\langle f, g\rangle_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Then $f_{(\psi)} \in L^2(\mathbb{R}^d)$ and $f_{(\psi)} = \|\psi\|^2_{L^2(\mathbb{R}^d)} f$ in $L^2(\mathbb{R}^d)$, which equivalently implies the reconstruction formula (7) in the weak sens of the Hilbert space $L^2(\mathbb{R}^d)$. □

Then we can present the following reproducing property for the multivariate wave-packet representations.

Corollary 5.5. *Let $\mathbb{H}$ be a compact subgroup of the multiplicative matrix group $\mathrm{GL}(d,\mathbb{R})$ with the normalized Haar measure $\sigma_{\mathbb{H}}$ and $\mathbb{W}(\mathbb{H})$ be the multivariate wave-packet group associated to $\mathbb{H}$ with the induced Haar measure $m_{\mathbb{W}(\mathbb{H})}$ by $\sigma_{\mathbb{H}}$. Let $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function and $\mathcal{H}_\psi$ be range of the multivariate transform $\mathcal{V}_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{W}(\mathbb{H}), m_{\mathbb{W}(\mathbb{H})})$. Then*

1. *$\mathcal{H}_\psi$ is a closed subspace of $L^2(\mathbb{W}(\mathbb{H}), m_{\mathbb{W}(\mathbb{H})})$.*
2. *$\mathcal{H}_\psi$ is the unique reproducing kernel Hilbert space (RKHS) over $\mathbb{W}(\mathbb{H})$ associated to the positive definite kernel $K_\psi : \mathbb{W}(\mathbb{H}) \times \mathbb{W}(\mathbb{H}) \to \mathbb{C}$ given by*

$$K_\psi[(A,\lambda),(A',\lambda')] := \langle D_A\pi(\lambda)\psi, D_{A'}\pi(\lambda')\psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(A,\lambda),(A',\lambda') \in \mathbb{W}(\mathbb{H})$.

Next corollary summarizes our recent results in terms of continuous frame theory [5, 38].

Corollary 5.6. *Let $\mathbb{H}$ be a compact subgroup of the multiplicative matrix group $\mathrm{GL}(d,\mathbb{R})$ and $\psi \in L^2(\mathbb{R}^d)$ be a non-zero window function. Then the multivariate wave-packet system*

$$\mathfrak{A}(\mathbb{H},\psi) := \{\Gamma_{\mathbb{H}}(A,\lambda)\psi : (A,\lambda) \in \mathbb{W}(\mathbb{H})\},$$

is a continuous tight frame for the Hilbert space $L^2(\mathbb{R}^d)$.

6. Analysis of multivariate wave-packet representations over compact subgroups of $\mathrm{GL}(d,\mathbb{R})$

Throughout this section, we study analytic aspects of compact subgroups of the multiplicative matrix group $\mathrm{GL}(d,\mathbb{R})$ in the framework of multivariate wave-packet analysis.

As it is proved in Theorem 5.2, just compact subgroups of the matrix group $\mathrm{GL}(d,\mathbb{R})$ are interesting from the L^2-theory and reproducing property of multivariate wave-packet representations. Roughly speaking, compact subgroups of $\mathrm{GL}(d,\mathbb{R})$ are highly important in the framework of multivariate covariant transforms and coherent state analysis over the Hilbert space $L^2(\mathbb{R}^d)$, since they guarantee that the multivariate coherent state and voice transforms over $L^2(\mathbb{R}^d)$ satisfy resolution of the identity formulas which are valid in the sense of the Hilbert function space $L^2(\mathbb{R}^d)$.

6.1. Wave packet transforms on $\mathbb{R}$. Let $d=1$. Then $\mathrm{GL}(d,\mathbb{R})=\mathbb{R}\setminus\{0\}$. It is easy to check that the only compact subgroups of the multiplicative group $\mathbb{R}\setminus\{0\}$ are $\{+1\}$ and $\{-1,+1\}$. Thus in this case, the classical wave-packet theory does not reproduce really a different analysis rather than the Gabor analysis, see also [19].

6.2. Wave packet transforms on $\mathbb{R}^d$ with $d>1$. The subgroup $\mathbb{H}=\mathrm{O}(d,\mathbb{R})$ is the most significant compact subgroup of $\mathrm{GL}(d,\mathbb{R})$. The compact subgroup $\mathrm{O}(d,\mathbb{R})$, or simply just $\mathrm{O}(d)$, is the multiplicative matrix group consists of all $d\times d$-orthogonal matrices. That is,

$$\mathrm{O}(d,\mathbb{R}) := \{A\in M_{d\times d}(\mathbb{R}) : A^TA=I_{d\times d}\}.$$

The compact group $\mathrm{O}(d)$ is a $\frac{d(d-1)}{2}$-dimensional real Lie group and it is non-connected. The probability (normalized Haar) measure over $\mathrm{O}(d)$ is given by

$$\int_{\mathrm{O}(d)}\phi(A)\mathrm{d}\sigma_{\mathrm{O}(d)}(A)=\int_{\mathbb{S}^{d-1}}\widetilde{\phi}(y)\mathrm{d}\lambda_{d-1}(y),$$

where λ_{d-1} is the normalized surface measure on $\mathbb{S}^{d-1}$, that is the standard unit sphere in $\mathbb{R}^d$, and the function $\widetilde{\phi}:\mathbb{S}^{d-1}\to\mathbb{C}$ is given by $\widetilde{\phi}(Ax):=\phi(A)$ for all $A\in\mathrm{O}(d)$ and a fixed point $x\in\mathbb{S}^{d-1}$.

Let K be a compact subgroup of $\mathrm{GL}(d,\mathbb{R})$ with the probability Haar measure σ_{K}. Then $\langle\cdot,\cdot\rangle_{\mathrm{K}}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ given by

$$(x,y)\mapsto\langle x,y\rangle_{\mathrm{K}}:=\int_{\mathrm{K}}\langle Ax,Ay\rangle\mathrm{d}\sigma_{\mathrm{K}}(A),$$

for all $x,y\in\mathbb{R}^d$, is a positive and symmetric bilinear from on $\mathbb{R}^d$. Also, it is a K-invariant form, that is

$$\langle Ax,Ay\rangle_{\mathrm{K}}=\langle x,y\rangle,$$

for all $x, y \in \mathbb{R}^d$ and $A \in \mathrm{K}$. Thus there exists a positive definite matrix $\mathbf{D} \in M_{d\times d}(\mathbb{R})$ such that

$$\langle x, y\rangle_{\mathrm{K}} = \langle x, \mathbf{D}y\rangle, \quad \forall x, y \in \mathbb{R}^d.$$

Let $\mathbf{D} = B^T B$ be the Cholesky factorization of D with B invertible. Then we deduce that $B\mathrm{K}B^{-1} \subset \mathrm{O}(d)$, or equivalently $\mathrm{K} \subset B^{-1}\mathrm{O}(d)B$. This implies that, up to conjugation, $\mathrm{O}(d)$ is the maximal compact subgroup of $\mathrm{GL}(d, \mathbb{R})$.

6.2.1. The orthogonal group. By the above argument and theoretical motivation, first we shall focus on analytic and constructive analysis of multivariate wave-packet representations over the compact subgroup $\mathbb{H} = \mathrm{O}(d)$.

In this case, the associated multivariate wave-packet group $\mathbb{W}(\mathbb{H})$ has the underlying manifold

$$\mathrm{O}(d) \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathrm{O}(d) \times \mathbb{R}^d \times \mathbb{R}^d,$$

which is equipped with the following group law

$$(A, x, \omega) \rtimes (A', x', \omega') = (AA', A'^{-1}x + x', A'\omega + \omega'),$$

for all $(A, x, \omega), (A', x', \omega') \in \mathbb{W}(\mathbb{H}) = \mathrm{O}(d) \rtimes (\mathbb{R}^d \times \mathbb{R}^d)$. Then $\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A, \lambda) = \mathrm{d}\sigma_{\mathrm{O}(d)}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$ is a Haar measure for the wave packet group $\mathbb{W}(\mathbb{H})$. The multivariate wave-packet representation

$$\Gamma_{\mathbb{H}} : \mathbb{W}(\mathbb{H}) = \mathrm{O}(d) \rtimes (\mathbb{R}^d \times \mathbb{R}^d) \to \mathcal{U}(L^2(\mathbb{R}^d))$$

is given by $\Gamma_{\mathbb{H}}(A, x, \omega) = D_A T_x M_\omega$ for all $(A, x, \omega) \in \mathbb{W}(\mathbb{H})$.

The multivariate wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ, is given by

$$\mathcal{V}_\psi f(A, x, \omega) = \langle f, \Gamma_{\mathbb{H}}(A, x, \omega)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle f, D_A T_x M_\omega \psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(A, x, \omega) \in \mathbb{W}(\mathbb{H})$. In integral terms we have

$$\mathcal{V}_\psi f(A, x, \omega) = \overline{\omega(x)} \int_{\mathbb{R}^d} f(y)e^{2\pi i\omega^T.A^{-1}y}\overline{\psi(A^{-1}y - x)}\mathrm{d}\mu_{\mathbb{R}^d}(y).$$

Corollary 5.3 guarantees the following Plancherel formula

$$\int_{\mathrm{O}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} |\langle f, \Gamma_{\mathbb{H}}(A, \lambda)\psi\rangle_{L^2(\mathbb{R}^n)}|^2 \mathrm{d}\sigma_{\mathrm{O}(d)}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^2_{L^2(\mathbb{R}^d)}\|f\|^2_{L^2(\mathbb{R}^d)}$$

which is equivalent to the following reconstruction formula in the sense of the Hilbert space $L^2(\mathbb{R}^d)$;

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \int_{\mathrm{O}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A, \lambda)\Gamma_{\mathbb{H}}(A, \lambda)\psi \; \mathrm{d}\sigma_{\mathrm{O}(d)}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda).$$

6.2.2. The special orthogonal group. For $d > 2$, the special orthogonal $\mathrm{SO}(d,\mathbb{R})$ or just $\mathrm{SO}(d)$ is given by

$$\mathrm{SO}(d) := \{A \in \mathrm{O}(d) : \det A = 1\}.$$

It is a connected and compact real Lie group.

In this case, the associated multivariate wave-packet group $\mathbb{W}(\mathbb{H})$ has the underlying manifold

$$\mathrm{SO}(d) \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathrm{SO}(d) \times \mathbb{R}^d \times \mathbb{R}^d,$$

which is equipped with the following group law

$$(A, x, \omega) \rtimes (A', x', \omega') = (AA', A'^{-1}x + x', A'\omega + \omega'),$$

for all $(A, x, \omega), (A', x', \omega') \in \mathbb{W}(\mathbb{H}) = \mathrm{SO}(d) \rtimes (\mathbb{R}^d \times \mathbb{R}^d)$. Then $\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A, \lambda) = \mathrm{d}\sigma_{\mathrm{SO}(d)}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda)$ is a Haar measure for the multivariate wave-packet group $\mathbb{W}(\mathbb{H})$. The wave packet representation

$$\Gamma_{\mathbb{H}} : \mathbb{W}(\mathbb{H}) = \mathrm{SO}(d) \rtimes (\mathbb{R}^d \times \mathbb{R}^d) \to \mathcal{U}(L^2(\mathbb{R}^d))$$

is given by $\Gamma_{\mathbb{H}}(A, x, \omega) = D_A T_x M_\omega$ for all $(A, x, \omega) \in \mathbb{W}(\mathbb{H})$.

The multivariate wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ, is given by

$$\mathcal{V}_\psi f(A, x, \omega) = \langle f, \Gamma_{\mathbb{H}}(A, x, \omega)\psi\rangle_{L^2(\mathbb{R}^d)} = \langle f, D_A T_x M_\omega \psi\rangle_{L^2(\mathbb{R}^d)},$$

for all $(A, x, \omega) \in \mathbb{W}(\mathbb{H})$. In integral terms we have

$$\mathcal{V}_\psi f(A, x, \omega) = \overline{\omega(x)} \int_{\mathbb{R}^d} f(y) e^{2\pi i \omega^T . A^{-1} y} \overline{\psi(A^{-1}y - x)} \mathrm{d}\mu_{\mathbb{R}^d}(y).$$

Corollary 5.3 guarantees the following Plancherel formula

$$\int_{\mathrm{SO}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} |\langle f, \Gamma_{\mathbb{H}}(A, \lambda)\psi\rangle_{L^2(\mathbb{R}^n)}|^2 \mathrm{d}\sigma_{\mathrm{SO}(d)}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^2_{L^2(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)}$$

which is equivalent to the following reconstruction formula in the sense of the Hilbert space $L^2(\mathbb{R}^d)$;

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \int_{\mathrm{SO}(d)} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A, \lambda) \Gamma_{\mathbb{H}}(A, \lambda)\psi \ \mathrm{d}\sigma_{\mathrm{SO}(d)}(A)\mathrm{d}\mu_{\mathbb{R}^d\times\widehat{\mathbb{R}^d}}(\lambda).$$

6.2.3. The maximal tori. A circle group is a linear (matrix) group isomorphic to $\mathbb{S}^1$. A torus (tori) is a direct sum of circle groups. Thus any torus is a compact connected Abelian Lie group. A maximal torus (tori) is a torus in a linear (matrix) group which is not contained in any other torus. The rank of a maximal tori T is the number r such that $\mathrm{T} = \oplus_{j=1}^r \mathbb{S}^1$.

The following proposition [26, 27] characterizes structure of a maximal tori of the special orthogonal group $\mathrm{SO}(d)$.

Proposition 6.1. *Let $d > 2$ and T be a maximal tori of $\mathrm{SO}(d)$. Then*

1. *If $d = 2n$ with $n \in \mathbb{N}$, then $\mathrm{T} = \oplus_{j=1}^r \mathrm{SO}(2)$.*
2. *If $d = 2n+1$ with $n \in \mathbb{N}$, then $\mathrm{T} = \left(\oplus_{j=1}^r \mathrm{SO}(2)\right) \oplus \{1\}$.*

In this case, the associated multivariate wave-packet group $\mathbb{W}(\mathrm{T})$ has the underlying manifold

$$\mathrm{T} \times \mathbb{R}^d \times \widehat{\mathbb{R}^d} = \mathrm{T} \times \mathbb{R}^d \times \mathbb{R}^d,$$

which is equipped with the following group law

$$(A, x, \omega) \rtimes (A', x', \omega') = (AA', A'^{-1}x + x', A'\omega + \omega'),$$

for all $(A, x, \omega), (A', x', \omega') \in \mathbb{W}(\mathbb{H}) = \mathrm{T} \rtimes (\mathbb{R}^d \times \mathbb{R}^d)$. Then $\mathrm{d}m_{\mathbb{W}(\mathbb{H})}(A, \lambda) = \mathrm{d}\sigma_{\mathrm{T}}(A)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda)$ is a Haar measure for the multivariate wave-packet group $\mathbb{W}(\mathbb{H})$. The multivariate wave-packet representation

$$\Gamma_{\mathbb{H}} : \mathbb{W}(\mathbb{H}) = \mathrm{T} \rtimes (\mathbb{R}^d \times \mathbb{R}^d) \to \mathcal{U}(L^2(\mathbb{R}^d))$$

is given by $\Gamma_{\mathbb{H}}(A, x, \omega) = D_A T_x M_\omega$ for all $(A, x, \omega) \in \mathbb{W}(\mathrm{T})$.

The multivariate wave-packet transform of $f \in L^2(\mathbb{R}^d)$ with respect to the window function ψ, is given by

$$\mathcal{V}_\psi f(A, x, \omega) = \langle f, \Gamma_{\mathrm{T}}(A, x, \omega)\psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, D_A T_x M_\omega \psi \rangle_{L^2(\mathbb{R}^d)},$$

for all $(A, x, \omega) \in \mathbb{W}(\mathrm{T})$. In integral terms we have

$$\mathcal{V}_\psi f(A, x, \omega) = \overline{\omega(x)} \int_{\mathbb{R}^d} f(y) e^{2\pi i \omega^T . A^{-1} y} \overline{\psi(A^{-1}y - x)} \mathrm{d}\mu_{\mathbb{R}^d}(y).$$

Corollary 5.3 guarantees the following Plancherel formula

$$\int_{\mathrm{T}} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} |\langle f, \Gamma_{\mathbb{H}}(A, \lambda)\psi \rangle_{L^2(\mathbb{R}^n)}|^2 \mathrm{d}\sigma_{\mathrm{T}}(A)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda) = \|\psi\|^2_{L^2(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)},$$

which is equivalent to the following reconstruction formula in the sense of the Hilbert space $L^2(\mathbb{R}^d)$;

$$f = \|\psi\|^{-2}_{L^2(\mathbb{R}^d)} \int_{\mathrm{T}} \int_{\mathbb{R}^d} \int_{\widehat{\mathbb{R}^d}} \mathcal{V}_\psi f(A, \lambda)\Gamma_{\mathbb{H}}(A, \lambda)\psi \; \mathrm{d}\sigma_{\mathrm{T}}(A)\mathrm{d}\mu_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}}(\lambda).$$

Concluding remarks. The main purpose of this article was dedicated to presenting a constructive admissibility criterion on closed subgroups of the general liner group $\mathrm{GL}(d,\mathbb{R})$ which guarantees square integrability of the associated multivariate wave-packet representations and hence a valid resolution of the identity operator in the sense of the Hilbert function space $L^2(\mathbb{R}^d)$.

Invoking topological and geometric structure of the real Lie group $\mathrm{GL}(d,\mathbb{R})$, there is a high degree of freedom in selecting an admissible subgroup $\mathbb{H}$ of $\mathrm{GL}(d,\mathbb{R})$. Among all closed subgroups of $\mathrm{GL}(d,\mathbb{R})$, just compact ones are admissible and hence they guarantee a square-integrable multivariate wave-packet representation and valid reconstruction formula.

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