Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 36 (2017), 477–480
DOI: 10.4171/ZAA/1597

A Characterization of Circles by Single Layer Potentials

Seyed M. Zoalroshd

Abstract. We give a characterization of circles by polynomial eigenfunctions of single layer potentials.

Keywords. Circle, single layer potential, polynomial eigenfunction

Mathematics Subject Classification (2010). Primary 31A25, secondary 31A99

1. Introduction

Let Ω be a smoothly bounded domain in the plane and ds be the arc-length measure supported on $\partial\Omega$. The single layer potential on $L^2(\partial\Omega, ds)$ is defined by

$$\mathcal{S}_{\partial\Omega} f(z) = -\frac{1}{2\pi} \int_{\partial\Omega} f(\zeta) \ln |z-\zeta| ds_\zeta$$

The operator $\mathcal{S}_{\partial\Omega}$ represents the potential associated with the electric field generated by a charge distribution on a surface $\partial\Omega$. The operator $\mathcal{S}_{\partial\Omega}$ is a self-adjoint Hilbert–Schmidt operator [6].

Throughout this paper $\mathbb{T}$ denotes the boundary of the unit disk $\mathbb{D}=\{z\in\mathbb{C}: |z| < 1\}$. We denote the zero set of polynomial p by $Z(p)$. It is easy to show that for the boundary curve being the unit circle, the eigenfunctions of the corresponding single layer potential are monomials. For the sake of completeness, we provide the simple calculations in here. For $n \in \mathbb{Z}^+$ and $z \in \mathbb{D}$,

$$\mathcal{S}_{\mathbb{T}}(z^n) = -\frac{1}{2\pi} \int_{\mathbb{T}} \zeta^n \ln |z-\zeta| \, ds_\zeta = -\frac{1}{4\pi} \int_{\mathbb{T}} \zeta^n \left[\ln\left(1 - \frac{z}{\zeta}\right) + \ln(1-\overline{z}\zeta) \right] ds_\zeta,$$

i.e.,

$$\mathcal{S}_{\mathbb{T}}(z^n) = -\frac{1}{4\pi} \int_{\mathbb{T}} \zeta^n \left[-\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{\zeta}\right)^k - \sum_{k=1}^{\infty} \frac{1}{k} (\overline{z}\zeta)^k \right] ds_\zeta = \frac{1}{4n\pi} \int_{\mathbb{T}} z^n \, ds_\zeta = \frac{1}{2n} z^n.$$

S. M. Zoalroshd: Vatterott College, Fairview Heights (IL), United States;
szoalros@mail.usf.edu

Similarly, for $n \in \mathbb{Z}^-$ and $z \in \mathbb{D}$, we have $\mathcal{S}_{\mathbb{T}}(z^n) = -\frac{1}{2n} z^n$ and $\mathcal{S}_{\mathbb{T}}(1) = 0$ for $n = 0$. Therefore, it follows from the continuity of the single layer potentials that

$$\mathcal{S}_{\mathbb{T}}(z^n) = \begin{cases} \frac{1}{2|n|} z^n, & n \in \mathbb{Z}^* \\ 0, & n = 0, \end{cases}$$

for $z \in \mathbb{T}$.

Ebenfelt et al. in [3] show that if the the exterior of the boundary curve Γ is Smirnov (see [2]) and $\mathcal{S}_\Gamma$ has a constant eigenfunction, then Γ must be a circle. A stronger version is given by Khavinson–Solynin–Vasillev in [5]. For the analog of these results in higher dimensions we refer to [4, 7].

In the present note we show that under a smoothness assumption on the boundary curve, only the circle allows the single layer potential to have polynomial eigenfunctions with zeros inside the disk. This can be considered as a generalization of the result given by Ebenfelt–Khavinson–Shapiro in [3].

2. Main results

Our approach is based on following characterizations (see [3, 5] for more detail):

Theorem 2.1 (Ebenfelt–Khavinson–Shapiro [3])**.** *Let Γ be a rectifiable Jordan curve and $T(z)$ the tangent vector to Γ defined a.e. on Γ. Suppose that*

$$\overline{T(z)} = H(z), \quad \textit{a.e. on } \Gamma,$$

where $H(z)$ stands for non-tangential boundary values of a bounded analytic function H in the exterior Ω_+ of Γ with $H(\infty) = 0$. Then, Γ must be a circle.

Theorem 2.2 (Khavinson–Solynin–Vassilev [5])**.** *Suppose that Γ, Ω_+ and H satisfy the conditions of the previous theorem, but H has a simple pole at a given finite point $z_0 \in \Omega_+$. Then*

$$\Gamma = \left\{ z = a\zeta + z_0 : \left| \zeta - \frac{p}{1-p^2} \right| = \frac{p^2}{1-p^2} \right\},$$

with some $a \in \mathbb{C}^*$ and $0 < p < 1$.

Theorem 2.3. *Assume Ω is a smoothly bounded domain in the plane. If $\mathcal{S}_{\partial\Omega}$ has a polynomial eigenfunction p with $Z(p) \subset \Omega$, then $\partial\Omega$ must be circle.*

Proof. Assume $\mathcal{S}_{\partial\Omega}(p) = \lambda p$ for some non-zero $\lambda \in \mathbb{R}$ and some polynomial $p(z)$ with $Z(p) \subset \Omega$. Since $\lambda p(z) = \int_{\partial\Omega} p(\zeta) \ln |z - \zeta| \, ds_\zeta$ for all $z \in \overline{\Omega}$, then

$$-2\lambda p'(z) = \int_{\partial\Omega} \frac{p(\zeta)}{\zeta - z} \, ds_\zeta = \int_{\partial\Omega} \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - z} d\zeta \quad \text{for } z \in \Omega.$$

Without loss of generality we may assume that Ω contains the origin (via an appropriate translation). Set $F(z) = \int_{\partial\Omega} \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - z} d\zeta$ on the exterior domain $\Omega_+ = \widehat{\mathbb{C}} \setminus \overline{\Omega}$. The function F is analytic in Ω_+ and $F(\infty) = 0$. By the Sokhotski–Plemelj jump theorem (see [6]), we find the following holds almost everywhere on $\partial\Omega$

$$\begin{aligned} 2\pi i p(z)\overline{T(z)} &= \lim_{\substack{t\to z,\\ t\in\Omega_+}} \int_{\partial\Omega} \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - t} d\zeta - \lim_{\substack{w\to z,\\ w\in\Omega}} \int_{\partial\Omega} \frac{p(\zeta)\overline{T(\zeta)}}{\zeta - w} d\zeta \\ &= \lim_{\substack{t\to z,\\ t\in\Omega_+}} F(t) + 2\lambda \lim_{\substack{w\to z,\\ w\in\Omega}} p'(w) \\ &= \lim_{\substack{w\to z,\\ w\in\Omega_+}} \Big[F(w) + 2\lambda p'(w)\Big], \end{aligned}$$

which can be rewritten as

$$\overline{T(z)} = \frac{1}{2\pi i} \lim_{\substack{w\to z,\\ w\in\Omega_+}} \left[\frac{F(w)}{p(w)} + 2\lambda \frac{p'(w)}{p(w)}\right].$$

The function $\Phi(w) = \frac{1}{2\pi i}\left(\frac{F(w)}{p(w)} + 2\lambda \frac{p'(w)}{p(w)}\right)$ is analytic on Ω_+, $\Phi(\infty) = 0$ and $\Phi \equiv \overline{T}$ a.e. on $\partial\Omega$. Thus, by Theorem 2.2, $\partial\Omega$ must be a circle. □

Corollary 2.4. *Suppose Ω contains the origin. If $\mathcal{S}_{\partial\Omega}$ has monomial eigenfunctions of the form z^n, then $\partial\Omega$ must be a circle.*

We conclude this paper with the following remark on logarithmic potentials. Recall that the logarithmic potential on a bounded domain Ω is defined by

$$(\mathcal{L}_\Omega f)(w) = \frac{1}{2\pi} \int_\Omega f(w) \ln|z - w|\, dA(w),$$

for $f \in L^2(\Omega, dA)$. The operator $\mathcal{L}_\Omega$ is a self-adjoint Hilbert–Schmidt operator on $L^2(\Omega, dA)$ (see [1]). We show that $\mathcal{L}_\Omega$ unlike $\mathcal{S}_{\partial\Omega}$, never has a polynomial eigenfunction.

Proposition 2.5. *The operator $\mathcal{L}_\Omega$ has no eigenfunction polynomial of z and $\overline{z}$.*

Proof. Assume to the contrary that there exists $\lambda \in \mathbb{C}$ and a polynomial $P = P(z, \overline{z})$, not identically zero, such that $\lambda P = \mathcal{L}_\Omega P$. We can find a polynomial $Q(z, \overline{z})$ so that $\Delta Q = P$. For $z \in \Omega$,

$$\begin{aligned} \lambda P(z, \overline{z}) &= \frac{1}{2\pi} \int_\Omega P(w, \overline{w}) \ln|w - z|\, dA(w) \\ &= \frac{1}{2\pi} \int_\Omega \Delta Q(w, \overline{w}) \ln|w - z|\, dA(w) \\ &= Q(z, \overline{z}) + \int_{\partial\Omega} \left(\frac{\partial Q}{\partial n_\zeta} \ln|\zeta - z| - Q(\zeta, \overline{\zeta}) \frac{\partial}{\partial n_\zeta} \ln|\zeta - z|\right) ds_\zeta. \end{aligned}$$

Taking $\overline{\partial}_z$-derivatives we obtain

$$\lambda\frac{\partial P}{\partial \overline{z}} = \frac{\partial Q}{\partial \overline{z}} + \frac{1}{2}\int_{\partial\Omega}\left[\frac{\partial Q}{\partial n_\zeta}\frac{1}{\overline{z}-\overline{\zeta}} - Q\frac{\partial}{\partial n_\zeta}\left(\frac{1}{\overline{z}-\overline{\zeta}}\right)\right] ds_\zeta.$$

Finally taking ∂_z-derivatives it follows that $4\lambda\Delta P = 4\Delta Q$. Since by assumption $P = \Delta Q$ we must have $\lambda\Delta P = P$. But this is a clear contradiction. □

Acknowledgement. I would like to express my sincere appreciation to my advisor, Professor Dmitry Khavinson, for his numerous comments and suggestions regarding this paper. I am also indebted to an anonymous referee for his (her) helpful comments.

References

[1] Anderson, J. M., Khavinson, D. and Lomonosov, V., Spectral properties of some integral operators arising in potential theory. *Quart. J. Math. Oxford Ser. (2)* 43 (1992)(4), 387 – 407.

[2] Duren, P. L., *Theory of H^p Spaces.* Pure Appl. Math. 38. New York: Academic Press 1970.

[3] Ebenfelt, P., Khavinson, D. and Shapiro, H. S., A free boundary problem related to single-layer potentials. *Ann. Acad. Sci. Fenn. Math.* 27 (2002)(1), 21 – 46.

[4] Fraenkel, L. E., *Introduction to Maximum Principles and Symmetry in Elliptic Problems.* Cambridge Tracts Math. 128, Cambridge: Cambridge Univ. Press 2000.

[5] Khavinson, D., Solynin, Y. and Vassilev, D., Overdetermined boundary value problems, quadrature domains and applications. *Comput. Methods Funct. Theory* 5 (2005)(1), 19 – 48.

[6] Kress, R., Maz'ya, V. and Kozlov, V., *Linear Integral Equations.* Appl. Math. Sci. 82. Berlin: Springer 1989.

[7] Shahgholian, H., A characterization of the sphere in terms of single-layer potentials. *Proc. Amer. Math. Soc.* 115 (1992), 1167 – 1168.

Received July 15, 2016; revised February 2, 2017