Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 37 (2018), 73–82
DOI: 10.4171/ZAA/1603

Global Well-Posedness for the Gross–Pitaevskii Equation with Pumping and Nonlinear Damping

Binhua Feng, Xiangxia Yuan and Jun Zheng

Abstract. This paper deals with the Cauchy problem for the Gross–Pitaevskii equation with pumping and nonlinear damping which describes the dynamics of pumped decaying Bose–Einstein condensates. This paper establishes global existence of solutions for general initial data in the energy space.

Keywords. Gross–Pitaevskii equation, global existence, nonlinear damping, pumping

Mathematics Subject Classification (2010). Primary 35J60, secondary 35Q55

1. Introduction

In this paper we study the Cauchy problem for the following Gross–Pitaevskii equation with linear pumping and nonlinear damping,

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + V(x)\psi + \lambda|\psi|^{2\alpha}\psi + i(a - b|\psi|^{2p})\psi, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N, \tag{1}$$

where $\psi = \psi(t,x)$ represents the wave function, $V(x)$ is the trapping potential, $a > 0$ is the pumping term, and $b > 0$ is the strength of the decaying term. Equation (1) was proposed by Keeling and Berloff [15] to study pumped decaying condensates, particularly the Bose–Einstein condensates (BEC) of exciton-polaritons. When $V = 0$, equation (1) describes the optical beam dynamics in nonlinear media, see [1].

B. Feng, X. Yuan: Department of Mathematics, Northwest Normal University, Gansu, 730070, P. R. China; binhuaf@163.com; 707740536@qq.com
J. Zheng: Department of Basic Course, Southwest Jiaotong University, Emeishan, 614202, Sichuan Province, P. R. China; School of Mathematics, Southwest Jiaotong University, Chengdu, 611756, Sichuan Province, P. R. China; zhengjun2014@aliyun.com

When $a = 0$ and $b > 0$, equation (1) appears in different physical contexts. For example, in considering the three-body interaction in collapsing Bose–Einstein condensates (BECs), within the realm of Gross–Pitaevskii theory, the emittance of particles from the condensate is described by the dissipative model involving a quintic nonlinear damping term [14]; in nonlinear optics, equation (1) with $V = 0$ describes the propagation of a laser pulse within an optical fiber under the influence of additional multi-photon absorption processes, see, e.g., [4, 12]. From a mathematical point of view, the last term in (1) is dissipative. Therefore, the energy of (1) is no longer conserved, in contrast to the usual case of Hamiltonian for nonlinear Schrödinger equations. Numerical studies of (1) can be found in [13, 16]; in particular, the nonlinear-damping continuation of singular solutions for (1) with critical and supercritical nonlinearities has been considered in [13]. When $V \equiv 0$, under some assumptions, Feng, Zhao and Sun [9] have showed that as $b \to 0$ the solution of (1) converges to that of (1) with $b = 0$. Some sufficient conditions for global existence of solutions to (1) have been established in [2, 3, 7, 8, 10, 11].

When $a > 0$ and $b > 0$, Sierra etc. in [17] explore numerically the behavior of solutions of (1). As far as we know, there are no any rigorously mathematical results about (1), despite the physical significance of the involved applications. The aim of this paper is to establish the global existence of solutions to (1). Due to the appearance of linear pumping, the result in [6] suggests that the solution ψ of (1) with $b = 0$ blows up in finite time for a sufficiently large. This bring some difficulties for our analysis.

To solve this problem, we set $\psi(t, x) = e^{at}u(t, x)$ in (1), then (1) can be transformed to

$$i\partial_t u = -\frac{1}{2}\Delta u + V(x)u + \lambda e^{2a\alpha t}|u|^{2\alpha}u - ibe^{2apt}|u|^{2p}u, \quad (t, x) \in [0, \infty) \times \mathbb{R}^N.$$

For mathematical interesting, we consider more general equation:

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + V(x)u + f(t)|u|^{2\alpha}u - ig(t)|u|^{2p}u, & (t, x) \in [0, \infty) \times \mathbb{R}^N, \\ u|_{t=0} = u_0 \in \Sigma, \end{cases} \tag{2}$$

where $N \geq 1$, $0 < \alpha, p < \frac{2}{N-2}$ ($0 < \alpha, p < \infty$, if $N = 1$ or $N = 2$). The external potential V is supposed to be an anisotropic harmonic confinement, i.e.,

$$V(x) = \frac{1}{2}\sum_{j=1}^{N} \omega_j^2 x_j^2, \quad \omega_j \in \mathbb{R}. \tag{3}$$

Denote by Σ the energy space associated to the harmonic potential, i.e.,

$$\Sigma = \{u \in H^1(\mathbb{R}^N) \text{ and } xu \in L^2(\mathbb{R}^N)\},$$

equipped with the following norm:

$$\|u\|_{\Sigma} := \|u\|_{L^2} + \|\nabla u\|_{L^2} + \|xu\|_{L^2}.$$

In order to state our main results, we first give the definition of solutions to (2).

Definition 1.1. A strong Σ-solution u of (2) on $[0,T]$ is a function

$$u \in C([0,T),\Sigma) \cap C^1([0,T],\Sigma^*)$$

such that $i\partial_t u = -\frac{1}{2}\Delta u + V(x)u + f(t)|u|^{2\alpha}u - ig(t)|u|^{2p}u$ for all $t \in [0,T]$ and $u|_{t=0} = u_0$, where Σ^* is the dual of the energy space Σ.

In the following, we shall establish the global existence for (1) and (2).

Theorem 1.2. *Let g be a positive continuous function defined in $[0,\infty)$, $f \in W^{1,\infty}_{loc}(0,\infty)$, $u_0 \in \Sigma$. Assume that V satisfies (3) and suppose further that*

(i) $0 < \alpha, p < \frac{2}{N}$ *or*
(ii) $\frac{2}{N} \le \alpha < p < \frac{2}{(N-2)_+}$ *or*
(iii) $\alpha = p = \frac{2}{N}$ *or*
(iv) $\frac{2}{N} < \alpha = p < \frac{2}{(N-2)_+}$ *and* $f(t) \ge -\frac{g(t)}{p}$ *for every* $t \in [0,\infty)$.

Then, the Cauchy problem (2) has a unique global solution $u \in C([0,\infty),\Sigma)$.

As a direct corollary of this theorem, we can obtain the global well-posedness for (1).

Corollary 1.3. *In either of the cases mentioned in Theorem 1.2, for any $\lambda \in \mathbb{R}$, $a > 0$, $b > 0$, and $\psi_0 \in \Sigma$, equation (1) has a unique global solution $\psi \in C([0,\infty),\Sigma)$.*

Remark 1.4. This result suggests that for every initial data, under the assumption $\alpha < p$, the corresponding solution is global no matter how the system gain energy. In addition, when $V(x) \equiv 0$, under the condition $\alpha < p$, a similar result on the global existence was established by Yokota in [18].

This paper is organized as follows: in Section 2, we will collect some lemmas such as the local well-posedness, and a-priori estimates for the solutions of (2). In Section 3, we will show Theorem 1.2.

2. Some lemmas

First, let us recall the local theory for the initial value problem (2). When $f(t) \equiv C$, $g(t) \equiv C$, in [2,5] and references therein, the authors showed that (2) is local well-posedness. For our case, since $f \in L^{\infty}_{loc}(0,\infty)$ and $g \in L^{\infty}_{loc}(0,\infty)$, we only need to take their L^{∞}-norms when the nonlinearities have to be estimated in some norms. Keeping this in mind and applying the method in [2, 5], one can show the local well-posedness of (2). For the sake of conciseness, we only state the results without detailed proof.

Proposition 2.1. *Let $u_0 \in \Sigma$, $0 < \alpha, p < \frac{2}{N-2}$, $f, g \in L^{\infty}_{loc}(0,\infty)$ and V satisfy (3). Then, there exists $T = T(\|u_0\|_{\Sigma})$ such that (2) admits a unique solution $u \in C([0,T],\Sigma) \cap C^1([0,T],\Sigma^*)$. Let $[0,T^*)$ be the maximal time interval on which u is well-defined. If $T^* < \infty$, then $\|\nabla u(t)\|_{L^2} \to +\infty$ as $t \to T^*$. In the case $\alpha = p = \frac{2}{N}$, if $T^* < \infty$, then*

$$\int_0^{T^*} \|u(t)\|_{L^{2+\frac{4}{N}}}^{2+\frac{4}{N}} dt = \infty.$$

In the following, in order to extend the obtained local in-time solution to arbitrary time intervals, we will derive several a-priori estimates.

Lemma 2.2. *Let $u(t) \in \Sigma$ be a solution of (2) defined on the maximal interval $[0,T^*)$, V satisfy (3), and $g(t) > 0$. Then*

$$\|u(t)\|_{L^2} \le \|u_0\|_{L^2}, \quad \forall\, t \in [0,T^*), \tag{4}$$

and

$$\int_0^{T^*} g(t) \int_{\mathbb{R}^N} |u(t,x)|^{2p+2} dx dt \le C(\|u_0\|_{L^2}). \tag{5}$$

Proof. We multiply (2) by $\bar{u}$ and integrate with respect to $x \in \mathbb{R}^N$. Taking the imaginary part, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |u(t,x)|^2 dx = -g(t) \int_{\mathbb{R}^N} |u(t,x)|^{2p+2} dx \le 0, \tag{6}$$

which implies that (4) holds. In addition, integrating (6) with respect to t, we can obtain (5). □

Lemma 2.3. *Let $u(t) \in \Sigma$ be a solution of (2) defined on the maximal interval $[0,T^*)$, $\alpha < p$, V satisfy (3), $f, k \in W^{\infty}_{loc}(0,\infty)$. Moreover, assume that k and g are two positive continuous functions defined in $[0,\infty)$ and $0 < k(t) < \frac{g(t)}{p(p+1)}$ for every $t \in [0,T^*)$. Then, for every $0 < T < T^*$,*

$$E(t) \le E(0) + C(T, \|u_0\|_{L^2}), \quad \forall\, t \in [0,T],$$

where

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx + \int_{\mathbb{R}^N} V(x)|u(t,x)|^2 dx + \frac{f(t)}{\alpha+1}\int_{\mathbb{R}^N} |u(t,x)|^{2\alpha+2} dx + k(t)\int_{\mathbb{R}^N} |u(t,x)|^{2p+2} dx.$$

Proof. We first assume that $u(t)$ is sufficiently regular and decaying so that all of the following formal manipulations can be carried out. Once the final result is established, a standard density argument allows to conclude that it also holds for $u \in C([0,T],\Sigma)$.

Since

$$\Delta \bar{u} = 2i\partial_t \bar{u} + 2V(x)\bar{u} + 2f(t)|u|^{2\alpha}\bar{u} + 2ig(t)|u|^{2p}\bar{u},$$

it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} |\nabla u(t)|^2 dx \\
&= -2\mathrm{Re}\int_{\mathbb{R}^N} \partial_t u \Delta \bar{u} dx \\
&= -2\mathrm{Re}\int_{\mathbb{R}^N} \partial_t u (2i\partial_t \bar{u} + 2V(x)\bar{u} + 2f(t)|u|^{2\alpha}\bar{u} + 2ig(t)|u|^{2p}\bar{u}) dx \\
&= -2\frac{d}{dt}\int_{\mathbb{R}^N} V|u|^2 dx - \frac{2f(t)}{\alpha+1}\frac{d}{dt}\int_{\mathbb{R}^N} |u|^{2\alpha+2} dx + 4g(t)\mathrm{Im}\int_{\mathbb{R}^N} \partial_t u |u|^{2p}\bar{u} dx.
\end{aligned} \tag{7}$$

For the last term, using (2) and integration by parts, we have

$$\begin{aligned}
&\mathrm{Im}\int_{\mathbb{R}^N} \partial_t u |u|^{2p}\bar{u} dx \\
&= \mathrm{Im}\int_{\mathbb{R}^N} \left(\frac{i}{2}\Delta u - iV(x)u - if(t)|u|^{2\alpha}u - g(t)|u|^{2p}u\right)|u|^{2p}\bar{u} dx \\
&= \frac{1}{2}\mathrm{Re}\int_{\mathbb{R}^N} \Delta u |u|^{2p}\bar{u} dx - \int_{\mathbb{R}^N} V(x)|u|^{2p+2} dx - f(t)\int_{\mathbb{R}^N} |u|^{2\alpha+2p+2} dx \\
&= -\frac{1}{2}\int_{\mathbb{R}^N} |u|^{2p}|\nabla u|^2 dx - p\int_{\mathbb{R}^N} |u|^{2p}|\nabla|u||^2 dx - \int_{\mathbb{R}^N} V(x)|u|^{2p+2} dx \\
&\quad - f(t)\int_{\mathbb{R}^N} |u|^{2\alpha+2p+2} dx.
\end{aligned} \tag{8}$$

To treat the last term in (8), we compute

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N} |u(t)|^q dx &= q\mathrm{Re}\int_{\mathbb{R}^N} \partial_t u |u|^{q-2}\bar{u} dx \\
&= -\frac{q}{2}\mathrm{Im}\int_{\mathbb{R}^N} \Delta u |u|^{q-2}\bar{u} dx - qg(t)\int_{\mathbb{R}^N} |u|^{q+2p} dx \\
&= \frac{q}{2}\int_{\mathbb{R}^N} \nabla |u|^{q-2} \cdot \mathrm{Im}(\nabla u \bar{u}) dx - qg(t)\int_{\mathbb{R}^N} |u|^{q+2p} dx,
\end{aligned} \tag{9}$$

where $q > 2$. By the same limit process as in [2], we can show that

$$\int_{\mathbb{R}^N} \nabla|u|^{q-2} \cdot \mathrm{Im}(\nabla u \bar{u})dx = (q-2)\int_{\mathbb{R}^N} |u|^{q-2}\mathrm{Re}(\bar{\phi}\nabla u) \cdot \mathrm{Im}(\bar{u}\nabla u)dx,$$

where ϕ is defined by,

$$\phi(t,x) := \begin{cases} |u(t,x)|^{-1}u(t,x) & \text{if } u(t,x) \neq 0, \\ 0 & \text{if } u(t,x) = 0. \end{cases}$$

In view of the identity

$$2\mathrm{Re}(\bar{\phi}\nabla u) \cdot \mathrm{Im}(\bar{u}\nabla u) = -|\mathrm{Re}(\bar{\phi}\nabla u) - \mathrm{Im}(\bar{u}\nabla u)|^2 + |\nabla u|^2,$$

we obtain:

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N} |u(t)|^q dx = &-\frac{q(q-2)}{4}\int_{\mathbb{R}^N} |u|^{q-2}|\mathrm{Re}(\bar{\phi}\nabla u) - \mathrm{Im}(\bar{u}\nabla u)|^2 dx \\ &+ \frac{q(q-2)}{4}\int_{\mathbb{R}^N} |u|^{q-2}|\nabla u|^2 dx - qg(t)\int_{\mathbb{R}^N} |u|^{q+2p}dx. \end{aligned} \tag{10}$$

Taking $q = 2p + 2$, we deduce from (7)–(10) that

$$\begin{aligned} \frac{d}{dt}E(t) = &\frac{f'(t)}{\alpha+1}\int_{\mathbb{R}^N} |u(t)|^{2\alpha+2}dx - g(t)\int_{\mathbb{R}^N} |u|^{2p}|\nabla u|^2 dx \\ &- 2pg(t)\int_{\mathbb{R}^N} |u|^{2p}|\nabla|u||^2 dx - 2g(t)\int_{\mathbb{R}^N} V|u|^{2p+2}dx \\ &- 2g(t)f(t)\int_{\mathbb{R}^N} |u|^{2\alpha+2p+2}dx - (2p+2)k(t)g(t)\int_{\mathbb{R}^N} |u|^{4p+2}dx \\ &+ p(p+1)k(t)\int_{\mathbb{R}^N} |u|^{2p}|\nabla u|^2 dx + k'(t)\int_{\mathbb{R}^N} |u(t)|^{2p+2}dx \\ &- p(p+1)k(t)\int_{\mathbb{R}^N} |u|^{2p}|\mathrm{Re}(\bar{\phi}\nabla u) - \mathrm{Im}(\bar{u}\nabla u)|^2 dx \\ \leq &\frac{|f'(t)|}{\alpha+1}\int_{\mathbb{R}^N} |u(t)|^{2\alpha+2}dx + |k'(t)|\int_{\mathbb{R}^N} |u(t)|^{2p+2}dx \\ &+ 2g(t)|f(t)|\int_{\mathbb{R}^N} |u|^{2\alpha+2p+2}dx - (2p+2)k(t)g(t)\int_{\mathbb{R}^N} |u|^{4p+2}dx \\ &+ (p(p+1)k(t) - g(t))\int_{\mathbb{R}^N} |u|^{2p}|\nabla u|^2 dx. \end{aligned} \tag{11}$$

On the other hand, since $\alpha < p$, we deduce from the interpolation inequality and the Young inequality with ε that

$$\|u\|_{L^{2\alpha+2p+2}}^{2\alpha+2p+2} \le \|u\|_{L^{2p+2}}^{(1-\frac{\alpha}{p})(2p+2)} \|u\|_{L^{4p+2}}^{\frac{\alpha}{p}(4p+2)} \le C(\varepsilon)\|u\|_{L^{2p+2}}^{2p+2} + \varepsilon\|u\|_{L^{4p+2}}^{4p+2}, \qquad (12)$$

$$\|u\|_{L^{2p+2}}^{2p+2} \le \|u\|_{L^2}\|u\|_{L^{4p+2}}^{2p+1} \le C(\varepsilon)\|u\|_{L^2}^2 + \varepsilon\|u\|_{L^{4p+2}}^{4p+2}, \qquad (13)$$

and

$$\|u\|_{L^{2\alpha+2}}^{2\alpha+2} \le \|u\|_{L^2}^{\frac{2p-\alpha}{p}} \|u\|_{L^{4p+2}}^{\frac{\alpha(2p+1)}{p}} \le C(\varepsilon)\|u\|_{L^2}^2 + \varepsilon\|u\|_{L^{4p+2}}^{4p+2}, \qquad (14)$$

for any $\varepsilon > 0$. Since k and g are two positive continuous functions defined in $[0,\infty)$, for any $T \in [0, T^*)$, there are M_1 and M_2 such that

$$k(t) \ge M_1 \quad \text{and} \quad g(t) \ge M_2 \quad \text{for all } t \in [0,T].$$

Therefore, we deduce from (11)–(14) that

$$\frac{d}{dt}E(t) \le C(\varepsilon)\|u(t)\|_{L^2}^2 + C(\varepsilon)\|u(t)\|_{L^{2p+2}}^{2p+2}. \qquad (15)$$

In addition, it follows from (5) that

$$M_2 \int_0^T \int_{\mathbb{R}^N} |u(t,x)|^{2p+2} dx dt \le \int_0^{T^*} g(t) \int_{\mathbb{R}^N} |u(t,x)|^{2p+2} dx dt \le C(\|u_0\|_{L^2}).$$

Therefore, integrating (15) with respect to t, we have

$$E(t) \le E(0) + C(T, \|u_0\|_{L^2}), \quad \text{for every } t \in [0,T].$$

This completes the proof. □

Lemma 2.4. *Let $u(t) \in \Sigma$ be a solution of* (2) *defined on the maximal interval $[0,T^*)$, $\alpha = p$, $p|f(t)| \le g(t)$ and V satisfy* (3)*. Then, for every $0 < T < T^*$,*

$$E_0(t) \le E_0(0), \quad \forall\, t \in [0,T],$$

where

$$E_0(t) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx + \int_{\mathbb{R}^N} V(x)|u(t,x)|^2 dx.$$

Proof. We first assume that $u(t)$ is sufficiently regular and decaying so that all of the following formal manipulations can be carried out. Once the final result is established, a standard density argument allows to conclude that it also holds for $u \in C([0,T],\Sigma)$.

Differentiating $E_0(t)$ and using equation (2) yields

$$\frac{d}{dt}E_0(t)=f(t)\int_{\mathbb{R}^N}|u|^{2p}\mathrm{Im}(\bar{u}\Delta u)dx+g(t)\int_{\mathbb{R}^N}|u|^{2p}\mathrm{Re}(\bar{u}\Delta u)dx-2g(t)\int_{\mathbb{R}^N}V(x)|u|^{2p+2}dx.$$

Using the same arguments as in the proof of Lemma 2.3, we infer

$$\begin{aligned}\frac{d}{dt}E_0(t) =& -2pf(t)\int_{\mathbb{R}^N}|u|^{2p}\nabla|u|\cdot\mathrm{Im}(\bar{\phi}\nabla u)dx-g(t)\int_{\mathbb{R}^N}|u|^{2p}|\nabla u|^2dx\\ &-2pg(t)\int_{\mathbb{R}^N}|u|^{2p}|\nabla|u||^2dx-2g(t)\int_{\mathbb{R}^N}V(x)|u|^{2p+2}dx.\end{aligned}$$

By using the Cauchy–Schwarz inequality and then the Young inequality, we infer

$$\begin{aligned}&2p|f(t)|\int_{\mathbb{R}^N}|u|^{2p}\nabla|u|\cdot\mathrm{Im}(\bar{\phi}\nabla u)dx\\ &\leq\frac{pf^2(t)}{2g(t)}\int_{\mathbb{R}^N}|u|^{2p}|\nabla u|^2dx+2pg(t)\int_{\mathbb{R}^N}|u|^{2p}|\nabla|u||^2dx.\end{aligned}$$

Hence, if $p|f(t)|\leq g(t)$, then we have $E_0(t)\leq E_0(0)<\infty$, for all $t\in[0,T]$. □

3. The proof of main theorem

Proof of Theorem 1.2. Case (1). In the case of L^2-subcritical, i.e., $0<\alpha,p<\frac{2}{N}$, the existence of global solutions for (2) depends only on the L^2-norm of the initial data. Therefore, the global well-posedness for (2) follows from (4) via the standard iterative argument, see [5] for details.

Case (2). By using interpolation and then the Young inequality with ε, we infer

$$\frac{|f(t)|}{\alpha+1}\|u\|_{L^{2\alpha+2}}^{2\alpha+2}\leq\frac{|f(t)|}{\alpha+1}\|u\|_{L^2}^{2(1-\frac{\alpha}{p})}\|u\|_{L^{2p+2}}^{\frac{\alpha}{p}(2p+2)}\leq C(\varepsilon)\frac{|f(t)|}{\alpha+1}\|u\|_{L^2}^2+\varepsilon\frac{|f(t)|}{\alpha+1}\|u\|_{L^{2p+2}}^{2p+2},$$

for any $\varepsilon>0$. Taking ε such that $\varepsilon\frac{|f(t)|}{\alpha+1}\leq k(t)$, we deduce from Lemma 2.3 that

$$\begin{aligned}\|\nabla u(t)\|_{L^2}^2&\leq E(t)-\frac{f(t)}{\alpha+1}\int_{\mathbb{R}^N}|u(t,x)|^{2\alpha+2}dx-k(t)\int_{\mathbb{R}^N}|u(t,x)|^{2p+2}dx\\ &\leq E(0)+C(T,\|u_0\|_{L^2})+C(\varepsilon)\frac{|f(t)|}{\alpha+1}\|u(t)\|_{L^2}^2\\ &\leq E(0)+C(T,\|u_0\|_{L^2}).\end{aligned}$$

Therefore, we deduce from the blow-up alternative in Proposition 2.1 that the solution of (2) is global.

Case (3). If $T^* < \infty$, it follows from Proposition 2.1 that

$$\int_0^{T^*} \int_{\mathbb{R}^N} |u(t,x)|^{2+\frac{4}{N}} dxdt = \infty.$$

This is a contradiction with (2). Hence, the conclusion follows.

Case (4). When $f(t) \geq 0$, this can be treated as in the proof of Case (2). When $-\frac{g(t)}{p} \leq f(t) \leq 0$, the conclusion follows from Lemma 2.4. □

4. Concluding remarks

In this paper, we only investigate the physically interesting case for model (1), that is $a > 0$ and $b > 0$. However, for the mathematical interest, one can consider the following general equation:

$$i\partial_t \psi = -\frac{1}{2}\Delta\psi + V(x)\psi + \lambda|\psi|^{2\alpha}\psi + ia|\psi|^{2p_1}\psi + ib|\psi|^{2p_2}\psi,$$

where $0 \leq p_1, p_2 < \frac{2}{N-2}$, $a, b \in \mathbb{R}$. Regarding this equation, many problems are unknown. For example, when $p_1 > 0$ or $p_2 > 0$, the Glassey's method cannot be applied to prove the existence of blow-up solutions. In the particular case, the existence of blow-up solutions has been proved in [7] by using the Merle's method. A more complete understanding of the possibility of finite time blow-up remains an interesting open problem. When $a > 0$, $b < 0$, $p_1, p_2 > 0$, under what conditions will the solutions blow up in finite time? And under what conditions will the solutions exist globally? The study of these problems requires to develop some new mathematical methods and will be the object of our future investigation.

Acknowledgement. The authors would like to thank the referee for his/her careful reading and valuable suggestions, which made this article more readable. This work is supported by NSFC Grants (No. 11601435, No. 11401478), Gansu Provincial Natural Science Foundation (1606RJZA010) and NWNU-LKQN-14-6.

References

[1] Allayarov, I. M. and Tsoy, E. N., Dynamics of fronts in optical media with linear gain and nonlinear losses. *Physics Letters A* 377 (2013), 550 – 554.

[2] Antonelli, P., Carles, R. and Sparber, C., On nonlinear Schrödinger type equations with nonlinear damping. *Int. Math. Res. Not.* 3 (2015), 740 – 762.

[3] Antonelli, P. and Sparber, C., Global well-posedness of cubic NLS with nonlinear damping. *Comm. Part. Diff. Eqn.* 35 (2010), 2310 – 2328.

[4] Biswas, A., Optical soliton perturbation with nonlinear damping and saturable amplifiers. *Math. Comput. Simulation* 56 (2001), 521 – 537.

[5] Cazenave, T., *Semilinear Schrödinger Equations.* Courant Lect. Notes Math. 10. Providence (RI): Amer. Math. Soc. 2003.

[6] Cazenave, T. and Snoussi, S., Finite-time blowup for some nonlinear complex Ginzburg–Landau equations. Preprint 2016 (available at arXiv:1602.05007).

[7] Mohamad, D., On the L^2-critical nonlinear Schrödinger equation with a nonlinear damping. *Comm. Pure Appl. Anal.* 13 (2014), 2377 – 2394.

[8] Feng, B. and Zhao, D., Global well-posedness for nonlinear Schrödinger equations with energy-critical damping. *Electron. J. Diff. Equ.* 6 (2015), 1 – 9.

[9] Feng, B., Zhao, D. and Sun, C., The limit behavior of solutions for the nonlinear Schrödinger equation including nonlinear loss/gain with variable coefficient. *J. Math. Anal. Appl.* 405 (2013), 240 – 251.

[10] Feng, B., Zhao, D. and Sun, C., On the Cauchy problem for the nonlinear Schrödinger equations with time-dependent linear loss/gain. *J. Math. Anal. Appl.* 416 (2014), 901 – 923.

[11] Feng, B., Zhao, D. and Sun, C., Homogenization for nonlinear Schrödinger equations with periodic nonlinearity and dissipation in fractional order spaces. *Acta Math. Sci. Ser. B Engl. Ed.* 35 (2015), 567 – 582.

[12] Fibich, G., Self-focusing in the damped nonlinear Schrödinger equation. *SIAM J. Appl. Math.* 61 (2001), 1680 – 1705.

[13] Fibich, G. and Klein, M., Nonlinear-damping continuation of the nonlinear Schrödinger equation - a numerical study. *Phys. D* 241 (2012), 519 – 527.

[14] Kagan, Y., Muryshev, A. E. and Shlyapnikov, G. V., Collapse and Bose–Einstein condensation in a trapped Bose gas with negative scattering length. *Phys. Rev. Lett.* 81 (1998), 933 – 937.

[15] Keeling, J. and Berloff, N. G., Spontaneous rotating vortex lattices in a pumped decaying condensate. *Phys. Rev. Lett.* 100 (2008), 250401.

[16] Passot, T., Sulem, C. and Sulem, P. L., Linear versus nonlinear dissipation for critical NLS equation. *Phys. D* 203 (2005), 167 – 184.

[17] Sierra, J., Kasimov, A., Markowich, P. and Weishäupl, R., On the Gross–Pitaevskii Equation with Pumping and Decay: Stationary States and Their Stability. *J. Nonlinear Sci.* 25 (2015), 709 – 739.

[18] Yokota, T., Monotonicity method applied to complex Ginzburg–Landau type equations. *J. Math. Anal. Appl.* 380 (2011), 455 – 466.

Received June 23, 2016; revised April 18, 2017