

On the Robin Problem with Indefinite Weight in Sobolev Spaces with Variable Exponents

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Abstract. The present paper is concerned with a Robin problem involving an indefinite weight in Sobolev spaces with variable exponents

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda V(x)|u|^{q(x)-2}u, & x \in \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} + a(x)|u|^{p(x)-2}u = 0. & x \in \partial\Omega \end{cases}$$

By means of the variational approach and Ekeland's principle, we establish that the above problem admits a non-trivial weak solution under appropriate conditions.

Keywords. Robin problem, Ekeland's variational principle, generalized Sobolev spaces, weak solution

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1. Introduction

Robin boundary conditions are a weighted combination of Dirichlet and Neuman boundary conditions and it is also called impedance boundary conditions, from their application in electromagnetic problems or convective boundary conditions from their application in heat transfer problems. Moreover Robin conditions are commonly used in solving Sturm–Liouville problems which appear in many contexts in sciences and engineering. In addition, it is a general form of the insulating boundary condition for convection-diffusion equations

$$u_x(0)c(0) - D\frac{\partial c(0)}{\partial x} = 0,$$

where D is the diffuse constant, u is the convective velocity at the boundary, c is the concentration and the convective and diffusive fluxes at the boundary sum

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to zero. The second term is a result of Fick's law of diffusion. In addition operators involving Lebesgue and Sobolev variable exponents are very interesting in many topic like electrorheological fluids (see [21]), elastic mechanics (see [22]), stationary thermo-rheological viscous flows of non-Newtonian fluids, image processing (See [5]) and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [1]). The $p(x)$ -Laplacian operator is nonhomogeneous and possess a more complicated structure then the classical p -Laplacian. Motivated by the above cited contributions, we study the existence of the weak solution for the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda V(x)|u|^{q(x)-2}u, & x \in \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} + a(x)|u|^{p(x)-2}u = 0. & x \in \partial\Omega \end{cases} \quad (1)$$

where $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the so-called $p(x)$ -Laplacian operator, Ω is a smooth bounded domain in \mathbb{R}^N , $\lambda > 0$, p, q are continuous functions on $\bar{\Omega}$, $a \in L^\infty(\partial\Omega)$ such that $\operatorname{ess\,inf}_{x \in \Omega} a(x) > 0$ and V is a given function in a generalized Lebesgue space $L^{s(x)}(\Omega)$ such that $V > 0$ in an open set $\Omega_0 \subset\subset \Omega$, where $|\Omega_0| > 0$.

The $p(x)$ -Laplacian problem involving Robin boundary conditions was studied by many authors in recent years, we mention that Deng in [6] considered the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x, u), & x \in \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} + a(x)|u|^{p(x)-2}u = 0. & x \in \partial\Omega \end{cases} \quad (2)$$

Applying the sub-supersolution method and the variational method, under appropriate assumptions on f , the author established the existence of $\lambda_* > 0$ such that the above problem has at least two positive solutions if $\lambda \in (0, \lambda_*)$, has at least one positive solution if $\lambda = \lambda_* < +\infty$ and has no positive solution if $\lambda > \lambda_*$. Deng et al in [7] investigated problem (1) under the particular case when $p(x) \equiv q(x)$ and $V(x) \equiv 1$, the authors established the existence of infinitely many eigenvalue sequences provided $p(x)$ is non constant and they presented some sufficient conditions for which there is no principal eigenvalue for the problem and the set of all eigenvalues is not closed.

Moreover problem (2) was investigated by Allaoui in [2], the author showed under appropriate conditions on f and using mountain pass theorem, the existence of a continuous spectrum of eigenvalues.

Meanwhile, elliptic problems involving operators in divergence form can be found in [4, 18]. Some other results dealing the $p(x)$ -Laplace operator and Sobolev spaces with variable exponents can be found in [14, 19].

Inspired by the above-mentioned papers, we study problem (1) under the assumptions

(H) $1 < q(x) < p(x) < N < s(x)$, for all $x \in \bar{\Omega}$, $V \in L^{s(x)}(\Omega)$ and $V > 0$ in $\Omega_0 \subset\subset \Omega$, with $|\Omega_0| > 0$.

Our main results established, the existence of a global minimizer of the Euler Lagrange functional associated to (1), in the second hand, we establish the existence of a continuous family of eigenvalues in a neighborhood of the origin. Note that condition (H) has never been used for Robin problems, moreover one of our result was investigated by Ekeland's variational problem which has never been used before for this kind of problem.

2. Abstract setting

In the sequel, we recall some results on the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ (for details, one can see [19, 20]), which will be needed later. Set

$$C_+(\bar{\Omega}) := \{h : h \in C(\bar{\Omega}), h(x) > 1, \text{ for all } x \in \bar{\Omega}\}.$$

For any $p \in C_+(\bar{\Omega})$, we denote by $1 < p^- := \min_{x \in \bar{\Omega}} p(x) \leq p^+ := \max_{x \in \bar{\Omega}} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable and } \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}.$$

We recall the following so-called Luxemburg norm on this space defined by the formula

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Clearly, when $p(x) = p$, a positive constant, the space $L^{p(x)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$ and the norm $|u|_{p(x)}$ reduces to the standard norm $\|u\|_{L^p} = (\int_{\Omega} |u|^p dx)^{\frac{1}{p}}$ in $L^p(\Omega)$.

Let $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, then the Hölder type inequality

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega), \quad v \in L^{p'(x)}(\Omega) \quad (3)$$

holds.

Moreover, if h_1, h_2 and $h_3 : \bar{\Omega} \rightarrow (1, \infty)$ are three Lipschitz continuous functions such that $\frac{1}{h_1(x)} + \frac{1}{h_2(x)} + \frac{1}{h_3(x)} = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$ the following inequality holds (see [10, Proposition 2.5]):

$$\left| \int_{\Omega} uvw dx \right| \leq \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-} \right) |u|_{h_1(x)} |v|_{h_2(x)} |w|_{h_3(x)}. \quad (4)$$

The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\rho_{p(x)}(u) := \int_{\Omega} |u|^{p(x)} dx,$$

and it satisfies the following propositions

Proposition 2.1 ([16, Proposition 1.2]). *For all $u, v \in L^{p(x)}(\Omega)$, we have*

1. $|u|_{p(x)} < 1$ (resp. $= 1, > 1$) $\Leftrightarrow \rho_{p(x)}(u) < 1$ (resp. $= 1, > 1$).
2. $\min \left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right) \leq \rho_{p(x)}(u) \leq \max \left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right)$.
3. $\rho_{p(x)}(u - v) \rightarrow 0 \Leftrightarrow |u - v|_{p(x)} \rightarrow 0$.

Proposition 2.2 ([8, Lemma 2.1]). *Let p and q two measurable functions such that $p \in L^{\infty}(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min \left(|u|_{p(x)q(x)}^{p^+}, |u|_{p(x)q(x)}^{p^-} \right) \leq \| |u|^{p(x)} \|_{q(x)} \leq \max \left(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+} \right).$$

For more details concerning the modular, one can see [11, 16].

Define also the variable exponent Sobolev space $X := W^{1,p(x)}(\Omega)$, by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = \inf \left\{ \mu > 0 : \int_{\Omega} \left(\left| \frac{\nabla u}{\mu} \right|^{p(x)} + \left| \frac{u}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\} \quad \text{for } u \in X,$$

$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

Note that, $(X, \|\cdot\|)$ is a separable and reflexive Banach space.

Let

$$\|u\|_a = \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{\nabla u}{\mu} \right|^{p(x)} dx + \int_{\partial\Omega} a(x) \left| \frac{u}{\mu} \right|^{p(x)} d\sigma \leq 1 \right\} \quad \text{for } u \in X.$$

Then, by [6, Theorem 2.1], $\|u\|_a$ is also a norm on X which is equivalent to $\|u\|$, moreover, if we define the so called modular which is defined by $I_a : X \rightarrow \mathbb{R}$ by

$$I_a(u) = \int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} a(x) |u|^{p(x)} d\sigma,$$

one has

Proposition 2.3 ([6, Proposition 2.4]).

1. $\|u\|_a < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow I_a(u) < 1$ (*resp.* $= 1, > 1$).
2. $\min(\|u\|_a^{p^-}, \|u\|_a^{p^+}) \leq I_a(u) \leq \max(\|u\|_a^{p^-}, \|u\|_a^{p^+})$.
3. $\|u_n\|_a \rightarrow 0$ (*resp.* $\rightarrow \infty$) $\Leftrightarrow I_a(u_n) \rightarrow 0$ (*resp.* $\rightarrow \infty$).

Proposition 2.4 ([15, Proposition 2.2]). *Let*

$$L(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} d\sigma,$$

then there holds true: The mapping $L' : X \rightarrow X^$ is a strictly monotone, continuous bounded homeomorphism and is of type (S_+) , namely $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow +\infty} L'(u_n)(u_n - u) \leq 0$, implies $u_n \rightarrow u$.*

In the sequel, let

$$\begin{cases} p^*(x) = \frac{Np(x)}{N - p(x)}, & p(x) < N, \\ p^*(x) = +\infty, & p(x) \geq N. \end{cases}$$

We point out that if $q \in C^+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then X is continuously and compactly embedded in $L^{q(x)}(\Omega)$.

3. Main results and auxiliary properties

Throughout the paper, the letters $c, c_i, i = 1, 2, \dots$ denote positive constants which may change from line to line.

In the sequel, denote by $s'(x)$ the conjugate exponent of the function $s(x)$ and put $\alpha(x) := \frac{s(x)q(x)}{s(x)-q(x)}$ then we have:

Remark 3.1. Under assumption **(H)**, one has $s'(x)q(x) < p^*(x)$, for all $x \in \overline{\Omega}$, $\alpha(x) < p^*(x)$, for all $x \in \overline{\Omega}$, so the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\alpha(x)}(\Omega)$ are compact and continuous.

Note that an eigenvalue for problem (1) satisfy the following definition.

Definition 3.2. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1), if there exists $u \in X \setminus \{0\}$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} a(x) |u|^{p(x)-2} u v d\sigma = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx,$$

for any $v \in X$ and we recall that if λ is an eigenvalue of the problem (1), then the corresponding $u \in X \setminus \{0\}$ is a weak solution of (1).

The first result in this paper is the following.

Theorem 3.3. *Assume hypothesis **(H)** is fulfilled. Then, there exists $\lambda^* > 0$, such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue of problem (1).*

In the second, we establish that the Euler–Lagrange functional associated to problem (1), has a global minimizer.

Theorem 3.4. *Assume that hypothesis **(H)** holds. Then any $\lambda > 0$ is an eigenvalue of problem (1).*

In order to formulate the variational problem (1), let us introduce the functionals Φ and $J : X \rightarrow \mathbb{R}$ defined by:

$$\Phi(u) = \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} d\sigma \quad \text{and} \quad J(u) = \int_{\Omega} \frac{V(x)}{q(x)} |u|^{q(x)} dx.$$

Using Proposition 2.2 and Remark 3.1, we mention that J is well defined and we have for all $u \in X$,

$$|J(u)| \leq \frac{1}{q^-} |V|_{s(x)} \| |u|^{q(x)} \|_{s'(x)} \leq \begin{cases} \frac{1}{q^-} |V|_{s(x)} |u|_{s'(x)q(x)}^{q^-}, & \text{if } |u|_{s'(x)q(x)} \leq 1, \\ \frac{1}{q^-} |V|_{s(x)} |u|_{s'(x)q(x)}^{q^+}, & \text{if } |u|_{s'(x)q(x)} > 1. \end{cases}$$

Let us introduce the Euler–Lagrange functional corresponding to problem (1) which is defined as

$$\Psi_{\lambda} : X \rightarrow \mathbb{R}, \quad \text{where} \quad \Psi_{\lambda}(u) := \Phi(u) - \lambda J(u).$$

We start with the following auxiliary property.

Proposition 3.5. *Under assumption **(H)**, $\Psi_{\lambda} \in C^1(X, \mathbb{R})$ is weakly lower semi-continuous and $u \in X$ is a critical point of Ψ_{λ} if and only if u is a weak solution for the problem (1).*

Proof. To show that $\Psi_{\lambda} \in C^1(X, \mathbb{R})$, we show that for all $\varphi \in X$,

$$\lim_{t \rightarrow 0^+} \frac{\Psi_{\lambda}(u + t\varphi) - \Psi_{\lambda}(u)}{t} = \langle d\Psi_{\lambda}(u), \varphi \rangle,$$

and $d\Psi_{\lambda} : X \rightarrow X^*$ continuous, where we denote by X^* the dual space of X .

For all $\varphi \in X$ we have

$$\begin{aligned}
 \lim_{t \rightarrow 0^+} \frac{J(u + t\varphi) - J(u)}{t} &= \frac{d}{dt} J(u + t\varphi)|_{t=0} \\
 &= \frac{d}{dt} \int_{\Omega} \frac{V(x)}{q(x)} |u + t\varphi|^{q(x)} dx |_{t=0} \\
 &= \int_{\Omega} \frac{\partial}{\partial t} \left(\frac{V(x)}{q(x)} |u + t\varphi|^{q(x)} \right) |_{t=0} dx \\
 &= \int_{\Omega} V(x) |u + t\varphi|^{q(x)-1} \operatorname{sgn}(u + t\varphi) \varphi |_{t=0} dx \\
 &= \int_{\Omega} V(x) |u + t\varphi|^{q(x)-2} (u + t\varphi) \varphi |_{t=0} dx \\
 &= \int_{\Omega} V(x) |u|^{q(x)-2} u \varphi dx \\
 &= \langle dJ(u), \varphi \rangle.
 \end{aligned}$$

The differentiation under the integral is allowed if for t close to zero. Indeed, for $|t| < 1$, we have:

$$|V(x)| |u + t\varphi|^{q(x)-2} (u + t\varphi) \varphi \leq |V(x)| (|u| + |\varphi|)^{q(x)-1} |\varphi| \in L^1(\Omega).$$

Because $u, \varphi \in X$ imply:

$$|u|, |\varphi| \in X \hookrightarrow L^{q(x)}(\Omega) \quad \text{and} \quad |\varphi| \in X \hookrightarrow L^{\alpha(x)}(\Omega).$$

Due to the fact that $V \in L^{s(x)}(\Omega)$, the conclusion is an immediate consequence of inequality (4). For $u \in X$ chosen we show that $dJ(u) \in X^*$. It is easy to see that $dJ(u)$ is linear.

Since there is a continuous embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, then

$$\|v\|_{\alpha(x)} \leq c \|v\|_a, \quad \text{for all } v \in X. \quad (5)$$

Using inequalities (4) and (5) we obtain

$$\begin{aligned}
 |\langle dJ(u), \varphi \rangle| &= \left| \int_{\Omega} V(x) |u|^{q(x)-2} u \varphi dx \right| \\
 &\leq \int_{\Omega} |V(x)| |u|^{q(x)-1} |\varphi| dx \\
 &\leq |V|_{s(x)} \| |u|^{q(x)-1} \|_{\frac{q(x)}{q(x)-1}} \|\varphi\|_{\alpha(x)} \\
 &\leq c |V|_{s(x)} \| |u|^{q(x)-1} \|_{\frac{q(x)}{q(x)-1}} \|\varphi\|_a.
 \end{aligned}$$

Hence there exists $c_1 = c |V(x)| \| |u|^{q(x)-1} \|_{\frac{q(x)}{q(x)-1}} > 0$ such that

$$|\langle dJ(u), \varphi \rangle| \leq c_1 \|\varphi\|_a.$$

Using the linearity of $dJ(u)$ and the above inequality we deduce that $dJ(u) \in X^*$. For the Fréchet differentiability we need the following Lemma

Lemma 3.6 ([3, Lemma 1]). *The map $u \in L^{q(x)}(\Omega) \rightarrow |u|^{q(x)-2}u \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$ is continuous.*

We conclude that J is Fréchet differentiable.

It is well known that Φ is well defined and continuously Gâteaux differentiable and its Gâteaux derivative at point $u \in X$ is given by

$$\langle d\Phi(u), v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} a(x) |u|^{p(x)-2} u v d\sigma, \quad \text{for all } v \in X.$$

We deduce that $\Psi_{\lambda} \in C^1(X, \mathbb{R})$ because $\Phi, J \in C^1(X, \mathbb{R})$. Moreover

$$\begin{aligned} \langle d\Psi_{\lambda}(u), v \rangle &= \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} a(x) |u|^{p(x)-2} u v d\sigma \\ &\quad - \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx, \quad \text{for all } v \in X. \end{aligned}$$

Let u be a critical point of Ψ_{λ} . Then we have $d\Psi_{\lambda}(u) = 0_{X^*}$ that is $\langle d\Psi_{\lambda}(u), v \rangle = 0$, for all $v \in X$, which yields

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} a(x) |u|^{p(x)-2} u v d\sigma = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx,$$

for all $v \in X$.

It follows that u is a weak solution for the problem (1). Now we assume that u is a weak solution, by Definition 3.2 it results that

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\partial\Omega} a(x) |u|^{p(x)-2} u v d\sigma = \lambda \int_{\Omega} V(x) |u|^{q(x)-2} u v dx,$$

for all $v \in X$.

That is $\langle d\Psi_{\lambda}(u), v \rangle = 0$, for all $v \in X$. We obtain $d\Psi_{\lambda}(u) = 0_{X^*}$. Hence u is a critical point of Ψ_{λ} . This completes the proof of Proposition 3.5. \square

The argument key for the proof of Theorem 3.3 is related to Ekeland's variational principle [9]. For this purpose, we show the existence of a *mountain* for Ψ_{λ} near the origin.

Lemma 3.7. *Suppose we are under hypotheses of Theorem 3.3. Then for all $\rho \in (0, 1)$, there exist $\lambda^* > 0$ and $b > 0$ such that for all $u \in X$ with $\|u\|_a = \rho$*

$$\Psi_{\lambda}(u) \geq b > 0 \quad \text{for all } \lambda \in (0, \lambda^*).$$

Proof. Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, then

$$|u|_{s'(x)q(x)} \leq c_2 \|u\|_a, \text{ for all } u \in X. \quad (6)$$

Let us assume that $\|u\|_a < \min(1, \frac{1}{c_2})$, where c_2 is the positive constant of inequality (6). Then, we have $|u|_{s'(x)q(x)} < 1$, using Hölder inequality (3), Proposition 2.3 and inequality (6), we deduce that for any $u \in X$ with $\|u\|_a = \rho$ the following inequalities hold true

$$\begin{aligned} \Psi_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} d\sigma - \frac{\lambda}{q^-} \int_\Omega V(x) |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \|u\|_a^{p^+} - \frac{\lambda}{q^-} |V|_{s(x)} \|u\|_{s'(x)}^{q^-} \\ &\geq \frac{1}{p^+} \|u\|_a^{p^+} - \frac{\lambda}{q^-} |V|_{s(x)} |u|_{s'(x)q(x)}^{q^-} \\ &\geq \frac{1}{p^+} \|u\|_a^{p^+} - \frac{\lambda}{q^-} |V|_{s(x)} c_2^{q^-} \|u\|_a^{q^-} \\ &= \frac{1}{p^+} \rho^{p^+} - \frac{\lambda}{q^-} c_2^{q^-} |V|_{s(x)} \rho^{q^-} \\ &= \rho^{q^-} \left(\frac{1}{p^+} \rho^{p^+ - q^-} - \frac{\lambda}{q^-} c_2^{q^-} |V|_{s(x)} \right). \end{aligned}$$

By the above inequality, we remark that if we define

$$\lambda^* = \frac{\rho^{p^+ - q^-}}{2p^+} \cdot \frac{q^-}{c_2^{q^-} |V|_{s(x)}}, \quad (7)$$

then for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\|_a = \rho$ there exists $b > 0$ such that

$$\Psi_\lambda(u) \geq b > 0.$$

The proof of Lemma 3.7 is complete. \square

The following result asserts the existence of a valley for Ψ_λ near the origin.

Lemma 3.8. *There exists $\phi \in X$ such that $\phi \geq 0$, $\phi \neq 0$ and $\Psi_\lambda(t\phi) < 0$, for $t > 0$ small enough.*

Proof. Assumption **(H)** implies $q(x) < p(x)$, for all $x \in \bar{\Omega}_0$. In the sequel, denote by $q_0^- = \inf_{\Omega_0} q(x)$ and by $p_0^- = \inf_{\Omega_0} p(x)$. Let ϵ_0 such that $q_0^- + \epsilon_0 < p_0^-$. On the other hand, since $q \in C(\bar{\Omega}_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q(x) - q_0^-| < \epsilon_0$, for all $x \in \Omega_1$. It follows that $q(x) \leq q_0^- + \epsilon_0 < p_0^-$, for all $x \in \Omega_1$.

Let $\phi \in C_0^\infty(\Omega)$ such that $\text{supp}(\phi) \subset \Omega_1 \subset \Omega_0$, $\phi = 1$ in a subset $\Omega'_1 \subset \text{supp}(\phi)$,

$0 \leq \phi \leq 1$ in Ω_1 . then we have:

$$\begin{aligned} \Psi_\lambda(t\phi) &= \int_{\Omega} \frac{1}{p(x)} |\nabla(t\phi)|^{p(x)} dx + \int_{\partial\Omega} \frac{a(x)}{p(x)} |t\phi|^{p(x)} d\sigma - \lambda \int_{\Omega} \frac{V(x)}{q(x)} |t\phi|^{q(x)} dx \\ &\leq \frac{1}{p_0^-} \left(\int_{\Omega_0} t^{p(x)} |\nabla\phi|^{p(x)} dx + \int_{\partial\Omega} t^{p(x)} a(x) |\phi|^{p(x)} d\sigma \right) - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\phi|^{q(x)} dx \\ &\leq \begin{cases} \frac{t^{p_0^-}}{p_0^-} I_a(\phi) - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\phi|^{q(x)} dx, & \text{if } \partial\Omega \cap \Omega_0 \neq \emptyset, \\ \frac{t^{p_0^-}}{p_0^-} \rho_{p(x)}(\nabla\phi) - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\phi|^{q(x)} dx, & \text{if } \partial\Omega \cap \Omega_0 = \emptyset. \end{cases} \end{aligned}$$

Using Proposition 2.1, one has

$$\begin{aligned} &\Psi_\lambda(t\phi) \\ &\leq \begin{cases} \frac{t^{p_0^-}}{p_0^-} I_a(\phi) - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\phi|^{q(x)} dx, & \text{if } \partial\Omega \cap \Omega_0 \neq \emptyset, \\ \frac{t^{p_0^-}}{p_0^-} \max \left(|\nabla\phi|_{p(x)}^{p^-}, |\nabla\phi|_{p(x)}^{p^+} \right) - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\phi|^{q(x)} dx, & \text{if } \partial\Omega \cap \Omega_0 = \emptyset. \end{cases} \\ &\leq \begin{cases} \frac{t^{p_0^-}}{p_0^-} I_a(\phi) - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\phi|^{q(x)} dx, & \text{if } \partial\Omega \cap \Omega_0 \neq \emptyset, \\ \frac{t^{p_0^-}}{p_0^-} \max \left(\|\phi\|^{p^-}, \|\phi\|^{p^+} \right) - \lambda \int_{\Omega_1} \frac{V(x)}{q(x)} t^{q(x)} |\phi|^{q(x)} dx, & \text{if } \partial\Omega \cap \Omega_0 = \emptyset. \end{cases} \end{aligned}$$

Therefore $\Psi_\lambda(t\phi) < 0$ for $t < \delta^{\frac{1}{p_0^- - q_0^- - \epsilon_0}}$ with

$$0 < \delta < \begin{cases} \min \left\{ 1, \frac{\frac{\lambda}{q_0^+} \int_{\Omega_1} V(x) |\phi|^{q(x)} dx}{I_a(\phi)} \right\}, & \text{if } \partial\Omega \cap \Omega_0 \neq \emptyset, \\ \min \left\{ 1, \frac{\frac{\lambda}{q_0^+} \int_{\Omega_1} V(x) |\phi|^{q(x)} dx}{\max(\|\phi\|^{p^+}, \|\phi\|^{p^-})} \right\}, & \text{if } \partial\Omega \cap \Omega_0 = \emptyset. \end{cases}$$

Since $\phi = 1$ in Ω'_1 , then $\|\phi\| > 0$ and Proposition 2.3 ensures that $I_a(\phi) > 0$, the proof of Lemma 3.8 is complete. \square

Proof of Theorem 3.3 completed. Let $\lambda^* > 0$ be defined as in (7) and $\lambda \in (0, \lambda^*)$. By Lemma 3.7 it follows that on the boundary of the ball centered at the origin and of radius ρ in X , denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} \Psi_\lambda > 0. \quad (8)$$

On the other hand, by Lemma 3.8, there exists $\phi \in X$ such that $\Psi_\lambda(t\phi) < 0$ for all $t > 0$ small enough. Moreover, using Hölder inequality (3), Proposition 2.3 and inequality (6) we deduce that for any $u \in B_\rho(0)$ we have

$$\Psi_\lambda(u) \geq \frac{1}{p^+} \|u\|_a^{p^+} - \frac{\lambda}{q^-} c_2^{q^-} |V|_{s(x)} \|u\|_a^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{B_\rho(0)} \Psi_\lambda < 0.$$

Let $0 < \epsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda - \inf_{B_\rho(0)} \Psi_\lambda$. Using the above information, the functional $\Psi_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$, is lower bounded on $\overline{B_\rho(0)}$ and $\Psi_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then by Ekeland's variational principle, there exists $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq \Psi_\lambda(u_\epsilon) \leq \underline{c} + \epsilon \\ 0 < \Psi_\lambda(u) - \Psi_\lambda(u_\epsilon) + \epsilon \cdot \|u - u_\epsilon\|_a, \quad u \neq u_\epsilon. \end{cases}$$

Since

$$\Psi_\lambda(u_\epsilon) \leq \inf_{B_\rho(0)} \Psi_\lambda + \epsilon \leq \inf_{B_\rho(0)} \Psi_\lambda + \epsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$. Now, we define $I_\lambda : \overline{B_\rho(0)} \rightarrow \mathbb{R}$ by $I_\lambda(u) = \Psi_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_a$. It is clear that u_ϵ is a minimum point of I_λ and thus

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{\Psi_\lambda(u_\epsilon + t \cdot v) - \Psi_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\|_a \geq 0$$

Letting $t \rightarrow 0$ it follows that $\langle d\Psi_\lambda(u_\epsilon), v \rangle + \epsilon \cdot \|v\|_a \geq 0$ and we infer that $\|d\Psi_\lambda(u_\epsilon)\|_a \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$\Psi_\lambda(w_n) \rightarrow \underline{c} < 0 \quad \text{and} \quad d\Psi_\lambda(w_n) \rightarrow 0_{X^*}. \quad (9)$$

It is clear that $\{w_n\}$ is bounded in X . Thus, there exists w in X such that, up to a subsequence, $\{w_n\}$ converges weakly to w in X . Since $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$ we deduce that there exists a compact embedding $E \hookrightarrow L^{\alpha(x)}(\Omega)$ and consequently $\{w_n\}$ converges strongly in $L^{\alpha(x)}(\Omega)$. For the strong convergence of $\{w_n\}$ in X , we need the following proposition:

Proposition 3.9.

$$\lim_{n \rightarrow \infty} \int_{\Omega} V(x) |w_n|^{q(x)-2} w_n (w_n - w) dx = 0.$$

Proof. Using Hölder inequality (3) we have:

$$\begin{aligned} \int_{\Omega} V(x)|w_n|^{q(x)-2}w_n(w_n - w)dx &\leq |V|_{s(x)}\|w_n\|^{q(x)-2}w_n(w_n - w)|_{s'(x)} \\ &\leq |V|_{s(x)}\|w_n\|^{q(x)-2}w_n\Big|_{\frac{q(x)}{q(x)-1}}\|w_n - w\|_{\alpha(x)}. \end{aligned}$$

Now if $\|w_n\|^{q(x)-2}w_n\Big|_{\frac{q(x)}{q(x)-1}} > 1$, by Proposition 2.2, we get

$$\|w_n\|^{q(x)-2}w_n\Big|_{\frac{q(x)}{q(x)-1}} \leq \|w_n\|_{q(x)}^{q^+}.$$

The compact imbedding $X \hookrightarrow L^{q(x)}(\Omega)$ ends the proof. \square

Since $d\Psi_{\lambda}(w_n) \rightarrow 0$ and w_n is bounded in X we have

$$\begin{aligned} |\langle d\Psi_{\lambda}(w_n), w_n - w \rangle| &\leq |\langle d\Psi_{\lambda}(w_n), w_n \rangle| + |\langle d\Psi_{\lambda}(w_n), w \rangle| \\ &\leq \|d\Psi_{\lambda}(w_n)\|_a \|w_n\|_a + \|d\Psi_{\lambda}(w_n)\|_a \|w\|_a. \end{aligned}$$

Moreover, using Proposition 3.9, one has $\lim_{n \rightarrow \infty} \langle d\Psi_{\lambda}(w_n), w_n - w \rangle = 0$. So

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\nabla w_n|^{p(x)-2} \nabla w_n (\nabla w_n - \nabla w) dx + \int_{\partial\Omega} a(x) |w_n|^{p(x)-2} w_n (w_n - w) d\sigma = 0.$$

Now, Proposition 2.4, ensures that $\{w_n\}$ converges strongly to w in X .

Since $\Psi_{\lambda} \in C^1(X, \mathbb{R})$, we conclude

$$d\Psi_{\lambda}(w_n) \rightarrow d\Psi_{\lambda}(w), \quad \text{as } n \rightarrow \infty. \quad (10)$$

Relations (9) and (10) show that $d\Psi_{\lambda}(w) = 0$ and thus w is a weak solution for problem (1). Moreover, by relation (9), it follows that $\Psi_{\lambda}(w) < 0$ and thus, w is a non-trivial weak solution for (1), since $\Psi_{\lambda}(|w|) = \Psi_{\lambda}(w)$ then problem (1) has a positive one. The proof of Theorem 3.3 is complete. \square

Proof of Theorem 3.4. Using Hölder inequality (3) for $\|u\|_a > 1$, one has

$$\begin{aligned} \Psi_{\lambda}(u) &= \int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\partial\Omega} a(x) |u|^{p(x)} d\sigma - \frac{\lambda}{q^-} \int_{\Omega} V(x) |u|^{q(x)} dx \\ &\geq \frac{1}{p^+} \|u\|_a^{p^-} - \frac{\lambda}{q^-} |V|_{s(x)} \|u\|^{q(x)}|_{s'(x)} \\ &\geq \frac{1}{p^+} \|u\|_a^{p^-} - \frac{\lambda}{q^-} |V|_{s(x)} \|u\|_{s'(x)q(x)}^{q^-} \\ &\geq \frac{1}{p^+} \|u\|_a^{p^-} - \frac{\lambda}{q^-} |V|_{s(x)} C^{q^-} \|u\|_a^{q^-} \rightarrow +\infty, \quad \text{as } \|u\|_a \rightarrow +\infty. \end{aligned}$$

As a conclusion, since Ψ_{λ} is weakly lower semi-continuous then it has a global minimizer which is solution of problem (1), moreover Lemma 3.8 ensures that this minimizer is non-trivial, which ends the proof. \square

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