Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 37 (2018), 39–50
DOI: 10.4171/ZAA/1601

Infinitely many Solutions for Klein–Gordon–Maxwell System with Potentials Vanishing at Infinity

Shang-Jie Chen and Lin Li

Abstract. In this paper, a nonlinear Klein–Gordon–Maxwell system with solitary waves solution is considered. Using critical point theory, we establish sufficient conditions for the existence of Infinitely many radial solitary waves solutions.

Keywords. Klein–Gordon–Maxwell system, variational methods, Fountain Theorem
Mathematics Subject Classification (2010). Primary 35J61, secondary 35C06, 35J20

1. Introduction and main results

Many recent papers show the application of global variational methods to the study of the interaction between matter and electromagnetic fields. A typical example is given by the following system of Klein–Gordon–Maxwell equations

$$\begin{cases} -\Delta u + [m^2 - (\omega + e\phi)^2]u = |u|^{q-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + e^2 \phi u^2 = -e\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1}$$

where m, ω and e are real constants, $u, \phi : \mathbb{R}^3 \to \mathbb{R}$. Such a system has been first introduced in [5] as a model describing solitary waves for the nonlinear stationary Klein–Gordon equation in the three-dimensional space interacting with the electrostatic field. Here m and e are the mass and the charge of the particle respectively, while ω denote the phase. The unknowns of the system are the field u associated to the particle and the electric potential ϕ. The presence of the nonlinear term simulates the interaction between many particles or external nonlinear perturbations.

S.-J. Chen, L. Li (Corresponding author): School of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, P.R. China; 11183356@qq.com; lilin420@gmail.com

Some existence and nonexistence results for Klein–Gordon–Maxwell equations (1) have been proved via modern variational methods under various hypotheses on the nonlinear term. We recall some of them as follows.

Benci and Fortunato [4, 5] were pioneered work with this system. They found the existence of infinitely many radially symmetric solutions for system (1) when $|m| > |\omega|$ and $4 < q < 6$. In [8], D'Aprile and Mugnai extended the interval of definition of the power in nonlinearity for the case $q \in (2, 4]$ provided $m\sqrt{\frac{q}{2} - 1} > \omega > 0$. Later, in [3], Azzollini, Pisani and Pomponio gave a small improvement with $q \in (2, 4)$. Under the conditions

- $3 \leq q < 6$ and $m > \omega > 0$,
- $2 < q < 3$ and $m\sqrt{(q-2)(4-q)} > \omega > 0$,

they proved that problem (1) adimits a nontrivial solution. In [2] the existence of a ground state solution (namely for solution which minimizes the action functional among all the solutions) for problem (1) was got under one of the conditions

- $4 \leq q < 6$ and $m > \omega$,
- $2 < q < 4$ and $m\sqrt{q-2} > \omega\sqrt{6-q}$,

Soon afterwards, it is improved by the result in [19] provided one of the following conditions is satisfied:

- $4 \leq q < 6$ and $m > \omega > 0$,
- $2 < q < 4$ and $m\left(1 + \frac{(4-q)^2}{4(q-2)}\right) > \omega > 0$.

In [9], the nonexistence results for system (1) were obtained for $q \leq 2$ or $q \geq 6$ and $m \geq \omega > 0$ respectively, where also more general nonlinear terms were considered.

Very recently, Cunha [7], Ding and Li [10], He [12], and Li and Tang [14] considered the following Klein–Gordon–Maxwell system with a non-constant potential and a general nonlinear terms:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi + u^2\phi = -\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{2}$$

where $\omega > 0$ is a constant, $V : \mathbb{R}^3 \to \mathbb{R}$, $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. Concretely, supposing that V satisfies some assumptions which contain the coercivity condition: $V(x) \to +\infty$ as $|x| \to \infty$, [10, 12, 14] established the existence of infinitely many high-energy solutions for problem (2), while the existence of nontrivial solution was proved in [7] with periodic potential V. Other related results about Klein–Gordon–Maxwell system on $\mathbb{R}^3$ can be found in [6, 13, 16].

Motivated by the works mentioned above, in this paper we consider the following Klein–Gordon–Maxwell system:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = K(x)|u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi + u^2\phi = -\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{$\mathcal{KGM}$}$$

where $\omega, p > 0$ are constants, $V, K : \mathbb{R}^3 \to \mathbb{R}$, are assumed to satisfy the following assumptions:

(V) $V(x) \in C(\mathbb{R}^3, \mathbb{R}^+)$ is radial, smooth, and there exists $\alpha \in (0, 2], a, A > 0$ such that
$$\frac{a}{1+|x|^\alpha} \leq V(x) \leq A.$$

(K) $K(x) \in C(\mathbb{R}^3, \mathbb{R}^+)$ is radial, smooth, and there exists $\beta, b > 0$ such that
$$0 < K(x) \leq \frac{b}{1+|x|^\beta}.$$

Before stating our main results, we give several notations. Let $C_0^\infty(\mathbb{R}^3)$ denote the collection of smooth functions with compact support and
$$C_{0,r}^\infty(\mathbb{R}^3) = \left\{u \in C_0^\infty(\mathbb{R}^3) \;\middle|\; u \text{ is radial}\right\}.$$

Denote respectively by $D_r^{1,2}(\mathbb{R}^3)$ and $H_r^1(\mathbb{R}^3; V)$ the Hilbert spaces defined as the completion of $C_{0,r}^\infty(\mathbb{R}^3)$ with respect to the following norms:
$$\|u\|_D := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}}, \quad \|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx\right)^{\frac{1}{2}}.$$

For our main result it is convenient to introduce the following quantities:
$$\sigma = \begin{cases} 6 - \frac{4\beta}{\alpha}, & \text{if } 0 < \beta < \alpha, \\ 2, & \text{otherwise.} \end{cases}$$

We state the main result.

Theorem 1.1. *Assume that condition* (V), (K) *hold for* $\alpha \in \left(0, \frac{4}{11}\right)$. *If* $p \in (4,6) \bigcap (\sigma, 6)$, *then problem* $(\mathcal{KGM})$ *has a sequence of solutions* $\{(u_n, \phi_n)\} \subset H_r^1(\mathbb{R}^3; V) \times D_r^{1,2}(\mathbb{R}^3)$ *satisfying*
$$\frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x)u_n^2 - |\nabla \phi_n|^2 - (2\omega + \phi_n)\phi_n u_n^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^3} K(x)|u_n|^p dx \to \infty.$$

Remark 1.2. The condition (V) and (K) were introduced by Ambrosetti, Felli and Malchiodi in [1] in the frame of nonlinear Schrödinger equations and were used in [15] where nontrivial solution were obtained for nonlinear Schrödinger–Poisson systems on $\mathbb{R}^N (N = 3, 4, 5)$. To the best of our knowledge, it seems that Theorem 1.1 is the first result for the Klein–Gordon–Maxwell system $(\mathcal{KGM})$ under the assumptions (V) and (K).

2. The variational framework and the proof of main results

In order to apply critical point theory, we first state two results on Sobolev embeddings, then reduce the problem of finding solutions of system ($\mathcal{KGM}$) to the one of seeking the critical points of a corresponding variational functional.

The first embedding results can be found in [15, Remark 2 and Lemma 2].

Proposition 2.1. *Let $\gamma := 1 - \frac{\alpha}{4}$, then $H_r^1(\mathbb{R}^3; V)$ is continuously embedded in $L^p(\mathbb{R}^3)$ for any $p \in [2 + \frac{\alpha}{\gamma}, 6]$. Furthermore, the embedding is compact for $p \in (2 + \frac{\alpha}{\gamma}, 6)$.*

We remark that if $\alpha \in \left(0, \frac{4}{11}\right)$, it follows that $\frac{12}{5} \in (2 + \frac{\alpha}{\gamma}, 6)$, which is important to ensure the solvability of the equation $-\Delta\phi + u^2\phi = -\omega u^2$ for $u \in H_r^1(\mathbb{R}^3; V)$.

Define for $q \geq 1$

$$L^q(\mathbb{R}^3; K) = \left\{ u : \mathbb{R}^3 \to \mathbb{R} \;\middle|\; u \text{ is measurable and } \int_{\mathbb{R}^3} K(x)|u|^q dx < +\infty \right\},$$

with norm

$$\|u\|_{q,K} = \left(\int_{\mathbb{R}^3} K(x)|u|^q dx \right)^{\frac{1}{q}}.$$

The second embedding results can be found in [18].

Proposition 2.2. *$H_r^1(\mathbb{R}^3; V)$ is continuously embedded in $L^p(\mathbb{R}^3; K)$ for any $p \in [\sigma, 6]$. Furthermore, the embedding is compact for $p \in (\sigma, 6)$.*

System ($\mathcal{KGM}$) has a variational structure. Indeed we consider the functional

$$\mathcal{J} : H_r^1(\mathbb{R}^3; V) \times D_r^{1,2}(\mathbb{R}^3) \to \mathbb{R}$$

defined by

$$\mathcal{J}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - |\nabla\phi|^2 - (2\omega + \phi)\phi u^2 \right) dx - \frac{1}{p} \int_{\mathbb{R}^3} K(x)|u|^p dx. \quad (3)$$

Evidently, the action functional $\mathcal{J}$ belongs to $C^1(H_r^1(\mathbb{R}^3; V) \times D_r^{1,2}(\mathbb{R}^3), \mathbb{R})$ and the partial derivatives in (u, ϕ) are given, for $\zeta \in H_r^1(\mathbb{R}^3; V), \eta \in D_r^{1,2}(\mathbb{R}^3)$, by

$$\left\langle \frac{\partial \mathcal{J}}{\partial u}(u, \phi), \zeta \right\rangle = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla\zeta + [V(x) - (2\omega + \phi)\phi]u\zeta - K(x)|u|^{p-2}u\zeta \right) dx,$$

$$\left\langle \frac{\partial \mathcal{J}}{\partial \phi}(u, \phi), \eta \right\rangle = \int_{\mathbb{R}^3} \left(-\nabla\phi \cdot \nabla\eta - (\phi + \omega)u^2\eta \right) dx.$$

Thus, we have the following result:

Proposition 2.3. *The pair (u, ϕ) is a weak solution of system $(\mathcal{KGM})$ if and only if it is a critical point of $\mathcal{J}$ in $H_r^1(\mathbb{R}^3; V) \times D_r^{1,2}(\mathbb{R}^3)$.*

The functional $\mathcal{J}$ is strongly indefinite, i.e. unbounded from below and from above on infinite dimensional spaces. To avoid this difficulty, we reduce the study of (3) to the study of a functional in the only variable u.

For any $u \in H_r^1(\mathbb{R}^3;V)$, let us consider the linear functional $T_u : D_r^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$T_u(v) = -\omega \int_{\mathbb{R}^3} u^2 v dx.$$

From $0 < \alpha < \frac{4}{11}$, we have $\frac{12}{5} \in (2 + \frac{\alpha}{\gamma}, 6)$. So by the Hölder inequality, Proposition 2.1 and the embedding $D_r^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we get

$$|T_u(v)| \leq \omega \|u^2\|_{\frac{6}{5}} \|v\|_6 = \omega \|u\|^2_{\frac{12}{5}} \|v\|_6 \leq C\|u\|^2 \|v\|_D,$$

where $\|u\|_q := \left(\int_{\mathbb{R}^3} |u|^q dx\right)^{\frac{1}{q}}$ is the norm of the usual Lebesgue space $L^q(\mathbb{R}^3)$, $(1 \leq q < \infty)$. So, T_u is continuous on $D_r^{1,2}(\mathbb{R}^3)$. Set

$$a(w, v) = \int_{\mathbb{R}^3} (\nabla w \cdot \nabla v + u^2 w v) dx, \quad w, v \in D_r^{1,2}(\mathbb{R}^3).$$

Again by the Hölder inequality, Proposition 2.1 and the embedding $D_r^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$,

$$a(w, v) \leq \|w\|_D \|v\|_D + \|u^2\|_{\frac{3}{2}} \|w\|_6 \|w\|_6 \leq \left(1 + C\|u\|_3^2\right) \|w\|_D \|v\|_D.$$

Thus, $a(w, v)$ is a bilinear (i.e. $a(w, v)$ is linear in w and v respectively), bounded (i.e. there exists a constant $C > 0$ such that for any $w, v \in D_r^{1,2}(\mathbb{R}^3)$, $|a(w, v)| \leq C\|w\|_D \|v\|_D$) and coercive (i.e. there exists a constant $\varrho > 0$ such that for any $w \in D_r^{1,2}(\mathbb{R}^3)$, $a(w, w) \geq \varrho \|w\|_D^2$). Hence, the Lax–Milgram theorem (see [11]) implies that for every $u \in H_r^1(\mathbb{R}^3; V)$, there exists a unique $\phi_u \in D_r^{1,2}(\mathbb{R}^3)$ such that

$$T_u(v) = a(\phi_u, v), \quad \text{for any } v \in D_r^{1,2}(\mathbb{R}^3),$$

that is, $-\omega \int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} (\nabla \phi_u \cdot \nabla v + u^2 \phi_u v)\, dx$. Using integration by parts, we get

$$-\omega \int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} \left(-\Delta \phi_u v + u^2 \phi_u v\right) dx, \quad \text{for any } v \in D_r^{1,2}(\mathbb{R}^3),$$

therefore,

$$-\Delta \phi_u + u^2 \phi_u = -\omega u^2, \tag{4}$$

in a weak sense.

Multiplying (4) by $\phi_u^+ := \max\{\phi_u, 0\}$ and using integration by parts, we get

$$\int_{\mathbb{R}^3} \left(|\nabla\phi_u^+|^2 + u^2\left(\phi_u^+\right)^2\right) dx = -\omega\int_{\mathbb{R}^3} u^2\phi_u^+ dx \le 0,$$

so that $\phi_u^+ \equiv 0$. if we multiply (4) by $(\omega+\phi_u)^-$ use integration by parts, which is an admissable test function since $\omega > 0$, we have

$$\int_{\phi_u<-\omega} |\nabla\phi_u|^2 dx = -\omega\int_{\phi_u<-\omega} u^2(\omega+\phi_u)^2 dx \le 0,$$

so that $(\omega+\phi_u)^- = 0$ where $u \neq 0$. Thus, we have $-\omega \le \phi_u \le 0$ on the set $\{x \,|\, u(x) \neq 0\}$.

We take inner product of (4) with ϕ_u and integrating by parts,

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx = -\int_{\mathbb{R}^3} \omega\phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \tag{5}$$

Since $-\omega \le \phi_u \le 0$, we have

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2 dx \le \omega\int_{\mathbb{R}^3} |\phi_u| u^2 dx = \omega\|u\|^2_{\frac{12}{5}}\|\phi_u\|_6 \le C\|u\|^2_{\frac{12}{5}}\|\phi_u\|_D.$$

Therefore,

$$\|\phi_u\|_D \le C\|u\|^2_{\frac{12}{5}} \le C\|u\|^2, \quad \text{and} \quad \int_{\mathbb{R}^3} |\phi_u|u^2 \le C\|u\|^4_{\frac{12}{5}} \le C\|u\|^4.$$

We have proved the following technical results.

Proposition 2.4. *For any fixed $u \in H_r^1(\mathbb{R}^3; V)$ be a radial function, there exists a unique $\phi = \phi_u \in D_r^{1,2}(\mathbb{R}^3)$ which solves the second equation of $(\mathcal{KGM})$ in a weak sense. Moreover,*

(i) *$-\omega \le \phi_u \le 0$ on the set $\{x \,|\, u(x) \neq 0\}$;*
(ii) *$\|\phi_u\|_D \le C\|u\|^2$ and $\int_{\mathbb{R}^3} |\phi_u|u^2 \le C\|u\|^4$.*

Remark 2.5. The proof is essentially contained on pages 5 and 6, and the result were already proved for a more general system in [17].

Using (5), we can rewrite $\mathcal{J}$ as a C^1 functional $\mathcal{I} : H_r^1(\mathbb{R}^3; V) \to \mathbb{R}$ given by

$$\mathcal{I}(u) = \mathcal{J}(u, \phi_u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2\right) dx - \frac{1}{p}\int_{\mathbb{R}^3} K(x)|u|^p dx,$$

while for $\mathcal{I}'$ we have

$$\langle \mathcal{I}'(u), v\rangle = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + V(x)uv - (2\omega+\phi_u)\phi_u uv - K(x)|u|^{p-2}uv\right) dx,$$

for all $v \in H_r^1(\mathbb{R}^3; V)$. Now we will look for its critical points since, as in [5], $(u, \phi) \in H_r^1(\mathbb{R}^3; V) \times D_r^{1,2}(\mathbb{R}^3)$ is a critical point for $\mathcal{J}$, if and only if u is a critical point for $\mathcal{I}$ with $\phi = \phi_u$, that is , a weak solution of

$$-\Delta u + V(x)u - (2\omega + \phi_u)\phi_u u = K(x)|u|^{p-2}u, \quad x \in \mathbb{R}^3.$$

In order to obtain infinitely many solutions of $(\mathcal{KGM})$, we shall use the following critical point theorem (see [20]).

Theorem 2.6 (Fountain Theorem). *Let X be a Banach space with the norm $\|\cdot\|$ and let X_j be a sequence of subspace of X with $\dim X_j < \infty$ for each $j \in \mathbb{N}$. Further, $X = \overline{\oplus_{j\in\mathbb{N}} X_j}$, the closure of the direct sum of all X_j. Set $Y_k = \oplus_{j=0}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider an even functional $\Phi \in C^1(X, \mathbb{R})$ (i.e. $\Phi(-u) = \Phi(u)$ for all $u \in X$). If, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(i) $a_k := \max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \le 0$,
(ii) $b_k := \inf_{u\in Z_k, \|u\|=r_k} \varphi(u) \to +\infty$ *as* $k \to \infty$,
(iii) *the Palais–Smale condition holds above* 0, *i.e. any sequence* $\{u_n\}$ *in* X *which satisfies* $\Phi(u_n) \to c > 0$ *and* $\Phi'(u_n) \to 0$ *contains a convergent subsequence,*

then Φ has an unbounded sequence of critical values.

We choose an orthogonal basis $\{e_j\}$ of $X := H_r^1(\mathbb{R}^3; V)$ and define $Y_k := \mathrm{span}\{e_1, \ldots, e_k\}$, $Z_k := Y_{k-1}^\perp$. To complete the proof of our theorems, we need the following lemma.

Lemma 2.7. *For any $\sigma < q < 2^*$, we have that*

$$\beta_q(k) := \sup_{u\in Z_k, \|u\|=1} \|u\|_{q,K} \to 0, \ \textit{as } k \to \infty.$$

Proof. It is clear that $0 \le \beta_q(k+1) \le \beta_q(k)$. Suppose that $\limsup \beta_q(k) = \beta_q > 0$ as $k \to \infty$. Then for any k sufficiently large, there exists $u_k \in Z_k$ with $\|u_k\| = 1$ and

$$\|u_k\|_{q,K} \ge \frac{\beta_q}{2}. \tag{6}$$

For any $u \in H_r^1(\mathbb{R}^3; V)$, since $\{e_j\}$ is an orthogonal basis of $H_r^1(\mathbb{R}^3; V)$, there exists a sequence $\{\alpha_j\} \subset \mathbb{R}$ satisfying $u = \sum_{j=1}^\infty \alpha_j e_j$, thus by the Schwartz inequality and the Parseval equality we have

$$|(u, u_k)| = \left|\left(\sum_{j=1}^\infty \alpha_j e_j, u_k\right)\right| = \left|\left(\sum_{j=k}^\infty \alpha_j e_j, u_k\right)\right| \le \left\|\sum_{j=k}^\infty \alpha_j e_j\right\| \|u_k\| = \sqrt{\sum_{j=k}^\infty \alpha_j^2} \to 0,$$

as $k \to \infty$, where $(\cdot, \cdot)$ denotes the inner product in $H_r^1(\mathbb{R}^3; V)$. Using the Riesz–Fréchet representation theorem, we obtain that $u_k \rightharpoonup 0$ in $H_r^1(\mathbb{R}^3; V)$ and thus $u_k \to 0$ in $L^q(\mathbb{R}^3; K)$ by Proposition 2.2. This is a contradiction to (6). $\square$

Consider $\Phi : H_r^1(\mathbb{R}^3; V) \to \mathbb{R}$ defined by

$$\Phi(u) := \mathcal{I}(u) = \frac{1}{2}\|u\|^2 - \frac{\omega}{2}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1}{p}\|u_k\|_{p,K}^p.$$

Evidently, Φ is even. First we show that under the condition (V) and (K), the functional Φ satisfies the geometric condition of Theorem 2.6. We have the following Lemma.

Lemma 2.8. *Assume that* (V), (K) *and* $p \in (4,6) \bigcap (\sigma, 6)$ *hold. Then for every* $k \in \mathbb{N}$*, there exists* $\rho_k > r_k > 0$*, such that*

(i) $a_k := \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$;
(ii) $b_k := \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \to +\infty$ *as* $k \to \infty$.

Proof. Due to Proposition 2.4(ii), we have

$$\Phi(u) \leq \frac{1}{2}\|u\|^2 + \frac{C}{4}\|u\|^4 - \frac{1}{p}\|u_k\|_{p,K}^p.$$

Since, on the finitely dimensional space Y_k all norms are equivalent, we have that $\Phi(u) \leq \frac{1}{2}\|u\|^2 + \frac{C}{4}\|u\|^4 - C\|u\|^p$. Since $p > 4$, it follows that

$$a_k := \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$$

for some $\rho_k > 0$ large enough.

We prove that Φ satisfies (ii). By Lemma 2.7, we have

$$\beta_p(k) \to 0, \quad \text{as } k \to \infty.$$

For each $k \geq 1$, taking

$$r_k = \left(\beta_p^p(k)\right)^{\frac{1}{2-p}},$$

one has $r_k \to +\infty$ as $k \to \infty$ since $p > 4$. Now, due to (i), Proposition 2.4, and the definition of $\beta_p(k)$, we have

$$\begin{aligned}
b_k &= \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \\
&\geq \inf_{u \in Z_k, \|u\| = r_k} \left[\frac{1}{2}\|u\|^2 - \frac{1}{p}\|u_k\|_{p,K}^p\right] \\
&\geq \inf_{u \in Z_k, \|u\| = r_k} \left[\frac{1}{2}\|u\|^2 - \frac{\beta_p^p(k)}{p}\|u\|^p\right] \\
&= \left(\frac{1}{2} - \frac{1}{p}\right) r_k^2 \to +\infty.
\end{aligned}$$

This proves (ii). $\square$

Lemma 2.9. *Assume that* (V), (K) *and* $p \in (4,6) \bigcap (\sigma, 6)$ *hold. Then the functional* Φ *satisfies the Palais–Smale condition at any level* $c > 0$.

Proof. We suppose that $\{u_n\}$ is a Palais–Smale sequence, that is for some $c \in \mathbb{R}^+$

$$\Phi(u_n) \to c, \quad \text{and} \quad \Phi'(u_n) \to 0 \quad (n \to \infty).$$

By Proposition 2.4(i),

$$\begin{aligned}\Phi(u_n) - \frac{1}{p}\langle \Phi'(u_n), u_n \rangle &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{2}{p}\right)\int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2 dx + \frac{1}{p}\int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2.\end{aligned}$$

Hence, $\{u_n\}$ is bounded. Now we shall prove $\{u_n\}$ contains a convergent subsequence. Without loss of generality, passing to a subsequence if necessary, there exists a $u \in H_r^1(\mathbb{R}^3; V)$ such that $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3; V)$, by Proposition 2.2, $u_n \to u$ in $L^p(\mathbb{R}^3; K)$. So we can get

$$\begin{aligned}&\left| \int_{\mathbb{R}^3} \big(K(x)|u_n|^{p-2}u_n - K(x)|u|^{p-2}u\big)(u_n - u)\, dx \right| \\ &\leq \int_{\mathbb{R}^3} \left(K(x)^{\frac{p-1}{p}}|u_n|^{p-1} + K(x)^{\frac{p-1}{p}}|u|^{p-1}\right) K(x)^{\frac{1}{p}}|u_n - u| dx \\ &\leq \left[\|u_n\|_{p,K}^{p-1} + \|u\|_{p,K}^{p-1}\right] \|u_n - u\|_{p,K}^{p} \\ &\leq C\left[\|u_n\|^{p-1} + \|u\|^{p-1}\right] \|u_n - u\|_{p,K}^{p} \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

We observe that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \to 0,$$

and we have

$$\begin{aligned}&\int_{\mathbb{R}^3} \left[(2\omega + \phi_{u_n})\phi_{u_n} u_n - (2\omega + \phi_u)\phi_u u\right](u_n - u) dx \\ &= 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u) dx + \int_{\mathbb{R}^3} (\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u) dx \to 0,\end{aligned}$$

as $n \to \infty$. Actually, by the Hölder inequality, the Sobolev inequality, and

Proposition 2.4(ii), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)dx\right| \\
&\le \left|\int_{\mathbb{R}^3}(\phi_{u_n}-\phi_u)u_n(u_n-u)dx\right| + \left|\int_{\mathbb{R}^3}\phi_u(u_n-u)^2dx\right| \\
&\le \|\phi_{u_n}-\phi_u\|_6\|u_n(u_n-u)\|_{\frac{6}{5}} + \|\phi_{u_n}\|_6\|(u_n-u)^2\|_{\frac{6}{5}} \\
&\le \|\phi_{u_n}-\phi_u\|_6\|u_n\|_{\frac{12}{5}}\|u_n-u\|_{\frac{12}{5}} + \|\phi_{u_n}\|_6\|u_n-u\|^2_{\frac{12}{5}} \\
&\le C\|\phi_{u_n}-\phi_u\|_D\|u_n\|\|u_n-u\|_{\frac{12}{5}} + C\|\phi_{u_n}\|_D\|u_n-u\|^2_{\frac{12}{5}} \\
&\le C(\|u_n\|^2+\|u\|^2)\|u_n\|\|u_n-u\|_{\frac{12}{5}} + C\|u_n\|^2\|u_n-u\|^2_{\frac{12}{5}}.
\end{aligned}$$

From the boundedness of $\{u_n\}$ in $H^1_r(\mathbb{R}^3;V)$ and Proposition 2.1, we know

$$\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_u u)(u_n-u)dx \to 0, \quad \text{as } n\to\infty.$$

Observe that the sequence $\{\phi^2_{u_n}u_n\}$ is bounded in $L^{\frac{3}{2}}(\mathbb{R}^3)$, since

$$\|\phi^2_{u_n}u_n\|_{\frac{3}{2}} \le \|\phi_{u_n}\|^2_6\|u_n\|_3 \le C\|\phi_{u_n}\|^2_D\|u_n\| \le C\|u_n\|^3,$$

so

$$\begin{aligned}
\left|\int(\phi^2_{u_n}u_n-\phi^2_u u)(u_n-u)\right| &\le \|\phi^2_{u_n}u_n-\phi^2_u u\|_{\frac{3}{2}}\|u_n-u\|_3 \\
&\le (\|\phi^2_{u_n}u_n\|_{\frac{3}{2}}+\|\phi^2_u u\|_{\frac{3}{2}})\|u_n-u\|_3 \to 0
\end{aligned}$$

Now, we can get

$$\begin{aligned}
\|u_n-u\|^2_E &= \langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle \\
&\quad + \int_{\mathbb{R}^3}[(2\omega+\phi_{u_n})\phi_{u_n}u_n-(2\omega+\phi_u)\phi_u u](u_n-u)dx \\
&\quad + \int_{\mathbb{R}^3}\left(K(x)|u_n|^{p-2}u_n-K(x)|u|^{p-2}u\right)(u_n-u)dx \\
&\to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

That is $u_n\to u$ in $H^1_r(\mathbb{R}^3;V)$ and the proof is complete. □

Proof of Theorem 1.1. By Lemma 2.8, the functional Φ satisfies the geometric conditions of Theorem 2.6. Lemma 2.9 implies that Φ satisfies the Palais–Smale condition. Hence, problem $(\mathcal{KGM})$ has infinitely many nontrivial solutions $(u_n,\phi_n)\in H^1_r(\mathbb{R}^3;V)\times D^{1,2}_r(\mathbb{R}^3)$. This completes the proof. □

Acknowledgement. Shang-Jie Chen is supported by Chongqing Basis and Frontier Research Project (No. cstc2016jcyjA0262). Lin Li is supported by Research Fund of National Natural Science Foundation of China (No. 11601046), Chongqing Science and Technology Commission (No. cstc2016jcyjA0310).

References

[1] Ambrosetti, A., Felli, V. and Malchiodi, A., Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity. *J. Eur. Math. Soc.* 7 (2005)(1), 117 – 144.

[2] Azzollini, A. and Pomponio, A., Ground state solutions for the nonlinear Klein–Gordon–Maxwell equations. *Topol. Methods Nonlinear Anal.* 35 (2010)(1), 33 – 42.

[3] Azzollini, A., Pisani, L. and Pomponio, A., Improved estimates and a limit case for the electrostatic Klein–Gordon–Maxwell system. *Proc. Roy. Soc. Edinburgh Sect. A* 141 (2011) (3), 449 – 463.

[4] Benci, V. and Fortunato, D., The nonlinear Klein–Gordon equation coupled with the Maxwell equations. Proceedings of the Third World Congress of Nonlinear Analysts, Part 9 (Catania, 2000). *Nonlinear Anal.* 47 (2001)(9), 6065 – 6072.

[5] Benci, V. and Fortunato, D., Solitary waves of the nonlinear Klein–Gordon equation coupled with the Maxwell equations. *Rev. Math. Phys.* 14 (2002)(4), 409 – 420.

[6] Benci, V. and Fortunato, D., Solitary waves in Abelian gauge theories. *Adv. Nonlinear Stud.* 8 (2008)(2), 327 – 352.

[7] Cunha, P. L., Subcritical and supercritical Klein–Gordon–Maxwell equations without Ambrosetti-Rabinowitz condition. *Diff. Integral Equ.* 27 (2014)(3–4), 387 – 399.

[8] D'Aprile, T. and Mugnai, D., Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations. *Proc. Roy. Soc. Edinburgh Sect. A* 134 (2004)(5), 893 – 906.

[9] D'Aprile, T. and Mugnai,D., Non-existence results for the coupled Klein–Gordon–Maxwell equations. *Adv. Nonlinear Stud.* 4 (2004)(3), 307 – 322.

[10] Ding, L. and Li, L., Infinitely many standing wave solutions for the nonlinear Klein–Gordon–Maxwell system with sign-changing potential. *Comput. Math. Appl.* 68 (2014)(5), 589 – 595.

[11] Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order.* Reprint of the 1998 edition. Classics Math. Berlin: Springer 2001.

[12] He, X., Multiplicity of solutions for a nonlinear Klein–Gordon–Maxwell system. *Acta Appl. Math.* 130 (2014), 237 – 250.

[13] Jeong, W. and Seok, J., On perturbation of a functional with the mountain pass geometry: applications to the nonlinear Schrödinger–Poisson equations and the nonlinear Klein–Gordon–Maxwell equations. *Calc. Var. Partial Diff. Equ.* 49 (2014)(1–2), 649 – 668.

[14] Li, L. and Tang, C.-L., Infinitely many solutions for a nonlinear Klein–Gordon–Maxwell system. *Nonlinear Anal.* 110 (2014), 157 – 169

[15] Mercuri, C., Positive solutions of nonlinear Schrödinger–Poisson systems with radial potentials vanishing at infinity. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 19 (2008)(3), 211 – 227.

[16] Mugnai, D., Solitary waves in Abelian gauge theories with strongly nonlinear potentials. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 27 (2010)(4), 1055 – 1071.

[17] Mugnai, D., Coupled Klein-Gordon and Born-Infeld-type equations: looking for solitary waves. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 460 (2004), 1519 – 1527.

[18] Opic, B. and Kufner, A., *Hardy-Type Inequalities*. Pitman Res. Notes Math. Ser. 219. Harlow: Longman 1990.

[19] Wang, F., Ground-state solutions for the electrostatic nonlinear Klein–Gordon–Maxwell system. *Nonlinear Anal.* 74 (2011)(14), 4796 – 4803.

[20] Willem, M., *Minimax Theorems*. Progr. Nonlinear Diff. Equ. Appl. 24. Boston (MA): Birkhäuser 1996.

Received July 6, 2016; revised March 9, 2017