

# Planar Traveling Waves of Mono-Stable Reaction-Diffusion Equations

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**Abstract.** This paper is concerned with planar traveling wavefronts of mono-stable reaction-diffusion equations in  $\mathbb{R}^n$  ( $n \geq 2$ ). We show that the large time behavior of the disturbed fronts can be controlled by two functions, which are the solutions of the specified nonlinear parabolic equations in  $\mathbb{R}^{n-1}$ , and the planar traveling fronts are asymptotically stable in  $L^\infty(\mathbb{R}^n)$  under ergodic perturbations, which include quasi-periodic and almost periodic ones as special cases.

**Keywords.** Planar traveling wavefronts, stability, super-solution and sub-solutions, mono-stable reaction-diffusion equations

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## 1. Introduction

In this paper, we consider the Cauchy problem of the following mono-stable reaction-diffusion equations

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, t > 0, \\ u(x, y, 0) = u_0(x, y), & x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, \end{cases} \quad (1)$$

where  $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial y^2}$  and  $n \geq 2$ . We assume that the initial data  $u_0$  is bounded and continuous on  $\mathbb{R}^n$  and that the function  $f$  is of mono-stable type.

In the above equation the term  $f(u)$  represents a source term with  $f(0) = f(1) = 0$ . Such equations have been derived to model problems arising from applied science, such as population dynamics, genetics, combustion and flame propagation. In these situations, one of most interesting and natural questions is the asymptotic behavior of solutions  $u(x, y, t)$  as  $t \rightarrow +\infty$ , in particular, the stability of traveling wavefronts. Here, the planar traveling wavefronts of (1)

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are the special solutions in the form of  $u(x, y, t) = \phi((x, y) \cdot \mathbf{e} + ct)$ , where  $c$  is the wave speed, and  $\mathbf{e}$  is the unit basis of  $\mathbb{R}^n$ . Without loss of generality, let us take  $\mathbf{e} = (0, \dots, 0, 1)$ . So, the planar wavefronts are the solutions to the 1-D equation

$$u_t = u_{yy} + f(u), \quad (2)$$

which, simply denoted by  $\phi(y + ct) = \phi(\xi)$  with  $\xi = y + ct$ , satisfy that

$$\begin{cases} \phi''(\xi) - c\phi'(\xi) + f(\phi(\xi)) = 0, & \xi \in \mathbb{R}, \\ \lim_{\xi \rightarrow -\infty} \phi(\xi) = 0, & \lim_{\xi \rightarrow \infty} \phi(\xi) = 1. \end{cases}$$

When  $f$  is of bistable type, namely,  $f'(0) < 0$  and  $f'(1) < 0$ , the traveling wave  $\phi(y + ct)$  with a specific wave speed  $c$  has been proved to uniquely exist up to shift, and further proved to be stable in different cases, cf. [1, 3–5, 11, 14, 17, 18, 25] and references therein. Notice that, in [18], Matano and Nara considered the Cauchy problem (1) under the condition that  $f$  is of bistable type. They proved that the planar front is asymptotically stable in  $L^\infty(\mathbb{R}^n)$  under spatially ergodic perturbation, and that the large time behavior of the disturbed planar front can be approximated by that of the mean curvature flow with a drift term for all large time up to  $+\infty$ .

When  $f(u)$  is mono-stable, i.e.,  $f(0) = f(1)$  and  $f'(1) < 0$  but  $f'(0) > 0$ , for example,  $f(u) = u(1 - u)$ , the equation (1) is just the so-called Fisher-KPP equation, which was first introduced and studied by Fisher [6] and Kolmogoroff, Petrovsky and Piscounoff [13] in 1937. The wavefronts  $\phi(y + ct)$  of (1) in this mono-stable case are obtained for  $c \geq c_* = 2\sqrt{f'(0)}$  (for example, see [9, 15, 16, 19, 20, 23]), and satisfy

$$\lim_{z \rightarrow -\infty} \phi(z)e^{-\lambda_1(c)z} = 1, \quad \lim_{z \rightarrow -\infty} \phi'(z)e^{-\lambda_1(c)z} = \lambda_1(c), \quad (3)$$

$$\lim_{z \rightarrow -\infty} \phi(z)|z|^{-1}e^{-\lambda^*(c_*)z} = 1, \quad \lim_{z \rightarrow -\infty} \phi'(z)|z|^{-1}e^{-\lambda^*(c_*)z} = \lambda^*(c_*), \quad (4)$$

where  $\lambda_1(c)$  is the smaller positive root of

$$\Delta_c(\lambda) := c\lambda - \lambda^2 - f'(0) = 0,$$

with  $c \geq c_* = 2\sqrt{f'(0)}$ , and  $\lambda^* = \lambda(c_*)$ . These wavefronts are further proved to be asymptotically stable in different designed solution spaces, for example, see [7, 10, 12, 15, 16, 19–23] and the references therein. However, when the initial perturbation  $u_0(x, y) - \phi(y)$  is ergodic in  $x$ , including quasi-periodic and almost periodic cases, the stability of such a wavefront has remained open for many years. Obviously, to attach this problem is interesting and significant. This will be the main target of the present paper.

Throughout this paper, we always assume that  $u_0(x, y) \geq 0$  and there exists  $M_0 > 0$  and  $\alpha \in (0, 1]$  such that

(F)  $f \in C^1(\mathbb{R})$ ,  $f(0) = f(1) = 0$ ,  $f'(0) > 0$ ,  $f'(1) < 0$ ,  $f(u) > 0$  for  $u \in (0, 1)$ ,  $f(u) < 0$  for  $u \in (1, \infty)$ ,  $0 < f'(0)u - f(u) < M_0 u^{1+\alpha}$ ,  $\forall u \in (0, 1)$ .

Inspired by [18], in this paper we will consider the stability of planar fronts under spatially ergodic perturbation. Due to the function  $f$  is of mono-stable type, there are something essentially different from [18] in the bistable case, particularly, here 0 is an unstable state, which essentially causes some difficulty to get the stability. In fact, we first need to construct different sub-supersolution from that of [18], then, in order to derive the  $\omega$ -limit of solution to (1), we then need technically to establish a crucial estimate.

Now let us now state our main results.

**Theorem 1.1** (Large time behavior). *Let  $n \geq 2$  and let (F) hold. Let  $u(x, y, t)$  be a solution problem (1) whose initial value  $u_0(x, y)$  is nonnegative, bounded and uniformly continuous on  $\mathbb{R}^n$  and satisfies*

$$\liminf_{y \rightarrow +\infty} \inf_{x \in \mathbb{R}^{n-1}} u_0(x, y) > 1 - \delta,$$

where  $0 < \delta < \varepsilon_0$  for some constant  $\varepsilon_0 \in (0, 1)$ .

1) When  $c > c^*$ , and  $u_0(x, y)$  satisfies

$$\lim_{y \rightarrow -\infty} u_0(x, y) e^{-\lambda_1(c)y} = B\varphi(x)$$

for some constant  $B > 0$ , where  $\lambda_1(c)$  is defined in (3) and  $0 < \kappa_0 \leq \varphi(x) \leq \kappa_1$ ,  $x \in \mathbb{R}$  ( $\kappa_0$  and  $\kappa_1$  are two constants), then there exist a constant  $T > 0$  and a smooth function  $\gamma(x, t)$  such that

- (i) for each  $t \in [T, +\infty)$  and  $x \in \mathbb{R}^{n-1}$ , one has  $u(x, y, t) = \phi(0)$  if and only if  $y = \gamma(x, t)$ ;
- (ii) it holds that

$$\lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathbb{R}^n} |u(x, y, t) - \phi(y - \gamma(x, t))| = 0;$$

(iii) for any  $\varepsilon > 0$ , there exists  $T_\varepsilon > 0$  such that

$$v_1(x, t) - \varepsilon \leq \gamma(x, t) \leq v_2(x, t) + \varepsilon, \quad t \geq T_\varepsilon,$$

where  $v_1(x, t)$  and  $v_2(x, t)$  satisfy

$$\begin{cases} v_{1t} = \Delta_x v_1 - k|\nabla_x v_1|^2 - c, & x \in \mathbb{R}^{n-1}, t > 0, \\ v_1(x, 0) = \gamma(x, T_\varepsilon), & x \in \mathbb{R}^{n-1}; \end{cases}$$

$$\begin{cases} v_{2t} = \Delta_x v_2 + k|\nabla_x v_2|^2 - c, & x \in \mathbb{R}^{n-1}, t > 0, \\ v_2(x, 0) = \gamma(x, T_\varepsilon), & x \in \mathbb{R}^{n-1}. \end{cases}$$

Here  $\Delta_x$  and  $\nabla_x$  denote the  $(n - 1)$ -dimensional Laplacian and the  $(n - 1)$ -dimensional gradient, respectively.  $k$  is defined as in Lemma 2.3.

2) When  $c = c^*$ , and  $u_0 > 0$  satisfies

$$\lim_{y \rightarrow -\infty} u_0(x, y) |y|^{-1} e^{-\lambda^*(c)y} = B\psi(x)$$

for some constant  $B > 0$ , where  $\lambda^*(c)$  is defined in (4), and  $0 < \iota_0 \leq \psi(x) \leq \iota_1$ ,  $x \in \mathbb{R}$  ( $\iota_0$  and  $\iota_1$  are two constants), then the above properties (i)–(iii) hold, where  $c$  is replaced by  $c^*$ .

The statement (i) of Theorem 1.1 shows that the  $\phi(0)$ -level surface of  $u(x, y, t)$  has a graphical representation  $y = \gamma(x, t)$ . The statement (ii) implies that the solution  $u(x, y, t)$  behaves like the function  $\phi(y - \gamma(x, t))$  for large  $t$ , thus the large time behavior of the solution  $u(x, y, t)$  is basically determined by the position of the  $\phi(0)$ -level surface  $\gamma(x, t)$ . The last statement shows that the behavior of  $\gamma(x, t)$  can be controlled by the two functions  $v_1(x, t)$  and  $v_2(x, t)$ .

In order to obtain the stability of planar wave, we need the following definition.

**Definition 1.2** (Unique ergodicity in the  $x$ -direction). A bounded uniformly continuous function  $p(x, y): \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}$  is called uniquely ergodic in the  $x$ -direction if there exists a unique probability measure on  $\mathbb{H}_p$ , namely,  $p(x, y)$  is  $\sigma_a$ -invariant for any  $a \in \mathbb{R}^{n-1}$ , where

$$\mathbb{H}_p := \overline{\{\sigma_a g | a \in \mathbb{R}^{n-1}\}}^{L^\infty(\mathbb{R}^{n-1})}, \quad (\sigma_a p)(x) = p(x + a),$$

and  $\overline{A}^X$  stands for the closure of a set  $A$  in the  $X$ -topology.

**Proposition 1.3.** [18, Corollary 2.12] *Let  $u_0(x, y)$  be a uniformly continuous bounded function on  $\mathbb{R}^{n-1} \times \mathbb{R}$  and uniquely ergodic in the  $x$ -direction. Then for each fixed  $t \geq 0$ , the solution  $u(x, y, t)$  of (1) is uniquely ergodic in the  $x$ -direction.*

**Theorem 1.4** (Stability of planar traveling waves). *In addition to the assumptions of Theorem 1.1, assume further that  $u_0(x, y)$  is uniquely ergodic in the  $x$ -direction. Then there exists a constant  $\mu \in \mathbb{R}$  such that*

$$\lim_{t \rightarrow \infty} \sup_{(x, y) \in \mathbb{R}^n} |u(x, y, t) - \phi(y + ct + \mu)| = 0.$$

From [18], we see that

$$\mathbb{P} \subset \mathbb{QP} \subset \mathbb{AP} \subset \mathbb{SE} \subset \mathbb{UE},$$

where  $\mathbb{P}, \mathbb{QP}, \mathbb{AP}, \mathbb{SE}, \mathbb{UE}$  denote, respectively, the sets of periodic functions, quasi-periodic functions, almost periodic functions, strictly ergodic functions and uniquely ergodic functions. Hence the above Theorem 1.4 is a general result.

It is also interesting to compare our result with those existing stability results for the mono-stable traveling waves studied in [10, 15, 16, 19, 20]. In [10], the problem under consideration is periodic, but the initial perturbation considered in our paper can be more general like ergodic perturbation. In [15, 19, 20], the periodic initial perturbations are not treated, and in order to apply the maximum principle, they need to restrict the initial data to be  $0 \leq u_0(x, y) \leq 1$ , but such a condition can be removed in our stability. However, our assumption on  $u_0(x, y)$  decay to 0 as  $y \rightarrow -\infty$  is stronger than those in [10, 15, 19, 20]. On the other hand, the proof approaches adopted in these papers are totally different. The methods used in [10] and [15, 19, 20] are super-solution and Green functions method respectively, but we use the  $\omega$ -limit and level set to prove the stability of planar fronts. Notice also that, here we give a new description about the large time behavior of the solution to (1). Unfortunately, we can not obtain the convergence rate by the  $\omega$ -limit method. In [16], for the special case with  $\phi(0) = \frac{1}{2}$ , the planar traveling waves for the mono-stable type equation was proved to be stable. Here we further extend and develop all results in [16], and our results on the asymptotic behave of perturbed planar waves as well as their stability are more specific than in [16]. We remark that, in papers [8, 10], they all considered the monostable case, but in this paper, we use a different method from [8, 10].

The rest of this paper is organized as follows. In Section 2, we consider problem (1) in one-dimension. Section 3 is concerned with the proofs of Theorems 1.1 and 1.4.

## 2. One dimensional problem and preliminary

In this section we first study the one dimensional problem (1) in a moving frame, i.e.,

$$\begin{cases} u_t - u_{zz} + cu_z = f(u), & z \in \mathbb{R}, t > 0, \\ u(z, 0) = u_0(z), & z \in \mathbb{R}. \end{cases} \quad (5)$$

Obviously, traveling wave solution  $\phi(z)$ , i.e., the solution of (2), is a stationary solution of (5).

Now, we construct a family of subsolutions. The proof of the following lemma is standard and thus we omit it.

**Lemma 2.1.** *Assume that (F) holds. Let  $\lambda_1(c)$  and  $\lambda_2(c)$  be the two positive roots of  $\Delta_c(\lambda) = 0$  with  $c > 2\sqrt{f'(0)}$ . For every  $\eta \in (1, \min\{1 + \alpha, \frac{\lambda_2(c)}{\lambda_1(c)}\})$ , there exists a positive number  $Q := Q(c, \eta) \geq 1$  such that for each  $q \geq Q$ , then*

$$\psi(z) := \max \left\{ 0, e^{\lambda_1(c)(z+z^-)} - qe^{\eta\lambda_1(c)(z+z^-)} \right\}, \quad \text{for } z^- \in \mathbb{R},$$

*is a subsolution to the traveling wave problem with speed  $c$ , i.e.,*

$$N[\psi](z) := c\psi'(z) - \psi''(z) - f(\psi(z)) \leq 0 \quad \text{for all } z \in \mathbb{R}.$$

The following supersolution and subsolution we construct here are different from that in [4], where Chen–Guo [4] considered the discrete quasilinear monostable equations.

**Lemma 2.2.** *Assume that (F) holds and  $\phi(z)$  is a monotone increasing solution of (2) with  $c \geq 2\sqrt{f'(0)}$ . Then there exist  $\varepsilon_0 \in (0, 1)$ ,  $\beta > 0$ , and  $\sigma \geq 1$  such that, for any  $\varepsilon \in (0, \varepsilon_0]$ , the functions defined by*

$$\begin{aligned} u^+(z, t) &= (1 + \varepsilon e^{-\beta t})\phi(z + \sigma\varepsilon(1 - e^{-\beta t})) \\ u^-(z, t) &= (1 - \varepsilon e^{-\beta t})\phi(z - \sigma\varepsilon(1 - e^{-\beta t})) \end{aligned}$$

are supersolution and subsolution of (5), respectively, i.e.,

$$\begin{aligned} L[u^+](z, t) &:= u_t^+ - u_{zz}^+ + cu_z^+ - f(u^+) \geq 0, \\ L[u^-](z, t) &:= u_t^- - u_{zz}^- + cu_z^- - f(u^-) \leq 0. \end{aligned}$$

*Proof.* We only study the supersolution because the subsolution can be proved similarly. By using the relation  $\phi'' - c\phi' + f(\phi) = 0$ , we have

$$\begin{aligned} L[u^+](z, t) &= -\varepsilon\beta e^{-\beta t}\phi + \sigma\varepsilon\beta e^{-\beta t}(1 + \varepsilon e^{-\beta t})\phi' \\ &\quad - (1 + \varepsilon e^{-\beta t})\phi'' + c(1 + \varepsilon e^{-\beta t})\phi' - f((1 + \varepsilon e^{-\beta t})\phi) \\ &= -\varepsilon\beta e^{-\beta t}\phi + \sigma\varepsilon\beta e^{-\beta t}(1 + \varepsilon e^{-\beta t})\phi' \\ &\quad + (1 + \varepsilon e^{-\beta t})f(\phi) - f((1 + \varepsilon e^{-\beta t})\phi). \end{aligned}$$

By the assumption (F), we take constants  $\beta > 0$  and  $\varepsilon_0 > 0$  such that

$$-f'(s) \geq \beta > 0, \quad \text{for } s \in [1 - 2\varepsilon_0, 1 + 2\varepsilon_0]. \quad (6)$$

In addition, we take  $M_1 > 0$  sufficiently large so that

$$\phi(\xi) \geq 1 - \varepsilon_0, \quad \xi \geq M_1. \quad (7)$$

Here we only consider the case that  $c > c^*$  and the other case that  $c = c^*$  can be similarly proved. As  $\lim_{\xi \rightarrow -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = 1$  and  $\lim_{\xi \rightarrow -\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c)$ , we can take  $M_2 > 0$  sufficiently large such that

$$\frac{1}{2} < \phi(\xi)e^{-\lambda_1(c)\xi} < \frac{3}{2}, \quad \phi'(\xi)e^{-\lambda_1(c)\xi} > \frac{1}{2}\lambda_1(c) \quad \text{for } \xi \leq -M_2. \quad (8)$$

Denote

$$\varrho := \min\{\phi'(\xi), -M_2 \leq \xi \leq M_1\} > 0.$$

Finally, choose  $\sigma \geq 1$  sufficiently large so that

$$\sigma \geq \max \left\{ 1, \frac{3\beta + 3\|f'\|_{L^\infty[0,1]}}{\beta\lambda_1(c)}, \frac{\beta + \|f'\|_{L^\infty[0,1]}}{\beta\varrho} \right\}.$$

Let  $\xi = z + \sigma\varepsilon(1 - e^{-\beta t})$ , then for any  $\xi \geq M_1$ , combining (6) and (7) and using the fact that  $\phi' > 0$ , we have

$$L[u^+] \geq \varepsilon\phi e^{-\beta t} \left( -\beta - \int_0^1 f'((1 + \varepsilon e^{-\beta t}\tau)\phi) d\tau \right) > 0.$$

For any  $\xi \leq -M_2$ , by using (8), we obtain

$$\varepsilon^{-1} e^{\beta t} e^{-\lambda_1(c)\xi} L[u^+] \geq \frac{1}{2} \sigma\beta\lambda_1(c) - \frac{3}{2}\beta - \frac{3}{2} \|f'\|_{L^\infty[0,1]} \geq 0.$$

For  $-M_2 \leq \xi \leq M_1$ , we have

$$\varepsilon^{-1} e^{\beta t} L[u^+] \geq \sigma\beta\varrho - \beta - \|f'\|_{L^\infty[0,1]} \geq 0.$$

In summary, we have  $L[u^+] \geq 0$ . This completes the proof.  $\square$

**Lemma 2.3** ([16, Lemma 2.2]). *Assume that (F) holds and  $\phi(z)$  is a monotone increasing solution of (2) with  $c \geq 2\sqrt{f'(0)}$ . Then there exists a constant  $k > 0$  depending only on  $f$  such that*

$$-k\phi'(\xi) \leq \phi''(\xi) \leq k\phi'(\xi), \quad \xi \in \mathbb{R}.$$

At the end of this section, we consider the following nonlinear parabolic equation

$$\begin{cases} v_t = \Delta_x v - k|\nabla_x v|^2 - c, & x \in \mathbb{R}^{n-1}, t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^{n-1}, \end{cases} \quad (9)$$

where constant  $k$  is defined as in Lemma 2.3,  $c$  is the wave speed, and  $\Delta_x$  and  $\nabla_x$  denote the  $(n-1)$ -dimensional Laplacian and the  $(n-1)$ -dimensional gradient, respectively. By using Definition 1.2, we have the following result.

**Lemma 2.4.** *Let  $v(x, t)$  be a solution to the problem (9) whose initial value  $v_0(x)$  is uniquely ergodic. Then*

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} |v(x, t) + (\mu + ct)| = 0$$

where  $\mu = \frac{1}{k} \ln \mu_*$ , and

$$\mu_* = \lim_{R \rightarrow \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \exp(-kv_0(x)) dx > 0$$

is the uniform mean of  $e^{-kv_0(x)}$  with a certain point  $a \in \mathbb{R}^{n-1}$ .

*Proof.* Let  $w(x, t) := \exp(-k(v(x, t) + ct))$ , then  $w$  satisfies the following Cauchy problem

$$\begin{cases} w_t = \Delta_x w, & x \in \mathbb{R}^{n-1}, t > 0, \\ w(x, 0) = \exp(-kv_0(x)), & x \in \mathbb{R}^{n-1}, \end{cases}$$

which implies

$$w(x, t) = \int_{\mathbb{R}^{n-1}} G(x - y, t) \exp(-kv_0(y)) dy,$$

where  $G(x, t)$  is the heat kernel on  $\mathbb{R}^{n-1}$  given by

$$G(x, t) = (4\pi t)^{-\frac{n-1}{2}} \exp\left(-|x|^2 \frac{1}{4t}\right).$$

Consequently, we have

$$v(x, t) = -\frac{1}{k} \ln \left( \int_{\mathbb{R}^{n-1}} G(x - y, t) \exp(-kv_0(y)) dy \right) - ct.$$

Since  $v_0(x)$  is uniquely ergodic, by [18, Remark 2.1], the function  $\exp(kv_0(x))$  has uniform mean in the sense that the following limit exists uniformly in  $a \in \mathbb{R}^{n-1}$  and is independent of  $a$ :

$$\mu^* = \lim_{R \rightarrow \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \exp(-kv_0(x)) dx > 0.$$

This implies

$$\int_{\mathbb{R}^{n-1}} G(x - y, t) \exp(-kv_0(y)) dy \rightarrow \mu^* \quad \text{as } t \rightarrow \infty,$$

uniformly in  $x \in \mathbb{R}^{n-1}$ . Thus, we obtain

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} \left| v(x, t) + \left( \frac{1}{k} \ln \mu^* + ct \right) \right| = 0$$

This completes the proof of the lemma.  $\square$

Similarly, we consider the following problem

$$\begin{cases} v_t = \Delta_x v + k|\nabla_x v|^2 - c, & x \in \mathbb{R}^{n-1}, t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^{n-1}, \end{cases} \quad (10)$$

where  $k > 0$  is defined as in Lemma 2.3.

**Lemma 2.5.** *Let  $v(x, t)$  be a solution to the problem (10) whose initial value  $v_0(x)$  is uniquely ergodic. Then there exists a constant  $\nu \in \mathbb{R}$  such that*

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} |v(x, t) + (\nu + ct)| = 0.$$



### 3. Proofs of Theorems 1.1 and 1.4

In this section, we consider problem (1) in  $\mathbb{R}^n$  and prove our main results. Firstly, we give some rough upper and lower bounds for the solution at large time, then introduce the notion of  $\omega$ -limit points of the solution and study the basic properties of  $\phi(0)$ -level surface of the solution. Finally, we construct a fine set of supersolutions and subsolutions and give the proofs of the main theorems.

Let us express the solutions  $u(x, y, t)$  of (1) in a moving frame, thus the planar waves can be viewed as stationary states. Let

$$u = u(x, z, t), \quad z = y + ct,$$

be the solution of the problem (1), namely,

$$\begin{cases} u_t = \Delta u - cu_z + f(u), & x \in \mathbb{R}^{n-1}, z \in \mathbb{R}, t > 0, \\ u(x, z, 0) = u_0(x, z), & x \in \mathbb{R}^{n-1}, \end{cases} \quad (11)$$

where  $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial z^2}$ . Throughout this section, we always assume that the initial value  $u_0$  is nonnegative, bounded and uniformly continuous on  $\mathbb{R}^n$ , and satisfies

$$\lim_{z \rightarrow -\infty} u_0(x, z)e^{-\lambda_1(c)z} = B\varphi(x), \quad \liminf_{z \rightarrow +\infty} \inf_{x \in \mathbb{R}^{n-1}} u_0(x, z) > 1 - \delta, \quad (12)$$

where  $\delta$  is defined as in Theorem 1.1. We first consider upper and lower bound of the solution of (11) at large time.

**Lemma 3.1.** *Let  $u(x, z, t)$  be the solution of (11). Then there exist constant  $z_*$ ,  $z^* \in \mathbb{R}$  such that*

$$\begin{aligned} \liminf_{t \rightarrow \infty} \inf_{x \in \mathbb{R}^{n-1}} u(x, z, t) &\geq \phi(z - z^*), \quad \text{uniformly in } z \in \mathbb{R}, \\ \limsup_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} u(x, z, t) &\leq \phi(z - z_*), \quad \text{uniformly in } z \in \mathbb{R}. \end{aligned}$$

*Proof.* We first show the upper bound. Let  $u^+(z, t)$  be defined as in Lemma 2.2. Then it suffices to show that there exist constants  $T > 0$  and  $z_0 \in \mathbb{R}$  such that

$$u(x, z, T) \leq (1 + \varepsilon_0)\phi(z - z_0) = u^+(z - z_0, 0). \quad (13)$$

Indeed, the comparison principle and (13) give  $u(x, z, t) \leq u^+(z - z_0, t - T)$  for  $t \geq T$ , which yield upper bound by letting  $t \rightarrow \infty$ .

We first consider the case  $c > c^*$ . Since  $f(u) < 0$  for  $u > 1$  by the assumption (F), we see from the comparison that

$$u(x, z, T) < 1 + \frac{\varepsilon_0}{2}, \quad (x, z) \in \mathbb{R}^n \quad (14)$$

for all sufficiently large  $T > 0$ . Note that  $\lambda_1(c) > 0$  and let  $\hat{z} = -\frac{1}{\lambda_1(c)} \ln(B\kappa_1)$ , then we have  $B\kappa_1 e^{\lambda_1(c)\hat{z}} = 1$ . From (12), we know that

$$\lim_{z \rightarrow -\infty} u_0(x, z + \hat{z}) e^{-\lambda_1(c)z} = B\varphi(x) e^{\lambda_1(c)\hat{z}} \leq 1,$$

thus there exists  $z_1 > 0$  such that  $u_0(x, z + \hat{z}) < (1 + \frac{\varepsilon_0}{2}) e^{\lambda_1(c)z}$  for  $z \leq -z_1$ . Then there exists  $z_2 \in \mathbb{R}$  such that

$$u_0(x, z) < \left(1 + \frac{\varepsilon_0}{2}\right) e^{\lambda_1(c)(z+z_2)} \quad \text{for } (x, z) \in \mathbb{R}^n.$$

On the other hand, let  $\bar{u}(x, z, t) = (1 + \frac{\varepsilon_0}{2}) e^{\lambda_1(c)(z+z_2)}$  and we have

$$\begin{aligned} \bar{u}_t - \Delta \bar{u} + c\bar{u}_z - f(\bar{u}) &= \left(1 + \frac{\varepsilon_0}{2}\right) (-\lambda_1^2(c) + c\lambda_1(c)) e^{\lambda_1(c)(z+z_2)} - f(\bar{u}) \\ &= f'(0)\bar{u} - f(\bar{u}) \\ &\geq 0, \end{aligned}$$

where we have used the facts that  $f'(0)\bar{u} - f(\bar{u}) \geq 0$  for  $\bar{u} \in (0, 1)$  and  $f(\bar{u}) \leq 0$  for  $\bar{u} \geq 1$ . So we obtain  $u(x, z, t) \leq (1 + \frac{\varepsilon_0}{2}) e^{\lambda_1(c)(z+z_2)}$  by the comparison principle. It then follows that there exists  $z_3 \in \mathbb{R}$  such that

$$u(x, z, t) \leq (1 + \varepsilon_0)\phi(z + z_2) \quad \text{for } z \leq z_3. \quad (15)$$

The assertion (13) then follows immediately by combining (14) and (15).

Next, we show the lower bound. Similar to the case for the upper bound, we only show that there exists constants  $T > 0$  and  $z^0 \in \mathbb{R}$  such that

$$u(x, z, T) \geq (1 - \varepsilon_0)\phi(z - z^0) = u^-(z - z^0, 0). \quad (16)$$

We first show that

$$\liminf_{z \rightarrow +\infty} \inf_{x \in \mathbb{R}^{n-1}} u(x, z, T) > 1 - \varepsilon_0 \quad (17)$$

for each  $T > 0$ . For this purpose, let us choose constants  $\beta, M$  such that

$$\liminf_{z \rightarrow +\infty} \inf_{x \in \mathbb{R}^{n-1}} u_0(x, z) > 1 - \delta > \beta > 1 - \varepsilon_0$$

and that

$$u_0(x, z) \geq \max\{0, \beta - Me^{-z}\}, \quad (x, z) \in \mathbb{R}^n.$$

Then the function  $w(x, z, t) = \max\{0, \beta - Me^{-(z-at)}\}$  is a subsolution of (11) if  $a > 0$  is chosen sufficiently large. Hence  $u(x, z, T) \geq \beta - Me^{-(z-aT)}$ . This proves (17). Next, it follows from (12) that there exists a constant  $z_4 \in \mathbb{R}$  such that

$$u_0(x, z + \hat{z} + 1) \geq e^{\lambda_1(c)z} \quad \text{for } z \leq z_4,$$

where  $\hat{z} = -\frac{1}{\lambda_1(c)} \ln B$ . Let

$$q = \max\{Q(c, \eta), e^{-(\eta-1)\lambda_1(c)z_4}\}, \quad \text{where } \eta = \frac{1 + \min\{1 + \alpha, \frac{\lambda_2(c)}{\lambda_1(c)}\}}{2}.$$

Then, when  $z > z_4$ , we have  $e^{\lambda_1(c)z} - qe^{\eta\lambda_1(c)z} < 0$ . Hence

$$u_0(x, z+1) \geq \max\{0, e^{\lambda_1(c)z} - qe^{\eta\lambda_1(c)z}\}.$$

The comparison principle then gives

$$u(x, z+1, t) \geq e^{\lambda_1(c)z} - qe^{\eta\lambda_1(c)z}.$$

As  $\lim_{z \rightarrow -\infty} \phi(z)e^{-\lambda_1(c)z} = 1$ , there exists  $z_5 \in \mathbb{R}$  such that  $e^{\lambda_1(c)(z+1)} - qe^{\eta\lambda_1(c)(z+1)} > \phi(z)$  for  $z \leq z_5$ . Consequently,

$$u(x, z, t) \geq \phi(z - \hat{z} - 2) \quad \text{for } z \leq z_5. \quad (18)$$

It is well-known that the solution  $u(x, z, t)$  of (11) has the following property:  $u(x, z, t) > 0$ , for all  $t > 0$  if the initial data  $u_0(x, z) \geq 0$  and  $u_0(x, z) \not\equiv 0$ . By using the above property, the assertion (16) then follows immediately by combining (17) and (18).

Next, we consider the case  $c = c^*$ . The upper bound is similar to the case that  $c > c^*$ . Next, we consider the lower bound. Similar to the case that  $c > c^*$ , it suffices to show that  $u_0(x, z) \geq (1 - \varepsilon_0)\phi(z - z^0)$  for all  $(x, z) \in \mathbb{R}^n$ . By the assumption of Theorem 1.1,  $\lim_{z \rightarrow -\infty} u_0(x, z)|z|^{-1}e^{-\lambda^*(c)z} = B\psi(x)$ , we have there exist a constant  $\tilde{z} \in \mathbb{R}$  such that

$$\lim_{z \rightarrow -\infty} u_0(x, z + \tilde{z})|z + \tilde{z}|^{-1}e^{-\lambda^*(c)z} = Be^{\lambda^*\tilde{z}}\psi > 2.$$

It follows from (4) that there exist a constant  $z_6 \in \mathbb{R}$  such that

$$u_0(x, z) \geq \phi(z - 2|\tilde{z}|) \quad \text{for } z \leq z_6.$$

Combining the above inequality and using (12) and  $u_0(x, z) > 0$ , it is easy to prove that  $u_0(x, z) \geq (1 - \varepsilon_0)\phi(z - z^0)$  for all  $(x, z) \in \mathbb{R}^n$ . This completes the proof.  $\square$

Now, we introduce the notion of  $\omega$ -limit points of the solution  $u(x, z, t)$  of (11), where we consider a sequence both in  $x$  and  $t$ . Then we show that any  $\omega$ -limit point is a planar wave under the assumption (12).

**Definition 3.2.** A function  $w(x, z, t)$  defined on  $\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$  is called an  $\omega$ -limit point of the solution  $u(x, z, t)$  of (11) if there exists a sequence  $\{(x_i, t_i)\}$  such that  $0 < t_1 < t_2 < \dots \rightarrow \infty$  and that

$$u(x + x_i, z, t + t_i) \rightarrow w(x, z, t) \quad \text{as } i \rightarrow \infty \quad \text{in } C_{loc}^{2,1}(\mathbb{R}^n \times \mathbb{R}).$$

The following remark tells us how to construct  $\omega$ -limit point, which is similar to what stated in [18].

**Remark 3.3.** Let  $u(x, z, t)$  be a solution of (11). Then for every sequence  $\{(x_i, t_i)\}$  with  $0 < t_1 < t_2 < \dots \rightarrow \infty$ , there exist a subsequence  $\{(x'_i, t'_i)\}$  and an  $\omega$ -limit point  $w(x, z, t)$  of  $u(x, z, t)$  such that

$$u(x + x'_i, z, t + t'_i) \rightarrow w(x, z, t), \quad \text{as } i \rightarrow \infty \quad \text{in } C_{loc}^{2,1}(\mathbb{R}^n \times \mathbb{R}).$$

Indeed, since  $u_0$  is bounded on  $\mathbb{R}^n$ , the assumption (F) and comparison principle imply that  $u(x, z, t)$  is bounded on  $\mathbb{R}^n \times [0, \infty)$ . Therefore, by  $L^p$ -estimates and Schauder's estimate, the solution  $u(x, z, t)$  belongs to  $C^{2+\alpha, 1+\frac{\alpha}{2}}(\mathbb{R}^n \times [\delta, T])$  for any  $0 < \delta < T$ . Furthermore,

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\mathbb{R}^n \times [\delta, T])} \leq C, \quad (19)$$

where  $C > 0$  is a constant independent of  $T > 0$ . Let  $\{Q_k\}_{k=1,2,\dots}$  be a sequence of compact subsets of  $\mathbb{R}^n \times \mathbb{R}$  satisfying

$$Q_1 \subset Q_2 \subset \dots \quad \text{and} \quad \lim_{k \rightarrow \infty} Q_k = \mathbb{R}^n \times \mathbb{R}.$$

Then, for each  $k$ , the sequence of functions  $\{u(x + x_i, z, t + t_i)\}_{i=1,2,\dots}$  is defined on  $Q_k$  for all large  $i$  and the restrictions of these functions onto  $Q_k$  is relatively compact in  $C^{2,1}(Q_k)$  by virtue of the estimate (19). By using diagonal argument, we can choose a subsequence  $\{(x'_i, t'_i)\}$  and a function  $w(x, z, t)$  defined on  $\mathbb{R}^n \times \mathbb{R}$  such that, for any  $k \geq 1$ , it holds that

$$\lim_{i \rightarrow \infty} \|u(x + x'_i, z, t + t'_i) - w(x, z, t)\|_{C^{2,1}(Q_k)} = 0,$$

which means  $u(x + x'_i, z, t + t'_i) \rightarrow w(x, z, t)$  as  $i \rightarrow \infty$  in  $C_{loc}^{2,1}(Q_k)$ .

In order to prove that any  $\omega$ -limit point is a planar wave, the following lemma is needed.

**Lemma 3.4.** Let  $u(x, z, t)$  be a function defined on  $\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$  and satisfies

$$u_t = \Delta u - cu_z + f(u), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

Assume further that there exist two constants  $z_*, z^* \in \mathbb{R}$  with  $z_* < z^*$ , such that

$$\phi(z - z^*) \leq u(x, z, t) \leq \phi(z - z_*), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}. \quad (20)$$

Then there exists a constant  $z_0 \in [z_*, z^*]$  such that

$$u(x, z, t) = \phi(z - z_0), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

*Proof.* The proof of this lemma is similar to [2, Theorem 3.1] and we only give outline of the proof. Define

$$u^s(x, z, t) = u(x + y, z + s, t + T), \quad y \in \mathbb{R}^{n-1}, \quad s, T \in \mathbb{R}.$$

Then  $\phi(z + s - z^*) \leq u^s(x, z, t) \leq \phi(z + s - z_*)$ . It follows from the monotonicity of  $\phi$  that there exists  $s \in \mathbb{R}$  satisfying

$$u^s(x, z, t) \geq u(x, z, t).$$

Indeed, let  $s = z^* - z_*$ , then  $u(x, z, t) \leq \phi(z - z_*) < \phi(z + s - z_*) \leq u^s(x, z, t)$ .

Now, let

$$s_* = \inf\{s \in \mathbb{R}, u^{s'} \geq u, \text{ in } \mathbb{R}^n \times \mathbb{R} \text{ for all } s' \geq s\}.$$

Since  $\lim_{z \rightarrow -\infty} \phi(z) = 0$ ,  $\lim_{z \rightarrow \infty} \phi(z) = 1$  and by using (20), there exists constant  $A_1 > 0$  such that

$$\begin{cases} u(x, z, t) \geq 1 - \gamma, & \text{for all } z \geq A_1 \text{ and } (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \\ u(x, z, t) \leq \gamma, & \text{for all } z \leq -A_1 \text{ and } (x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \end{cases}$$

Note that, without loss of generality, one can assume  $\gamma \in (0, \frac{1}{2})$ . Let  $A = \max\{A_1, \frac{z^* - z_*}{2}\}$ , then  $u(x, z, t) \geq 1 - \gamma$  for  $z \geq A$  and  $u(x, z, t) \leq \gamma$  for  $z \leq -A$ . It is easy to see that  $s_* \leq 2A$  and  $s_* > -\infty$ . Assume that  $s_* > 0$  and call  $S = \{(x, z, t) \in \mathbb{R}^n \times \mathbb{R}, -A \leq z \leq A\}$ . If  $\inf_S(u^{s_*} - u) > 0$ , then there exists  $\eta_0 \in (0, s_*)$  such that  $u^{s_* - \eta} \geq u$  is  $S$  for all  $\eta \in [0, \eta_0]$ . Denote  $E = \{(x, z, t) \in \mathbb{R}^n \times \mathbb{R}, z > A\}$  and  $v = u^{s_* - \eta} - u$ , we see that  $v \geq 0$  on  $\partial E = \{z = A\}$  and satisfies

$$v_t - \Delta v + cv_z = f(u^{s_* - \eta}) - f(u) \geq -Bv$$

for some constant  $B$  (remember that  $f \in C^1(\mathbb{R})$  and  $0 \leq u^{s_* - \eta}, u \leq 1$ ). The parabolic maximum principle implies that  $v(x, z, t) \geq 0$  in  $E$ . Similarly, we can prove that  $v \geq 0$  for  $z \leq -A$ . Therefore,  $u^{s_* - \eta} \geq u$  in  $\mathbb{R}^n \times \mathbb{R}$  for all  $\eta \in [0, \eta_0]$ . This contradicts the minimality of  $s_*$ . It follows then that

$$\inf_S(u^{s_*} - u) = 0.$$

As a consequence, there exist  $z_\infty \in [-A, A]$  and a sequence  $(x_n, z_n, t_n)_{n \in \mathbb{N}}$  such that

$$z_n \rightarrow z_\infty \quad \text{and} \quad u^{s_*}(x_n, z_n, t_n) - u(x_n, z_n, t_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Set  $u_n(x, z, t) = u(x + x_n, z + z_n, t + t_n)$ . Up to extraction of a subsequence, the functions  $u_n$  converge locally and uniformly to a solution  $u_\infty$  of  $u_t - \Delta u + cu_z - f(u) = 0$  such that

$$v(x, z, t) = u_\infty(x + y, z + s_*, t + T) - u(x, z, t) \geq 0, \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

and  $v(0, z_\infty, 0) = 0$ . It follows from the strong parabolic maximum principle that  $v(x, z, t) = 0$  for all  $t \leq 0$  and then  $v \equiv 0$  in  $\mathbb{R}^n \times \mathbb{R}$  by uniqueness of the solution of the Cauchy problem (11). Thus  $u_\infty(0, 0, 0) = u_\infty(ky, ks_*, kT)$  for all  $k \in \mathbb{Z}$ . But  $u_\infty(ky, ks_*, kT) \rightarrow 1$  as  $k \rightarrow \infty$  since  $s_* > 0$ . This is a contradiction.

Thus,  $s_* \leq 0$ , whence

$$u^0(x, z, t) = u(x + y, z, t + T) \geq u(x, z, t).$$

Since  $T \in \mathbb{R}$  and  $y \in \mathbb{R}^{n-1}$  are arbitrary, we conclude that  $u$  depends on  $z$  only, namely,  $u(x, z, t) = \phi(z - z_0)$  for some  $z_0 \in \mathbb{R}$ . This completes the proof.  $\square$

From Lemma 3.1, any  $\omega$ -limit point of  $u$  satisfies

$$\phi(z - z^*) \leq w(x, z, t) \leq \phi(z - z_*), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}$$

for some constant  $z^*, z_* \in \mathbb{R}$ . Applying the above Lemma, we immediately have the following result.

**Lemma 3.5.** *Let  $u(x, z, t)$  be a solution of (11). Then every  $\omega$ -limit point  $w(x, z, t)$  of  $u$  is a planar wave, that is, there exists a constant  $z_0 \in \mathbb{R}$  such that*

$$w(x, z, t) = \phi(z - z_0), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

Now, we derive estimate for the derivatives of the solution of (11).

**Lemma 3.6** (Monotonicity in  $z$ ). *Let  $u(x, z, t)$  be a solution of (11). Then for any constant  $R > 0$ , there exists a constant  $T > 0$  such that*

$$\inf_{x \in \mathbb{R}^{n-1}, |z| \leq R, t \geq T} u_z(x, z, t) > 0.$$

By using Lemma 3.5 and similar to [18, Lemma 4.7], one can easily prove the above result. By using the above Lemmas 3.6 and 3.1, the following corollary is obtained.

**Corollary 3.7.** *Let  $u(x, z, t)$  be a solution of (11). Then there exists a constant  $T > 0$  such that*

$$\inf_{(x, z, t) \in D} u_z(x, z, t) > 0,$$

where

$$D = \left\{ (x, z, t) \in \mathbb{R}^n \times [T, \infty), \frac{\phi(0)}{2} \leq u(x, z, t) \leq \frac{1 + \phi(0)}{2} \right\}.$$

**Lemma 3.8** (Decay of  $x$ -derivatives). *Let  $u(x, z, t)$  be a solution of (11). Then for any constant  $R > 0$ , it holds that*

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}, |z| \leq R} |u_{x_i}(x, z, t)| = 0, \quad \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}, |z| \leq R} |u_{x_i x_j}(x, z, t)| = 0$$

for each  $1 \leq i, j \leq n - 1$ .

Note that there is no assumption about the function  $f$  in the proof of [18, Lemma 4.9], so the above Lemma can be proved similar to that of [18, Lemma 4.9] and we omit it here.

Next we study the  $\phi(0)$ -level surface of the solution of (11). From Corollary 3.7 and Lemma 3.8, we can derive the following lemma that the  $\phi(0)$ -level surface of the solution  $u(x, z, t)$  has a graphical representation  $z = \Gamma(x, t)$  for all  $t$ .

**Lemma 3.9.** *Let  $u(x, z, t)$  be a solution of (11) and let  $T > 0$  be as defined in Corollary 3.7. Then there exists a smooth bounded function  $\Gamma(x, t)$  such that*

$$u(x, z, t) = \phi(0) \quad \text{if and only if} \quad z = \Gamma(x, t) \quad (21)$$

for any  $(x, t) \in \mathbb{R}^{n-1} \times [T, \infty)$ . Furthermore, the following estimates hold:

(i) for each  $1 \leq i, j \leq n - 1$ ,

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} |\Gamma_{x_i}(x, t)| = 0, \quad \lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} |\Gamma_{x_i x_j}(x, t)| = 0;$$

(ii) there exists a constant  $M > 0$  such that, for each  $1 \leq i, j, k \leq n - 1$ ,

$$\sup_{x \in \mathbb{R}^{n-1}} |\Gamma_{x_i x_j x_k}(x, t)| \leq M \quad \text{for } t \geq T.$$

*Proof.* Since

$$D = \left\{ (x, z, t) \in \mathbb{R}^n \times [T, \infty), \frac{\phi(0)}{2} \leq u(x, z, t) \leq \frac{1 + \phi(0)}{2} \right\}$$

is bounded in the  $z$ -direction by virtue of Lemma 3.1 and the facts  $\phi(-\infty) = 0$ ,  $\phi(+\infty) = 1$ , we can define a bounded function  $\Gamma(x, t)$  satisfying (21) thanks to Corollary 3.7. Here  $\Gamma(x, t)$  is smooth by the implicit function theorem, since  $u(x, z, t)$  is smooth for  $t > 0$ . The other estimates follows from Lemma 3.8 and we omit it here. This completes the proof.  $\square$

The following lemma shows that the large time behavior of the solution can be essentially determined by the  $\phi(0)$ -level surface  $\Gamma(x, t)$ .

**Lemma 3.10.** *Let  $u(x, z, t)$  be a solution of (11) and let  $\Gamma(x, t)$  be as defined in Lemma 3.9. Then it holds that*

$$\lim_{t \rightarrow \infty} \sup_{(x, z) \in \mathbb{R}^n} |u(x, z, t) - \phi(z - \Gamma(x, t))| = 0.$$

*Proof.* We prove this lemma by contradiction method. If the above claim does not hold, there exists a constant  $\delta > 0$  and a sequence  $\{(x_k, z_k, t_k)\}$  such that  $0 < t_1 < t_2 < \dots \rightarrow \infty$  and that

$$|u(x_k, z_k, t_k) - \phi(z_k - \Gamma(x_k, t_k))| \geq \delta. \quad (22)$$

On the other hand, by virtue of Lemma 11 and boundedness of  $\Gamma(x, t)$ , we can choose constants  $R > 0$  and  $T > 0$  such that

$$\sup_{x \in \mathbb{R}^{n-1}, |z| \geq R, t \geq T} |u(x, z, t) - \phi(z - \Gamma(x, t))| < \delta,$$

which mean that  $\{z_k\}$  is bounded. We can choose subsequence of  $\{(x_k, z_k, t_k)\}$ , which we denote again by  $\{(x_k, z_k, t_k)\}$  such that

$$\begin{aligned} z_\infty &:= \lim_{k \rightarrow \infty} z_k, & \gamma_\infty &:= \lim_{k \rightarrow \infty} \Gamma(x_k, t_k) \\ u(x + x_k, z, t + t_k) &\rightarrow w(x, z, t) \quad \text{as } k \rightarrow \infty \quad \text{in } C_{loc}^{2,1}(\mathbb{R}^n \times \mathbb{R}), \end{aligned}$$

where  $w$  is some  $\omega$ -limit point of  $u$ . This and (22) shows that

$$|w(0, z_\infty, 0) - \phi(z_\infty - \gamma_\infty)| = \lim_{k \rightarrow \infty} |u(x_k, z_k, t_k) - \phi(z_k - \gamma(x_k, t_k))| \geq \delta. \quad (23)$$

On the other hand, since we have

$$w(0, \gamma_\infty, 0) = \lim_{k \rightarrow \infty} u(x_k, \Gamma(x_k, t_k), t_k) = \phi(0).$$

Lemma 3.5 implies that  $w(x, z, t) \equiv \phi(z - \gamma_\infty)$ . This contradicts to (23). Thus, the proof is complete.  $\square$

By setting  $y = z - ct$  and  $\gamma(x, t) = \Gamma(x, t) - ct$ , we obtain the statements (i),(ii) of Theorem 1.1 from Lemmas 3.9 and 3.10. Thus it remains to prove the statement (iii). This will be done at the end of this section.

In the following, we construct supersolutions of (11). For this purpose, we consider the Cauchy problem in the form

$$\begin{cases} v_t = \Delta_x v - k|\nabla_x v|^2, & x \in \mathbb{R}^{n-1}, t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^{n-1}, \end{cases} \quad (24)$$

where constant  $k$  is defined as in Lemma 2.3.



**Lemma 3.11** (Supersolution). *For any constant  $\varepsilon \in (0, \varepsilon_0]$ , there exist smooth functions  $p(t)$  and  $q(t)$  satisfying*

$$p(0) > 0, \quad q(0) = 0, \quad 0 \leq p(t), q(t) \leq \varepsilon \quad \text{for } t \geq 0, \quad (25)$$

such that for any function  $v(x, t)$  satisfying (24), the function defined by

$$u^+(x, z, t) = (1 + p(t))\phi(z - v(x, t) + q(t)),$$

satisfies

$$L[u^+] := u_t^+ - \Delta u^+ + cu_z^+ - f(u^+) \geq 0, \quad (x, z) \in \mathbb{R}^n, \quad t > 0.$$

*Proof.* The proof of this lemma is similar to that of Lemma 2.2. Define a constant  $k > 0$  as in Lemma 2.3. Then by using the relation  $\phi'' - c\phi' + f(\phi) = 0$  and Lemma 2.3, we have

$$\begin{aligned} L[u^+] &= [(-v_t + \Delta_x v)\phi' - |\nabla_x v|^2 \phi''] (1 + p(t)) + p'(t)\phi + (1 + p(t))q'(t)\phi' \\ &\quad - (1 + p(t))(\phi'' - c\phi') - f((1 + p(t))\phi) \\ &\geq (1 + p(t))(k\phi' - \phi'')|\nabla_x v|^2 + p'(t)\phi + (1 + p(t))q'(t)\phi' \\ &\quad + (1 + p(t))f(\phi) - f((1 + p(t))\phi) \\ &\geq \left( \frac{p'(t)}{p(t)}\phi + \frac{q'(t)}{p(t)}\phi' - \phi \int_0^1 f'((1 + \tau p(t))\phi) d\tau \right) p(t) \end{aligned}$$

By the assumption (F), we take constants  $0 < K < 1$  and  $\varepsilon_0 > 0$  such that

$$-f'(s) \geq K > 0, \quad s \in [1 - 2\varepsilon_0, 1 + 2\varepsilon_0]. \quad (26)$$

In addition, we take  $M_1 > 0$  sufficiently large so that

$$\phi(\xi) \geq 1 - \varepsilon_0, \quad \xi \geq M_1. \quad (27)$$

As  $\lim_{\xi \rightarrow -\infty} \phi(\xi)e^{-\lambda_1(c)\xi} = 1$  and  $\lim_{\xi \rightarrow -\infty} \phi'(\xi)e^{-\lambda_1(c)\xi} = \lambda_1(c)$ , we can take  $M_2 > 0$  sufficiently large such that

$$\frac{1}{2} < \phi(\xi)e^{-\lambda_1(c)\xi} < \frac{3}{2}, \quad \phi'(\xi)e^{-\lambda_1(c)\xi} > \frac{1}{2}\lambda_1(c) \quad \text{for } \xi \leq -M_2. \quad (28)$$

Denote

$$\begin{aligned} H(t) &= \min \left\{ \frac{K^2 \varepsilon^2}{16C_0^2}, t^{-2} \right\}, \quad \varrho := \min \{ \phi'(\xi), -M_2 \leq \xi \leq M_1 \} > 0, \\ C_0 &\geq \max \left\{ 1, \frac{3K + 3\|f'\|_{L^\infty[0,1]}}{\lambda_1(c)}, \frac{K + \|f'\|_{L^\infty[0,1]}}{\varrho} \right\}. \end{aligned}$$

Finally, we choose functions  $p(t), q(t) \in C^\infty$  satisfying

$$H(t) \leq Kp(t) \leq 2H(t), \quad K|p'(t)| \leq 2|H'(t)|, \quad q(t) = C_0 \int_0^t p(s) ds.$$

Then (25) holds provided that

$$p(0) \leq \frac{K\varepsilon^2}{16C_0^2}, \quad 0 < p(t) \leq \frac{K\varepsilon^2}{8C_0^2} \leq \varepsilon, \quad 0 \leq q(t) \leq C_0 \int_0^\infty p(s) ds \leq \varepsilon.$$

Let  $\xi = z - v(x, t) + q(t)$ , then for any  $\xi \geq M_1$ , combining (26) and (27) and using the fact that  $\phi' > 0$ , we have

$$L[u^+] \geq \left( \frac{p'(t)}{p(t)} - \int_0^1 f'((1 + \tau p(t))\phi) d\tau \right) p(t)\phi \geq \left( \frac{p'(t)}{p(t)} + K \right) p(t)\phi \geq 0,$$

since we have  $\sup_{t \geq 0} \frac{|p'(t)|}{p(t)} \leq \sup_{t \geq 0} \frac{2|H'(t)|}{Kp(t)} \leq \sup_{t \geq 0} \frac{2|H'(t)|}{H(t)} = \frac{K\varepsilon}{C_0} < K$ . For any  $\xi \leq -M_2$ , by using (28), we obtain

$$e^{-\lambda_1(c)\xi} p(t)^{-1} L[u^+] \geq \frac{1}{2} C_0 \lambda_1(c) - \frac{3K}{2} - \frac{3}{2} \|f'\|_{L^\infty[0,1]} \geq 0.$$

For any  $-M_2 \leq \xi \leq M_1$ , we have

$$L[u^+] \geq (C_0 \varrho - K - \|f'\|_{L^\infty[0,1]}) p(t) \geq 0.$$

In summary, we have  $L[u^+] \geq 0$ . This completes the proof.  $\square$

In order to construct subsolutions of (11) we consider the problem of the form

$$\begin{cases} v_t = \Delta_x v + k|\nabla_x v|^2, & x \in \mathbb{R}^{n-1}, t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^{n-1}, \end{cases} \quad (29)$$

where constant  $k$  is defined as in Lemma 2.3.

**Lemma 3.12** (Subsolution). *For any constant  $\varepsilon \in (0, \varepsilon_0]$ , there exist smooth functions  $p(t)$  and  $q(t)$  satisfying*

$$p(0) > 0, \quad q(0) = 0, \quad 0 \leq p(t), q(t) \leq \varepsilon_0 \quad \text{for } t \geq 0,$$

such that for any function  $v(x, t)$  satisfying (29), the function defined by

$$u^-(x, z, t) = (1 - p(t))\phi(z - v(x, t) - q(t)),$$

satisfies

$$L[u^-] := u_t^- - \Delta u^- + cu_z^- - f(u^-) \leq 0, \quad (x, z) \in \mathbb{R}^n, t > 0.$$

The proof of Lemma 3.12 is similar to that of Lemma 3.11 and we omit it here.

**Lemma 3.13** (Lower bound of  $\Gamma(x, t)$ ). *Let  $u(x, z, t)$  be a solution of (11) and let  $\Gamma(x, t)$  be as defined in Lemma 3.9. Then for any  $\varepsilon > 0$ , there exists a constant  $T > 0$  such that the function  $v_1(x, t)$  defined by*

$$\begin{cases} v_{1t} = \Delta_x v_1 - k|\nabla_x v_1|^2, & x \in \mathbb{R}^{n-1}, t > 0, \\ v_1(x, 0) = \Gamma(x, T), & x \in \mathbb{R}^{n-1}, \end{cases} \quad (30)$$

satisfies

$$\Gamma(x, t) \geq v_1(x, t - T) - \varepsilon, \quad t \geq T.$$

*Proof.* From Corollary 3.7, we can choose constants  $T > 0$  and  $L > 0$  such that

$$\begin{aligned} \inf_{(x, z, t) \in D} u_z(x, z, t) &\geq L, \\ D &= \left\{ (x, z, t) \in \mathbb{R}^n \times [T, \infty), \frac{\phi(0)}{2} \leq u(x, z, t) \leq \frac{1 + \phi(0)}{2} \right\}. \end{aligned}$$

For  $\hat{\varepsilon} := \min \left\{ \varepsilon_0, \frac{1}{1 + \|\phi'\|_{L^\infty}} \min \left\{ L\varepsilon, \frac{1 + \phi(0)}{2} \right\} \right\}$ , we choose functions  $p(t)$  and  $q(t)$  satisfying

$$p(0) > 0, \quad q(0) = 0, \quad 0 \leq p(t), q(t) \leq \hat{\varepsilon} \quad \text{for } t \geq 0,$$

as in Lemma 3.11. Then it follows from Lemma 3.10 that, by choosing  $T$  larger if necessary,

$$u(x, z, T) \leq (1 + p(0))\phi(z - \Gamma(x, T)). \quad (31)$$

For such  $T$ , we define  $v_1(x, t)$  as a function satisfying (30). Then Lemma 3.11 shows that the function  $u^+(x, z, t)$  given by

$$u^+(x, z, t) = (1 + p(t - T))\phi(z - v_1(x, t) + q(t - T))$$

is a supersolution of (11) for  $t > T$ . Due to that (31) implies that  $u(x, z, T) \leq u^+(x, z, T)$ , the comparison principle gives that  $u(x, z, t) \leq u^+(x, z, t)$  for  $t \geq T$ . Then by virtue of  $p(t), q(t) \leq \hat{\varepsilon}$ , we have

$$\begin{aligned} u(x, v_1(x, t - T), t) - \phi(0) &\leq u^+(x, v_1(x, t - T), t) - \phi(0) \\ &= (1 + p(t - T))\phi(q(t - T)) - \phi(0) \\ &\leq (1 + \|\phi'\|_{L^\infty})\hat{\varepsilon} \\ &\leq \min \left\{ L\varepsilon, \frac{1 + \phi(0)}{2} \right\}. \end{aligned}$$

Noting that  $u(x, \Gamma(x, t), t) = \phi(0)$ , we get

$$\begin{aligned} L\varepsilon &\geq u(x, v_1(x, t - T), t) - u(x, \Gamma(x, t - T), t) \\ &\geq \inf_{u \in [\phi(0), \frac{1+\phi(0)}{2}], t \geq T} u_z \times (v_1(x, t - T) - \Gamma(x, t - T)) \\ &\geq L(v_1(x, t - T) - \Gamma(x, t - T)), \end{aligned}$$

which implies that  $\Gamma(x, t) \geq v_1(x, t - T) - \varepsilon$  for  $t > T$ . This completes the proof.  $\square$

Similarly, we have the following result.

**Lemma 3.14** (Upper bound of  $\Gamma(x, t)$ ). *Let  $u(x, z, t)$  be a solution of (11) and let  $\Gamma(x, t)$  be as defined in Lemma 3.9. Then for any  $\varepsilon > 0$ , there exists a constant  $T > 0$  such that the function  $v_2(x, t)$  defined by*

$$\begin{cases} v_{2t} = \Delta_x v_2 + k|\nabla_x v_2|^2, & x \in \mathbb{R}^{n-1}, t > 0, \\ v_2(x, 0) = \Gamma(x, T), & x \in \mathbb{R}^{n-1}, \end{cases}$$

satisfies

$$\Gamma(x, t) \leq v_2(x, t - T) + \varepsilon, \quad t \geq T.$$

Now we are ready to complete the proof of Theorem 1.1.

*Proof of Theorem 1.1.* The statements (i),(ii) of Theorem 1.1 are directly from Lemmas 3.9 and 3.10, respectively. Thus we only prove the statement (iii).

By Lemmas 3.13 and 3.14, the large time behavior of the  $\phi(0)$ -level surface  $\Gamma(x, t)$  of the solution  $u(x, z, t)$  of (11) can be controlled by the two functions  $v_1(x, t)$  and  $v_2(x, t)$ . This means that the  $\phi(0)$ -level surface  $\gamma(x, t) = \Gamma(x, t) - ct$  of the solution  $u(x, y, t)$  of (1) can be controlled by the two functions  $v_1(x, t)$  and  $v_2(x, t)$ , where  $v_1(x, t)$  and  $v_2(x, t)$  are the solution of (9) and (10), respectively. Hence the statement (iii) of Theorem 1.1 follows from Lemmas 3.13 and 3.14. This completes the proof of Theorem 1.1.  $\square$

Next, we prove Theorem 1.4. Firstly, similar to [18, Lemma 4.15], one can prove the  $\phi(0)$ -level surface  $\Gamma(x, t)$  remains uniquely ergodic for all large  $t \geq 0$ .

**Lemma 3.15** (Ergodicity of  $\phi(0)$ -level surface). *Let  $u(x, z, t)$  be a solution of (11) and assume that  $u_0(x, z)$  is uniquely ergodic in the  $x$ -direction. Then the  $\phi(0)$ -level surface  $\Gamma(x, t)$  defined in Lemma 3.9 is uniquely ergodic for each  $t \geq T$ , where  $T > 0$  is the constant in Lemma 3.9.*

Secondly, note that if  $v_1(x, t)$  and  $v_2(x, t)$  are the solution of (9) and (10), respectively, then  $v_1(x, t) \leq v_2(x, t)$ . Indeed, let  $w(x, t) = v_1(x, t) - v_2(x, t)$ , then  $w(x, t)$  satisfies

$$\begin{cases} w_t(x, t) - \Delta w(x, t) = -|\nabla_x v_1(x, t)|^2 - |\nabla_x v_2(x, t)|^2 \leq 0, & x \in \mathbb{R}^{n-1}, t > 0, \\ w_0(x) = 0, & x \in \mathbb{R}^{n-1}. \end{cases}$$

By the comparison principle of parabolic equation, we see that  $v_1(x, t) \leq v_2(x, t)$ .

*Proof of Theorem 1.4.* Let  $T > 0$  and  $\Gamma(x, t)$  be defined in Lemma 3.9, then  $\Gamma(x, t)$  is uniquely ergodic for each  $t \geq T$  from Lemma 3.15. Combining Lemmas 2.4 and 2.5, one can easily prove the Theorem 1.4.  $\square$

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