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Stability of Global Bounded Solutions to a Nonautonomous Nonlinear Second Order Integro-Differential Equation

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Abstract. We study the long-time behavior as time goes to infinity of global bounded weak solutions to the following integro-differential equation

$$\ddot{u} + k * \dot{u} + \nabla E(u) = g,$$

in finite dimensions, where the nonlinear potential E satisfies the Łojasiewicz inequality near some equilibrium point. Based on an appropriate new Lyapunov function and Łojasiewicz inequality we prove that any global bounded weak solution converges to a steady state. We also obtain the rate of convergence according to the Łojasiewicz exponent and the time-dependent right-hand side g.

Keywords. Integro-differential equation, nonlinear equation, stability, exponential decay, polynomial decay, Lyapunov function, Łojasiewicz inequality

Mathematics Subject Classification (2010). Primary 45J05, secondary 45M05, 45M10

1. Introduction

In this paper we study the long-time behavior of global bounded solutions of a nonlinear second order evolutionary equation with memory damping of the form

$$\ddot{u} + k * \dot{u} + \nabla E(u) = g, \quad t \geq 0. \tag{1}$$

Here, $\nabla E(u)$ is the gradient of the scalar function $E \in C^2(\mathbb{R}^n)$, $k \in L^1(\mathbb{R}^+)$ is a nonegative kernel, $k * u(t) = \int_0^t k(t-s)u(s)ds$, and the forcing term g tends to 0 with exponential or polynomial decay rate. A typical example for the kernel k we have in mind is given by

$$k(s) = s^{-\alpha} e^{-\beta s}, \quad s > 0, \tag{2}$$

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for some $\alpha \in [0, 1)$, $\beta > 0$.

For this type of singular kernel, we show that, if g tends to 0 sufficiently fast at infinity, the dissipation given by the memory term is strong enough to guarantee convergence to a steady state for any global weak bounded solution of (1), assuming that the function E satisfies the Łojasiewicz inequality near some ω-limit point (see below for the definition of the Łojasiewicz inequality).

We show also that the so-called Łojasiewicz exponent θ and the decay conditions on g determine the decay rate of the solution to the steady state. If $\theta = \frac{1}{2}$ and g decays to 0 exponentially, then the solution u converges exponentially to its limit, and if $\theta \in (0, \frac{1}{2})$ and g decays to 0 polynomially, then the convergence rate of u is polynomial.

The proof is based on the construction of an appropriate new Lyapunov functional, differential inequalities, and on the Łojasiewicz inequality which was proved by Łojasiewicz for analytic functions defined on the finite dimensional space $\mathbb{R}^n$ [10–12]. This inequality was generalized first by L. Simon [13], then by A. Haraux and M. A. Jendoubi [7–9] to functionals defined on infinite-dimensional Banach spaces where convergence to equilibrium as $t \to \infty$ was obtained by the well known Łojasiewicz-Simon approach for bounded solutions of the heat and wave equations with linear dissipation and analytic nonlinearity.

The problem (1) has already been investigated in several papers under different additional assumptions. The main difficulty in treating such a problem is due to the presence of the memory term. For the type of kernel k and nonlinearity E as above, we note that there are up to now two techniques to construct an appropriate Lyapunov functional which allows one to apply the Łojasiewicz inequality in order to obtain a convergence result. The first technique goes back to C. Dafermos [5], and this technique was adaptated by S. Aizicovici and E. Feireisl [1] in order to obtain a convergence result for a phase-field model with memory (see also S. Aizicovici and H. Petzeltová [2]), and then by R. Chill and E. Fašangová [4] in order to obtain a convergence results for bounded solutions of equation (1), the autonomous equation in infinite dimensions case, where the dissipation is both frictional and with memory

$$\ddot{u} + \dot{u} + k * \dot{u} + \nabla E(u) = 0, \quad t \geq 0.$$

Later, R. Zacher and V. Vergara [14] have developed a second technique to find Lyapunov functions for ordinary differential equations of order less than 1, and of order between 1 and 2 in time, which combined with the Łojasiewicz inequality leads to a proof of convergence of global, bounded solutions to a single steady state. In [15], Zacher has obtaind a convergence result for global bounded strong solutions of equation (1). In his proof, Zacher used the Łojasiewicz inequality together with the method of higher order energies.

The present work extends the result of [4] to the nonautonomous case with damping of memory type only. The proof of the current result is easier than the one in [15] since we do not need the method of higher order energies. Moreover, we obtain here the rate of decay betwen the solution and its limit. Finally, our result is established under weaker conditions on g than those imposed by Zacher.

2. Assumptions and main results

We consider here the nonautonomous case; we assume that, for some $\delta > 0$, the function $g \in L^1(\mathbb{R}^+, \mathbb{R}^n) \cap L^2(\mathbb{R}^+, \mathbb{R}^n)$ satisfies the polynomial condition

$$\sup_{t \in \mathbb{R}^+} (1+t)^{1+\delta} \int_t^{\infty} \|g(s)\|^2 \, ds < \infty, \tag{G1}$$

or the exponential condition

$$\sup_{t \in \mathbb{R}^+} e^{\delta t} \int_t^{\infty} \|g(s)\|^2 \, ds < \infty. \tag{G2}$$

The kernel k is assumed to be positive, convex and integrable on $(0, \infty)$, and there exists a constant $C > 0$ such that

$$dk'(s) + Ck'(s)ds \geq 0, \tag{3}$$

where dk' is the distributional derivative of k'.

Remark 2.1. (i) The kernels k in (2) satisfy the condition (3).

(ii) If we integrate the inequality (3), we obtain an inequality which will be used in the sequel:

$$0 \leq k(s)ds \leq -k_0 k'(s)ds \leq k_1 dk'(s) \quad \text{on } (0, \infty),$$

for some $k_0,\ k_1 \geq 0$.

Definition 2.2. For $T > 0$ we say that a function $u \in H^2([0,T], \mathbb{R}^n)$ is a solution of (1) on $[0,T]$ if (1) holds a.e. on $[0,T]$. A function $u \in H^2_{loc}(\mathbb{R}^+, \mathbb{R}^n)$ is called a global solution of (1) if for any $J = [0,T],\ T > 0$, the function $u|_J$ is a solution of (1) on J.

We recall that the ω-limit set of a global solution u of (1) is defined by

$$\omega(u) = \{\phi \in \mathbb{R}^n : \text{ there exists } t_n \to \infty \text{ such that } \lim_{n\to\infty} u(t_n) = \phi\}.$$

Our first main result reads as follows.

Theorem 2.3. *Let* $u \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R}^n)$ *be a global bounded solution of equation* (1). *Suppose that*

(i) *the kernel* k *is positive, convex, integrable on* $(0, \infty)$ *and satisfies* (3);

(ii) *the function* $g \in L^1(\mathbb{R}^+, \mathbb{R}^n) \cap L^2(\mathbb{R}^+, \mathbb{R}^n)$ *satisfies either* (G1) *or* (G2);

(iii) *there exists some* $\phi \in \omega(u)$ *such that* E *satisfies the Lojasiewicz inequality near* ϕ, *i.e. there are constants* $\theta \in (0, \frac{1}{2}]$ *and* $\sigma, \beta > 0$ *such that*

$$|E(x) - E(\phi)|^{1-\theta} \leq \beta \|\nabla E(x)\|, \quad \text{for all } x \in \mathbb{R}^n \text{ with } \|x - \phi\| \leq \sigma. \tag{4}$$

Then

$$\|\dot{u}(t)\| + \|u(t) - \phi\| \longrightarrow 0 \quad \text{as } t \to \infty.$$

From the proof of Theorem 2.3 and by using differential inequalities we obtain also the rate of the convergence of the solution u to the steady state ϕ.

Theorem 2.4. *Under the assumptions of Theorem* 2.3, *the following assertions hold:*

(i) *If* $\theta \in (0, \frac{1}{2})$ *and* g *satisfies the polynomial decay* (G1), *then there exist constants* $C, \xi > 0$ *such that for all* $t \geq 0$ *we have*

$$\|u(t) - \phi\| \leq C(1 + t)^{-\xi},$$

where

$$\xi = \begin{cases} \inf\left\{\dfrac{\theta}{1 - 2\theta}, \dfrac{\delta}{2}\right\} & \text{if } g \neq 0, \\ \dfrac{\theta}{1 - 2\theta} & \text{if } g = 0. \end{cases}$$

(ii) *If* $\theta = \frac{1}{2}$ *and* g *satisfies the exponential growth* (G2), *then there exist constants* $C, \kappa > 0$ *such that*

$$\|u(t) - \phi\| \leq Ce^{-\kappa t}, \quad t \geq 0.$$

3. The convergence result

We denote by C (sometime C_i) a generic positive constant which may vary from line to line, which may depend on $\|k\|_{L^1(\mathbb{R}^+)}$, but which can be chosen independently of $t \in \mathbb{R}^+$. We start our proof by citing and providing some Lemmas. The first one is a technical lemma. Its proof can be found in [4].

Lemma 3.1. *Assume that the kernel k is positive, convex, integrable on* $(0, \infty)$ *and satisfies* (3). *Let $v \in L^2(\mathbb{R}^+, \mathbb{R}^n)$ and define $\eta(t,s) = \int_{t-s}^t v(\rho)d\rho$, $0 \le s \le t$. Then*

$$\lim_{t\to\infty} \int_0^t (-k')(s)\|\eta(t,s)\|^2 ds = \lim_{t\to\infty} \int_0^t k(s)\|\eta(t,s)\| ds = 0$$
$$\lim_{s\to 0^+} sk(s) = \lim_{s\to 0^+} s^2 k'(s) = 0.$$

The following lemma is needed in the construction of the Lyapunov energy. Its proof can be found in [4].

Lemma 3.2. *Let H be a Hilbert space and $k \in L^1_{loc}(\mathbb{R}^+)$ be positive and convex. Let $v \in L^\infty_{loc}(\mathbb{R}^+, H)$ and define $\eta(t,s) = \int_{t-s}^t v(r)dr$, $0 \le s \le t$. Then, for almost every $t \ge 0$,*

$$\begin{aligned}(k * v(t), v(t))_H = &\frac{d}{dt}\left(\frac{1}{2}\int_0^t (-k')(s)\|\eta(t,s)\|_H^2 ds + k(t)\|\eta(t,t)\|_H^2\right)\\ &+ \frac{1}{2}\int_0^t \|\eta(t,s)\|_H^2 dk'(s) + \frac{1}{2}(-k'(t))\|\eta(t,t)\|_H^2.\end{aligned}$$

3.1. Lyapunov function. The crucial step point for our proof is to find a Lyapunov functional. Using Lemma 3.2, we begin by the basic energy estimate

Lemma 3.3. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by*

$$\begin{aligned}\Phi(t) = &\frac{1}{2}\|\dot{u}(t)\|^2 + E(u(t)) + \frac{1}{2}\int_0^t (-k')(s)\|\eta(t,s)\|^2 ds\\ &+ k(t)\|\eta(t,t)\|^2 + \int_t^\infty (g(s), \dot{u}(s))ds,\end{aligned}$$

where $\eta(t,s) = \int_{t-s}^t \dot{u}(r)dr = u(t) - u(t-s)$, $0 \le s \le t$. Then, for almost every $t \in \mathbb{R}^+$, we have

$$\frac{d}{dt}\Phi(t) = -\frac{1}{2}\int_0^t \|\eta(t,s)\|^2 dk'(s) + \frac{1}{2}k'(t)\|\eta(t,t)\|^2 \le 0.$$

Proof. Taking the inner product of (1) with $\dot{u}$ and using Lemma 3.2 for $H = \mathbb{R}^n$, we obtain the result. □

The dissipation given by the basic energy estimates is not strong enough to prove the convergence with the Łojasiewicz approach. To overcome the difficulty due to the weaker dissipation, it is necessary to introduce a suitable perturbation which serves to control some terms and to produce some new dissipation. Indeed, we prove two lemmas, the first one serves to control $\|\dot{u}\|$ and the second serves to control $\|\nabla E(u)\|$. Let $\varepsilon > 0$ be a real positive number which will be fixed in the sequel.

Lemma 3.4. *Let $I : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by*

$$I(t) = -(\dot{u}(t), \int_0^t k(s)\eta(t,s)ds) + \frac{1}{2}\| \int_0^t k(s)\eta(t,s)ds\|^2 + \frac{1}{2}\int_t^\infty \|g(s)\|^2 ds$$
$$- \frac{1}{2}\int_0^t (-k')(s)\|\eta(t,s)\|^2 ds - k(t)\|\eta(t,t)\|^2, \quad t \in \mathbb{R}^+.$$

Then there exists $t_0 > 0$ such that for almost every $t \geq t_0$

$$\frac{d}{dt}I(t) \leq \left\{-\int_0^t k(s)ds + \frac{\varepsilon}{2}(1+\|k\|^2_{L^1(\mathbb{R}^+)})\right\}\|\dot{u}(t)\|^2 + \frac{\varepsilon}{8}\|\nabla E(u)\|^2$$
$$+ C_1\left(1+\frac{1}{\varepsilon}\right)\left\{\int_0^t \|\eta(t,s)\|^2 dk'(s) - k'(t)\|\eta(t,t)\|^2\right\}.$$

Proof. Taking the derivative of I and using the fact that $\frac{d}{dt}\int_0^t k(s)\eta(t,s)ds = k(t)\eta(t,t) + \dot{u}(t)\int_0^t k(s)ds - k * \dot{u}(t)$, we have

$$\frac{d}{dt}I(t) = -(\ddot{u}(t) + k * \dot{u}(t) - k(t)\eta(t,t) - \dot{u}(t)\int_0^t k(s)ds, \int_0^t k(s)\eta(t,s)ds)$$
$$- (\dot{u}(t), k(t)\eta(t,t) + \dot{u}(t)\int_0^t k(s)ds - k * \dot{u}(t)) - \frac{1}{2}\|g(t)\|^2$$
$$- \frac{d}{dt}\left\{\frac{1}{2}\int_0^t (-k')(s)\|\eta(t,s)\|^2 ds + k(t)\|\eta(t,t)\|^2\right\}.$$

Using Lemma 3.2 and equation (1), we get almost every $t \geq 0$

$$\frac{d}{dt}I(t) = -(\ddot{u}(t) + k * \dot{u}(t) - k(t)\eta(t,t) - \dot{u}(t)\int_0^t k(s)ds, \int_0^t k(s)\eta(t,s)ds)$$
$$- (\dot{u}(t), k(t)\eta(t,t) + \dot{u}(t)\int_0^t k(s)ds - k * \dot{u}(t)) - \frac{1}{2}\|g(t)\|^2$$
$$- (k * \dot{u}(t), \dot{u}(t)) + \frac{1}{2}\int_0^t \|\eta(t,s)\|^2 dk'(s) + \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2$$
$$= -(-\nabla E(u(t)) + g(t) - k(t)\eta(t,t) - \dot{u}(t)\int_0^t k(s)ds, \int_0^t k(s)\eta(t,s)ds)$$
$$- \int_0^t k(s)ds\|\dot{u}(t)\|^2 - (\dot{u}(t), k(t)\eta(t,t)) - \frac{1}{2}\|g(t)\|^2$$
$$+ \frac{1}{2}\int_0^t \|\eta(t,s)\|^2 dk'(s) + \frac{1}{2}(-k'(t))\|\eta(t,t)\|^2$$

Next we estimate some terms in $\frac{d}{dt}I(t)$. The Cauchy–Schwarz inequality implies

$$\begin{aligned}(\nabla E(u), \int_0^t k(s)\eta(t,s)ds) &\le \frac{\varepsilon}{8}\|\nabla E(u(t))\|^2 + \frac{2}{\varepsilon}\left(\int_0^t \sqrt{k(s)}\sqrt{k(s)}\|\eta(t,s)\|ds\right)^2\\ &\le \frac{\varepsilon}{8}\|\nabla E(u(t))\|^2 + \frac{2\|k\|_{L^1(\mathbb{R}^+)}}{\varepsilon}\int_0^t k(s)\|\eta(t,s)\|^2 ds\\ &\le \frac{\varepsilon}{8}\|\nabla E(u(t))\|^2 + \frac{2C\|k\|_{L^1(\mathbb{R}^+)}}{\varepsilon}\int_0^t \|\eta(t,s)\|^2 dk'(s),\\ (g(t), \int_0^t k(s)\eta(t,s)ds) &\le \frac{1}{2}\|g(t)\|^2 + \frac{C\|k\|_{L^1(\mathbb{R}^+)}}{2}\int_0^t \|\eta(t,s)\|^2 dk'(s),\end{aligned}$$

$$\begin{aligned}&(\dot{u}(t)\int_0^t k(s)ds, \int_0^t k(s)\eta(t,s)ds)\\ &\le \frac{\varepsilon\|k\|^2_{L^1(\mathbb{R}^+)}}{2}\|\dot{u}(t)\|^2 + \frac{1}{2\varepsilon}\left(\int_0^t \sqrt{k(s)}\sqrt{k(s)}\|\eta(t,s)\|ds\right)^2\\ &\le \frac{\varepsilon\|k\|^2_{L^1(\mathbb{R}^+)}}{2}\|\dot{u}(t)\|^2 + \frac{C\|k\|_{L^1(\mathbb{R}^+)}}{2\varepsilon}\int_0^t \|\eta(t,s)\|^2 dk'(s).\end{aligned}$$

It follows from the assumption on k that there exists $t_0 > 0$ such that for all $t \ge t_0$ the Cauchy–Schwarz inequality implies

$$(\dot{u}(t), k(t)\eta(t,t)) \le \frac{\varepsilon}{2}\|\dot{u}(t)\|^2 + \frac{k^2(t)}{2\varepsilon}\|\eta(t,t)\|^2 \le \frac{\varepsilon}{2}\|\dot{u}(t)\|^2 + \frac{C(-k'(t))}{\varepsilon}\|\eta(t,t)\|^2.$$

Also, for $t \ge t_0$

$$\begin{aligned}(k(t)\eta(t,t), \int_0^t k(s)\eta(t,s)ds) &\le \frac{1}{2}\left(\int_0^t \sqrt{k(s)}\sqrt{k(s)}\|\eta(t,s)\|ds\right)^2 + \frac{k^2(t)}{2}\|\eta(t,t)\|^2\\ &\le \frac{C\|k\|_{L^1(\mathbb{R}^+)}}{2}\int_0^t \|\eta(t,s)\|^2 dk'(s) + C(-k'(t))\|\eta(t,t)\|^2.\end{aligned}$$

Hence, putting these estimates together we obtain the result. □

Lemma 3.5. *Let $J : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by*

$$J(t) = (\nabla E(u(t)), \dot{u}(t) - \int_0^t k(s)\eta(t,s)ds) + \int_t^\infty \|g(s)\|^2 ds, \quad t \in \mathbb{R}^+.$$

Then for all $t \ge t_0$

$$\frac{d}{dt}J(t) \le -\frac{1}{4}\|\nabla E(u(t))\|^2 + C_2\left\{\|\dot{u}(t)\|^2 + \int_0^t \|\eta(t,s)\|^2 dk'(s) - k'(t)\|\eta(t,t)\|^2\right\}$$

Proof. Take the derivative of J and use equation (1),

$$\begin{aligned}
\frac{d}{dt}J(t) &= (\nabla E(u(t)), \ddot{u}(t) + k * \dot{u}(t) - k(t)\eta(t,t) - \dot{u}(t)\int_0^t k(s)ds) \\
&\quad + (\nabla^2 E(u(t))\dot{u}(t), \dot{u}(t) - \int_0^t k(s)\eta(t,s)ds) - \|g(t)\|^2 \\
&= (\nabla E(u(t)), -\nabla E(u(t)) + g(t) - k(t)\eta(t,t) - \dot{u}(t)\int_0^t k(s)ds) \\
&\quad + (\nabla^2 E(u(t))\dot{u}(t), \dot{u}(t) - \int_0^t k(s)\eta(t,s)ds) - \|g(t)\|^2 \\
&= -\|\nabla E(u(t))\|^2 + (\nabla E(u(t)), g(t) - k(t)\eta(t,t) - \dot{u}(t)\int_0^t k(s)ds) \\
&\quad + (\nabla^2 E(u(t))\dot{u}(t), \dot{u}(t) - \int_0^t k(s)\eta(t,s)ds) - \|g(t)\|^2.
\end{aligned}$$

Next we estimate some terms in $\frac{d}{dt}J(t)$. The Cauchy–Schwarz inequality implies, for all $t \geq 0$

$$(\nabla E(u(t)), g(t)) \leq \frac{1}{4}\|\nabla E(u(t))\|^2 + \|g(t)\|^2.$$

$$(\nabla E(u(t)), \dot{u}(t)\int_0^t k(s)ds) \leq \frac{1}{4}\|\nabla E(u(t))\|^2 + \|k\|^2_{L^1(\mathbb{R}^+)}\|\dot{u}(t)\|^2.$$

$$(\nabla E(u(t)), k(t)\eta(t,t)) \leq \frac{1}{4}\|\nabla E(u(t))\|^2 + C(-k'(t))\|\eta(t,t)\|^2.$$

By the boundedness of u, the continuity of E, and the Cauchy–Schwarz inequality, for all $t \geq 0$

$$(\nabla^2 E(u(t))\dot{u}(t), \dot{u}(t)) \leq C\|\dot{u}(t)\|^2.$$

$$(\nabla^2 E(u(t))\dot{u}(t), \int_0^t k(s)\eta(t,s)ds) \leq C\big(\|\dot{u}(t)\|^2 + \int_0^t \|\eta(t,s)\|^2 dk'(s)\big).$$

The claim follows by combining these estimates with $\frac{d}{dt}J(t)$. □

Now, thanks to the last three lemmas, we construct the suitable Lyapunov functional. Let $H_0 : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$\begin{aligned}
&H_0(t) \\
&= \Phi(t) + \varepsilon^2 I(t) + \varepsilon^3 J(t) \\
&= \frac{1}{2}\|\dot{u}(t)\|^2 + E(u(t)) + \frac{1-\varepsilon^2}{2}\int_0^t (-k')(s)\|\eta(t,s)\|^2 ds + (1-\varepsilon^2)k(t)\|\eta(t,t)\|^2 \\
&\quad -\varepsilon^2(\dot{u}(t), \int_0^t k(s)\eta(t,s)ds) + \frac{\varepsilon^2}{2}\|\int_0^t k(s)\eta(t,s)ds\| \\
&\quad +\varepsilon^3(\nabla E(u(t)), \dot{u}(t) - \int_0^t k(s)\eta(t,s)ds) + \frac{\varepsilon^2+2\varepsilon^3}{2}\int_t^\infty \|g(s)\|^2 ds + \int_t^\infty (g(s), \dot{u}(s))ds
\end{aligned}$$

Then, keeping in mind the last tree lemmas, for almost every $t \geq t_0$

$$\begin{aligned}
\frac{d}{dt}H_0(t) &\leq -\frac{1}{2}\int_0^t \|\eta(t,s)\|^2 dk'(s) + \frac{1}{2}k'(t)\|\eta(t,t)\|^2 - \frac{\varepsilon^3}{4}\|\nabla E(u(t))\|^2 \\
&\quad + \varepsilon^2\Big\{-\int_0^t k(s)ds + \frac{\varepsilon}{2}\big(1+\|k\|^2_{L^1(\mathbb{R}^+)}\big)\Big\}\|\dot{u}(t)\|^2 + \frac{\varepsilon^3}{8}\|\nabla E(u(t))\|^2 \\
&\quad + \varepsilon^2 C_1\Big(1+\frac{1}{\varepsilon}\Big)\Big\{\int_0^t \|\eta(t,s)\|^2 dk'(s) - k'(t)\|\eta(t,t)\|^2\Big\} \\
&\quad + \varepsilon^3 C_2\Big\{\|\dot{u}(t)\|^2 + \int_0^t \|\eta(t,s)\|^2 dk'(s) - k'(t)\|\eta(t,t)\|^2\Big\} \\
&\leq \varepsilon^2\Big\{-\int_0^{t_0} k(s)ds + \frac{\varepsilon}{2}\big(1+2C_2+\|k\|^2_{L^1(\mathbb{R}^+)}\big)\Big\}\|\dot{u}(t)\|^2 - \frac{\varepsilon^3}{8}\|\nabla E(u(t))\|^2 \\
&\quad + \Big\{-\frac{1}{2}+\varepsilon C_1(1+\varepsilon)+\varepsilon^3 C_2\Big\}\Big(\int_0^t \|\eta(t,s)\|^2 dk'(s) - k'(t)\|\eta(t,t)\|^2\Big).
\end{aligned}$$

Thus, if we choose $\varepsilon > 0$ small enough, we see that there exists a constant $C_3 > 0$ such that for almost every $t \geq t_0$,

$$\frac{d}{dt}H_0(t) \leq -C_3\Big\{\|\dot{u}(t)\|^2 + \int_0^t \|\eta(t,s)\|^2 dk'(s) + \|\nabla E(u(t))\|^2 - k'(t)\|\eta(t,t)\|^2\Big\} \leq 0. \quad (5)$$

3.2. Properties of the ω-limit set. If u is a global bounded solution of (1), $\omega(u)$ is non-empty, compact, and connected [6]. Moreover, since our equation has a continuous Lyapunov functional, we prove the following lemma which is fundamental for the proof of Theorem 2.3.

Lemma 3.6. *Let u be a global solution of* (1) *and assume that the assumptions of Theorem* 2.3 *hold. Then*

(i) $\dot{u} \in L^2(\mathbb{R}^+, \mathbb{R}^n)$ *and* $\int_0^t \|\eta(t,s)\|^2 dk'(s), \int_0^t (-k'(s))\|\eta(t,s)\|^2 ds \in L^1(\mathbb{R}^+)$.

(ii) *The function E is constant on $\omega(u)$, and*

$$E(\phi) = \lim_{t\to\infty} E(u(t)) = E_\infty = const < \infty \quad \textit{for all } \phi \in \omega(u).$$

(iii) $\lim_{t\to\infty}\|\dot{u}(t)\| = 0$.

(iv) *The ω-limit set of u is a subset of the set of stationary solutions.*

Proof. The two functions Φ and H_0 are decreasing for large t ($t \geq t_0$) and (since $g \in L^1(\mathbb{R}^+, \mathbb{R}^n) \cap L^2(\mathbb{R}^+, \mathbb{R}^n)$, $u \in W^{1,\infty}(\mathbb{R}^+, \mathbb{R}^n)$ and E is continuous) bounded from below. Then, the following limits exist

$$\lim_{t\to\infty} H_0(t) = \inf_{t\geq 0} H_0(t) = H_\infty,$$
$$\lim_{t\to\infty} \Phi(t) = \inf_{t\geq 0} \Phi(t) = \Phi_\infty.$$

From this, the inequality (5), and the fact that $\int_0^t(-k'(s))\|\eta(t,s)\|^2 ds \leq C\int_0^t\|\eta(t,s)\|^2 dk'(s)$ we obtain (i).

Next, let $\phi \in \omega(u)$ and $t_n \nearrow \infty$ such that $u(t_n) \to \phi$. Since $\dot{u} \in L^2(\mathbb{R}^+, \mathbb{R}^n)$, we have

$$u(t_n+s) = \left(u(t_n) + \int_{t_n}^{t_n+s} \dot{u}(\rho)\, d\rho\right) \to \phi \quad \text{for every } s \in [0,1].$$

Hence, by the continuity of E, $E(u(t_n{+}s)) \to E(\phi)$ for every $s\in[0,1]$. The dominated convergence theorem yields $E(\phi) = \lim_{n\to\infty}\int_0^1 E(u(t_n{+}s))ds$. Therefore, by integrating $\Phi(t_n+\cdot)$ over $(0,1)$, we obtain

$$E(\phi) = \lim_{n\to\infty}\int_0^1 \Phi(t_n+s)ds = \Phi_\infty.$$

Here we have used (i), the fact that $k \in L^1(\mathbb{R}^+)$, $g \in L^2(\mathbb{R}^+, \mathbb{R}^n)$, and the boundedness of η. Since ϕ was chosen arbitrarily in $\omega(u)$, this implies that E is constant on $\omega(u)$. Moreover, by the relative compactness of the orbit of u, we see that

$$\lim_{t\to\infty} E(u(t)) = \Phi_\infty = E_\infty.$$

From this, (i), the definition of Φ, Lemma 3.1, the assumtions on k and g, and the boundedness of η, we obtain (iii).

In order to prove (iv), let $\phi \in \omega(u)$ and choose $t_n \to \infty$ such that $u(t_n) \to \phi$. We have already seen that this implies $u(t_n+s) \to \phi$ for every $s \in [0,1]$. Hence

$$\nabla E(u(t_n+s)) \to \nabla E(\phi) \quad \text{for every } s \in [0,1]. \tag{6}$$

Using the dominated convergence theorem and the equation (1),

$$\begin{aligned}
\nabla E(\phi) &= \int_0^1 \nabla E(\phi) d\tau = \lim_{n\to\infty}\int_0^1 \nabla E(u(t_n+\tau))d\tau \\
&= \lim_{n\to\infty}\int_0^1 \big(-\ddot{u} - k * \dot{u} + g\big)(t_n+\tau)d\tau \\
&= \lim_{n\to\infty}(\dot{u}(t_n) - \dot{u}(t_n+1)) + \lim_{n\to\infty}\int_{t_n}^{t_n+1}(-k*\dot{u}(\tau) + g(\tau))d\tau \\
&= 0,
\end{aligned}$$

by (i),(iii), and the integrability of k and g. □

3.3. Proof of the convergence result. After the previous preparation, we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. Let $H : \mathbb{R}^+ \to \mathbb{R}$ be the function given by

$$H(t) = H_0(t) - E_\infty.$$

By (5), for almost every $t \geq t_0$

$$\frac{d}{dt}H(t) \leq -C_3\left\{\|\dot{u}(t)\|^2 + \int_0^t \|\eta(t,s)\|^2 dk'(s) + \|\nabla E(u(t))\|^2 - k'(t)\|\eta(t,t)\|^2\right\} \leq 0. \quad (7)$$

Then $H(t)$ is decreasing. In addition, using (iii) and Lemma 3.1, we get $\lim_{t\to\infty} H(t) = 0$. Then $H(t)$ is nonnegative for large t ($t > t_0$).

Now, we consider two possibilities. If the function g satisfies the polynomial growth (G1), then, for θ as in Theorem 2.3, let $\theta_0 \in (0, \theta]$ be such that

$$(1+\delta)(1-\theta_0) > 1, \quad \text{that is} \quad \theta_0 < \frac{\delta}{1+\delta}.$$

Note that (4) is satisfied with θ replaced by θ_0. Then, by applying the Cauchy–Schwarz inequality, Young's inequality $\left(a^{1-\theta_0}b^{1-\theta_0} \leq C\left(a + b^{\frac{1-\theta_0}{\theta_0}}\right),\ a, b \geq 0\right)$, and the fact that $\| \int_0^t k(s)\eta(t,s)\, ds\| \leq C\big(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\big)^{\frac{1}{2}}$, we obtain for all $t \geq 0$

$$\begin{aligned} H(t)^{1-\theta_0} \leq C\Bigg\{ & \|\dot{u}(t)\|^{2(1-\theta_0)} + |E(u(t)) - E_\infty|^{(1-\theta_0)} + \|\nabla E(u(t))\| + \|\dot{u}(t)\|^{\frac{1-\theta_0}{\theta_0}} \\ & + (k(t)\|\eta(t,t)\|^2)^{(1-\theta_0)} + \left(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\right)^{\frac{1}{2}2(1-\theta_0)} \\ & + \left(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\right)^{\frac{1-\theta_0}{2\theta_0}} + \left(\int_t^\infty (g(s), \dot{u}(s))ds\right)^{(1-\theta_0)} \\ & + \left(\int_t^\infty \|g(s)\|^2\, ds\right)^{(1-\theta_0)} \Bigg\}. \end{aligned} \quad (8)$$

Next, we will control the term $\big(\int_t^\infty (g(s), \dot{u}(s))ds\big)^{(1-\theta_0)}$ by the other terms of the right-hand side of the inequality (8). Using the Cauchy–Schwarz inequality and Young's inequality,

$$2\int_t^\infty (g(s), \dot{u}(s))ds \leq \frac{1}{C_3}\int_t^\infty \|g(s)\|^2 ds + C_3\int_t^\infty \|\dot{u}(s)\|^2 ds,$$

where C_3 is given by (7). In addition, $H(t)$ positive and decreasing to 0. Then, by (7), for almost every $t \geq t_0$, $\int_t^\infty \|\dot{u}(s)\|^2 \leq \frac{1}{C_3}H(t)$. Hence,

$$2\int_t^\infty (g(s), \dot{u}(s))ds \leq \frac{1}{C_3}\int_t^\infty \|g(s)\|^2 ds + H(t).$$

Using this inequality, H, the Cauchy–Schwarz inequality, Young's inequality, and the fact that $\| \int_0^t k(s)\eta(t,s)\, ds\| \le C(\int_0^t(-k')(s)\|\eta(t,s)\|^2\, ds)^{\frac{1}{2}}$ we obtain

$$\begin{aligned}\int_t^\infty (g(s),\dot u(s))ds \le C\Big\{ &\|\dot u(t)\|^2 + |E(u(t)) - E_\infty| + \int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds \\ &+ k(t)\|\eta(t,t)\|^2 + \|\nabla E(u(t))\|^{\frac{1}{1-\theta_0}} + \|\dot u(t)\|^{\frac{1}{\theta_0}} \\ &+ \Big(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\Big)^{\frac{1}{2\theta_0}} + \int_t^\infty \|g(s)\|^2\, ds\Big\}\end{aligned}$$

for almost every $t \ge t_0$. From this inequality and Young's inequality, it follows that

$$\begin{aligned}(\int_t^\infty (g(s),\dot u(s))ds)^{1-\theta_0} \le C\Big\{ &\|\dot u(t)\|^{2(1-\theta_0)} + |E(u(t)) - E_\infty|^{(1-\theta_0)} \\ &+ \|\dot u(t)\|^{\frac{1-\theta_0}{\theta_0}} + \Big(\int_0^t (-k')(s)\|\eta(t,s)\|^2 ds\Big)^{\frac{1-\theta_0}{2\theta_0}} \\ &+ (k(t)\|\eta(t,t)\|^2)^{(1-\theta_0)} + \Big(\int_t^\infty \|g(s)\|^2 ds\Big)^{(1-\theta_0)} \\ &+ \Big(\int_0^t (-k')(s)\|\eta(t,s)\|^2 ds\Big)^{\frac{1}{2}2(1-\theta_0)} + \|\nabla E(u(t))\|\Big\},\end{aligned}$$

which together with (8) implies that, for almost every $t \ge t_0$,

$$\begin{aligned}H(t)^{1-\theta_0} \le C\Big\{ &\|\dot u(t)\|^{2(1-\theta_0)} + |E(u(t)) - E_\infty|^{(1-\theta_0)} + \|\nabla E(u(t))\| + \|\dot u(t)\|^{\frac{1-\theta_0}{\theta_0}} \\ &+ \Big(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\Big)^{\frac{1-\theta_0}{2\theta_0}} + (k(t)\|\eta(t,t)\|^2)^{\frac{1}{2}2(1-\theta_0)} \\ &+ \Big(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\Big)^{\frac{1}{2}2(1-\theta_0)} + \Big(\int_t^\infty \|g(s)\|^2\, ds\Big)^{(1-\theta_0)}\Big\}.\end{aligned}$$

By Lemma 3.6(iii) and Lemma 3.1, choosing $t_0 > 0$ sufficiently large so as to ensure that both $\|\dot u(t)\| \le 1$, $\int_0^t(-k')(s)\|\eta(t,s)\|^2\, ds \le 1$, $k(t)\|\eta(t,t)\|^2 \le 1$, for all $t \ge t_0$. Taking into account that $\frac{1-\theta_0}{\theta_0} \ge 1$ and $2(1-\theta_0) \ge 1$, it follows that for almost every $t \ge t_0$ we have

$$\begin{aligned}H(t)^{1-\theta_0} \le C\Big\{ &\|\dot u(t)\| + |E(u(t)) - E_\infty|^{(1-\theta_0)} + \|\nabla E(u(t))\| + (k(t)\|\eta(t,t)\|^2)^{\frac{1}{2}} \\ &+ \Big(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\Big)^{\frac{1}{2}} + \Big(\int_t^\infty \|g(s)\|^2\, ds\Big)^{(1-\theta_0)}\Big\}.\end{aligned}$$

If there exists $\tau \geq t_0$ such that $H(\tau) = 0$, then $H(t) = 0$ for all $t \geq \tau$. By the inequality (7) we obtain $\|\dot{u}(t,.)\| = 0$ for all $t \geq \tau$. In this case u is a stationary solution and, in particular, a convergent solution. We may therefore suppose in the following that

$$H(t) > 0 \quad \text{for all } t \geq t_0.$$

Let ϕ, σ be as in assumption (4), and let $t_1 > t_0$ be such that $\|u(t_1) - \phi\| < \sigma$. Let

$$t_2 = \sup\Big\{t \geq t_1 : \sup_{s\in[t_1,t]} \|u(s) - \phi\| \leq \sigma\Big\}.$$

By continuity of u, $t_2 > t_1$. By (4)

$$|E(u(t)) - E_\infty|^{(1-\theta_0)} \leq C\|\nabla E(u(t))\|, \quad \text{for every } t \in [t_1, t_2).$$

Thus we obtain

$$\begin{aligned} H(t)^{1-\theta_0} \leq C\Bigg\{ &\|\dot{u}(t)\| + \|\nabla E(u(t))\| + (k(t)\|\eta(t,t)\|^2)^{\frac{1}{2}} \\ &+ \Big(\int_0^t (-k')(s)\|\eta(t,s)\|^2\, ds\Big)^{\frac{1}{2}} + \Big(\int_t^\infty \|g(s)\|^2\, ds\Big)^{(1-\theta_0)} \Bigg\} \end{aligned} \tag{9}$$

for almost every $t \in [t_1, t_2)$. Now, using (7),(9),(G1), we have

$$\begin{aligned} &-\frac{d}{dt}H(t)^{\theta_0} \\ &= -\theta_0 H(t)^{\theta_0-1}\frac{d}{dt}H(t) \\ &\geq \frac{C\big(\|\dot{u}(t)\|^2 + \int_0^t \|\eta(t,s)\|^2 dk'(s) + \|\nabla E(u(t))\|^2 - k'(t)\|\eta(t,t)\|^2\big)}{\|\dot{u}\| + \|\nabla E(u)\| + k(t)^{\frac{1}{2}}\|\eta(t,t)\| + \big(\int_0^t (-k')(s)\|\eta(t,s)\|^2 ds\big)^{\frac{1}{2}} + (1+t)^{-(1+\delta)(1-\theta_0)}} \\ &\geq \frac{C\big(\|\dot{u}(t)\|^2 + \int_0^t \|\eta(t,s)\|^2 dk'(s) + \|\nabla E(u(t))\|^2 - k'(t)\|\eta(t,t)\|^2\big)}{\|\dot{u}\| + \|\nabla E(u)\| + (-k'(t))^{\frac{1}{2}}\|\eta(t,t)\| + \big(\int_0^t \|\eta(t,s)\|^2 dk'(s)\big)^{\frac{1}{2}} + (1+t)^{-(1+\delta)(1-\theta_0)}} \\ &\geq C\Bigg(\|\dot{u}(t)\| + \|\nabla E(u(t))\| + \Big(\int_0^t \|\eta(t,s)\|^2 dk'(s)\Big)^{\frac{1}{2}} + (-k'(t))^{\frac{1}{2}}\|\eta(t,t)\|\Bigg) \\ &\quad - C(1+t)^{-(1+\delta)(1-\theta_0)}. \end{aligned} \tag{10}$$

From the above inequality and from the fact that the term $-\frac{d}{dt}H(t)^{\theta_0} + C(1+t)^{-(1+\delta)(1-\theta_0)}$ is integrable we obtain that $\|\dot{u}\|$ is integrable on $[t_1, t_2)$. Moreover, for $t \in [t_1, t_2)$,

$$\begin{aligned} \|u(t) - \phi\| &\leq \|u(t_1) - \phi\| + \int_{t_1}^t \|\dot{u}(s)\| ds \\ &\leq \|u(t_1) - \phi\| + C\left\{H(t_1)^{\theta_0} + (1+t_1)^{-(1+\delta)(1-\theta_0)+1}\right\}. \end{aligned}$$

The three terms on the right-hand side of this inequality tend to 0, if t_1 tends to ∞. This implies that $t_2=\infty$ if t_1 is chosen large enough. In fact, if this was not true, then we could find a sequence $(t_1^n)\nearrow\infty$ such that $\lim_{n\to\infty}\|u(t_1^n)-\phi\|=0$ and such that the corresponding t_2^n are finite. By definition $\|u(t_2^n)-\phi\|=\sigma$, and by the above inequality, $\lim_{n\to\infty}\|u(t_2^n)-\phi\|=0$. The compactness of the range of u now yields a contradiction.

Hence $t_2=\infty$ if t_1 is large enough and then $\|\dot u\|$ is integrable on $[t_1,\infty)$. In particular, $\lim_{t\to\infty}u(t)$ exists.

Finally, when the growth condition in g is exponential, we replace, in the inequality (10), the term $(1+t)^{-(1+\delta)(1-\theta_0)}$ by the term $e^{-\delta t(1-\theta_0)}$ which is integrable too, and then the same conclusion holds. This completes the proof of Theorem 2.3. □

4. Convergence rate

Now we shall prove the exponential or polynomial decay of solutions to equation (1), depending on the Łojasiewicz exponent and the decay conditions on g. The following lemma is used in the proof of the polynomial convergence rate. Its proof can be found in [3].

Lemma 4.1. *Let $\zeta\in W^{1,1}_{loc}(\mathbb{R}^+,\mathbb{R}^+)$. We suppose that there exist constants $K_1>0$, $K_2\geq 0$, $k>1$ and $\lambda>0$ such that for almost every $t\geq 0$ we have*

$$\frac{d}{dt}\zeta(t)+K_1\zeta(t)^k\leq K_2(1+t)^{-\lambda}.$$

Then there exists a positive constant m such that

$$\zeta(t)\leq m(1+t)^{-\nu},\quad \text{where } \nu=\min\left\{\frac{1}{k-1},\frac{\lambda}{k}\right\}.$$

Proof of Theorem 2.4. We proceed in two steps.

Step 1 (Polynomial decay). First, we note that the inequalities (9) and (10) are satisfied when θ_0 is replaced by the initial exponent θ given by (4). By using (9) together with (G1) and Young's inequality, we obtain for almost every $t\in[t_1,\infty)$

$$\begin{aligned}H(t)^{2(1-\theta)}\leq C\bigg(&\|\dot u(t)\|^2+\|\nabla E(u(t))\|^2+k(t)\|\eta(t,t)\|^2\\&+\int_0^t(-k')(s)\|\eta(t,s)\|^2\,ds+(1+t)^{-2(1+\delta)(1-\theta)}\bigg).\end{aligned}$$

From this inequality, (7), and the assumption on k, we obtain the following differential inequality for almost every $t \geq t_1$

$$C_4 \frac{d}{dt} H(t) + H(t)^{2(1-\theta)} \leq C(1+t)^{-2(1+\delta)(1-\theta)}. \tag{11}$$

Then we may apply Lemma 4.1 in order to obtain

$$H(t) \leq C(1+t)^{-\gamma}, \quad \text{for every } t \geq t_1 \tag{12}$$

where $\gamma = \min\{\frac{1}{1-2\theta}, 1+\delta\}$. Using again (7), we have $-\frac{d}{dt}H(t) \geq C\|\dot{u}(t)\|^2$, for almost every $t \geq t_1$. Integrating this inequality over $[t, 2t]$ $(t \geq t_1)$, using (12) and the fact that $H(t) \geq 0$, we obtain

$$\int_t^{2t} \|\dot{u}(s)\|^2 \, ds \leq C(1+t)^{-\gamma}.$$

Note that for every $t \in \mathbb{R}^+$, $\int_t^{2t} \|\dot{u}(s)\| \, ds \leq t^{\frac{1}{2}} \left(\int_t^{2t} \|\dot{u}(s)\|^2 \, ds \right)^{\frac{1}{2}}$. It follows that

$$\int_t^{2t} \|\dot{u}(s)\| \, ds \leq C(1+t)^{\frac{1-\gamma}{2}} \quad \text{for every } t \geq t_1.$$

Therefore we obtain

$$\int_t^{\infty} \|\dot{u}(s)\| \, ds \leq \sum_{k=0}^{\infty} \int_{2^k t}^{2^{k+1} t} \|\dot{u}(s)\| \, ds \leq C \sum_{k=0}^{\infty} (2^k t)^{\frac{1-\gamma}{2}} \leq C(1+t)^{\frac{1-\gamma}{2}}.$$

Then for all $t \geq t_1$

$$\|u(t) - \phi\| \leq C \int_t^{\infty} \|\dot{u}(s)\| \, ds \leq C(1+t)^{-\xi}, \quad \text{where } \xi = \min\left\{ \frac{\theta}{1-2\theta}, \frac{\delta}{2} \right\}.$$

Step 2 (Exponential decay). Suppose that $\theta = \frac{1}{2}$ and that g satisfies the exponential growth (G2). Then (11) becomes

$$\frac{d}{dt} H(t) \leq -C_5 H(t) + C_6 e^{-\delta t},$$

where $C_5 = \frac{1}{C_4}$ and C_4 can be chosen large enough to ensure that $C_5 < \delta$. Now let

$$K(t) = H(t) - C_6 e^{-C_5 t} \int_0^t e^{-(\delta - C_5)s} \, ds.$$

Then

$$\frac{d}{dt} K(t) = \frac{d}{dt} H(t) - C_6 e^{-\delta t} + C_5 C_6 e^{-C_5 t} \int_0^t e^{-(\delta - C_5)s} \, ds \leq -C_5 K(t).$$

This yields $K(t) \leq K(0)e^{-C_5 t}$, and therefore

$$H(t) \leq e^{-C_5 t}\big(K(0) + C_6 \int_0^t e^{-(\delta - C_5)s}\, ds\big) \leq Ce^{-C_5 t}. \tag{13}$$

On the other hand, from the inequality (10) (when g satisfies the exponential decay and $\theta_0 = \theta = \frac{1}{2}$) we have for almost every $t \geq t_1$

$$-\frac{d}{dt}H(t)^{\frac{1}{2}} + Ce^{-\frac{\delta t}{2}} \geq C\|\dot{u}(t)\|.$$

Integrating this inequality over the interval $[t, \infty)$ $(t \geq t_1)$, we obtain

$$\|u(t) - \phi\| \leq \int_t^\infty \|\dot{u}(s)\|\, ds \leq CH(t)^{\frac{1}{2}} + Ce^{-\frac{\delta t}{2}}.$$

This inequality together with the inequality (13) implies the claim. □

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