Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 37 (2018), 101–126
DOI: 10.4171/ZAA/1605
© European Mathematical Society

A Grobman–Hartman Theorem for Differential Equations with Piecewise Constant Arguments of Mixed Type

Manuel Pinto and Gonzalo Robledo

Abstract. We obtain sufficient conditions for the existence of a uniformly and Hölder continuous homeomorphism between the solutions of a linear differential system with piecewise constant argument of generalized type and the solutions of a perturbed family. The main tool is a recently introduced definition of exponential dichotomy.

Keywords. Differential equations, piecewise constants arguments, topological equivalence, exponential dichotomy

Mathematics Subject Classification (2010). Primary 34A30, secondary 34D09

1. Introduction

The purpose of this note is to obtain conditions ensuring the topological equivalence between the solutions of the systems with piecewise constant arguments of mixed (i.e., alternately advanced/delayed) type:

$$\dot{x}(t) = A(t)x(t) + A_0(t)x(\gamma(t)) + f(t, x(t), x(\gamma(t))), \tag{1}$$

and

$$\dot{y}(t) = A(t)y(t) + A_0(t)y(\gamma(t)), \tag{2}$$

where $t \mapsto \gamma(t)$ is a piecewise constant function (more details will be given later).

In what follows, we denote respectively by $||\cdot||$ and $|\cdot|$ a matrix and vector norm. Further, we will assume that the $n \times n$ matrix functions $A(\cdot)$ and $A_0(\cdot)$ and $f\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ verify the following properties:

(A1) There exist positive constants M and M_0 such that

$$\sup_{-\infty<t<+\infty} ||A(t)|| \leq M \quad \text{and} \quad \sup_{-\infty<t<+\infty} ||A_0(t)|| \leq M_0,$$

M. Pinto, G. Robledo: Departmento de Matemáticas, Universidad de Chile, Casilla 653 Santiago, Chile; pintoj.uchile@gmail.com; grobledo@uchile.cl

(A2) there exists a positive constant μ such that

$$|f(t,x,y)| \leq \mu \quad \text{for any } (t,x,y) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n,$$

(A3) there exist positive constants ℓ_1 and ℓ_2 such that if $x, x', y, y' \in \mathbb{R}^n$

$$|f(t,x,y) - f(t,x',y')| \leq \ell_1|x - x'| + \ell_2|y - y'| \quad \text{for any } t \in \mathbb{R}.$$

1.1. Differential equations with piecewise constant arguments. The study of systems with piecewise constant arguments begin with the work of Myshkis [23], which considers the integer part $\gamma(t) = [t]$, this case and other variations were usually known as DEPCA (Differential Equations with Piecewise Constant Argument). A generalization was made by Akhmet [1–4], which introduces the DEPCAG (Differential Equations with Piecewise Constant Generalized Argument) by considering two sequences $\{t_i\}_{i\in\mathbb{Z}}$ and $\{\zeta_i\}_{i\in\mathbb{Z}}$, which satisfy:

(B1) $t_i < t_{i+1}$ and $t_i \leq \zeta_i \leq t_{i+1}$ for any $i \in \mathbb{Z}$,
(B2) $t_i \to \pm\infty$ as $i \to \pm\infty$,
(B3) $\gamma(t) = \zeta_i$ for $t \in [t_i, t_{i+1})$,
(B4) $0 < \theta_0 \leq t_{i+1} - t_i \leq \theta$ for any $i \in \mathbb{Z}$.

Several functions $\gamma(t)$ satisfying **(B1)**–**(B4)** have been constructed with the integer part. For example, $\gamma(t) = \alpha h[\frac{t}{\alpha h}]$ induce the sequences $t_k = \zeta_k = k\alpha h$ and $\gamma(t) = m[\frac{t+j}{m}]$ $(m > j > 0)$ induce the sequences $t_k = mk - j$ and $\zeta_k = mk$.

Definition 1.1 (Akhmet and Yilmaz [6, p. 25], Wiener[35, p. 4])**.** A continuous function $u(t)$ is a solution of (1) or (2) if:

(i) The derivative $u'(t)$ exists at each point $t \in \mathbb{R}$ with the possible exception of the points t_i, $i \in \mathbb{Z}$, where the one side derivatives exists;
(ii) The equation is satisfied for $u(t)$ on each interval (t_i, t_{i+1}), and it holds for the right derivative of $u(t)$ at the points t_i.

DEPCAG systems combine properties of continuous and discrete systems. Indeed, the continuity of a solution $t \mapsto u(t)$ implies that the system can be seen as an ODE on (t_i, t_{i+1}) whose solutions at $t = t_{i+1}$ are defined by a recursive relation. In spite that this recursivity plays an important role, we emphasize that it not describe completely the system's behavior. In consequence, the study of topics as: existence and uniqueness of solutions, continuity with respect to initial conditions [13], variation of parameters [4,28], stability [4,34], asymptotic equivalence [3,27], almost periodic solutions [4,12,38] has been fashioned along the classical qualitative theories of ODE and difference equations. Nevertheless, the qualitative theory of DEPCAG is still incomplete due – in part – to the difficulties arised by **(B1)**–**(B4)**. See in [4,16,35] for details.

In particular, the study of system (2) and its properties is more complicated than ODE and difference equations because its transition matrix $Z(t,\tau)$, namely, a matrix function satisfying

$$\frac{\partial Z}{\partial t}(t,\tau) = A(t)Z(t,\tau) + A_0(t)Z(\gamma(t),\tau) \quad \text{and} \quad Z(\tau,\tau) = I \tag{3}$$

can be constructed only under certain conditions (see Section 3 for details).

Finally, DEPCAG systems have been used in applied problems as neural networks [6, 14], control systems [29], population dynamics [20], fisheries [10], numerical approximation of differential equations [16, 18].

1.2. Topological equivalence and Dichotomies. The Hartman–Grobman theorem proves the existence of a local homeomorphism $h\colon U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ between the solutions of the nonlinear autonomous system $y' = g(y)$, around an hyperbolic equilibrium $x^* \in U$ and the linear one $x' = Dg(x^*)x$. The concept of topological equivalence was introduced by Palmer [24] in order to generalize this theorem to a global and nonautonomous framework.

Definition 1.2. The systems (1) and (2) are topologically equivalent if there exists a function $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with the properties

(i) For each fixed $t \in \mathbb{R}$, $u \mapsto H(t,u)$ is an homeomorphism of $\mathbb{R}^n$,
(ii) $H(t,u) - u$ is bounded in $\mathbb{R} \times \mathbb{R}^n$,
(iii) if $x(t)$ is a solution of (1), then $H[t,x(t)]$ is a solution of (2),

In addition, the function $L(t,u) = H^{-1}(t,u)$ has properties (i)–(iii) also.

The concepts of strongly and Hölder topologically equivalence were introduced by Shi and Xiong [32], who realized that, in several cases of topological equivalence, the maps $u \mapsto H(t,u)$ and $u \mapsto L(t,u)$ could have properties stronger than continuity.

Definition 1.3. The systems (1) and (2) are:

(a) Strongly topologically equivalent if they are topologically equivalent and H and L are uniformly continuous for all t.
(b) Hölder topologically equivalent if the are topologically equivalent and there exists constants $C_1 > 1$, $D_1 > 1$, $C_2 \in (0,1)$ and $D_2 \in (0,1)$ such that:

$$|H(t,\xi) - H(t,\xi')| \leq C_1 |\xi - \xi'|^{C_2} \quad \text{and} \quad |L(t,\nu) - L(t,\nu')| \leq D_1 |\nu - \nu'|^{D_2} \tag{4}$$

for any couple (ξ,ξ') and (ν,ν') verifying $|\xi - \xi'| < 1$ and $|\nu - \nu'| < 1$.

Contrary to the autonomous case, it is worth to emphasize that it does not exists an ubiquitous definition of hyperbolicity in the non-autonomous framework and the property of *dichotomy* plays a key role, being the exponential dichotomy the most usual one. Let us recall the following definition in a DEPCAG framework:

Definition 1.4 (Akhmet [5,6], Pinto et al. [15])**.** The linear DEPCAG (2) has an α-exponential dichotomy on $\mathbb{R}$ if there exists a projection P, constants $K \geq 1$ and $\alpha > 0$, such that the transition matrix $Z(t,s)$ stated in (3) verifies

$$||Z_p(t,s)|| \leq Ke^{-\alpha|t-s|} \tag{5}$$

where $Z_p(t,s)$ is defined by

$$Z_p(t,s) = \begin{cases} Z(t,0)PZ(0,s) & \text{if } t \geq s \\ -Z(t,0)\{I-P\}Z(0,s) & \text{if } s > t. \end{cases} \tag{6}$$

The dichotomy property plays a key role in the topological equivalence theory since (5),(6) allows an explicit construction of the homeomorphisms H and L of Definition 1.2 as done in [9, 19, 24, 32, 37] for ODE systems. This idea has been extended for difference equations [8,11,22,25], impulsive equations [21] and time scales [7, 30, 36]. Nevertheless, the generalization to functional differential equations face several difficulties, mainly due to the fact that some solutions do not have backward continuation. Some preliminar results have been obtained for autonomous delay equations in [17, 33] and for some nonautonomous cases in [31]. As DEPCAG can be seen as a particular case of functional diferential equations and a set of conditions ensuring backward and forward continuation of the solutions has been deduced (see Section 3 for details), we are able to generalize the topological equivalence results by following a dichotomic approach.

1.3. Outline. The main results are stated in Section 2 and their proofs are developed from the Sections 3 to 6. The Section 7 is devoted to some byproducts and an application to Liapunov's stability.

2. Main results

Before state our main results, we will asume that

(C) There exist $\nu^+ > 0$ and $\nu^- > 0$ such that $A(t)$ and $A_0(t)$ satisfy:

$$\sup_{k\in\mathbb{Z}} \exp\Big(\int_{t_k}^{\zeta_k} |A(s)|\,ds\Big) \int_{t_k}^{\zeta_k} |A_0(s)|\,ds \leq \nu^+ < 1, \tag{7}$$

and

$$\sup_{k\in\mathbb{Z}} \exp\Big(\int_{\zeta_k}^{t_{k+1}} |A(s)|\,ds\Big) \int_{\zeta_k}^{t_{k+1}} |A_0(s)|\,ds \leq \nu^- < 1. \tag{8}$$

Notice that **(A1)** and **(B4)** imply that

$$\rho(A) \triangleq \sup_{k\in\mathbb{Z}} \exp\Big(\int_{t_k}^{t_{k+1}} |A(s)|\,ds\Big) < +\infty. \tag{9}$$

Theorem 2.1. *If* (2) *has a transition matrix* $Z(t,0)$ *satisfying the exponential dichotomy* (5), *conditions* **(A)**, **(B)** *and* **(C)** *are satisfied and*

$$2(\ell_1+\ell_2)K\rho(A)e^{\alpha\theta}<\alpha, \tag{10}$$

$$F_1(\theta)(M_0+\ell_2)\theta=v<1, \quad \textit{with} \quad F_1(\theta)=\frac{e^{(M+\ell_1)\theta}-1}{(M+\ell_1)\theta}, \tag{11}$$

$$F_0(\theta)M_0\theta=\tilde{v}<1, \quad \textit{with} \quad F_0(\theta)=\frac{e^{M\theta}-1}{M\theta}, \tag{12}$$

then (1) *and* (2) *are strongly topologically equivalent.*

Theorem 2.2. *If* (2) *has a transition matrix* $Z(t,0)$ *satisfying the exponential dichotomy* (5), *conditions* **(A)**, **(B)**, **(C)**, (10)–(12) *are satisfied and*

$$\alpha<M+\min\Big\{\ell_1+\frac{M_0+\ell_2}{1-v}e^{(M+\ell_1)\theta},\frac{M_0}{1-\tilde{v}}e^{M\theta}\Big\}, \tag{13}$$

then (1) *and* (2) *are Hölder strongly topologically equivalent.*

Remark 2.3. We emphasize in the difficulty – at least in the framework of the proofs of Theorems 2.1 and 2.2 – to obtain conditions ensuring that H and L are Lipschitz homeomorphisms for any t.

Remark 2.4. The inequalities (11) and (12) imply the existence and uniqueness of the solutions of (1) and (2). This fact is proved in [13, Section 2]. In addition, (11) and (12) are always satisfied for small values of θ (note that $F_i(\theta)\to 1$ when $\theta\to 0$ for $i=1,2$). Finally, it will be useful to denote by $t\mapsto x(t,\tau,\xi)$ as the unique solution of (1) passing through ξ at $t=\tau$. By uniqueness of solutions of (1), we know that

$$x\big(s,t,x(t,\tau,\xi)\big)=x\big(s,\tau,\xi\big). \tag{14}$$

The inequality (13) is extremely important to prove the Hölder continuity. Note that it is always satisfied when $\alpha<M$.

2.1. Contribution of this work. To the best of our knowledge, the unique result in a DEPCA framework has been obtained by G. Papaschinopoulos [26, Proposition 1] by introducing an ad-hoc definition of exponential dichotomy restricted to $\gamma(t)=[t]$. We generalize this result in several ways:

i) Theorems 2.1 and 2.2 consider a generic piecewise constant argument including the particular delayed case $\gamma(t)=[t]$.
ii) We obtain results sharper than topological equivalence, namely, strongly and Hölder topological equivalence.

iii) We use a variation of parameters formula on $\mathbb{R}$, which combined with exponential dichotomy allow a global treatment (i.e., not restricted to intervals).

iv) Our results don't need to assume that $x' = A(t)x$ has the classical exponential dichotomy [7,24,32] and allows limit cases as $A(t) = 0$ for any $t \in \mathbb{R}$.

v) The smallness of $A_0(\cdot)$ is not always necessary as in [26], for example, a threshold between θ and M_0 ensuring $v < 1$ can be constructed.

2.2. Structure of the proofs. In spite that our proofs are inspired in the classical ODE context, we point out that we need to employ several tools developed in the recent years for the study of DEPCA systems. The proofs involve several steps:

a) Section 3 recalls results of linear DEPCAG systems stated in the literature [4–6,15,28], which are included in order to make the article self contained.

b) Section 4 introduces a result of continuity of the solutions of (1) and (2) with respect to the initial conditions.

c) By using the results of Sections 2 and 3, we construct a biunivocal correspondence between the solutions of (1) and (2) in Section 5. To that end, we are inspired in the approach developed by Palmer [24].

d) In the Section 6, we prove the continuity properties of the biunivocal correspondence stated above. In this step, we follow and adapt some ideas of Shi et al. [32].

3. Linear systems: some results

Let $\Phi(t)$ be the Cauchy matrix of

$$x' = A(t)x. \tag{15}$$

We will assume that $\Phi(0) = I$. The transition matrix will be denoted by $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$. The following $n \times n$ matrices are introduced in [4,28]:

$$J(t,\tau) = I + \int_{\tau}^{t} \Phi(\tau,s)A_0(s)\,ds, \tag{16}$$

$$E(t,\tau) = \Phi(t,\tau) + \int_{\tau}^{t} \Phi(t,s)A_0(s)\,ds = \Phi(t,\tau)J(t,\tau). \tag{17}$$

Given a set of $n \times n$ matrices $\mathcal{Q}_k$ $(k = 1,\dots,m)$, we will consider the product in the backward and forward sense as follows:

$$\prod_{k=1}^{\leftarrow m} \mathcal{Q}_k = \begin{cases} \mathcal{Q}_m \cdots \mathcal{Q}_1 & \text{if } m \geq 1 \\ I & \text{if } m < 1 \end{cases} \quad \text{and} \quad \prod_{k=1}^{\rightarrow m} \mathcal{Q}_k = \begin{cases} \mathcal{Q}_1 \cdots \mathcal{Q}_m & \text{if } m \geq 1 \\ I & \text{if } m < 1 \end{cases}$$

For any $k \in \mathbb{Z}$, we define $I_k = [t_k, t_{k+1})$ and for any $t \in \mathbb{R}$, we define $i(t) \in \mathbb{Z}$ as the unique integer such that $t \in I_i$.

Lemma 3.1. *For any s and t, it follows that*

$$|\gamma(s) - t| \leq \theta + |t - s|, \tag{18}$$

where θ is the same stated in **(B4)**.

Proof. As $s \in [t_{i(s)}, t_{i(s)+1})$, it follows that $\gamma(s) = \zeta_{i(s)}$. Now **(B1)** implies that

$$t_{i(s)} - t_{i(s)+1} \leq \zeta_{i(s)} - t_{i(s)+1} < \gamma(s) - s < \zeta_{i(s)} - t_{i(s)} < t_{i(s)+1} - t_{i(s)}$$

and **(B4)** implies that $|\gamma(s) - s| \leq \theta$. Finally, (18) follows from $|\gamma(s) - t| \leq |\gamma(s) - s| + |s - t|$. □

An important consequence of **(C)** is the following result:

Lemma 3.2 ([28, Lemma 4.3])**.** *If* (7) *is verified, it follows that*

$$|\Phi(t, s)| \leq \rho(A) \quad \text{for any} \quad t, s \in I_i.$$

and $J(t, s)$ is nonsingular for any $t, s \in I_i$.

A distinguished feature of DEPCAG systems is that their solutions could be noncontinuable in several cases. In this context, **(C)** is introduced in [28] in order to ensure the continuability of the solutions of (2) to $(-\infty, +\infty)$. Furthermore, **(C)** and Lemma 3.2 imply that $J(t, s)$ and $E(t, s)$ are nonsingular for any $t, s \in I_i$, which allow to construct the transition matrix for (2) and to derive the variation of parameters formula.

Proposition 3.3 ([28, p. 239])**.** *For any $t \in I_j, \tau \in I_i$, the solution of* (2) *with $z(\tau) = \xi$ is defined by*

$$z(t) = Z(t, \tau)\xi,$$

where $Z(t, \tau)$ is defined by

$$\begin{aligned} Z(t, \tau) = E(t, \zeta_j)E(t_j, \zeta_j)^{-1} \prod_{k=i+2}^{\leftarrow j} E(t_k, \gamma(t_{k-1}))E(t_{k-1}, \gamma(t_{k-1}))^{-1} \\ \times E(t_{i+1}, \gamma(\tau))E(\tau, \gamma(\tau))^{-1}, \end{aligned} \tag{19}$$

when $t > \tau$ and by

$$\begin{aligned} Z(t, \tau) = E(t, \zeta_j)E(t_j, \zeta_j)^{-1} \prod_{k=i+2}^{\rightarrow j} E(t_k, \gamma(t_k))E(t_k, \gamma(t_{k-1}))^{-1} \\ \times E(t_i, \gamma(\tau))E(\tau, \gamma(\tau))^{-1}, \end{aligned} \tag{20}$$

when $t < \tau$.

Remark 3.4. A consequence of Proposition 3.3 is that the $Z(\cdot,\cdot)$ verifies

$$Z(t,\tau)Z(\tau,s) = Z(t,s) \quad \text{and} \quad Z(t,s) = Z(s,t)^{-1}. \tag{21}$$

In addition, by using the facts

$$E(\tau,\tau) = I \quad \text{and} \quad \frac{\partial E}{\partial t}(t,\tau) = A(t)E(t,\tau) + A_0(t)$$

combined with Proposition 3.3, we can deduce that (3) is verified.

Proposition 3.5 ([28, Theorem 3.1])**.** *For any $j > i$, $t \in I_j$ and $\tau \in I_i$, the solution of*

$$\dot{x}(t) = A(t)x(t) + A_0(t)x(\gamma(t)) + g(t), \tag{22}$$

with $x(\tau) = \xi$ is defined by

$$\begin{aligned} x(t) = {} & Z(t,\tau)\xi + \int_{\tau}^{\zeta_i} Z(t,\tau)\Phi(\tau,s)g(s)\,ds + \sum_{r=i+1}^{j}\int_{t_r}^{\zeta_r} Z(t,t_r)\Phi(t_r,s)g(s)\,ds \\ & + \sum_{r=i}^{j-1}\int_{\zeta_r}^{t_{r+1}} Z(t,t_{r+1})\Phi(t_{r+1},s)g(s)\,ds + \operatorname{Sgn}(t-\zeta_j)\int_{\min\{\zeta_j,t\}}^{\max\{\zeta_j,t\}} \Phi(t,s)g(s)\,ds, \end{aligned}$$

when $\tau \in [t_i,\zeta_i)$.

Definition 3.6. Given $t \in (\zeta_j, t_{j+1})$ and $Z_p(t,\tau)$ introduced in (6), let us define the Green function corresponding to (2) in the interval $(-\infty,\infty)$

$$\widetilde{G}(t,s) = \begin{cases} Z_p(t,t_r)\Phi(t_r,s) & \text{if } s \in [t_r,\zeta_r) \quad \text{for any } r \in \mathbb{Z}, \\ Z_p(t,t_{r+1})\Phi(t_{r+1},s) & \text{if } s \in [\zeta_r,t_{r+1}) \quad \text{for any } r \in \mathbb{Z}\setminus\{j\}, \\ \Phi(t,s) & \text{if } s \in [\zeta_j,t), \\ 0 & \text{if } s \in [t,t_{j+1}), \end{cases}$$

and if $t \in [t_j,\zeta_j]$

$$\widetilde{G}(t,s) = \begin{cases} Z_p(t,t_r)\Phi(t_r,s) & \text{if } s \in [t_r,\zeta_r) \quad \text{for any } r \in \mathbb{Z}\setminus\{j\}, \\ Z_p(t,t_{r+1})\Phi(t_{r+1},s) & \text{if } s \in [\zeta_r,t_{r+1}) \quad \text{for any } r \in \mathbb{Z}, \\ 0 & \text{if } s \in [t_j,t), \\ -\Phi(t,s) & \text{if } s \in [t,\zeta_j), \end{cases}$$

Note that $\widetilde{G}$ takes into account delayed and advanced intervals.

Proposition 3.7. *If* (2) *has an α-exponential dichotomy* (5), *then* $\widetilde{G}$ *satisfies*

$$|\widetilde{G}(t,s)| \le K\rho^* e^{-\alpha|t-s|}, \quad \textit{where} \quad \rho^* = \rho(A)e^{\alpha\theta}. \tag{23}$$

By using Propositions 3.5 and 3.7 combined with Definition 3.6, the following result has been proved by Akhmet & Yilmaz [5,6] and Pinto et al. [15]:

Proposition 3.8. *If DEPCAG* (2) *has an α-exponential dichotomy and the series*

$$\sum_{r=-\infty}^{0} PZ(0,t_r)\int_{t_r}^{\zeta_r} \Phi(t_r,s)\,ds, \qquad \sum_{r=-\infty}^{0} PZ(0,t_{r+1})\int_{\zeta_r}^{t_{r+1}} \Phi(t_{r+1},s)\,ds, \tag{24}$$

and

$$\sum_{r=0}^{+\infty}(I-P)Z(0,t_r)\int_{t_r}^{\zeta_r}\Phi(t_r,s)\,ds, \quad \sum_{r=0}^{+\infty}(I-P)Z(0,t_{r+1})\int_{\zeta_r}^{t_{r+1}}\Phi(t_{r+1},s)\,ds, \tag{25}$$

are absolutely convergent, then for each bounded function $t \mapsto g(t)$, the system (22) *has a unique solution bounded on $\mathbb{R}$, defined by*

$$x_g^*(t) = \int_{-\infty}^{\infty} \widetilde{G}(t,s)g(s)\,ds$$

and the map $g \mapsto x_g^*$ is Lipschitz satisfying $|x_g^*|_\infty \le \frac{2K\rho^*}{\alpha|g|_\infty}$.

Remark 3.9. The convergence of series (24),(25) can be ensured by imposing additional properties to the sequence $\{t_r\}_r$. For example, by α-exponential dichotomy (5) combined with $Z(0,0)=I$ and Lemma 3.2, we conclude that

$$\sum_{r=k}^{+\infty}\left|(I-P)Z(0,t_{r+1})\int_{\zeta_r}^{t_{r+1}} \Phi(t_{r+1},s)\,ds\right| \le K\rho(A)\sum_{r=k}^{+\infty} e^{-\alpha|t_{r+1}|},$$

and the second series of (25) converges if the series $S_n = \sum_{r=k}^{n} e^{-\alpha|t_{r+1}|}$ $(n>k)$ is convergent. Now, the convergence of S_n can be ensured in several cases. For example, by **(B4)** there exists $\theta_0 > 0$ such that

$$\theta_0 \le t_{r+1} - t_r \quad \text{for any } r \in \mathbb{Z},$$

we have that the series S_n is dominated by a geometric one.

4. Continuity with respect to initial conditions

The following result generalizes Gronwall's inequality to DEPCAG:

Proposition 4.1 ([13, Lemma 2.1]). *Let* $u,\tilde{\eta}_i\colon \mathbb{R} \to [0,+\infty)$ $i = 1,2$ *be continuous functions and* $\tilde{C} > 0$. *Suppose that for all* $t \geq \tau$, *the inequality*

$$u(t) \leq \tilde{C} + \int_{\tau}^{t} \{\tilde{\eta}_1(s)u(s) + \tilde{\eta}_2(s)u(\gamma(s))\}\, ds$$

holds. If

$$w = \sup_{i\in\mathbb{N}} \int_{t_i}^{\zeta_i} \tilde{\eta}_2(s)e^{\int_s^{\zeta_i}\tilde{\eta}_1(r)\,dr}\, ds < 1,$$

then for any $t \geq \tau$ *it follows that*

$$u(t) \leq \tilde{C}\exp\Big(\int_{\tau}^{t} \tilde{\eta}_1(s)\, ds + \frac{1}{1-w}\int_{\tau}^{t} \Big[\tilde{\eta}_2(s)e^{\int_{t_i(s)}^{\gamma(s)}\tilde{\eta}_1(r)\,dr}\Big]\, ds\Big).$$

Similarly as in the ODE context, Gronwall's inequality is a key tool in the proof of continuity with respect to the initial conditions:

Lemma 4.2. *Let* $t \mapsto x(t,\tau,\xi)$ *and* $t \mapsto x(t,\tau,\xi')$ *be the solutions of* (1) *passing respectively through* ξ *and* ξ' *at* $t = \tau$. *If* (11) *is verified, then it follows that*

$$|x(t,\tau,\xi') - x(t,\tau,\xi)| \leq |\xi - \xi'|e^{p_1|t-\tau|} \tag{26}$$

where p_1 *is defined by*

$$p_1 = \eta_1 + \frac{\eta_2 e^{\eta_1\theta}}{1-v} \quad \text{with} \quad \eta_1 = M + \ell_1, \quad \eta_2 = M_0 + \ell_2 \tag{27}$$

and $v \in [0,1)$ *is defined by* (11).

Proof. Without loss of generality, we will assume that $t > \tau$, the case corresponding to $t < \tau$ can be proved similary and is left to the reader.

Firstly, let us consider the case $t_i < \tau < t < t_{i+1}$ for some $i \in \mathbb{Z}$, then notice that **(A1)** and **(A3)** imply

$$\begin{aligned}&|x(t,\tau,\xi') - x(t,\tau,\xi)| \\ &\leq |\xi - \xi'| + \int_{\tau}^{t} \big\{\eta_1|x(s,\tau,\xi') - x(s,\tau,\xi)| + \eta_2|x(\gamma(s),\tau,\xi') - x(\gamma(s),\tau,\xi)|\big\}\, ds.\end{aligned}$$

As (11) implies that $\int_{t_i}^{\zeta_i} \eta_2 e^{\eta_1(\zeta_i - s)}\, ds = \frac{\eta_2}{\eta_1}\big(e^{\eta_1(\zeta_i - t_i)} - 1\big) \leq v < 1$, then Proposition 4.1 combined with $\zeta_i - t_i \leq \theta$ for any $i \in \mathbb{Z}$ imply (26) for any $t \in (\tau, t_{i+1}]$. In particular, at $t = t_{i+1}$, we have that

$$|x(t_{i+1},\tau,\xi') - x(t_{i+1},\tau,\xi)| \leq |\xi' - \xi|\exp\left(\left\{\eta_1 + \frac{\eta_2 e^{\eta_1\theta}}{1-v}\right\}(t_{i+1} - \tau)\right). \tag{28}$$

Let us consider $t \in (t_{i+1}, t_{i+2}]$ and notice that uniqueness of the solutions imply

$$x(t, t_{i+1}, x(t_{i+1}, \tau, \xi)) = x(t, \tau, \xi), \tag{29}$$

$$x(\gamma(t), t_{i+1}, x(t_{i+1}, \tau, \xi)) = x(\gamma(t), \tau, \xi). \tag{30}$$

As in the previous step, we can observe that

$$\begin{aligned} &|x(t,\tau,\xi') - x(t,\tau,\xi)| \\ &\leq |x(t_{i+1},\tau,\xi') - x(t_{i+1},\tau,\xi)| \\ &\quad + \int_{t_{i+1}}^{t} \left\{\eta_1 |x(s,\tau,\xi') - x(s,\tau,\xi)| + \eta_2 |x(\gamma(s),\tau,\xi') - x(\gamma(s),\tau,\xi)|\right\} ds \end{aligned} \tag{31}$$

for any $t \in (t_{i+1}, t_{i+2}]$. By applying Lemma 4.2 to (31) combined with (27)–(29), we can deduce that

$$|x(t,\tau,\xi') - x(t,\tau,\xi)| \leq |x(t_{i+1},\tau,\xi') - x(t_{i+1},\tau,\xi)| e^{p_1(t-t_{i+1})} \leq |\xi' - \xi| e^{p_1(t-\tau)}$$

for any $t \in (t_{i+1}, t_{i+2}]$ and we can verify in a recursive way see that (26) follows for any $t \geq \tau$. □

The proof of the next result is similar and is left to the reader.

Lemma 4.3. *Let $t \mapsto y(t,\tau,\nu)$ and $t \mapsto y(t,\tau,\nu')$ be the solutions of* (2) *passing respectively through ν and ν' at $t = \tau$. If* (12) *is satisfied, then:*

$$|y(t,\tau,\nu') - y(t,\tau,\nu)| \leq |\nu - \nu'| e^{p_2|t-\tau|} \quad \textit{with} \quad p_2 = M + \frac{M_0 e^{M\theta}}{1 - \tilde{v}}, \tag{32}$$

where $\tilde{v} \in [0,1)$ is defined by (12).

5. Correspondence between bounded solutions

Lemma 5.1. *For any solution $x(t,\tau,\xi)$ of* (1) *passing through ξ at $t = \tau$, there exists a unique bounded solution $t \mapsto \chi(t;(\tau,\xi))$ of*

$$\dot{z}(t) = A(t)z(t) + A_0(t)z(\gamma(t)) - f(t, x(t,\tau,\xi), x(\gamma(t),\tau,\xi)). \tag{33}$$

Proof. By Proposition 3.8 with $g(t) = -f(t, x(t,\tau,\xi), x(\gamma(t),\tau,\xi))$, we have that

$$\chi(t;(\tau,\xi)) = -\int_{-\infty}^{\infty} \widetilde{G}(t,s) f(s, x(s,\tau,\xi), x(\gamma(s),\tau,\xi))\, ds$$

is the unique bounded solution of (33). □

Remark 5.2. By uniqueness of solutions of (1) and equation (14) with $s = t$ and $s = \gamma(t)$, we know that

$$x\big(t, t, x(t, \tau, \xi)\big) = x\big(t, \tau, \xi\big) \quad \text{and} \quad x\big(\gamma(t), t, x(t, \tau, \xi)\big) = x\big(\gamma(t), \tau, \xi\big),$$

this fact implies that system (33) can be written as

$$\dot{z}(t) = A(t)z(t) + A_0(t)z(\gamma(t)) - f(t, x(t, t, x(t, \tau, \xi)), x(\gamma(t), t, x(t, \tau, \xi)))$$

and Lemma 5.1 implies the identity

$$\chi(t; (\tau, \xi)) = \chi(t; (t, x(t, \tau, \xi))). \tag{34}$$

Lemma 5.3. *For any solution $y(t, \tau, \nu)$ of (2) passing through ν at $t = \tau$, there exists a unique bounded solution $t \mapsto \vartheta(t; (\tau, \nu))$ of*

$$\dot{w}(t) = A(t)w(t) + A_0(t)w(\gamma(t)) + f(t, y(t, \tau, \nu) + w(t), y(\gamma(t), \tau, \nu) + w(\gamma(t))). \tag{35}$$

Proof. Let BC be the Banach space of bounded and continuous functions $\varphi\colon \mathbb{R} \to \mathbb{R}^n$ with supremun norm. By Proposition 3.8, we know that the map $\Gamma\colon BC \to BC$:

$$\Gamma\varphi(t) = \int_{-\infty}^{\infty} \widetilde{G}(t, s) f(s, y(s, \tau, \nu) + \varphi(s), y(\gamma(s), \tau, \nu) + \varphi(\gamma(s)))\, ds,$$

is well defined. Now, notice that **(A3)** and (23) imply

$$|\Gamma\varphi(t) - \Gamma\phi(t)| \leq \frac{2K\rho^*}{\alpha}(\ell_1 + \ell_2)||\varphi - \phi||,$$

and (10) implies that Γ is a contraction, having a unique fixed point satisfying

$$\vartheta(t; (\tau, \nu)) = \int_{\mathbb{R}} \widetilde{G}(t, s) f\big(s, y(s, \tau, \nu) + \vartheta(s; (\tau, \nu)), y(\gamma(s), \tau, \nu) + \vartheta(\gamma(s); (\tau, \nu))\big)\, ds \tag{36}$$

and the reader can easily verify that is a bounded solution of (35). □

Remark 5.4. Similarly as in Remark 5.2, the reader can verify that

$$\vartheta(t; (\tau, \nu)) = \vartheta(t; (t, y(t, \tau, \nu))). \tag{37}$$

Lemma 5.5. *There exists a unique function $H\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, satisfying:*

(i) *$H(t, x) - x$ is bounded in $\mathbb{R} \times \mathbb{R}^n$,*

(ii) *For any solution $t \mapsto x(t)$ of (1), then $t \mapsto H[t, x(t)]$ is a solution of (2) verifying*

$$|H[t, x(t)] - x(t)| \leq 2\mu K \rho^* \alpha^{-1} \tag{38}$$

Proof. The proof will be decomposed in several steps.

Step 1. *Existence of* H*:* Let us define the function $H\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ as follows

$$H(t,\xi) = \xi + \chi(t;(t,\xi)) = \xi - \int_{-\infty}^{\infty} \widetilde{G}(t,s) f(s, x(s,t,\xi), x(\gamma(s),t,\xi))\, ds \quad (39)$$

and **(A2)** combined with (23) imply $|H(t,\xi)-\xi| \leq 2\mu K\rho^*\alpha^{-1}$.

By replacing (t,ξ) by $(t, x(t,\tau,\xi))$ in (39), we have that

$$H[t, x(t,\tau,\xi)] = x(t,\tau,\xi) + \chi(t;(t, x(t,\tau,\xi))).$$

Now, by (34), we have

$$H[t, x(t,\tau,\xi)] = x(t,\tau,\xi) + \chi(t;(\tau,\xi)) \quad (40)$$

or equivalently

$$H[t, x(t,\tau,\xi)] = x(t,\tau,\xi) - \int_{\mathbb{R}} \widetilde{G}(t,s) f(s, x(s,\tau,\xi), x(\gamma(s),\tau,\xi))\, ds. \quad (41)$$

Finally, it is easy to verify that $t \mapsto H[t, x(t,\tau,\xi)]$ is solution of (2).

Step 2. *Uniqueness of* H*:* Let us suppose that there exists another map $\widetilde{H}$ satisfying properties (i) and (ii), this implies that $\widetilde{H}[t, x(t,\tau,\xi)]$ is solution of (2) and

$$\hat{z}(t,\xi) = \widetilde{H}[t, x(t,\tau,\xi)] - x(t,\tau,\xi)$$

is a bounded solution of (33). Nevertheless, as (33) has a unique bounded solution, we can conclude that $\hat{z}(t) = \chi(t;(\tau,\xi))$ and (40) implies that

$$\widetilde{H}[t, x(t,\tau,\xi)] = x(t,\tau,\xi) + \chi(t;(\tau,\xi)) = H[t, x(t,\tau,\xi)]. \qquad \square$$

Lemma 5.6. *There exists a unique function* $L\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$*, satisfying:*

(i) $L(t,y) - y$ *is bounded in* $\mathbb{R}\times\mathbb{R}^n$*,*

(ii) *For any solution* $t\mapsto y(t)$ *of* (2)*, we have that* $t\mapsto L[t,y(t)]$ *is a solution of* (1) *verifying*

$$|L[t,y(t)] - y(t)| \leq 2\mu K\rho^*\alpha^{-1}. \quad (42)$$

Proof. The existence and uniqueness of the function L satisfying (i),(ii) can be proved in a similar way. Indeed, L is defined by

$$L(t,\nu) = \nu + \vartheta(t;(t,\nu)),$$

where

$$\vartheta(t;(t,\nu)) = \int_{\mathbb{R}} \widetilde{G}(t,s) f(s, y(s,t,\nu) + \vartheta(s;(t,\nu)), y(\gamma(s),t,\nu) + \vartheta(\gamma(s);(t,\nu)))\, ds.$$

As before, by using (37), for $y(t) = y(t, \tau, \nu)$ we can define

$$L[t, y(t)] = y(t, \tau, \nu) + \vartheta(t; (t, y(t, \tau, \nu))) = y(t, \tau, \nu) + \vartheta(t; (\tau, \nu)). \tag{43}$$

Finally, (42) can be deduced by describing $L[t, y(t)]$ as follows

$$L[t, y(t)] = y(t) + \int_{-\infty}^{\infty} \widetilde{G}(t, s) f(s, L[s, y(s)], L[\gamma(s), y(\gamma(s))])\, ds. \tag{44}$$

This finishes the proof. □

Lemma 5.7. *For any solution $x(t)$ of* (1) *and $y(t)$ of* (2)*, it follows that*

$$L[t, H[t, x(t)]] = x(t) \quad \text{and} \quad H[t, L[t, y(t)]] = y(t) \quad \text{for any fixed } t.$$

Proof. We prove only the first identity, the other one can be deduced similarly.

Let $t \mapsto x(t) = x(t, \tau, \xi)$ be a solution of (1). By using Lemma 5.5, we know that $H[t, x(t)]$ is solution of (2). Moreover, by Lemma 5.6, we can see that $t \mapsto J[t, x(t)] = L[t, H[t, x(t)]]$ is solution of (1). Notice that

$$J[t, x(t)] = H[t, x(t)] + \vartheta(t; (t, H[t, x(t)]))$$

where $t \mapsto \vartheta(t; (t, H[t, x(t)]))$ is the unique bounded solution of the system

$$\dot{w}(t) = A(t)w(t) + A_0(t)w(\gamma(t)) + f(t, H[t, x(t)] + w(t), H[\gamma(t), x(\gamma(t))] + w(\gamma(t))).$$

By using Lemma 5.6 with $H[t, x(t)]$ instead of $y(t)$, we have that

$$J[t, x(t)] = H[t, x(t)] + \int_{-\infty}^{\infty} \widetilde{G}(t, s) f(s, J[s, x(s)], J[\gamma(s), x(\gamma(s))])\, ds.$$

Upon inserting (41) in the identity above, we have that

$$J[t, x(t)] - x(t) = \int_{\mathbb{R}} \widetilde{G}(t, s) \Big\{ f\big(s, J[s, x(s)], J[\gamma(s), x(\gamma(s))]\big) - f\big(s, x(s), x(\gamma(s))\big) \Big\} ds,$$

which implies the inequality

$$|J[t, x(t)] - x(t)| \leq \frac{2K\rho^*}{\alpha} \big(\ell_1 + \ell_2\big) |J[\cdot, x(\cdot)] - x(\cdot)|_\infty$$

and (10) implies that $J[t, x(t)] = L[t, H[t, x(t)]] = x(t)$. □

Lemma 5.8. *For any fixed t and any couple $(\xi, \nu) \in \mathbb{R}^n \times \mathbb{R}^n$, it follows that*

$$L(t, H(t, \xi)) = \xi \quad \text{and} \quad H(t, L(t, \nu)) = \nu. \tag{45}$$

Proof. By using Lemma 5.7, we have that

$$L[t, H[x(t, \tau, \xi)] = x(t, \tau, \xi) \quad \text{for any } t \in \mathbb{R}.$$

Now, if we consider the particular case $\tau = t$, we obtain the first identity of (45). The second one can be deduced similarly. □

Remark 5.9. The maps $\xi \mapsto H(t, \xi)$ and $\nu \mapsto L(t, \nu)$ satisfy properties (ii),(iii) of Definition 1.2, which follows from Lemmas 5.5–5.7. In addition, Lemma 5.8 says that $u \mapsto L(t, u) = H^{-1}(t, u)$ for any $t \in \mathbb{R}$. In consequence, the last step is to prove the uniform continuity of the maps, which will be made in the next section.

6. Proof of main results

6.1. Proof of Theorem 2.1. We only have to prove that the maps $\xi \mapsto H(t, \xi)$ and $\nu \mapsto L(t, \nu)$ are uniformly continuous.

Lemma 6.1. *The map $\xi \to H(t, \xi)$ is uniformly continuous for any t.*

Proof. As the identity is uniformly continuous, we only need to prove that the map $\xi \to \chi(t; (t, \xi))$ is uniformly continuous.

Let ξ and ξ' be two initial conditions of (1). Notice that (39) implies

$$\begin{aligned}
&\chi(t; (t, \xi)) - \chi(t; (t, \xi')) \\
&= -\int_{-\infty}^{t} \widetilde{G}(t, s)\{f(s, x(s, t, \xi), x(\gamma(s), t, \xi)) - f(s, x(s, t, \xi'), x(\gamma(s), t, \xi'))\} ds \\
&\quad - \int_{t}^{\infty} \widetilde{G}(t, s)\{f(s, x(s, t, \xi), x(\gamma(s), t, \xi)) - f(s, x(s, t, \xi'), x(\gamma(s), t, \xi'))\} ds \\
&= -\, I_1 + I_2.
\end{aligned} \tag{46}$$

Given a positive constant L, we divide I_1 and I_2 as follows:

$$I_1 = \int_{-\infty}^{t-L} \cdots + \int_{t-L}^{t} \cdots = I_{11} + I_{12} \quad \text{and} \quad I_2 = \int_{t}^{t+L} \cdots + \int_{t+L}^{\infty} \cdots = I_{21} + I_{22}.$$

By using **(A2)** combined with Proposition 3.7, we can see that the integrals I_{11} and I_{22} are always finite since

$$|I_{11}| \leq \frac{2K\mu\rho^*}{\alpha} e^{-\alpha L} \quad \text{and} \quad |I_{22}| \leq \frac{2K\mu\rho^*}{\alpha} e^{-\alpha L}.$$

Now, by **(A3)** and Proposition 3.7, we have that

$$\begin{aligned}|I_{12}| &\leq \int_{t-L}^{t} K\rho^* e^{-\alpha(t-s)}\ell_1|x(s,t,\xi)-x(s,t,\xi')|\,ds \\ &\quad + \int_{t-L}^{t} K\rho^* e^{-\alpha(t-s)}\ell_2|x(\gamma(s),t,\xi)-x(\gamma(s),t,\xi')|\,ds \\ &\leq \int_{0}^{L} K\rho^* e^{-\alpha u}\ell_1|x(t-u,t,\xi)-x(t-u,t,\xi')|\,du \\ &\quad + \int_{0}^{L} K\rho^* e^{-\alpha u}\ell_2|x(\gamma(t-u),t,\xi)-x(\gamma(t-u),t,\xi')|\,du.\end{aligned}$$

On the other hand, by Lemma 4.2, we have that

$$0 \leq |x(t-u,t,\xi)-x(t-u,t,\xi')| \leq |\xi-\xi'|e^{p_1 L} \quad \text{for any } u\in[0,L].$$

Similarly, by using Lemmas 3.1 and 4.2, we have that

$$0 \leq |x(\gamma(t-u),t,\xi)-x(\gamma(t-u),t,\xi')| \leq |\xi-\xi'|e^{p_1(\theta+L)} \quad \text{for any } u\in[0,L].$$

The reader can deduce that the inequalities above imply

$$|I_{12}| \leq D|\xi-\xi'| \quad \text{with} \quad D=\frac{K\rho^* e^{p_1 L}}{\alpha}(1-e^{-\alpha L})(\ell_1+\ell_2 e^{p_1\theta}). \tag{47}$$

Analogously, we can deduce that

$$|I_{21}| \leq D|\xi-\xi'|. \tag{48}$$

For any $\varepsilon>0$, we can choose $L \geq \alpha^{-1}\ln\left(\frac{8K\mu\rho^*}{\alpha\varepsilon}\right)$, which implies $|I_{11}|+|I_{22}|<\frac{\varepsilon}{2}$. This fact combined with (47),(48) imply

$$\forall\varepsilon>0 \ \exists\delta=\frac{\varepsilon}{4D} \quad \text{such that} \quad |\xi-\xi'|<\delta \ \Rightarrow \ |\chi(t;(t,\xi))-\chi(t;(t,\xi'))|<\varepsilon$$

and the uniform continuity follows. □

Lemma 6.2. *The map $\nu \mapsto L(t,\nu)$ is uniformly continuous for any t.*

Proof. We only need to prove that the map $\nu \mapsto \vartheta(t;(t,\nu))$ is uniformly continuous. Let ν and ν' be two initial conditions of (2) and define

$$\Delta=\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu')).$$

By using (36), we can see that Δ can be written as follows:

$$\begin{aligned}\Delta &= \int_{-\infty}^{t} \widetilde{G}(t,s)\big\{f(s,y(s,t,\nu)+\vartheta(s;(t,\nu)),y(\gamma(s),t,\nu)+\vartheta(\gamma(s);(t,\nu)))\\ &\quad - f(s,y(s,t,\nu')+\vartheta(s;(t,\nu')),y(\gamma(s),t,\nu')+\vartheta(\gamma(s);(t,\nu')))\big\}\,ds\\ &\quad + \int_{t}^{\infty} \widetilde{G}(t,s)\big\{f(s,y(s,t,\nu)+\vartheta(s;(t,\nu)),y(\gamma(s),t,\nu)+\vartheta(\gamma(s);(t,\nu)))\\ &\quad - f(s,y(s,t,\nu')+\vartheta(s;(t,\nu')),y(\gamma(s),t,\nu')+\vartheta(\gamma(s);(t,\nu')))\big\}\,ds\\ &= J_1 + J_2. \end{aligned} \tag{49}$$

As before, we divide J_1 and J_2 as follows:

$$J_1 = \int_{-\infty}^{t-\tilde{L}} \cdots + \int_{t-\tilde{L}}^{t} \cdots = J_{11} + J_{12}, \quad J_2 = \int_{t}^{t+\tilde{L}} \cdots + \int_{t+\tilde{L}}^{\infty} \cdots = J_{21} + J_{22}.$$

By **(A2)** and Proposition 3.7, it is straightforward to verify that

$$|J_{11}| \leq \frac{2K\rho^*\mu}{\alpha} e^{-\alpha\tilde{L}} \quad \text{and} \quad |J_{22}| \leq \frac{2K\rho^*\mu}{\alpha} e^{-\alpha\tilde{L}}.$$

Let us define

$$||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_\infty = \sup_{s\in(-\infty,\infty)} |\vartheta(s;(t,\nu)) - \vartheta(s;(t,\nu'))|, \tag{50}$$

and notice that **(A3)** and Proposition 3.7 implies:

$$\begin{aligned}|J_{12}| &\leq \frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_\infty\\ &\quad + K\rho^*\ell_1 \int_0^{\tilde{L}} e^{-\alpha u}|y(t-u,t,\nu) - y(t-u,t,\nu')|\,ds\\ &\quad + K\rho^*\ell_2 \int_0^{\tilde{L}} e^{-\alpha u}|y(\gamma(t-u),t,\nu) - y(\gamma(t-u),t,\nu')|\,du.\end{aligned}$$

By using Lemma 4.3, we know that

$$|y(t-u,t,\nu) - y(t-u,t,\nu')| \leq |\nu-\nu'|e^{p_2\tilde{L}} \quad \text{for any } u \in [0,\tilde{L}]$$

and by using again Lemmatas 4.3 and 3.1, we have

$$|y(\gamma(t-u),t,\nu) - y(\gamma(t-u),t,\nu')| \leq |\nu-\nu'|e^{p_2(\theta+\tilde{L})} \quad \text{for any } u \in [0,\tilde{L}]$$

and the reader can deduce that

$$|J_{12}| \leq \frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_\infty + \tilde{D}|\nu-\nu'|$$

with $\tilde{D} = K\rho^* e^{p_2 L}\alpha^{-1}(1-e^{-\alpha\tilde{L}})(\ell_1+\ell_2 e^{p_2\theta})$. In addition, the following inequality can be proved in a similar way

$$|J_{21}| \leq \frac{K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty + \tilde{D}|\nu-\nu'|.$$

By using the inequalities stated above combined with (10), he have

$$|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))|$$
$$\leq \frac{4K\rho^*\mu}{\alpha}e^{-\alpha\tilde{L}} + 2\tilde{D}|\nu-\nu'| + \frac{2K\rho^*}{\alpha}(\ell_1+\ell_2)||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_\infty,$$

and we obtain

$$|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))| \leq \frac{4K\rho^*\mu e^{-\alpha\tilde{L}}}{\alpha(1-\Gamma^*)} + \frac{2\tilde{D}}{1-\Gamma^*}|\nu-\nu'| \quad \text{with} \quad \Gamma^* = \frac{2K\rho^*}{\alpha}(\ell_1+\ell_2).$$

Finally, for any $\varepsilon > 0$, we choose $\tilde{L} \geq \ln\left(\frac{8K\mu\rho^*}{\alpha\varepsilon(1-\Gamma^*)}\right)^{\frac{1}{\alpha}}$, which implies the uniform continuity since

$$\forall\varepsilon>0\ \exists\delta = \frac{(1-\Gamma^*)\varepsilon}{4\tilde{D}} : \quad |\nu-\nu'|<\delta \ \Rightarrow\ |\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))|<\varepsilon. \qquad \square$$

6.2. Proof of Theorem 2.2. As before, we only have to prove that the maps $\xi \mapsto H(t,\xi)$ and $\nu \mapsto L(t,\nu)$ defined in the Section 5 are Hölder continuous.

Lemma 6.3. *If $|\xi-\xi'|<1$, there exists $C_1>1$ such that*

$$|H(t,\xi)-H(t,\xi')| \leq C_1|\xi-\xi'|^{\frac{\alpha}{p_1}} \quad \text{for any fixed } t\in\mathbb{R},$$

with $p_1>\alpha$ defined by (27).

Proof. Firstly, we will study the map $\xi \mapsto \chi(t;(t,\xi))$ by using the identity

$$\chi(t;(t,\xi))-\chi(t;(t,\xi')) = -I_1+I_2,$$

described by (46). Nevertheless, this time we consider the integrals I_1 and I_2:

$$I_1 = \int_{-\infty}^{t-T}\cdots + \int_{t-T}^{t}\cdots = I_{11}+I_{12}, \quad I_2 = \int_{t}^{t+T}\cdots + \int_{t+T}^{\infty}\cdots = I_{21}+I_{22},$$

where $T = \frac{1}{p_1}\ln\left(\frac{1}{|\xi-\xi'|}\right)$. Now, the reader can easily verify that

$$e^{-\alpha T} = |\xi-\xi'|^{\frac{\alpha}{p_1}} \quad \text{and} \quad e^{p_1 T} = |\xi-\xi'|^{-1}, \tag{51}$$

which combined with (23) implies that

$$|I_{11}| \leq \frac{2\mu K\rho^*}{\alpha}|\xi-\xi'|^{\frac{\alpha}{p_1}} \quad \text{and} \quad |I_{22}| \leq \frac{2\mu K\rho^*}{\alpha}|\xi-\xi'|^{\frac{\alpha}{p_1}}$$

By using **(A3)**, Proposition 3.7 and Lemma 4.2 we have that

$$\begin{aligned}|I_{21}| &\leq \int_t^{t+T} K\rho^* e^{-\alpha(s-t)}\ell_1|x(s,t,\xi)-x(s,t,\xi')|\,ds \\ &\quad+\int_t^{t+T} K\rho^* e^{-\alpha(s-t)}\ell_2|x(\gamma(s),t,\xi)-x(\gamma(s),t,\xi')|\,ds \\ &\leq |\xi-\xi'|K\rho^*\ell_1\int_t^{t+T} e^{(p_1-\alpha)(s-t)}\,ds+|\xi-\xi'|K\rho^*\ell_2\int_t^{t+T} e^{-\alpha(s-t)}e^{p_1|\gamma(s)-t|}\,ds.\end{aligned}$$

By using Lemma 3.1, we can see that

$$|I_{21}| \leq \big\{\ell_1+\ell_2 e^{p_1\theta}\big\}|\xi-\xi'|K\rho^*\int_t^{t+T} e^{(p_1-\alpha)(s-t)}\,ds.$$

By (13), we have $p_1 > \alpha$. This fact combined with (51) implies:

$$|I_{21}| \leq \frac{K\rho^*}{p_1-\alpha}\big\{\ell_1+\ell_2 e^{p_1\theta}\big\}|\xi-\xi'|^{\frac{\alpha}{p_1}}.$$

Finally, as we can be obtain a similar estimation for I_{12}. By using $\alpha < p_1$ and $|\xi-\xi'|<1$, we can conclude that

$$|H(t,\xi)-H(t,\xi')| \leq \left(1+\frac{2K\rho^*}{p_1-\alpha}\big\{\ell_1+\ell_2 e^{p_1\theta}\big\}+\frac{4\mu K\rho^*}{\alpha}\right)|\xi-\xi|^{\frac{\alpha}{p_1}}.$$

and the lemma follows. □

Lemma 6.4. *If $|\nu-\nu'|<1$, there exists $D_1>1$ such that*

$$|L(t,\nu)-L(t,\nu')| \leq D_1|\nu-\nu'|^{\frac{\alpha}{p_2}}, \quad \textit{for any fixed } t,$$

where $p_2 > \alpha$ is defined in (32).

Proof. We will start by studying the map $\nu \to \vartheta(t;(t,\nu))$. Let us recall the identity

$$|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))| = J_1+J_2,$$

described by (49). As before, we divide J_1 and J_2 as follows:

$$J_1=\int_{-\infty}^{t-\tilde{T}}\cdots+\int_{t-\tilde{T}}^{t}\cdots = J_{11}+J_{12}, \quad J_2=\int_t^{t+\tilde{T}}\cdots+\int_{t+\tilde{T}}^{\infty}\cdots = J_{21}+J_{22},$$

with $\tilde{T}=\frac{1}{p_2}\ln\big(\frac{1}{|\nu-\nu'|}\big)$. In addition, we can prove as before that

$$|J_{11}| \leq \frac{2\mu K}{\alpha}|\nu-\nu'|^{\frac{\alpha}{p_2}} \quad \text{and} \quad |J_{22}| \leq \frac{2\mu K}{\alpha}|\nu-\nu'|^{\frac{\alpha}{p_2}}.$$

By using **(A3)**, (50), Proposition 3.7 and Lemma 4.3 we can deduce that

$$
\begin{aligned}
|J_{12}| \leq & \int_{t-\tilde{T}}^{t} K\rho^{*}e^{-\alpha(t-s)}\ell_{1}|y(s,t,\nu)-y(s,t,\nu')|\,ds \\
& + \int_{t-\tilde{T}}^{t} K\rho^{*}e^{-\alpha(t-s)}\ell_{2}|y(\gamma(s),t,\nu)-y(\gamma(s),t,\nu')|\,ds \\
& + \int_{t-\tilde{T}}^{t} K\rho^{*}e^{-\alpha(t-s)}\ell_{1}|\vartheta(s;(t,\nu))-\vartheta(s;(t,\nu'))|\,ds \\
& + \int_{t-\tilde{T}}^{t} K\rho^{*}e^{-\alpha(t-s)}\ell_{2}|\vartheta(\gamma(s);(t,\nu))-\vartheta(\gamma(s);(t,\nu'))|\,ds \\
\leq & \frac{K\rho^{*}}{p_{2}-\alpha}\Big\{\ell_{1}+\ell_{2}e^{p_{2}\theta}\Big\}|\nu-\nu'|^{\frac{\alpha}{p_{2}}}+\frac{K\rho^{*}}{\alpha}(\ell_{1}+\ell_{2})||\vartheta(\cdot;(t,\nu))-\vartheta(\cdot;(t,\nu'))||_{\infty},
\end{aligned}
$$

where $p_2 > \alpha$ since (13). A similar bound can be deduced for $|J_{21}|$ and we obtain

$$
\begin{aligned}
||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_{\infty} \leq & \Big(\frac{2K\rho^{*}}{p_{2}-\alpha}\Big\{\ell_{1}+\ell_{2}e^{p_{2}\theta}\Big\}+\frac{4\mu K}{\alpha}\Big)|\nu-\nu'|^{\frac{\alpha}{p_{2}}} \\
& + \frac{2K\rho^{*}}{\alpha}(\ell_{1}+\ell_{2})||\vartheta(\cdot;(t,\nu)) - \vartheta(\cdot;(t,\nu'))||_{\infty}.
\end{aligned}
$$

Now, by using (10) and Γ^{*} as in the previous proof, we conclude that

$$
|\vartheta(t;(t,\nu))-\vartheta(t;(t,\nu'))| \leq \underbrace{\big(1-\Gamma^{*}\big)^{-1}\Big(\frac{2K\rho^{*}}{p_{2}-\alpha}\Big\{\ell_{1}+\ell_{2}e^{p_{2}\theta}\Big\}+\frac{4\mu K}{\alpha}\Big)}_{=E}|\nu-\nu'|^{\frac{\alpha}{p_{2}}},
$$

and the lemma follows with $C = 1 + E$. □

7. Some consequences and applications

7.1. Limits cases study. In the case $A_0(t) = 0$, the system (1) becomes:

$$
x'(t) = A(t)x(t) + f(t,x(t),x(\gamma(t))), \tag{52}
$$

and **(C)** is always verified. In addition, (2) becomes the ODE system (15), $J(t,\tau) = I$, $E(t,\tau) = Z(t,\tau) = \Phi(t,\tau)$ and $\widetilde{G}(t,s)$ becomes:

$$
G(t,s) = \begin{cases} \Phi(t)P\Phi^{-1}(s) & \text{if } t \geq s \\ -\Phi(t)(I-P)\Phi^{-1}(s) & \text{if } s > t. \end{cases}
$$

Now, it is easy to prove the following result:

Corollary 7.1. *If there exists two constants $\tilde{K} \geq 1$, $\tilde{\alpha} > 0$ such that*

$$||G(t,s)|| \leq \tilde{K} e^{-\tilde{\alpha}(|t-s|)}, \tag{53}$$

conditions **(A)** *and* **(B)** *are satisfied and*

$$2(\ell_1 + \ell_2)\tilde{K} < \tilde{\alpha}, \tag{54}$$

$$F_1(\theta)\ell_2\theta = v_0 < 1, \quad \textit{with} \quad F_1(\theta) = \frac{e^{M\theta} - 1}{M\theta}, \tag{55}$$

then (15) *and* (52) *are strongly topologically equivalent. Moreover, if $M > \tilde{\alpha}$, they are Hölder topologically equivalent.*

When $A(t) = 0$, the systems (1),(2) becomes

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(\gamma(t)) + f(t, x(t), x(\gamma(t))), &(56)\\ \dot{y}(t) &= A_0(t)y(\gamma(t)). &(57) \end{aligned}$$

In this context, the reader can verify that $\Phi(t,\tau) = I$ and

$$J(t,\tau) = E(t,\tau) = I + \int_\tau^t A_0(s)\,ds.$$

The limit case $A(t) = 0$ also modifies the corresponding definitions of $Z(t,s)$ and $\widetilde{G}(t,s)$ with $\rho^* = e^{\alpha\theta}$ and it is easy to prove:

Corollary 7.2. *If* (57) *has a transition matrix $Z(t,0)$ satisfying the exponential dichotomy* (5)*, conditions* **(A)** *and* **(B)** *are satisfied and*

$$2(\ell_1 + \ell_2)Ke^{\alpha\theta} < \alpha \quad \textit{and} \quad (M_0 + \ell_2)\theta = \tilde{u}_0 < 1 \tag{58}$$

$$\tilde{F}_1(\theta)(M_0 + \ell_2)\theta = \tilde{v}_0 < 1, \quad \textit{with} \quad \tilde{F}_1(\theta) = \frac{e^{\ell_1\theta} - 1}{\ell_1\theta}, \tag{59}$$

then (56) *and* (57) *are strongly topologically equivalent. In addition, if*

$$\alpha < \min\left\{\ell_1 + \frac{M_0 + \ell_2}{1 - \tilde{v}_0}e^{\ell_1\theta}, \frac{M_0}{1 - \tilde{u}_0}\right\},$$

then they are Hölder topologically equivalent.

Proof. We only need to prove that **(C)** is satisfied with $A(t) = 0$. Indeed, notice that $\rho(A) = 1$ combined with **(A1)** and (58) imply (7),(8) since

$$\max_{k\in\mathbb{Z}}\left\{\int_{t_k}^{\zeta_k} |A_0(s)|\,ds, \int_{\zeta_k}^{t_{k+1}} |A_0(s)|\,ds\right\} \leq M_0\theta < \tilde{u}_0 < 1. \qquad \square$$

7.2. Application to stability. In [19,21], it has been pointed out that strong topological equivalence preserves Liapunov's stability of the zero solution. In order to extend these results to the DEPCAG case, we will assume that:

(A4) The function f verifies $f(t,0,0)=0$ for any t,

and a direct consequence of **(A4)** is that the origin is a solution of (1).

Definition 7.3. The origin of (1) (resp. (2)) is:

(a) Uniformly stable if for any $u_0>0$, there exists $\delta(\varepsilon)>0$ such that $|u_0|<\delta$ implies $|x(t,\tau,u_0)|<\varepsilon$ (resp. $|y(t,\tau,u_0)|<\varepsilon$) for any $t\geq\tau$.

(b) Uniformly asymptotically stable if is uniformly stable and there is a $\delta_0>0$ such that for every $\varepsilon>0$ and $\tau>0$ there exists $T(\varepsilon)>0$ such that $|x(t,\tau,u_0)|<\varepsilon$ (resp. $|y(t,\tau,u_0)|<\varepsilon$) for any $t\geq\tau+T$ whenever $|u_0|<\delta_0$.

Theorem 7.4. *Under the assumptions of Theorem* 2.1*, assume that* **(A4)** *is satisfied.*

(i) *If the origin is a uniformly stable solution of* (2)*, then is also a uniformly stable solution of* (1) *and vice versa.*

(ii) *If the origin is a uniformly asymptotically stable solution of* (2)*, then is also a uniformly asymptotically stable solution of* (1) *and vice versa.*

Proof. Let us recall that for any solution $t\mapsto x(t,\tau,\xi)$ of (1), there exists a unique solution $t\mapsto y(t,\tau,\nu)$ of (2) such that $x(t,\tau,\xi)=L[t,y(t,\tau,\nu)]$, where $\xi=L(\tau,\nu)$. Now, notice that $L[t,0]=0$ for any $t\in\mathbb{R}$. Indeed, by (41) combined with **(A4)** we have $H[t,0]=H[t,x(t,\tau,0)]=0$ for any t and we can see that

$$L[t,0]=L[t,H[t,0]]=0 \quad \text{for any } t\in\mathbb{R}.$$

Uniform continuity of $L[t,\cdot]$ says that for any $\varepsilon>0$, exists $\eta(\varepsilon)>0$ such that

$$|y(t,\tau,\nu)|<\eta \;\Rightarrow\; |L[t,y(t,\tau,\nu)]|<\varepsilon \quad \text{for any fixed } t. \tag{60}$$

By uniform stability of the zero solution of (2) and considering η as in (60), it follows that, there exists $\tilde{\delta}(\eta(\varepsilon))>0$ such that

$$|\nu|<\tilde{\delta} \;\Rightarrow\; |y(t,\tau,\nu)|<\eta. \tag{61}$$

By uniform continuity of $H[\tau,\cdot]$ and considering $\tilde{\delta}>0$ from (61), there exists $\delta(\tilde{\delta}(\eta(\varepsilon)))=\delta(\varepsilon)$ such that

$$|L[\tau,\nu]|\leq\delta \;\Rightarrow\; |H[\tau,L[\tau,\nu]]|=|\nu|<\tilde{\delta}. \tag{62}$$

Finally, by coupling (60)–(62) we have that for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that $|L[\tau,\nu]|<\delta$ implies $|L[t,y(t,\tau,\nu)]|<\varepsilon$ for any $t\geq\tau$ and (i) follows.

The proof of (ii) is similar: by uniform asymptotic stability of the zero solution of (2) and using $\eta > 0$ from (60), there exists $\hat{\delta} > 0$ such that

$$\forall \tau > 0 \ \exists T(\eta) > 0 \quad \text{such that} \quad |y(t,\tau,\nu)| < \eta \quad \forall t > T + \tau \text{ when } |\nu| < \hat{\delta}. \tag{63}$$

By uniform continuity of $H[\tau,\cdot]$ and considering $\hat{\delta} > 0$ from (63), there exists $\delta(\varepsilon) > 0$ such that (62) is satisfied. By using this inequality combined with (63) and (60), we can conclude the existence of $\delta > 0$ such that for any $\varepsilon > 0$ and $\tau > 0$ there exists $T = T(\eta(\varepsilon))$ such that

$$|L[t, y(t,\tau,\nu)]| < \varepsilon \quad \text{for any } t > T + \tau \text{ when } |L[\tau,\nu]| < \delta$$

and the uniform stability follows. □

Theorem 7.5. *Under the assumptions of Theorem* 2.2*, assume that* **(A4)** *is satisfied.*

If the origin is a uniformly (asymptotically) stable solution of (2)*, then there exists* $\ell \in (0,1)$ *such that for any* $\varepsilon \in (0,\ell)$ *we have positive constants* $\delta(\varepsilon)$*,* P_i *and* Q_i *(*$i = 1,2$*) such that any solution* $t \mapsto y(t,\tau,\nu)$ *of* (2) *with* $|\nu| < \delta$*, verifies*

$$P_1|y(t,\tau,\nu)|^{P_2} \leq |L[t, y(t,\tau,\nu)]| \leq Q_1|y(t,\tau,\nu)|^{Q_2} \quad \text{for any } t \geq t. \tag{64}$$

Conversely, if the origin is a uniformly (asymptotically) stable solution of (1)*, there exists* $\ell \in (0,1)$ *such that for any* $\varepsilon \in (0,\ell)$ *we have positive constants* $\delta(\varepsilon)$*,* $\tilde{P}_i$ *and* $\tilde{Q}_i$ *(*$i = 1,2$*) such that any solution* $t \mapsto x(t,\tau,\nu)$ *of* (1) *with* $|\xi| < \delta$*, verifies*

$$\tilde{P}_1|x(t,\tau,\xi)|^{\tilde{P}_2} \leq |H[t, x(t,\tau,\xi)]| \leq \tilde{Q}_1|x(t,\tau,\xi)|^{\tilde{Q}_2}. \tag{65}$$

Proof. Let $\ell = \left(\frac{1}{D_1}\right)^{\frac{1}{D_2}} < 1$, with $D_1 > 1$ and $D_2 \in (0,1)$ stated in (4). As the origin of (2) is uniformly asymptotically stable, for any $\varepsilon \in (0,\ell)$, exists $\delta(\varepsilon) > 0$ such that $|\nu| < \delta$ implies $|y(t,\tau,\nu)| < \varepsilon < 1$ for any $t \geq \tau$.

By Theorem 2.2 combined with $L[t,0] = H[t,0] = 0$ we have the inequalities

$$|L[t, y(t,\tau,\nu)]| \leq D_1|y(t,\tau,\nu)|^{D_2} < 1 \quad \text{for any fixed } t \geq \tau,$$

$$|y(t,\tau,\nu)| = |H[t, L[t, y(t,\tau,\nu)]]| \leq C_1|L[t, y(t,\tau,\nu)]|^{C_2} \quad \text{for any fixed } t \geq \tau.$$

Now, (64) is obtained with $P_1 = \left(\frac{1}{C_1}\right)^{\frac{1}{C_2}}$, $P_2 = \left(\frac{1}{C_2}\right)$, $Q_1 = D_1$ and $Q_2 = D_2$. The inequality (65) can be deduced in a similar way. □

Acknowledgement. M. Pinto was supported by FONDECYT Regular (grants 1170466 and 1120709). G. Robledo was supported by MATHAMSUD Regional Cooperation Program (16-04 STADE).

References

[1] Akhmet, M. U., On the integral manifolds of the differential equation with piecewise constant argument of generalized type. In: *Differential & Difference Equations and Applications* (Proceedings Melbourne (FL) 2005; eds.: R. P. Agarwal et al.). New York: Hindawi 2006, pp. 11 – 20.

[2] Akhmet, M. U., On the reduction principle for differential equations with piecewise constant argument of generalized type. *J. Math. Anal. Appl.* 336 (2007), 646 – 663.

[3] Akhmet, M. U., Asymptotic behavior of solutions of differential equations with piecewise constant arguments. *Appl. Math. Lett.* 21 (2008), 951 – 956.

[4] Akhmet, M. U., *Nonlinear Hybrid Continuous/Discrete-Time Models.* Paris: Atlantis Press 2011.

[5] Akhmet, M. U., Exponentially dichotomous linear systems of differential equations with piecewise constant argument. *Discontin. Nonlinearity Complex.* 1 (2012), 337 – 352.

[6] Akhmet, M. U. and Yilmaz, E., *Neural Networks with Discontinuous/Impact Activations. Nonlinear Systems and Complexity.* New York: Springer 2014.

[7] Aulbach, B. and Wanner, T., The Hartman–Grobman theorem for Carathéodory-type differential equations in Banach spaces. *Nonlinear Anal.* 40 (2000), 91 – 104.

[8] Aulbach, B. and Wanner, T., Topological simplification of nonautonomous difference equations. *J. Difference Equ. Appl.* 12 (2006), 283 – 296.

[9] Barreira, L. and Valls, C., A simple proof of the Grobman–Hartman theorem for nonuniformly hyperbolic flows. *Nonlinear Anal.* 74 (2011), 7210 – 7225.

[10] Byrne H. and Gourley, S., Global convergence in a reaction-diffusion equation with piecewise constant argument, *Math. Comp. Modelling* 34 (2001), 403 – 409.

[11] Castañeda, A. and Robledo, G., A topological equivalence result for a family of nonlinear difference systems having generalized exponential dichotomy. *J. Difference Equ. Appl.* 22 (2016), 1271 – 1291.

[12] Castillo, S. and Pinto, M., Existence and stability of almost periodic solutions to differential equations with piecewise constant argument. *Electron. J. Diff. Equ.* 58 (2015), 1 – 15.

[13] Chiu, K. S. and Pinto, M., Periodic solutions of differential equations with a general piecewise constant argument and applications. *Electron. J. Qual. Theory Diff. Equ.* 46 (2010), 1 – 20.

[14] Chiu, K. S., Pinto, M. and Jeng, J. C., Existence and global convergence of periodic solutions in recurrent neural network with a general piecewise alternately advanced and retarded argument. *Acta Appl. Math.* 133 (2014), 133 – 152.

[15] Coronel, A., Maulén, C., Pinto, M. and Sepúlveda, D., Dichotomies and asymptotic equivalence in alternately advanced and delayed differential systems. *J. Math. Anal. Appl.* 450 (2017), 1434 – 1458.

[16] Dai, L., *Nonlinear Dynamics of Piecewise Constants Systems and Implementation of Piecewise Constants Argument.* Singapore: World Scientific 2008.

[17] Farkas, G., A Hartman–Grobman result for retarded functional differential equations with an application to the numerics around hyperbolic equilibria. *Z. Angew. Math. Phys.* 52 (2001), 421 – 432.

[18] Györi, I., Hartung, F. and Turi, J., Numerical approximations for a class of differential equations with time and state-dependent delays. *Appl. Math. Lett.* 8 (1995), 19 – 24.

[19] Jiang, L., Generalized exponential dichotomy and global linearization. *J. Math. Anal. Appl.* 315 (2006), 474 – 490.

[20] Kartal, S. and Gurcan, F., Stability and bifurcations analysis of a competition model with piecewise constant arguments. *Math. Meth. Appl. Sci.* 38 (2015), 1855 – 1866.

[21] López-Fenner, J. and Pinto, M., On a Hartman linearization theorem for a class of ODE with impulse effect. *Nonlinear Anal.* 38 (1999), 307 – 325.

[22] Kurzweil, J. and Papaschinopoulos, G., Topological equivalence and structural stability for linear difference equations. *J. Diff. Equ.* 89 (1991), 89 – 94.

[23] Myshkis, A. D., Some problems in the theory of differential equations with deviating argument (in Russian). *Uspekhi Mat. Nauk* 32 (1977), 173 – 202.

[24] Palmer, K. J., A generalization of Hartman's linearization theorem. *J. Math. Anal. Appl.* 41 (1973), 753 – 758.

[25] Papaschinopoulos, G., Exponential dichotomy, topological equivalence and structural stability for differential equations with piecewise constant argument. *Analysis* 14 (1994), 239 – 247.

[26] Papaschinopoulos, G., A linearization result for a differential equation with piecewise constant argument. *Analysis* 16 (1996), 161 – 170.

[27] Pinto, M., Asymptotic equivalence of nonlinear and quasi-linear differential equations with piecewise constant arguments. *Math. Comp. Modelling* 49 (2009), 1750 – 1758.

[28] Pinto, M., Cauchy and Green matrices type and stability in alternately advanced and delayed differential systems. *J. Difference Equ. Appl.* 17 (2011), 235 – 254.

[29] Pinto, M. and Robledo, G., Controllability and observability for a linear time varying system with piecewise constant delay, *Acta Appl. Math.* 136 (2015), 193 – 216.

[30] Pötzsche, C., Topological decoupling, linearization and perturbation on inhomogeneous time scales. *J. Diff. Equ.* 245 (2008), 1210 – 1242.

[31] Pötzsche, C. and Russ, E., Topological decoupling and linearization of nonautnomous evolution equations. *Discrete Contin. Dyn. Syst. Ser. S* 9 (2016), 1235 – 1268

[32] Shi, J. and Xiong, K., On Hartman's linearization theorem and Palmer's linearization theorem. *J. Math. Anal. Appl.* 192 (1995), 813 – 832.

[33] Sternberg, N., A Hartman–Grobman theorem for a class of retarded functional differential equations. *J. Math. Anal. Appl.* 176 (1993), 156 – 165.

[34] Veloz, T. and Pinto, M., Existence, computability and stability for solutions of the diffusion equation with general piecewise constant argument. *J. Math. Anal. Appl.* 426 (2015), 330 – 339.

[35] Wiener, J., *Generalized Solutions of Functional Differential Equations*. Singapore: World Scientific 1993.

[36] Xia, Y., Cao, J. and Han, M., A new analytical method for the linearization of dynamic equation on measure chains. *J. Diff. Equ.* 235 (2007), 527 – 543.

[37] Xia, Y., Wang, R., Kou, K. and O'Regan, D., On the linearization theorem for nonautonomous differential equations. *Bull. Sci. Math.* 139 (2015) 829 – 846.

[38] Yuan, R., On Favard's theorems. *J. Diff. Equ.* 249 (2010), 1884 – 1916.

Received May 4, 2016; revised February 8, 2017