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# An Easy Approach to Distributions and Operational Calculus

*Vakhtang Lomadze*

**Abstract.** The Mikusinski space is introduced as the smallest extension of the space of classical continuous functions in which the integration operator is bijective. (This can be viewed as a very small part of the field of Mikusinski operators.) It is shown that the space of Schwartz distributions (of finite order) can be defined as a quotient of the Mikusinski space. A remarkable property of Mikusinski functions is that they admit multiplication by all rational functions. It is demonstrated that this multiplication provides a natural simple basis for Heaviside's operational calculus.

**Keywords.** Mikusinski functions, distributions, operational calculus

**Mathematics Subject Classification (2010).** 44A40, 44A45

## 1. Introduction

Let $I$ be an interval of real axis (with more than one point), and let $C(I)$ be the space of all complex-valued continuous functions defined on $I$. Assuming (without loss of generality) that the interval contains 0, for every continuous function $u \in C(I)$, let $J(u)$ denote the continuous function defined by

$$J(u)(x) = \int_0^x u(\alpha)d\alpha, \quad x \in I.$$

The integral operator $J : C(I) \to C(I)$ is injective. But it is not bijective; in other words, not every continuous function is an integral.

We construct an extension $M(I) \supseteq C(I)$ and an integral operator on it (that agrees with $J$ on $C(I)$ and) that is bijective. A differentiation operator is defined to be the inverse of the integration one. Any element of $M(I)$ is an iterated derivative of a continuous function. (So that $M(I)$ is the smallest extension with the property above.) What is not normal with $M(I)$ is that the

V. Lomadze: A. Razmadze Mathematical Institute, Mathematics Department of I. Javakhishvili Tbilisi State University, Georgia; vakhtang.lomadze@tsu.ge

iterated derivatives of constant functions are not zero, and a natural idea is to "kill" all of them. Doing this, we get exactly the Schwartz distribution space.

Elements of $M(I)$ are called Mikusinski functions. The point of these functions is that they admit multiplication by all rational functions, and this makes them very convenient to work with. We show that the representation of $\mathcal{D}'_{fin}(I)$ in terms of Mikusinski functions provides a natural simple basis for Heaviside's operational calculus.

In what follows, $t$ will be an indeterminate, and $\mathbb{C}[[t]]$ will stand for the ring of complex coefficient convergent formal series in $t$. (We remind that a formal series $\sum_{i\geq 0} b_i t^i$ is said to be convergent if $\sum_{i\geq 0} |b_i|\varepsilon^i < +\infty$ for some positive real $\varepsilon$.) The fraction field of $\mathbb{C}[[t]]$ will be denoted by $\mathbb{C}((t))$, and its elements will be referred to as convergent Laurent series. By the theorem on units (see [4]), a convergent formal series is invertible if and only if it has nonzero free coefficient. It is immediate from this that every convergent Laurent series can be written as $g/t^n$ with $g \in \mathbb{C}[[t]]$ and $n \in \mathbb{Z}_+$. We put $s = t^{-1}$. It is worth mentioning that $\mathbb{C}((t))$ contains as a subfield the rational function field $\mathbb{C}(s)$, which is the fraction field of the polynomial ring $\mathbb{C}[s]$. Following Mikusinski [3], if $c$ is a constant, we shall write $\{c\}$ to denote the function that is $c$ everywhere on $I$.

## 2. Mikusinski functions

Define the Mikusinski space $M(I)$ to be the inductive limit of the sequence

$$C(I) \xrightarrow{J} C(I) \xrightarrow{J} C(I) \xrightarrow{J} C(I) \xrightarrow{J} \cdots .$$

One can represent a Mikusinski function as a pair $(u, m)$, where $u \in C(I)$ and $m \in \mathbb{Z}_+$. Two such pairs $(u, m)$ and $(v, n)$ represent the same Mikusinski function if and only if

$$J^n u = J^m v.$$

**Remark 2.1.** As is known, Mikusinski (see [3]) developed an approach to the notion of generalized functions that is based on using the convolution ring of continuous functions defined on $\mathbb{R}_+$. From this ring, which is commutative and without zero divisors, Mikusinski goes on to define the fraction field, elements of which are called operators. In the case when $I = \mathbb{R}_+$, there is a canonical mapping from our Mikusinski functions to Mikusinski's operators, namely, the mapping

$$(u, m) \mapsto \frac{u}{\{1\}^m}.$$

It is easily seen that this is an embedding.

Obviously, the map $u \mapsto (u, 0)$ is injective. This permits us to make the identification

$$u = (u, 0).$$

We extend the integration operator $J$ to Mikusinski functions by defining

$$J(u, m) = (Ju, m),$$

and the differentiation operator $D$ by defining

$$D(u, m) = (u, m + 1).$$

Notice that

$$DJ = id \quad \text{and} \quad JD = id.$$

Thus, both of the operators $J : M(I) \to M(I)$ and $D : M(I) \to M(I)$ are bijective, and are inverse to each other.

Observe that if $g = \sum_{i \geq 0} b_i t^i$ is a convergent formal series and $u$ is a continuous function on $I$, then the series

$$\sum_{i \geq 0} b_i J^i(u)$$

converges uniformly on every compact subinterval of $I$ that contains 0. We therefore can define the product $gu$ by setting

$$gu = \sum_{i \geq 0} b_i J^i(u).$$

Given a convergent Laurent series $g/t^n$ and a Mikusinski function $(u, m)$, we set

$$g/t^n \cdot (u, m) = (gu, m + n).$$

One can easily verify that this multiplication is well-defined and that it makes $M(I)$ a *linear space* over $\mathbb{C}((t))$. (We shall need, in fact, the linear space structure over the rational function field $\mathbb{C}(s)$.)

We want to note that the multiplication by $t$ is the same as $J$ and the multiplication by $s$ is the same as $D$.

Concluding the section, remark that every Mikusinski function $w$ can be written in the form $w = s^m u$ (or, what is the same, in the form $w = D^m u$), where $u \in C(I)$ and $m \in \mathbb{Z}_+$.

## 3. Distributions

Distributions have been introduced by Schwartz, basically, in order to be able to differentiate all continuous functions (see [5, p. 72]). Sebastião e Silva (see [6]) gave an appealing axiomatic characterization of distributions of finite order. He showed that finite order distributions on $I$ may be introduced as elements of a linear space $E(I)$ for which an embedding $\imath : C(I) \to E(I)$ and an operator $D : E(I) \to E(I)$ are defined such that

(S1) If $f \in C(I)$ is a continuously differentiable function, then $D(\imath f) = \imath(f')$;

(S2) Every $\xi \in E(I)$ can be written as $\xi = D^m(\imath f)$ for some $f \in C(I)$ and $m \in \mathbb{Z}_+$;

(S3) If $\xi \in E(I)$, then $D(\xi) = 0$ if and only if $\xi = \imath(\{c\})$ for some constant $c$.

Silva showed that the Schwartz distribution space $\mathcal{D}'_{fin}(I)$ is a model for the above axioms and that all other models are isomorphic to it. (For the Silva theory of distributions, the reader may refer [6] and also the books [1,2].)

Let $N(I)$ denote the subspace of $M(I)$ spanned by functions $D^m\{1\}$, $m \geq 1$. (Remark that $N(I) = \{f\{1\} \mid f \in s\mathbb{C}[s]\}$.)

Postulating that constant functions must have zero derivatives, we are led to the following definition.

**Definition 3.1.** Define the distribution space $S(I)$ to be

$$S(I) \;=\; M(I)/N(I).$$

(The terminology will be justified in a moment.)

Since $N(I)$ is a $\mathbb{C}[s]$-submodule of $M(I)$, $S(I)$ has the structure of a module over $\mathbb{C}[s]$. In particular, we can multiply distributions by $s$.

The differential operator of $M(I)$ induces a differential operator of $S(I)$. We shall denote it by the same letter $D$. Thus,

$$D(w(\text{mod } N(I))) = (Dw)(\text{mod } N(I)), \quad w \in M(I).$$

To proceed further, we need the following lemma.

**Lemma 3.2.** *We have:*

$$C(I) \cap N(I) = \{0\}.$$

*Proof.* Assume there is $u \in C(I)$ that is not zero and belongs to $N(I)$. We then have

$$u = (a_0 s^n + \cdots + a_{n-1}s)\{1\}$$

with $n \geq 1$, $a_i \in \mathbb{C}$ and $a_0 \neq 0$. Multiplying both of the sides by $t^n$, we get

$$t^n u = (a_0 + a_1 t + \cdots + a_{n-1}t^{n-1})\{1\}.$$

It follows that

$$t^n u - (a_1 t + \cdots + a_{n-1} t^{n-1})\{1\} = \{a_0\}.$$

On the left, we have a continuous function having value 0 at 0. Hence, $a_0 = 0$, which is a contradiction. The proof is complete. □

Define the canonical map $\jmath : C(I) \to S(I)$ by the formula

$$\jmath(u) = u \pmod{N(I)}.$$

It is immediate from the previous lemma, that $\jmath$ is injective.

**Theorem 3.3.** *The triple $(S(I), D, \jmath)$ satisfies the Silva axioms.*

*Proof.* (S1): Let $u$ be a continuously differentiable function. Then, by the Newton–Leibniz formula, $u = Ju' + u(0)\{1\}$. Consequently

$$Du \pmod{N(I)} = (u' + u(0)D\{1\}) \pmod{N(I)} = u' (\text{mod } N(I)) = \jmath(u').$$

(S2): Every distribution can be written as $s^m u (\text{mod } N(I))$ with continuous function $u$ and nonnegative integer $m$, and we have

$$s^m u \pmod{N(I)} = D^m (u \pmod{N(I)}) = D^m(\jmath u).$$

(S3): Assume that a distribution $\xi = s^m u (\text{mod } N(I))$ is such that $D\xi = 0$. Then

$$s^{m+1} u \in N(I).$$

This means that there exist complex numbers $a_0, a_1, a_2, \ldots$ such that all but a finite number of them are zero and such that

$$s^{m+1} u = s(a_0 + a_1 s + a_2 s^2 + \cdots)\{1\}.$$

Multiplying both sides by $t^{m+1}$, from this we get

$$u = (a_0 t^m + a_1 t^{m-1} + \cdots + a_m)\{1\} + (a_{m+1} s + \cdots)\{1\}.$$

Since the left side is an ordinary continuous function, we can see that $a_k = 0$ for all $k \geq m + 1$. We therefore have

$$s^m u = a_0 + (a_1 s + \cdots + a_m s^m)\{1\}.$$

It follows that

$$\xi = a_0 (\text{mod } N(I)) = \jmath(\{a_0\}).$$

The proof is complete. □

**Corollary 3.4.** *$S(I)$ is canonically isomorphic to $\mathcal{D}'_{fin}(I)$.*

**Remark 3.5.** 1) The isomorphism $S(I) \simeq \mathcal{D}'_{fin}(I)$ can be established directly. Indeed, for each $u \in C(I)$, let $T_u$ be the corresponding Schwartz distribution. We have seen that if $u \in C(I)$, then $D\jmath(u) = 0$ if and only if $u$ is a constant function. This, in turn, implies that $D^m\jmath(u) = 0$ if and only if $u$ is a polynomial function of degree $\leq m$. It follows that the mapping

$$D^m\jmath(u) \;\mapsto\; D^m T_u$$

is well-defined and injective. The surjectivity is clear.

2) As is known, Schwartz distributions defined on a compact interval have finite order. Therefore, the Schwartz space $\mathcal{D}'(I)$, can be defined as the projective limit

$$\lim_{\leftarrow} S([\alpha, \beta]),$$

where $[\alpha, \beta]$ runs over all compact subintervals of $I$ that contain 0.

## 4. Heaviside's operational calculus

In his seminal works in electromagnetic theory, O. Heaviside developed formal rules for dealing with linear constant coefficient differential equations, much of which he arrived at intuitively. A mathematical basis for his operational calculus was done by mathematicians with the aid of the Laplace transform, which however has some defects. (Laplace transforms methods need to add restrictions to the growth of the functions considered; one also should worry about one-sidedness and convergence.) A satisfactory theory of operational calculus was described later by Mikusinski in his textbook [3]. (It should be noted, however, that the Mikusinski approach is applicable only to differential equations defined on $\mathbb{R}_+$.)

The goal of this section is to demonstrate that with the representation of distribution space as a quotient of the Mikusinski space we have a natural, simple explanation of Heaviside's operational calculus.

Assume we have a linear constant coefficient differential equation

$$f(D)\xi = \omega \qquad (\xi \in S(I)),$$

where $f \in \mathbb{C}[s]$ and $\omega \in S(I)$.

A particular solution of this equation can be found very easily. Indeed, if $\omega$ is represented by a Mikusinski function $w$, then clearly

$$\xi = \frac{w}{f} \bmod N(I)$$

is a solution. So, we only need to consider the homogeneous case.

Given $g \in \mathbb{C}[[t]]$, define its (inverse) Laplace transform $L(g)$ by setting

$$L(g) = g\{1\}.$$

**Remark 4.1.** Let $\mathcal{L}$ be the conventional Laplace transform. The connection of our $L$ with $\mathcal{L}^{-1}$ is as follows. If $g = \sum_i b_i t^i$, then

$$L(g) = \sum b_i \frac{x^i}{i!} \quad \text{and} \quad \mathcal{L}^{-1}(tg) = \mathcal{L}^{-1}\left(\frac{1}{s}\sum b_i \frac{1}{s^i}\right) = \sum b_i \frac{x^i}{i!}.$$

And we see that

$$L(g) = \mathcal{L}^{-1}(tg).$$

Let us identify continuous functions with the corresponding distributions.

**Theorem 4.2.** *Let $f \in \mathbb{C}[s]$ be a polynomial of degree $d \geq 1$. Then, the solutions of the linear differential equation*

$$f(D)\xi = 0, \qquad \xi \in S(I)$$

*are given by the formula*

$$\xi = L\left(\frac{sr}{f}\right),$$

*where $r$ runs over the polynomials in $\mathbb{C}[s]$ of degree $\leq d-1$.*

*Proof.* It is easily seen that all these functions are solutions. Indeed,

$$f(D)\xi = f\frac{sr}{f}\{1\} = sr\{1\} = 0.$$

To show the converse, we proceed as we did in the proof of Theorem 3.3. Assume that $\xi$ is a solution of our equation and assume that a Mikusinski function $s^m u$ represents it. Then

$$fs^m u \in N(I).$$

This means that there exist complex numbers $a_0, a_1, a_2, \ldots$ such that all but a finite number of them are zero and such that

$$fs^m u = s(a_0 + a_1 s + a_2 s^2 + \cdots)\{1\}.$$

Multiplying both sides by $t^{d+m}$, from this we get

$$(ft^d)u = (a_0 t^{d+m-1} + \cdots + a_{d+m-1})\{1\} + (a_{d+m}s + \cdots)\{1\}.$$

Since the left side is an ordinary continuous function, it follows that $a_k = 0$ for all $k \geq d+m$. We therefore have

$$s^m u = \frac{s(a_0 + a_1 s + \cdots + a_{d+m-1}s^{d+m-1})}{f}\{1\}.$$

By the Euclidean division theorem, there exists a polynomial $r$ of degree $\leq d-1$ such that

$$\frac{a_0 + a_1 s + \cdots + a_{d+m-1} s^{d+m-1}}{f} \equiv \frac{r}{f} \bmod \mathbb{C}[s].$$

So that

$$\xi = L\left(\frac{sr}{f}\right).$$

The proof is complete. □

## References


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