Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 37 (2018), 127–149
DOI: 10.4171/ZAA/1606
© European Mathematical Society

Sharp Estimates of the Norms of Embeddings between Besov Spaces

Oscar Domínguez

Abstract. Optimal estimates are obtained for the rates of blow up of the norm of the embedding $B^s_{p,r} \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{q})}_{q,r}$ with $p < q$ as $s \to n\big(\frac{1}{p} - \frac{1}{q}\big)+$. We also show optimal limiting embeddings between Besov spaces and Lipschitz spaces.

Keywords. Besov spaces, embedding norms, interpolation, Lipschitz spaces
Mathematics Subject Classification (2010). 46E35, 46B70

1. Introduction

The analysis of the behaviour of the embedding constants associated to embeddings of Sobolev type as the smoothness parameters approach critical values has attracted much recent attention. The motivation comes from the sharp form of the Sobolev embedding theorem obtained by Bourgain, Brézis and Mironescu [7]. Namely let Q be a cube in $\mathbb{R}^n$ and let $p \geq 1, \frac{1}{2} \leq s < 1$ and $sp < n$. Assume that f is defined on Q such that

$$\int_Q f = 0 \quad \text{and} \quad \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy < \infty. \tag{1}$$

Then

$$\|f|L^{\frac{pn}{n-sp}}(Q)\|^p \leq c_n \frac{1-s}{(n-sp)^{p-1}} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy. \tag{2}$$

Here the constant c_n depends only on n. Afterwards, this result was extended to all values $s \in (0,1)$ by Maz'ya and Shaposhnikova [34] (see also [32]). More precisely, they proved the following sharp inequality: let $p \geq 1, 0 < s < 1$, and $sp < n$. Assume that f satisfies the conditions given in (1). Then

$$\|f|L^{\frac{pn}{n-sp}}(Q)\|^p \leq c_{p,n} \frac{s(1-s)}{(n-sp)^{p-1}} \int_Q \int_Q \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, dxdy \tag{3}$$

O. Domínguez: Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain; oscar.dominguez@ucm.es

where the constant $c_{p,n}$ depends only on p and n. A detailed study of related problems may be found as well in [33, 10.2].

The natural extension of the inequalities (2) and (3) to higher-order smoothness $0 < s < \frac{n}{p}$ was studied by Karadzhov, Milman and Xiao [30] and Edmunds, Evans and Karadzhov [16, 17] with the help of the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ defined by the modulus of smoothness $\omega_k(f,t)_p$ of a function $f \in L^p(\mathbb{R}^n)$ (see (4) and (5) below).

Concerning sharp estimates of the norms of embeddings for Besov spaces when the smoothness parameters are approaching some critical values, we also refer to the book by Triebel [39, 4.4.2] where certain embeddings into the space of continuous functions are investigated.

Before we state our main results, we introduce all the function spaces that we shall use in the paper.

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{T}^n$ is an n-dimensional torus, $\mathbb{T} = \mathbb{T}^1$. Throughout this paper Ω denotes a subset of $\mathbb{R}^n$ of one of the following two types: Ω is either $\mathbb{R}^n$ or a bounded Lipschitz domain in $\mathbb{R}^n$ (see [39, pp. 63–64]). If X and Y are two Banach spaces, then the symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. All unimportant positive constants will be denoted by c, occasionally with subscripts; its value may vary from line to line.

Let $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. Let $W^k_p(\Omega)$ be the Sobolev space given by the completion of $C^\infty_0(\Omega)$ with respect to the semi-norm

$$\|f|W^k_p(\Omega)\| = \max_{|\beta|=k} \|D^\beta f|L^p(\Omega)\|.$$

Here $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n_0$ stands for some multi-index,

$$|\beta| = \beta_1 + \cdots + \beta_n$$

and D^β are classical derivatives, that is,

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}.$$

We use the notation $W^k_p(\mathbb{T}^n)$ to mean the periodic counterpart which is obtained by replacing $L^p(\Omega)$ by $L^p(\mathbb{T}^n)$.

In this paper we always work with the Besov spaces $B^s_{p,q}(\Omega)$ given by the modulus of smoothness (see (5) below). However, in the literature one can find several equivalent approaches to introduce Besov spaces. For instance, using the Fourier transform, decomposition methods, interpolation, … but the equivalence constants may depend on the involved parameters s, p, q and the dimension n of the underlying domain. As a consequence, studying sharp embedding constants, it is important to fix a natural norm on Besov spaces. Next we recall the definition of Besov spaces through the modulus of smoothness.

Given $h \in \mathbb{R}^n$, we let $\Omega_h = \{x : x+\beta h \in \Omega \text{ for all } 0 \le \beta \le 1\}$. For $x \in \Omega_{kh}$, we introduce

$$\Delta_h^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(x+ih)$$

and define the k-th order modulus of smoothness of $f \in L^p(\Omega)$ by

$$\omega_k(f,t)_p = \sup_{|h|\le t} \|\Delta_h^k f | L^p(\Omega_{kh})\|, \quad t > 0. \tag{4}$$

In the case $k = 1$ we simply write $\omega(f,t)_p$ instead of $\omega_1(f,t)_p$. In the definition of $\omega_k(f,t)_p$ for a periodic function $f \in L^p(\mathbb{T}^n)$, the norm is taken over all $\mathbb{T}^n$.

Let $s > 0, \alpha \in \mathbb{R}$ and $1 \le q \le \infty$. We take $k \in \mathbb{N}$ such that $k > s$. The Besov space $B_{p,q}^{s,\alpha}(\Omega)$ consists of all $f \in L^p(\Omega)$ having a finite semi-norm

$$\|f|B_{p,q}^{s,\alpha}(\Omega)\|_k = \left(\int_0^\infty \left(t^{-s}(1+|\log t|)^\alpha \omega_k(f,t)_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{5}$$

(with the usual modification if $q = \infty$). Any choice of $k > s$ will define the same space with equivalent norms where the corresponding equivalence constants depend on s. This justifies the subscript k in our notation for Besov semi-norms (5). The spaces $B_{p,q}^{s,\alpha}(\Omega)$ have classical smoothness s and an additional logarithmic smoothness of exponent α. In particular, if $\alpha = 0$ we recover the classical Besov spaces $B_{p,q}^{s}(\Omega)$ equipped with the semi-norm $\|\cdot|B_{p,q}^{s}(\Omega)\|_k$.

This approach also allows us to introduce the Besov spaces $B_{p,q}^{0,\alpha}(\Omega)$ involving only logarithmic smoothness. The space $B_{p,q}^{0,\alpha}(\Omega)$ is formed by all $f \in L^p(\Omega)$ such that

$$\|f|B_{p,q}^{0,\alpha}(\Omega)\|_k = \left(\int_0^1 \left((1+|\log t|)^\alpha \omega_k(f,t)_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{6}$$

is finite. Here $k \in \mathbb{N}$. These spaces were already investigated by DeVore, Riemenschneider and Sharpley [14] and are attracting a lot of attention in the last years as can be seen in the papers [5, 9–11, 15, 40] and the references given there. Note that the case of interest is when $\alpha \ge -\frac{1}{q}$ if $q < \infty$ ($\alpha > 0$ if $q = \infty$), otherwise it is not hard to check that $B_{p,q}^{0,\alpha}(\Omega) = L^p(\Omega)$.

Putting $s = k = 1$ and $\alpha = 0$ in the Besov semi-norm (5), it is clear that the only functions satisfying that this condition is finite are constant functions. One can overcome this obstruction with the help of the additional parameter α. In particular, the logarithmic Lipschitz space $\mathrm{Lip}_{p,q}^{(1,-\alpha)}(\Omega)$ is formed by all functions $f \in L^p(\Omega)$ for which

$$\|f|\mathrm{Lip}_{p,q}^{(1,-\alpha)}(\Omega)\| = \left(\int_0^1 \left(t^{-1}(1-\log t)^{-\alpha} \omega(f,t)_p\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{7}$$

is finite. See [23, 24]. Here, $\alpha > \frac{1}{q}$ if $q < \infty$ ($\alpha \geq 0$ if $q = \infty$). This restriction is natural as otherwise we obtain trivial spaces.

It is worthwhile to remark that the integral in (6) (respectively, (7)), which corresponds to the semi-norm on $B^{0,\alpha}_{p,q}(\Omega)$ (respectively, $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega)$), is defined on the interval $(0,1)$ in contrast to the integral on the larger interval $(0,\infty)$ which defines the semi-norm on the Besov space $B^{s,\alpha}_{p,q}(\Omega)$ for $s > 0$ (cf. (5)). This modification becomes necessary to avoid trivial spaces.

Analogously, one can follow the same approach to introduce the periodic counterparts $B^{s,\alpha}_{p,q}(\mathbb{T}^n)$, $B^{0,\alpha}_{p,q}(\mathbb{T}^n)$ and $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\mathbb{T}^n)$ of the above spaces.

The classical Sobolev embedding theorem between Besov spaces (see, e.g., [35, 6.3]) claims that

$$B^s_{p,r}(\Omega) \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{q})}_{q,r}(\Omega) \quad \text{if } 1 \leq p < q \leq \infty,\ 1 \leq r \leq \infty,\ s > n\left(\frac{1}{p} - \frac{1}{q}\right). \tag{8}$$

Before proceeding further, some comments are in order. Working with Fourier-analytically defined Besov spaces on $\mathbb{R}^n$, the previous embedding holds for all $s \in \mathbb{R}$ (see, e.g., [3, Theorem 6.5.1], [38, Theorem 2.8.1]). Clearly, we need the restriction $s \geq n\big(\frac{1}{p} - \frac{1}{q}\big)$ when we deal with Besov spaces given by the modulus of smoothness. Furthermore, it was proved recently in [20, Corollary 2.8, (2.20)] and [11, Theorem 3.7] that if $s = n\big(\frac{1}{p} - \frac{1}{q}\big)$ then the embedding holds with a loss of $\frac{1}{\min\{q,r\}}$ in the exponent of the logarithmic smoothness of the source space. To be more precise, the following embedding between Besov spaces of generalized smoothness holds

$$B^{n(\frac{1}{p}-\frac{1}{q}),\alpha+\frac{1}{\min\{q,r\}}}_{p,r}(\mathbb{R}^n) \hookrightarrow B^{0,\alpha}_{q,r}(\mathbb{R}^n), \quad \alpha > -\frac{1}{r}. \tag{9}$$

The corresponding result for Besov spaces on $\mathbb{T}^n$ also holds true. On the other hand, the order k with $k > s$ of the modulus of smoothness used in the Besov semi-norms (5) on $B^s_{p,r}(\Omega)$ and $B^{s-n(\frac{1}{p}-\frac{1}{q})}_{q,r}(\Omega)$ will also play a role in the embedding constant of (8). Then we will refer to the values $s = n(\frac{1}{p} - \frac{1}{q})$ and $s = k$ as the limiting cases of the embedding given in (8).

The main aim of this paper is to investigate the behaviour of the embedding constant of the embedding (8) as the smoothness parameter s approaches the critical values $n\big(\frac{1}{p} - \frac{1}{q}\big)$ and k. This question was already studied by Kolyada and Lerner [32, Theorem 1.1] in the case of spaces defined over $\mathbb{R}^n$ with indices $1 \leq p < q < \infty, 1 \leq r < \infty$ and $k = 1$. Under these assumptions, they stated that if $n\big(\frac{1}{p} - \frac{1}{q}\big) < s < 1$ then

$$\|f|B^{s-n(\frac{1}{p}-\frac{1}{q})}_{q,r}(\mathbb{R}^n)\|_1 \leq c_{p,q,r,n} \frac{(1-s)^{\frac{1}{\max\{p,r\}}}}{\Big(s-n\big(\frac{1}{p}-\frac{1}{q}\big)\Big)^{\frac{1}{r}}} \|f|B^s_{p,r}(\mathbb{R}^n)\|_1, \quad f \in B^s_{p,r}(\mathbb{R}^n) \tag{10}$$

where the constant $c_{p,q,r,n}$ does not depend on s and f (but may depend on p, q, r and n). As they showed in [32, Remark 3.2] the exponent $\frac{1}{\max\{p,r\}}$ in (10) is sharp in general. However, there is an inaccuracy in the exponent of the term $s - n\big(\frac{1}{p} - \frac{1}{q}\big)$. Indeed, we show here that for any $f \in B^s_{p,r}(\mathbb{R}^n)$,

$$\|f|B^{s-n(\frac{1}{p}-\frac{1}{q})}_{q,r}(\mathbb{R}^n)\|_1 \le c_{p,q,r,n} \frac{(1-s)^{\frac{1}{\max\{p,r\}}}}{\Big(s - n\big(\frac{1}{p} - \frac{1}{q}\big)\Big)^{\frac{1}{\min\{q,r\}}}} \|f|B^s_{p,r}(\mathbb{R}^n)\|_1, \tag{11}$$

and the exponent $\frac{1}{\min\{q,r\}}$ in (11) is optimal in general. The approach that we follow is completely different from the one in [32] and it is based on ideas of interpolation theory which work for higher-order Besov spaces defined over $\mathbb{R}^n$, as well as on $\mathbb{T}^n$ and bounded Lipschitz domains in $\mathbb{R}^n$. In addition, it allows us to cover the extreme cases when $r = \infty$ or $q = \infty$. In this latter case, the constant in (11) is replaced by $c_{p,r,n}(1-s)^{\frac{1}{\max\{p,r\}}}(s - \frac{n}{p})^{-1}$.

It is known that there exist limiting relationships between Sobolev and Besov (semi-) norms. In particular, it was proved in [6] that for any $f \in W^1_p(\mathbb{R}^n)$,

$$\lim_{s\to 1-} (1-s) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy = c\,\|\nabla f|L^p(\mathbb{R}^n)\|^p. \tag{12}$$

The corresponding result when $s \to 0+$ was achieved by Maz'ya and Shaposhnikova [34], who observed that

$$\lim_{s\to 0+} s \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy = c\,\|f|L^p(\mathbb{R}^n)\|^p. \tag{13}$$

As we shall point out (see Remark 3.4 below), it is a simple consequence of (12) and (13) that the Sobolev-type inequalities

$$\|f|B^{1-n(\frac{1}{p}-\frac{1}{q})}_{q,p}(\mathbb{R}^n)\|_1 \le c_1\|\nabla f|L^p(\mathbb{R}^n)\|$$

and

$$\|f|L^q(\mathbb{R}^n)\| \le c_2\|f|B^{n(\frac{1}{p}-\frac{1}{q})}_{p,q}(\mathbb{R}^n)\|_1$$

can be considered as the limiting cases of (11) when $s \to 1-$ and $s \to n\big(\frac{1}{p}-\frac{1}{q}\big)+$, respectively.

We also study the limiting cases of the embedding (8), that is, $s = n\big(\frac{1}{p} - \frac{1}{q}\big)$ or $s = k = 1$. Here, we recall that k denotes the order of the modulus of smoothness used in the semi-norm (5) on the involved spaces in (8). As mentioned above, spaces of generalized smoothness arise in a natural way when we investigate the case $s = n\big(\frac{1}{p} - \frac{1}{q}\big)$ (see (9)). In this paper, we derive the corresponding embedding when $s = k = 1$. This is done with the help of the spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega)$ (see (7)) and applying extrapolation methods to the inequality (11). In this case, we lose $\frac{1}{\max\{p,r\}}$ in the exponent of the logarithmic smoothness of the target space. More specifically, we obtain that

$$\mathrm{Lip}^{(1,-\alpha)}_{p,r}(\Omega) \hookrightarrow B^{1-n(\frac{1}{p}-\frac{1}{q}),-\alpha+\frac{1}{\max\{p,r\}}}_{q,r}(\Omega). \tag{14}$$

In addition we show that the losses of logarithmic smoothness in (9) and (14) are indeed necessary because these embeddings are optimal. To get this we rely on realization results for the modulus of smoothness and integrability theorems for Fourier series.

The paper is organized as follows. Section 2 contains necessary information concerning real interpolation and extrapolation that we need in the paper. In Section 3 we study the rates of blow up of the embedding constant of the embedding $B^s_{p,r}(\Omega) \hookrightarrow B^{s-n(\frac{1}{p}-\frac{1}{q})}_{q,r}(\Omega), p < q$, and we derive some sharp limiting embeddings between smoothness spaces.

2. Preliminaries

Let (A_0, A_1) be a pair of Banach spaces that are compatible in the sense that both A_0 and A_1 are continuously embedded in some common Banach space. The K-functional for (A_0, A_1) is defined, for $t > 0$ and $f \in A_0 + A_1$, by

$$K(t, f) = K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \{\|f_0|A_0\| + t\|f_1|A_1\|\}.$$

In some specific situations, it is customary to work with a modified K-functional in which $\|\cdot|A_0\|$ and/or $\|\cdot|A_1\|$ are semi-norms. All the results of this section hold as well for such a modification.

For $0 < \theta < 1$ and $1 \le q \le \infty$, the real interpolation space $(A_0, A_1)_{\theta,q}$ is the set of all $f \in A_0 + A_1$ such that

$$\|f|(A_0, A_1)_{\theta,q}\| = \left(\int_0^\infty \left(t^{-\theta} K(t, f)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \tag{15}$$

is finite (as usual, the integral should be replaced by the supremum if $q = \infty$). See [2, 3, 8, 38].

For later use, we recall some properties of the spaces $(A_0, A_1)_{\theta,q}$. Since $K(t, f; A_0, A_1) = tK(t^{-1}, f; A_1, A_0)$, we have that

$$\|f|(A_0, A_1)_{\theta,q}\| = \|f|(A_1, A_0)_{1-\theta,q}\|. \tag{16}$$

See also [3, Theorem 3.4.1].

The following sharp reiteration formulas turn out to be an essential key in the sequel. Let $0 < \theta, \sigma < 1$ and $1 \leq p, q \leq \infty$. Then, there are positive constants c_1, c_2 which are independent of θ and f such that

$$\begin{aligned} & c_1(1-\theta)^{-\frac{1}{\max\{p,q\}}} \|f|(A_0,A_1)_{\theta\sigma,q}\| \\ & \leq \|f|(A_0,(A_0,A_1)_{\sigma,p})_{\theta,q}\| \\ & \leq c_2\left((1-\sigma)^{-\frac{1}{p}} + (1-\theta)^{-\frac{1}{\min\{p,q\}}}\right) \|f|(A_0,A_1)_{\theta\sigma,q}\| \end{aligned} \tag{17}$$

For the proof we refer to [25, Theorem 3.1 and Remark 3.2] (see also [30, Theorem 3]).

Real interpolation spaces can also be introduced by using the J-functional which is defined, for $t > 0$ and $f \in A_0 \cap A_1$, by

$$J(t,f) = J(t,f;A_0,A_1) = \max\{\|f|A_0\|, t\|f|A_1\|\}.$$

Let $(A_0,A_1)_{\theta,q;J}$ be the space of all $f \in A_0 + A_1$ for which

$$\|f|(A_0,A_1)_{\theta,q;J}\| = \inf\left(\int_0^\infty \left(t^{-\theta}J(t,u(t))\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty, \tag{18}$$

where the infimum is taken over all representations

$$f = \int_0^\infty u(t)\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1) \tag{19}$$

with $u(t)$ is a strongly measurable function taking values in $A_0 \cap A_1$. The equivalence theorem (see [3, Theorem 3.3.1]) yields that

$$(A_0,A_1)_{\theta,q} = (A_0,A_1)_{\theta,q;J} \tag{20}$$

with equivalence of norms. Furthermore, the exact dependence on θ and q of the equivalence constants in (20) was studied in [12]. In particular, we have

$$c_1\|f|(A_0,A_1)_{\theta,q;J}\| \leq \theta(1-\theta)\|f|(A_0,A_1)_{\theta,q}\| \leq c_2\|f|(A_0,A_1)_{\theta,q;J}\| \tag{21}$$

where the constants c_1 and c_2 are independent of f and θ. The latter estimates were shown in [12, Corollary 3.4] for the discrete versions of the K- and J-spaces, but it is readily seen that these discrete norms are equivalent to the corresponding norms given by (15) and (18) with constants independent of θ. Hence, (21) holds.

The extensions of the interpolation methods which are obtained by replacing $t^{-\theta}$ in the definitions (15) and (18) by more general non-negative measurable functions $\mathfrak{g}$ on $(0,\infty)$ are also important. We can define the spaces $(A_0,A_1)_{\mathfrak{g},q}$ and $(A_0,A_1)_{\mathfrak{g},q;J}$ as the sets of all $f \in A_0 + A_1$ for which

$$\|f|(A_0,A_1)_{\mathfrak{g},q}\| = \left(\int_0^\infty \left(\mathfrak{g}(t)K(t,f)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \tag{22}$$

and

$$\|f|(A_0,A_1)_{\mathfrak{g},q;J}\| = \inf\left(\int_0^\infty \big(\mathfrak{g}(t)J(t,u(t))\big)^q\frac{dt}{t}\right)^{\frac{1}{q}} < \infty$$

where the infimum is taken over all representations (19), respectively. Our main interest is when $\mathfrak{g}$ has the shape

$$\mathfrak{g}(t) = \begin{cases} t^{-\theta_0}(1-\log t)^{\alpha_0} & \text{for } 0 < t \le 1, \\ t^{-\theta_1}(1+\log t)^{\alpha_\infty} & \text{for } 1 < t < \infty, \end{cases}$$

with $0 \le \theta_0, \theta_1 \le 1$ and $\alpha_0, \alpha_\infty \in \mathbb{R}$. In the particular case $\mathfrak{g}(t) = t^{-\theta}(1+|\log t|)^\alpha$ (i.e., $\theta_0 = \theta_1 = \theta$ and $\alpha_0 = \alpha_\infty = \alpha$), we simply write $(A_0,A_1)_{\theta,q,\alpha}$ and $(A_0,A_1)_{\theta,q,\alpha;J}$. See [18, 19, 22].

The modulus of smoothness $\omega_k(f,t)_p$ given in (4) is intimately connected to the K-functional associated to the pair $(L^p(\Omega), W^k_p(\Omega))$. In fact, there are positive constants c_1 and c_2 depending only on p, k and Ω such that

$$c_1\,\omega_k(f,t)_p \le K(t^k, f; L^p(\Omega), W^k_p(\Omega)) \le c_2\,\omega_k(f,t)_p, \quad t > 0. \tag{23}$$

See [28, Theorem 1] or [2, page 341] (for $\Omega = \mathbb{R}^n$). By the definition of the real interpolation space (15) and (23), we derive that

$$(L^p(\Omega), W^k_p(\Omega))_{s,q} = B^{sk}_{p,q}(\Omega) \tag{24}$$

with equivalence constants which are independent of s. More precisely, there exist positive constants c_1 and c_2 which depend only on p, q, k and Ω for which

$$c_1\,\|f|(L^p(\Omega), W^k_p(\Omega))_{s,q}\| \le \|f|B^{sk}_{p,q}(\Omega)\|_k \le c_2\,\|f|(L^p(\Omega), W^k_p(\Omega))_{s,q}\|. \tag{25}$$

Since (23) also hold for periodic functions, one can replace Ω by $\mathbb{T}^n$ in the previous estimates.

It turns out that Besov spaces of generalized smoothness can be characterized as interpolation spaces involving logarithmic weights. Indeed, it is an immediate consequence of (23) and (22) that

$$(L^p(\Omega), W^k_p(\Omega))_{s,q,\alpha} = B^{sk,\alpha}_{p,q}(\Omega). \tag{26}$$

In the particular case $\alpha = 0$ we recover the formula (24).

Assume now that (A_0, A_1) is pair with $A_1 \hookrightarrow A_0$. We shall also need the following limiting K- and J- spaces. For $\tau \in \mathbb{R}$, we define $(A_0,A_1)_{(0,\tau),q}$ as the collection of all $f \in A_0$ for which

$$\|f|(A_0,A_1)_{(0,\tau),q}\| = \left(\int_0^1 \big((1-\log t)^\tau K(t,f)\big)^q\frac{dt}{t}\right)^{\frac{1}{q}} < \infty \tag{27}$$

and $(A_0, A_1)_{(0,\tau),q;J}$ is formed by all $f \in A_0$ which admit a representation

$$f = \int_0^1 u(t)\frac{dt}{t} \quad \text{(convergence in } A_0\text{)} \tag{28}$$

where $u(t)$ is a strongly measurable function taking values in A_1 with

$$\int_0^1 \big((1-\log t)^\tau J(t,u(t))\big)^q \frac{dt}{t} < \infty.$$

The norm in $(A_0, A_1)_{(0,\tau),q;J}$ is given by

$$\|f|(A_0,A_1)_{(0,\tau),q;J}\| = \inf \left(\int_0^1 \big((1-\log t)^\tau J(t,u(t))\big)^q \frac{dt}{t}\right)^{\frac{1}{q}}$$

where the infimum is taken over all representations (28). See [10, 13].

In sharp contrast to classical interpolation, the equivalence result (20) does not hold true for the spaces $(A_0, A_1)_{(0,\tau),q}$ and $(A_0, A_1)_{(0,\tau),q;J}$. In fact, it was proved in [15, Theorem 3.3] that if $\tau > -\frac{1}{q}$ then

$$(A_0, A_1)_{(0,\tau),q} = (A_0, A_1)_{(0,\tau+1),q;J}. \tag{29}$$

Let

$$\mathfrak{g}(t) = \begin{cases} (1-\log t)^\tau & \text{for } 0 < t \le 1, \\ t^{-\theta} & \text{for } 1 < t < \infty, \end{cases}$$

where $0 \le \theta < 1$ and $\tau \in \mathbb{R}$. It is not hard to check that

$$(A_0, A_1)_{(0,\tau),q;J} = (A_0, A_1)_{\mathfrak{g},q;J} \tag{30}$$

with equivalence of norms.

An example of limiting K-space is the space $B^{0,\alpha}_{p,q}(\Omega)$ (cf. (6)). To be more precise, we have

$$(L^p(\Omega), W^k_p(\Omega))_{(0,\alpha),q} = B^{0,\alpha}_{p,q}(\Omega). \tag{31}$$

This follows immediately from (27) and (23).

Let us recall the extrapolation constructions of scales formed by real interpolation spaces. See [27, 29]. Let $1 \le q \le \infty, 0 \le \theta < 1$ and $j_0 \in \mathbb{N}$ such that $\theta + 2^{-j} < 1$ for all $j \ge j_0$. Let (M_j) be a sequence formed by positive numbers such that $\sum_{j=0}^\infty (\frac{1}{M_j})^{q'} < \infty$, where $\frac{1}{q} + \frac{1}{q'} = 1$. The $\Sigma^{(q)}$ sum of the scale $\{M_j (A_0, A_1)_{\theta+2^{-j},q}\}$ is defined by all elements $f \in A_0 + A_1$ which admit a representation

$$f = \sum_{j=j_0}^\infty g_j, \quad g_j \in (A_0, A_1)_{\theta+2^{-j},q}, \tag{32}$$

with

$$\sum_{j=j_0}^\infty \big(M_j \|g_j|(A_0, A_1)_{\theta+2^{-j},q}\|\big)^q < \infty. \tag{33}$$

Furthermore, $\Sigma^{(q)}(M_j(A_0,A_1)_{\theta+2^{-j},q})$ becomes a Banach space under the norm given by

$$\|f|\Sigma^{(q)}(M_j(A_0,A_1)_{\theta+2^{-j},q})\| = \inf\left(\sum_{j=j_0}^{\infty}\big(M_j\|g_j|(A_0,A_1)_{\theta+2^{-j},q}\|\big)^q\right)^{\frac{1}{q}} \tag{34}$$

where the infimum is taken over all possible representations (32) such that (33) holds. Note that the spaces $\Sigma^{(q)}(M_j(A_0,A_1)_{\theta+2^{-j},q})$ are independent of j_0, with equivalence of norms. Similarly, one can introduce the constructions $\Sigma^{(q)}(M_j(A_0,A_1)_{\theta+2^{-j},q;J})$ which are obtained by using the spaces $(A_0,A_1)_{\theta+2^{-j},q;J}$ in (32)–(34).

It turns out that $\Sigma^{(q)}$-spaces can be characterized as interpolation spaces. In the special case that $(M_j)=(2^{j\alpha})$ with $\alpha>0$ and $0<\theta<\theta_0<1$, the following formula holds with equivalence of norms [29, Theorem 4.2]

$$\Sigma^{(q)}(2^{j\alpha}(A_0,A_1)_{\theta+2^{-j},q}) = (A_0,A_1)_{\mathfrak{g},q} \tag{35}$$

where

$$\mathfrak{g}(t) = \begin{cases} t^{-\theta}(1-\log t)^{\alpha} & \text{for } 0<t\le 1,\\ t^{-\theta_0} & \text{for } 1<t<\infty.\end{cases}$$

Consequently, since $\mathfrak{g}(t)\le ct^{-\theta}(1+|\log t|)^{\alpha}, t>0$, we get the embedding

$$(A_0,A_1)_{\theta,q,\alpha} \hookrightarrow \Sigma^{(q)}(2^{j\alpha}(A_0,A_1)_{\theta+2^{-j},q}). \tag{36}$$

Working with extrapolation of J-spaces, the corresponding formula (35) with $\theta=0$ reads

$$\Sigma^{(q)}(2^{j\alpha}(A_0,A_1)_{2^{-j},q;J}) = (A_0,A_1)_{\mathfrak{g},q;J} \tag{37}$$

where

$$\mathfrak{g}(t) = \begin{cases} (1-\log t)^{\alpha} & \text{for } 0<t\le 1,\\ t^{-\theta_0} & \text{for } 1<t<\infty.\end{cases}$$

See [29, Remark 2.6].

Suppose that $\sum_{j=0}^{\infty} M_j^q<\infty$. Let $0<\theta\le 1$ and $j_0\in\mathbb{N}$ such that $\theta-2^{-j}>0$ for $j\ge j_0$. The $\Delta^{(q)}$ space of the scale $\{M_j(A_0,A_1)_{\theta-2^{-j},q}\}$ is formed by all elements $f\in\bigcap_{j=j_0}^{\infty}(A_0,A_1)_{\theta-2^{-j},q}$ satisfying that

$$\|f|\Delta^{(q)}\big(M_j(A_0,A_1)_{\theta-2^{-j},q}\big)\| = \left(\sum_{j=j_0}^{\infty}\big(M_j\|f|(A_0,A_1)_{\theta-2^{-j},q}\|\big)^q\right)^{\frac{1}{q}} \tag{38}$$

is finite. It is readily seen that the spaces $\Delta^{(q)}\big(M_j(A_0,A_1)_{\theta-2^{-j},q}\big)$ are independent of j_0, in the sense of equivalent norms.

The corresponding characterization of $\Delta^{(q)}$-spaces as K-spaces reads as follows

$$\Delta^{(q)}\big(M_j(A_0, A_1)_{\theta-2^{-j},q}\big) = (A_0, A_1)_{\mathfrak{g},q} \tag{39}$$

where

$$\mathfrak{g}(t) = \Big(\sum_{j=j_0}^{\infty} M_j^q t^{2^{-j}q}\Big)^{\frac{1}{q}} t^{-\theta}, \quad t > 0. \tag{40}$$

The proof of (39) can easily be obtained by using Fubini and the definition of the K-space (22).

3. Sharp embedding constant for Besov spaces and limiting embeddings

Let $1 \le p < q \le \infty, 1 \le r \le \infty, k \in \mathbb{N}$ and $\gamma = n\big(\frac{1}{p} - \frac{1}{q}\big) < s < k$. We are concerned with the embedding $B^s_{p,r}(\Omega) \hookrightarrow B^{s-\gamma}_{q,r}(\Omega)$, as well as its periodic counterpart. First, we study the asymptotic behaviour of the embedding constant as $s \to k-$ and $s \to \gamma+$.

Theorem 3.1. *Let $s > 0, 1 < p < q < \infty$ and $1 \le r \le \infty$. Put $\gamma = n\big(\frac{1}{p} - \frac{1}{q}\big)$. Take any $k \in \mathbb{N}$ such that $s < k$. Assume that $\gamma < s$, then there exists a positive constant $c_{p,q,r,k,n}$ which depends only on p, q, r, k and n such that*

$$\|f|B^{s-\gamma}_{q,r}(\Omega)\|_k \le c_{p,q,r,k,n} \frac{(k-s)^{\frac{1}{\max\{p,r\}}}}{(s-\gamma)^{\frac{1}{\min\{q,r\}}}} \|f|B^s_{p,r}(\Omega)\|_k \tag{41}$$

for all $f \in B^s_{p,r}(\Omega)$. The same result holds for periodic functions defined on $\mathbb{T}^n$.

Proof. Let θ be given by the equation $s = k\theta + \gamma(1-\theta)$. We interpolate by the real interpolation method the well-known embeddings

$$B^\gamma_{p,q}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{and} \quad W^k_p(\Omega) \hookrightarrow B^{k-\gamma}_{q,p}(\Omega) \tag{42}$$

(see [4, 36]) which imply that

$$\|f|(L^q(\Omega), B^{k-\gamma}_{q,p}(\Omega))_{\theta,r}\| \le c\|f|(B^\gamma_{p,q}(\Omega), W^k_p(\Omega))_{\theta,r}\| \tag{43}$$

where c is a positive constant that is independent of θ and f (but may depend on p, q, r, k and n). Using (24), (16) and (17), we derive

$$\begin{aligned}
\|f|(B^\gamma_{p,q}(\Omega), W^k_p(\Omega))_{\theta,r}\| &\le c_1\|f|((L^p(\Omega), W^k_p(\Omega))_{\frac{\gamma}{k},q}, W^k_p(\Omega))_{\theta,r}\| \\
&= c_1\|f|(W^k_p(\Omega), (W^k_p(\Omega), L^p(\Omega))_{1-\frac{\gamma}{k},q})_{1-\theta,r}\| \\
&\le c_2\Big(\Big(\frac{\gamma}{k}\Big)^{-\frac{1}{q}} + \theta^{-\frac{1}{\min\{q,r\}}}\Big)\|f|(W^k_p(\Omega), L^p(\Omega))_{(1-\theta)(1-\frac{\gamma}{k}),r}\| \\
&= c_2\Big(\Big(\frac{\gamma}{k}\Big)^{-\frac{1}{q}} + \theta^{-\frac{1}{\min\{q,r\}}}\Big)\|f|(L^p(\Omega), W^k_p(\Omega))_{\frac{s}{k},r}\| \\
&\le c_3\Big(\Big(\frac{\gamma}{k}\Big)^{-\frac{1}{q}} + \theta^{-\frac{1}{\min\{q,r\}}}\Big)\|f|B^s_{p,r}(\Omega)\|_k
\end{aligned}$$

where in the last equivalence we have used (25). To characterize the interpolation norm on the left-hand side in (43) we can proceed similarly to derive

$$(1-\theta)^{-\frac{1}{\max\{p,r\}}}\|f|B^{s-\gamma}_{q,r}(\Omega)\|_k \le c_4\|f|(L^q(\Omega),B^{k-\gamma}_{q,p}(\Omega))_{\theta,r}\|$$

where the constant $c_4>0$ does not depend on θ.

Finally, it is not hard to check that there is $c_5>0$, which is independent of s with $\gamma<s<k$, such that

$$(1-\theta)^{\frac{1}{\max\{p,r\}}}\left(\left(\frac{\gamma}{k}\right)^{-\frac{1}{q}}+\theta^{-\frac{1}{\min\{q,r\}}}\right)\le c_5(k-s)^{\frac{1}{\max\{p,r\}}}(s-\gamma)^{-\frac{1}{\min\{q,r\}}}$$

which yields the desired estimate (41).

The proof in the case that we work with periodic functions can be carried out in a similar way. □

Remark 3.2. The previous result is also true in the limiting case $p=1$ with $k=1$ if we work with Besov spaces defined over $\mathbb{R}^n$ with $n\ge 2$ because the embedding $W^1_p(\mathbb{R}^n)\hookrightarrow B^{1-\gamma}_{q,p}(\mathbb{R}^n)$ holds (see [31]).

Remark 3.3. The corresponding result to Theorem 3.1 when $q=\infty$ reads as follows: Let $s>0, 1<p<\infty, 1\le r\le\infty$ and $k\in\mathbb{N}$ with $k>s$. Put $\gamma=\frac{n}{p}$. If $\gamma<s$, then there exists a positive constant $c_{p,r,k,n}$ depending only on p,r,k and n such that

$$\|f|B^{s-\frac{n}{p}}_{\infty,r}(\Omega)\|_k\le c_{p,r,k,n}\frac{(k-s)^{\frac{1}{\max\{p,r\}}}}{s-\frac{n}{p}}\|f|B^s_{p,r}(\Omega)\|_k,\quad f\in B^s_{p,r}(\Omega).$$

The proof is similar to that given in Theorem 3.1, but now we interpolate between the well-known embedding

$$B^{\frac{n}{p}}_{p,1}(\Omega)\hookrightarrow L^\infty(\Omega)\quad\text{and}\quad W^k_p(\Omega)\hookrightarrow B^{k-\frac{n}{p}}_{\infty,p}(\Omega)$$

(see [26,31]).

Remark 3.4. Assume that $0<s<1$ and $1<p<\infty$. In [32, Proposition 2.3] it is shown that

$$\begin{aligned}(nv_n)^{\frac{1}{p}}2^{-n-2}\|f|B^s_{p,p}(\mathbb{R}^n)\|_1 &\le \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dxdy\right)^{\frac{1}{p}}\\ &\le ((n+p)v_n)^{\frac{1}{p}}\|f|B^s_{p,p}(\mathbb{R}^n)\|_1.\end{aligned}$$

Here v_n denotes the volume of the unit ball in $\mathbb{R}^n$. Hence our semi-norm $\|\cdot|B^s_{p,p}(\mathbb{R}^n)\|_1$ is uniformly equivalent with respect to s to that used in [6,7,34].

Taking limits when $s \to \gamma+$ with $r = q$ in the inequality (41) and applying (13), we get that

$$\|f|L^q(\mathbb{R}^n)\| \le c\|f|B^{\gamma}_{p,q}(\mathbb{R}^n)\|_1 \tag{44}$$

for any $f \in B^{\gamma}_{p,q}(\mathbb{R}^n)$. Analogously, when $s \to 1-$, it follows from (12) that

$$\|f|B^{1-\gamma}_{q,p}(\mathbb{R}^n)\|_1 \le c\|\nabla f|L^p(\mathbb{R}^n)\| \tag{45}$$

for any $f \in W^1_p(\mathbb{R}^n)$. Then, we have shown that the embeddings (44) and (45) are simple consequences of the inequality (41) using the limits of Besov norms given by (12) and (13). Conversely, the proof of Theorem 3.1 exhibits that (41) follows from (44) and (45) (cf. (42)) together with the sharp reiteration formula (17). Hence, in some sense, the sharp inequality (41) turns out to be equivalent to the classical embeddings (44) and (45).

In the rest of the paper we study the limiting cases of the embedding (8) when $s = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and $s = k = 1$. First, we investigate the case $s = n\left(\frac{1}{p} - \frac{1}{q}\right)$. Let us recall that the following embedding holds (see (9))

$$B^{n\left(\frac{1}{p}-\frac{1}{q}\right),\alpha+\frac{1}{\min\{q,r\}}}_{p,r}(\mathbb{T}^n) \hookrightarrow B^{0,\alpha}_{q,r}(\mathbb{T}^n).$$

Here $1 < p < q < \infty, 1 \le r \le \infty$ and $\alpha > -\frac{1}{r}$. Note that there is a loss of logarithmic smoothness in order to have $f \in B^{0,\alpha}_{q,r}(\mathbb{T}^n)$. This result is the best possible in general as we state next.

Proposition 3.5. *Let $1 < p < q \le r < \infty$ and $\alpha > -\frac{1}{r}$. For any $\varepsilon > 0$ there is a function $f \in B^{\frac{1}{p}-\frac{1}{q},\alpha+\frac{1}{q}-\varepsilon}_{p,r}(\mathbb{T})$ such that $f \notin B^{0,\alpha}_{q,r}(\mathbb{T})$.*

In order to show Proposition 3.5 we will need the technical results given in Lemmas 3.6 and 3.7 below. Lemma 3.6 is due to Askey and Wainger [1] and consists of a generalization of the well-known Hardy-Littlewood theorem for Fourier series with monotone coefficients (cf. [41, Chapter XII, (6.6)]). On the other hand, Lemma 3.7 is a realization result of the modulus of smoothness in terms of the partial Fourier series. See [37].

As usual, $[\cdot]$ denotes the greatest integer function, $S_l f$ stands for the l-th partial sum of the Fourier series f and $S_l' f$ is its first derivative.

Lemma 3.6. *Let $1 < p < \infty$ and let $\sum_{j=1}^{\infty} a_j \cos(jx)$ be the Fourier series of an integrable function f.*

(i) *If the sequences (a_j) and (λ_j) are such that $\sum_{\nu=j}^{\infty} |a_\nu - a_{\nu+1}| \le c\lambda_j, j \in \mathbb{N}$, for some $c > 0$ which is independent of j, then there is $K_1 > 0$ for which*

$$\|f|L^p(\mathbb{T})\|^p \le K_1 \sum_{j=1}^{\infty} j^{p-2}\lambda_j^p.$$

(ii) *If* (a_j) *is a nonnegative sequence, then there is* $K_2 > 0$ *such that*

$$\sum_{j=1}^{\infty} \left(\sum_{\nu=[\frac{j}{2}]}^{j} a_\nu \right)^p j^{-2} \le K_2 \|f|L^p(\mathbb{T})\|^p.$$

Lemma 3.7. *Let* $1 < p < \infty$*. Assume* $f \in L^p(\mathbb{T})$*. Then,*

$$\begin{aligned} & c_1 \left(\|f - S_l f|L^p(\mathbb{T})\| + l^{-1} \|S_l' f|L^p(\mathbb{T})\| \right) \\ & \le \omega \left(f, l^{-1}\right)_p \\ & \le c_2 \left(\|f - S_l f|L^p(\mathbb{T})\| + l^{-1} \|S_l' f|L^p(\mathbb{T})\| \right) \quad \text{for all } l \in \mathbb{N}. \end{aligned}$$

Proof of Proposition 3.5. Take $\beta \in \mathbb{R}$ such that

$$-\frac{1}{q} - \frac{1}{r} - \alpha \le \beta < \min \left\{ -\frac{1}{q} - \frac{1}{r} - \alpha + \varepsilon, -\frac{1}{q} \right\}.$$

We consider the Fourier series $f(x) \sim \sum_{j=1}^{\infty} a_j \cos(jx)$, $x \in \mathbb{T}$, with $a_j = j^{-\frac{1}{q'}}(1+\log j)^\beta$, $j \in \mathbb{N}$. Since the sequence (a_j) is monotonically decreasing to zero we have $\sum_{\nu=j}^{\infty} |a_\nu - a_{\nu+1}| \le c a_j$. Then, applying Lemma 3.6(i) we derive

$$\|f|L^q(\mathbb{T})\|^q \le K_1 \sum_{j=1}^{\infty} \left(j^{-\frac{1}{q'}} (1+\log j)^\beta \right)^q j^{q-2} = K_1 \sum_{j=1}^{\infty} (1+\log j)^{\beta q} \frac{1}{j} < \infty$$

since $\beta + \frac{1}{q} < 0$. Hence, $f \in L^q(\mathbb{T})$.

We proceed to estimate $\omega(f,t)_q$ with the help of Lemma 3.7. We have for $l \in \mathbb{N}$ that

$$\begin{aligned} & \omega(f,l^{-1})_q \\ & \ge c(\|f - S_l f|L^q(\mathbb{T})\| + l^{-1} \|S_l' f|L^q(\mathbb{T})\|) \\ & \ge c \left\| \sum_{j=l+1}^{\infty} j^{-\frac{1}{q'}} (1+\log j)^\beta \cos(jx) |L^q(\mathbb{T}) \right\| + c\, l^{-1} \left\| \sum_{j=1}^{l} j^{1-\frac{1}{q'}} (1+\log j)^\beta \cos(jx) |L^q(\mathbb{T}) \right\| \end{aligned}$$

where we have used the boundedness of the conjugate function in the last inequality (see, e.g., [21, Theorem 3.5.6]). Using now Lemma 3.6(ii), we get

$$\begin{aligned} \omega(f,l^{-1})_q & \ge c \left(\sum_{j=2l}^{\infty} \left(j^{-\frac{1}{q'}} (1+\log j)^\beta \right)^q j^{q-2} \right)^{\frac{1}{q}} + c\, l^{-1} \left(\sum_{j=1}^{l} \left(j^{1-\frac{1}{q'}} (1+\log j)^\beta \right)^q j^{q-2} \right)^{\frac{1}{q}} \\ & = c \left(\sum_{j=2l}^{\infty} (1+\log j)^{\beta q} \frac{1}{j} \right)^{\frac{1}{q}} + c\, l^{-1} \left(\sum_{j=1}^{l} \left(j (1+\log j)^\beta \right)^q \frac{1}{j} \right)^{\frac{1}{q}} \\ & \ge c \left((1+\log l)^{\beta + \frac{1}{q}} + l^{-1} l (1+\log l)^\beta \right) \\ & \ge c (1+\log l)^{\beta + \frac{1}{q}} \end{aligned}$$

because $\beta + \frac{1}{q} < 0$. By monotonicity, we derive $\omega(f,t)_q \geq c(1-\log t)^{\beta+\frac{1}{q}}$ for $0 < t < 1$. Consequently,

$$\|f|B^{0,\alpha}_{q,r}(\mathbb{T})\|_1 = \left(\int_0^1 \big((1-\log t)^{\alpha}\omega(f,t)_q\big)^r \frac{dt}{t}\right)^{\frac{1}{r}} \geq c\left(\int_0^1 (1-\log t)^{(\alpha+\beta+\frac{1}{q})r}\frac{dt}{t}\right)^{\frac{1}{r}} = \infty$$

since $\alpha+\beta+\frac{1}{q}+\frac{1}{r}\geq 0$. Then, $f \notin B^{0,\alpha}_{q,r}(\mathbb{T})$.

On the other hand, using again Lemma 3.6(i), we obtain

$$\|f|L^p(\mathbb{T})\|^p \leq K_1 \sum_{j=1}^{\infty}\big(j^{-\frac{1}{q'}}(1+\log j)^{\beta}\big)^p j^{p-2} = K_1\sum_{j=1}^{\infty}\big(j^{-(\frac{1}{p}-\frac{1}{q})}(1+\log j)^{\beta}\big)^p \frac{1}{j} < \infty,$$

which implies that $f \in L^p(\mathbb{T})$. We proceed to estimate its modulus of smoothness of first order with respect to the $L^p(\mathbb{T})$-norm. We have

$$\begin{aligned}
\omega(f,l^{-1})_p &\leq c(\|f - S_l f|L^p(\mathbb{T})\| + l^{-1}\|S_l' f|L^p(\mathbb{T})\|)\\
&\leq c\, l^{-(\frac{1}{p}-\frac{1}{q})}(1+\log l)^{\beta} + c\left(\sum_{j=l+1}^{\infty}\big(j^{-\frac{1}{q'}}(1+\log j)^{\beta}\big)^p j^{p-2}\right)^{\frac{1}{p}}\\
&\quad + c\, l^{-1}\left(\sum_{j=1}^{l}\big(l^{1-\frac{1}{q'}}(1+\log l)^{\beta}\big)^p j^{p-2}\right)^{\frac{1}{p}}\\
&\leq c\Big(l^{-(\frac{1}{p}-\frac{1}{q})}(1+\log l)^{\beta} + l^{-1} l^{\frac{1}{q}}(1+\log l)^{\beta} l^{1-\frac{1}{p}}\Big)\\
&\leq c\, l^{-(\frac{1}{p}-\frac{1}{q})}(1+\log l)^{\beta}
\end{aligned}$$

and then

$$\omega(f,t)_p \leq c\, t^{\frac{1}{p}-\frac{1}{q}}(1-\log t)^{\beta}, \quad 0 < t < 1.$$

Therefore,

$$\begin{aligned}
&\left(\int_0^{\infty}\big(t^{-(\frac{1}{p}-\frac{1}{q})}(1+|\log t|)^{\alpha+\frac{1}{q}-\varepsilon}\omega(f,t)_p\big)^r\frac{dt}{t}\right)^{\frac{1}{r}}\\
&\leq c\|f|L^p(\mathbb{T})\| + \left(\int_0^1\big(t^{-(\frac{1}{p}-\frac{1}{q})}(1-\log t)^{\alpha+\frac{1}{q}-\varepsilon}\omega(f,t)_p\big)^r\frac{dt}{t}\right)^{\frac{1}{r}}\\
&\leq c\,\|f|L^p(\mathbb{T})\| + c\left(\int_0^1 (1-\log t)^{(\alpha+\frac{1}{q}-\varepsilon+\beta)r}\frac{dt}{t}\right)^{\frac{1}{r}}\\
&< \infty
\end{aligned}$$

since $\alpha+\frac{1}{q}-\varepsilon+\beta+\frac{1}{r}<0$. Hence, $f \in B^{\frac{1}{p}-\frac{1}{q},\alpha+\frac{1}{q}-\varepsilon}_{p,r}(\mathbb{T})$. The proof is complete. □

Remark 3.8. We claim that the exponent $\frac{1}{\min\{q,r\}}$ in Theorem 3.1 is sharp in general. Indeed, assume that $q < r$ and that there exists q_0 satisfying that $q < q_0$ and

$$\|f|B^{s-(\frac{1}{p}-\frac{1}{q})}_{q,r}(\mathbb{T})\|_k \leq c\frac{1}{\big(s-\big(\frac{1}{p}-\frac{1}{q}\big)\big)^{\frac{1}{q_0}}}\|f|B^s_{p,r}(\mathbb{T})\|_k, \quad f\in B^s_{p,r}(\mathbb{T}), \tag{46}$$

where c is uniform with respect to $s \to \big(\frac{1}{p}-\frac{1}{q}\big)+$. Note that $k > \frac{1}{p}-\frac{1}{q}$. We observe that (46) can be rewritten in terms of interpolation spaces via (25). Namely, we have

$$\|f|(L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{(s-(\frac{1}{p}-\frac{1}{q}))k^{-1},r}\| \leq c\frac{1}{\big(s-\big(\frac{1}{p}-\frac{1}{q}\big)\big)^{\frac{1}{q_0}}}\|f|(L^p(\mathbb{T}),W^k_p(\mathbb{T}))_{\frac{s}{k},r}\|$$

for all $f\in B^s_{p,r}(\mathbb{T})$ and $s \to \big(\frac{1}{p}-\frac{1}{q}\big)+$. In particular, for all $j\in\mathbb{N}$ it holds that

$$\|f|(L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{2^{-j},r}\| \leq c\, 2^{\frac{j}{q_0}}\|f|(L^p(\mathbb{T}),W^k_p(\mathbb{T}))_{(\frac{1}{p}-\frac{1}{q})k^{-1}+2^{-j},r}\|. \tag{47}$$

Suppose first that $q_0 \leq r$. Let $\alpha > -\frac{1}{r}$. Multiplying both sides of (47) by $2^{j\alpha}$ and using the $\Sigma^{(r)}$-extrapolation approach (see (34)), we derive that

$$\Sigma^{(r)}\Big(2^{j(\alpha+\frac{1}{q_0})}(L^p(\mathbb{T}),W^k_p(\mathbb{T}))_{(\frac{1}{p}-\frac{1}{q})k^{-1}+2^{-j},r}\Big)\hookrightarrow\Sigma^{(r)}\big(2^{j\alpha}(L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{2^{-j},r}\big). \tag{48}$$

Since $\alpha > -\frac{1}{q_0}$, it follows from (36) that

$$\begin{aligned}&\Sigma^{(r)}\Big(2^{j(\alpha+\frac{1}{q_0})}(L^p(\mathbb{T}),W^k_p(\mathbb{T}))_{(\frac{1}{p}-\frac{1}{q})k^{-1}+2^{-j},r}\Big)\\ &\hookleftarrow (L^p(\mathbb{T}),W^k_p(\mathbb{T}))_{(\frac{1}{p}-\frac{1}{q})k^{-1},r,\alpha+\frac{1}{q_0}} = B^{\frac{1}{p}-\frac{1}{q},\alpha+\frac{1}{q_0}}_{p,r}(\mathbb{T})\end{aligned} \tag{49}$$

where in the last equivalence we have used (26).

Next we deal with the right-hand side space in (48). Let

$$\mathfrak{g}(t) = \begin{cases}(1-\log t)^{\alpha+1} & \text{for } 0<t\leq 1,\\ t^{-\alpha_0} & \text{for } 1<t<\infty,\end{cases}$$

for any $0<\alpha_0<1$. Applying (21), (37), (29)–(31), we get

$$\begin{aligned}\Sigma^{(r)}\big(2^{j\alpha}(L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{2^{-j},r}\big) &= \Sigma^{(r)}\big(2^{j(\alpha+1)}(L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{2^{-j},r;J}\big)\\ &= (L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{\mathfrak{g},r;J}\\ &= (L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{(0,\alpha+1),r;J}\\ &= (L^q(\mathbb{T}),W^k_q(\mathbb{T}))_{(0,\alpha),r}\\ &= B^{0,\alpha}_{q,r}(\mathbb{T}).\end{aligned} \tag{50}$$

Then, by (48)–(50), we get the embedding

$$B^{\frac{1}{p}-\frac{1}{q},\alpha+\frac{1}{q_0}}_{p,r}(\mathbb{T}) \hookrightarrow B^{0,\alpha}_{q,r}(\mathbb{T})$$

which contradicts the statement of Proposition 3.5. Hence, (46) is not true.

If $q_0 > r$, the proof follows from the fact that $\big(s-\big(\frac{1}{p}-\frac{1}{q}\big)\big)^{-\frac{1}{q_0}} < \big(s-\big(\frac{1}{p}-\frac{1}{q}\big)\big)^{-\frac{1}{r}}$ whenever $s \to \big(\frac{1}{p}-\frac{1}{q}\big)+$ and the case given above.

Next we deal with the limiting case $s = k = 1$ in the embedding (8) with the help of the logarithmic Lipschitz spaces $\mathrm{Lip}^{(1,-\alpha)}_{p,q}(\Omega)$ (cf. (7)).

Theorem 3.9. *Let $1 < p < q < \infty, 1 \le r \le \infty$ and $\alpha > \frac{1}{r}$. Assume that $\gamma = n\big(\frac{1}{p}-\frac{1}{q}\big) < 1$. Then*

$$\mathrm{Lip}^{(1,-\alpha)}_{p,r}(\Omega) \hookrightarrow B^{1-\gamma,-\alpha+\frac{1}{\max\{p,r\}}}_{q,r}(\Omega).$$

The periodic counterpart of the previous embedding also holds true.

Proof. A similar argument used to get (47) from (46) can be applied to obtain from (41) the inequality

$$2^{\frac{j}{\max\{p,r\}}}\|f|(L^q(\Omega),W^1_q(\Omega))_{1-\gamma-2^{-j},r}\| \le c\|f|(L^p(\Omega),W^1_p(\Omega))_{1-2^{-j},r}\|, \quad j \ge j_0. \quad (51)$$

Here, j_0 is chosen so that $2^{-j_0} < 1-\gamma$ and c is constant which is independent of f and j. Multiplying both sides of the previous inequality by $2^{-j\alpha}$ and applying the $\Delta^{(r)}$-extrapolation approach (cf. (38)), we arrive at

$$\begin{aligned}&\Delta^{(r)}\big(2^{-j\alpha}(L^p(\Omega),W^1_p(\Omega))_{1-2^{-j},r}\big)\\ &\hookrightarrow \Delta^{(r)}\big(2^{-j(\alpha-\frac{1}{\max\{p,r\}})}(L^q(\Omega),W^1_q(\Omega))_{1-\gamma-2^{-j},r}\big)\end{aligned} \quad (52)$$

Next we deal only with the case $r < \infty$. The modifications for the case $r = \infty$ are usual. According to (39) and (40) we have

$$\Delta^{(r)}\big(2^{-j\alpha}(L^p(\Omega),W^1_p(\Omega))_{1-2^{-j},r}\big) = (L^p(\Omega),W^1_p(\Omega))_{\mathfrak{g},r} \quad (53)$$

with $\mathfrak{g}(t) = \Big(\sum_{j=j_0}^{\infty} 2^{-j\alpha r} t^{2^{-j}r}\Big)^{\frac{1}{r}} t^{-1}$. For $0 < t < 1$, by monotonicity, the function $\mathfrak{g}(t)$ can be estimated by

$$\begin{aligned}\int_0^{2^{-j_0}} t^{\sigma r}\sigma^{(\alpha-\frac{1}{r})r}d\sigma &\le c\int_0^{2^{-j_0}} e^{-\sigma r(1-\log t)}\sigma^{(\alpha-\frac{1}{r})r}d\sigma\\ &= c\int_0^{2^{-j_0}(1-\log t)} e^{-\sigma r}\Big(\frac{\sigma}{1-\log t}\Big)^{(\alpha-\frac{1}{r})r}\frac{d\sigma}{1-\log t}\\ &= c(1-\log t)^{-\alpha r}\int_0^{2^{-j_0}(1-\log t)} e^{-\sigma r}\sigma^{(\alpha-\frac{1}{r})r}d\sigma\\ &= c(1-\log t)^{-\alpha r}\end{aligned}$$

since $\int_0^\infty e^{-\sigma r}\sigma^{(\alpha-\frac{1}{r})r}d\sigma = \int_1^\infty u^{-r}(\log u)^{(\alpha-\frac{1}{r})r}\frac{du}{u} < \infty$. On the other hand, if $t \geq 1$ we have the elementary estimates

$$2^{-j_0\alpha r}t^{2^{-j_0}r} \leq \sum_{j=j_0}^\infty 2^{-j\alpha r}t^{2^{-j}r} \leq t^{2^{-j_0}r}\sum_{j=j_0}^\infty 2^{-j\alpha r} \leq c\,t^{2^{-j_0}r}.$$

Hence $\mathfrak{g}$ can be estimated from above by $\mathfrak{h}$ where

$$\mathfrak{h}(t) = \begin{cases} t^{-1}(1-\log t)^{-\alpha} & \text{for } 0 < t < 1, \\ t^{-(1-2^{-j_0})} & \text{for } t \geq 1, \end{cases}$$

and then,

$$\mathrm{Lip}^{(1,-\alpha)}_{p,r}(\Omega) \hookrightarrow (L^p(\Omega), W^1_p(\Omega))_{\mathfrak{h},r} \hookrightarrow (L^p(\Omega), W^1_p(\Omega))_{\mathfrak{g},r} \tag{54}$$

where the left-hand side embedding is clear by (23).

Analogously, we treat the right-hand side space in (52). Then,

$$\Delta^{(r)}\left(2^{-j(\alpha-\frac{1}{\max\{p,r\}})}(L^q(\Omega), W^1_q(\Omega))_{1-\gamma-2^{-j},r}\right) \hookrightarrow (L^q(\Omega), W^1_q(\Omega))_{\mathfrak{f},r}$$

where

$$\mathfrak{f}(t) = \begin{cases} t^{-(1-\gamma)}(1-\log t)^{-\alpha+\frac{1}{\max\{p,r\}}} & \text{for } 0 < t < 1, \\ t^{-(1-\gamma-2^{-j_0})} & \text{for } t \geq 1. \end{cases}$$

In particular,

$$\Delta^{(r)}\left(2^{-j(\alpha-\frac{1}{\max\{p,r\}})}(L^q(\Omega), W^1_q(\Omega))_{1-\gamma-2^{-j},r}\right) \hookrightarrow B^{1-\gamma,-\alpha+\frac{1}{\max\{p,r\}}}_{q,r}(\Omega) \tag{55}$$

because $t^{-(1-\gamma)}(1+\log t)^{-\alpha+\frac{1}{\max\{p,r\}}} \leq c\,t^{-(1-\gamma-2^{-j_0})}$ for $t \geq 1$. Finally, the desired result follows from (52)–(55). □

Next we show the optimality in general of the previous result.

Proposition 3.10. *Let $1 < r \leq p < q < \infty$ and $\alpha > \frac{1}{r}$. Given any $\varepsilon > 0$, there exists $f \in \mathrm{Lip}^{(1,-\alpha)}_{p,r}(\mathbb{T})$ such that $f \notin B^{1-\frac{1}{p}+\frac{1}{q},-\alpha+\frac{1}{p}+\varepsilon}_{q,r}(\mathbb{T})$.*

Proof. Take $\delta \in \mathbb{R}$ satisfying that

$$\max\left\{\alpha - \frac{1}{r} - \frac{1}{p} - \varepsilon, -\frac{1}{p}\right\} \leq \delta < \alpha - \frac{1}{r} - \frac{1}{p}$$

and we consider the Fourier series $f(x) \sim \sum_{j=1}^\infty a_j\cos(jx)$, $x \in \mathbb{T}$, where $a_j = j^{-(2-\frac{1}{p})}(1+\log j)^\delta$, $j \in \mathbb{N}$. By Lemma 3.6(i),

$$\|f|L^p(\mathbb{T})\|^p \leq K_1\sum_{j=1}^\infty \left(j^{-(2-\frac{1}{p})}(1+\log j)^\delta\right)^p j^{p-2} = K_1\sum_{j=1}^\infty j^{-p}(1+\log j)^{\delta p}\frac{1}{j} < \infty.$$

Therefore, $f \in L^p(\mathbb{T})$. We shall estimate $\omega(f,t)_p$ by using Lemmas 3.7 and 3.6(i), then

$$\begin{aligned}
\omega(f,l^{-1})_p &\leq c\big(\|f-S_l f|L^p(\mathbb{T})\| + l^{-1}\|S_l' f|L^p(\mathbb{T})\|\big)\\
&\leq c\left\|\sum_{j=l+1}^{\infty} j^{-(2-\frac{1}{p})}(1+\log j)^{\delta}\cos(jx)|L^p(\mathbb{T})\right\|\\
&\quad + c\,l^{-1}\left\|\sum_{j=1}^{l} j^{-(1-\frac{1}{p})}(1+\log j)^{\delta}\cos(jx)|L^p(\mathbb{T})\right\|\\
&\leq c\,l^{-1}(1+\log l)^{\delta} + c\left(\sum_{j=l+1}^{\infty} j^{-p}(1+\log j)^{\delta p}\frac{1}{j}\right)^{\frac{1}{p}}\\
&\quad + c\,l^{-1}\left(\sum_{j=1}^{l}(1+\log j)^{\delta p}\frac{1}{j}\right)^{\frac{1}{p}}\\
&\leq c\,l^{-1}(1+\log l)^{\delta+\frac{1}{p}}
\end{aligned}$$

since $\delta > -\frac{1}{p}$. So,

$$\omega(f,t)_p \leq c\,t(1-\log t)^{\delta+\frac{1}{p}}, \quad 0<t<1. \tag{56}$$

Further, the function f also belongs to $L^q(\mathbb{T})$ because

$$\|f|L^q(\mathbb{T})\|^q \leq K_1\sum_{j=1}^{\infty}\big(j^{-(2-\frac{1}{p})}(1+\log j)^{\delta}\big)^q j^{q-2} = K_1\sum_{j=1}^{\infty}\big(j^{-(1-\frac{1}{p}+\frac{1}{q})}(1+\log j)^{\delta}\big)^q\frac{1}{j} < \infty,$$

and applying Lemma 3.6(ii) we can estimate its first $L^q(\mathbb{T})$-modulus of smoothness,

$$\begin{aligned}
&\omega(f,l^{-1})_q\\
&\geq c\big(\|f-S_l f|L^q(\mathbb{T})\| + l^{-1}\|S_l' f|L^q(\mathbb{T})\|\big)\\
&\geq c\left(\sum_{j=2l}^{\infty}\big(j^{-(2-\frac{1}{p})}(1+\log j)^{\delta}\big)^q j^{q-2}\right)^{\frac{1}{q}} + c\,l^{-1}\left(\sum_{j=1}^{l}\big(j^{-(1-\frac{1}{p})}(1+\log j)^{\delta}\big)^q j^{q-2}\right)^{\frac{1}{q}}\\
&= c\left(\sum_{j=2l}^{\infty}\big(j^{-(1-\frac{1}{p}+\frac{1}{q})}(1+\log j)^{\delta}\big)^q\frac{1}{j}\right)^{\frac{1}{q}} + c\,l^{-1}\left(\sum_{j=1}^{l}\big(j^{\frac{1}{p}-\frac{1}{q}}(1+\log j)^{\delta}\big)^q\frac{1}{j}\right)^{\frac{1}{q}}\\
&\geq c\,l^{-(1-\frac{1}{p}+\frac{1}{q})}(1+\log l)^{\delta}.
\end{aligned}$$

Then,

$$\omega(f,t)_q \geq c\,t^{1-\frac{1}{p}+\frac{1}{q}}(1-\log t)^{\delta}, \quad 0<t<1. \tag{57}$$

By (56) we derive that

$$\int_0^1 \big(t^{-1}(1-\log t)^{-\alpha}\omega(f,t)_p\big)^r \frac{dt}{t} \le c \int_0^1 (1-\log t)^{(-\alpha+\delta+\frac{1}{p})r} \frac{dt}{t} < \infty,$$

since $-\alpha+\delta+\frac{1}{p}+\frac{1}{r}<0$. Consequently, $f \in \mathrm{Lip}^{(1,-\alpha)}_{p,r}(\mathbb{T})$. On the other hand, by (57) we have

$$\begin{aligned} &\int_0^\infty \big(t^{-(1-\frac{1}{p}+\frac{1}{q})}(1+|\log t|)^{-\alpha+\frac{1}{p}+\varepsilon}\omega(f,t)_q\big)^r \frac{dt}{t} \\ &\ge \int_0^1 \big(t^{-(1-\frac{1}{p}+\frac{1}{q})}(1-\log t)^{-\alpha+\frac{1}{p}+\varepsilon}\omega(f,t)_q\big)^r \frac{dt}{t} \\ &\ge c\int_0^1 (1-\log t)^{(-\alpha+\frac{1}{p}+\varepsilon+\delta)r}\frac{dt}{t} = \infty \end{aligned}$$

because $-\alpha+\frac{1}{p}+\varepsilon+\delta+\frac{1}{r}\ge 0$. This implies that $f \notin B^{1-\frac{1}{p}+\frac{1}{q},-\alpha+\frac{1}{p}+\varepsilon}_{q,r}(\mathbb{T})$. The proof is finished. □

Remark 3.11. The exponent $\frac{1}{\max\{p,r\}}$ in Theorem 3.1 is sharp in general. This fact was already shown in [32, Remark 3.2] for $k=1$. We can also show it from the extrapolation argument given in the proof of Theorem 3.9 together with Proposition 3.10. Indeed, assume that $r<p$ and that there exists $p_0<p$ such that $\|f|B^{s-\frac{1}{p}+\frac{1}{q}}_{q,r}(\mathbb{T})\|_1 \le c(1-s)^{\frac{1}{p_0}}\|f|B^s_{p,r}(\mathbb{T})\|_1$ for all $f\in B^s_{p,r}(\mathbb{T})$, where the constant c is independent of s with $s\to 1-$. Then, starting from the sharp estimate (51) but now with the exponent $\frac{1}{p_0}$ on its left-hand side, one can follow the proof of Theorem 3.9 to arrive at $\mathrm{Lip}^{(1,-\alpha)}_{p,r}(\mathbb{T}) \hookrightarrow B^{1-\frac{1}{p}+\frac{1}{q},-\alpha+\frac{1}{p_0}}_{q,r}(\mathbb{T})$, which contradicts the statement of Proposition 3.10.

Acknowledgement. The author has been supported in part by the Spanish Ministerio de Economía y Competitividad (MTM2017-84058-P (AEI/FEDER, UE)) and by the FPU grant AP2012-0779 of the Ministerio de Economía y Competitividad.

It is a pleasure to thank Professors Fernando Cobos and Hans Triebel for valuable comments.

The author would like to thank the referees for their useful comments which have led to improve the paper.

References

[1] Askey, R. and Wainger, S., Integrability theorems for Fourier series. *Duke Math. J.* 33 (1966), 223 – 228.

[2] Bennett, C. and Sharpley, R., *Interpolation of Operators.* Boston: Academic Press 1988.

[3] Bergh, J. and Löfström, J., *Interpolation Spaces. An Introduction.* Berlin: Springer 1976.

[4] Besov, O. V., Il'in, V. P. and Nikolskiĭ, S. M., *Integral Representations of Functions and Imbedding Theorems. Vols. 1, 2.* New York: Halsted Press 1978, 1979.

[5] Besov, O. V., On spaces of functions of smoothness zero. *Sb. Math.* 203 (2012), 1077 – 1090.

[6] Bourgain, J., Brézis, H. and Mironescu, P., Another look at Sobolev spaces. In: *Optimal Control and Partial Differential Equations.* In honour of Professor Alain Bensoussan's 60th birthday (Eds.: J. L. Menaldi et al.). Amsterdam: IOS Press 2001, pp. 439 – 455.

[7] Bourgain, J., Brézis, H. and Mironescu, P., Limiting embedding theorem for $W^{s,p}$ when $s \uparrow 1$ and applications. *J. Analyse Math.* 87 (2002), 77 – 101.

[8] Brudnyĭ, Yu. A. and Krugljak, N. Ya., *Interpolation Functors and Interpolation Spaces. Vol. 1.* Amsterdam: North-Holland 1991.

[9] Cobos, F. and Domínguez, O., Embeddings of Besov spaces of logarithmic smoothness. *Studia Math.* 223 (2014), 193 – 204.

[10] Cobos, F. and Domínguez, O., Approximation spaces, limiting interpolation and Besov spaces. *J. Approx. Theory* 189 (2015), 43 – 66.

[11] Cobos, F. and Domínguez, O., On Besov spaces of logarithmic smoothness and Lipschitz spaces. *J. Math. Anal. Appl.* 425 (2015), 71 – 84.

[12] Cobos, F., Fernández-Cabrera, L. M., Karadzhov, G. E. and Kühn, T., On the dependence of parameters in the equivalence theorem for the real method. *Comment. Math.* 55 (2015), 79 – 87.

[13] Cobos, F., Fernández-Cabrera, L. M., Kühn, T. and Ullrich, T., On an extreme class of real interpolation spaces. *J. Funct. Anal.* 256 (2009), 2321 – 2366.

[14] DeVore, R. A., Riemenschneider, S. D. and Sharpley, R. C., Weak interpolation in Banach spaces. *J. Funct. Anal.* 33 (1979), 58 – 94.

[15] Domínguez, O., Tractable embeddings of Besov spaces into small Lebesgue spaces. *Math. Nachr.* 289 (2016), 1739 – 1759.

[16] Edmunds, D. E., Evans, W. D. and Karadzhov, G. E., Sharp estimates of the embedding constants for Besov spaces. *Rev. Mat. Complut.* 19 (2006), 161 – 182.

[17] Edmunds, D. E., Evans, W. D. and Karadzhov, G. E., Sharp estimates of the embedding constants for Besov spaces $b^s_{p,q}, 0 < p < 1$. *Rev. Mat. Complut.* 20 (2007), 445 – 462.

[18] Evans, W. D. and Opic, B., Real interpolation with logarithmic functors and reiteration. *Canad. J. Math.* 52 (2000), 920 – 960.

[19] Evans, W. D., Opic, B. and Pick, L., Real interpolation with logarithmic functors. *J. Inequal. Appl.* 7 (2002), 187 – 269.

[20] Gogatishvili, A., Opic, B., Tikhonov, S. and Trebels, W., Ulyanov-type inequalities between Lorentz–Zygmund spaces. *J. Fourier Anal. Appl.* 20 (2014), 1020 – 1049.

[21] Grafakos, L., *Classical Fourier Analysis.* New York: Springer 2008.

[22] Gustavsson, J., A function parameter in connection with interpolation of Banach spaces. *Math. Scand.* 42 (1978), 289 – 305.

[23] Haroske, D. D., On more general Lipschitz spaces. *Z. Anal. Anwend.* 19 (2000), 781 – 799.

[24] Haroske, D. D., *Envelopes and Sharp Embeddings of Function Spaces.* Boca Raton (FL): Chapman & Hall 2007.

[25] Holmstedt, T., Interpolation of quasi-normed spaces. *Math. Scand.* 26 (1970), 177 – 199.

[26] Jawerth, B., Some observations on Besov and Lizorkin–Triebel spaces. *Math. Scand.* 40 (1977), 94 – 104.

[27] Jawerth, B. and Milman, M., Extrapolation theory with applications. *Mem. Amer. Math. Soc.* 89 (1991), no. 440.

[28] Johnen, H. and Scherer, K., On the equivalence of the K-functional and moduli of continuity and some applications. In: *Constructive Theory of Functions of Several Variables* (Proceedings Oberwolfach 1976; eds.: W. Schempp et al.). Lect. Notes Math. 571. Berlin: Springer 1977, pp. 119 – 140.

[29] Karadzhov, G. E. and Milman, M., Extrapolation theory: new results and applications. *J. Approx. Theory* 133 (2005), 38 – 99.

[30] Karadzhov, G. E., Milman, M. and Xiao, J., Limits of higher-order Besov spaces and sharp reiteration theorems. *J. Funct. Anal.* 221 (2005), 323 – 339.

[31] Kolyada, V. I., On relations between moduli of continuity in different metrics. *Proc. Steklov Inst. Math.* 181 (1989), 127 – 148.

[32] Kolyada, V. I. and Lerner, A. K., On limiting embeddings of Besov spaces. *Studia Math.* 171 (2005), 1 – 13.

[33] Maz'ya, V., *Sobolev Spaces with Applications to Elliptic Partial Differential Equations.* Berlin: Springer 2011.

[34] Maz'ya, V. and Shaposhnikova, T., On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.* 195 (2002), 230 – 238.

[35] Nikolskiĭ, S. M., *Approximation of Functions of Several Variables and Imbedding Theorems.* Berlin: Springer 1975.

[36] Peetre, J., Espaces d'interpolation et théorème de Soboleff (in French). *Ann. Inst. Fourier* 16 (1966), 279 – 317.

[37] Simonov, B. and Tikhonov, S., Sharp Ul'yanov-type inequalities using fractional smoothness. *J. Approx. Theory* 162 (2010), 1654 – 1684.

[38] Triebel, H., *Interpolation Theory, Function Spaces, Differential Operators.* Amsterdam: North-Holland 1978.

[39] Triebel, H., *Theory of Function Spaces. III.* Basel: Birkhäuser 2006.

[40] Triebel, H., Comments on tractable embeddings and function spaces of smoothness near zero. Report 2013.

[41] Zygmund, A., *Trigonometric Series.* Cambridge: Cambridge Univ. Press 1968.

Received April 18, 2016; revised September 7, 2017