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Topological Structure of Solution Sets for Semilinear Evolution Inclusions

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Abstract. This paper deals with a semilinear evolution inclusion involving a non-densely defined closed linear operator satisfying the Hille–Yosida condition and source term of multivalued type in Banach spaces. The topological structure of the set of solutions is investigated in the case that semigroup is noncompact. It is shown that the solution set is nonempty, compact and an R_δ-set. It is proved on compact intervals and then, using the inverse limit method, obtained on non-compact intervals. As a sample of application, we consider a parabolic partial differential inclusion at end of the paper.

Keywords. Evolution inclusions, solution sets, noncompact semigroup, topological structure, Hille–Yosida condition

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1. Introduction

Consider the following semilinear evolution inclusion on compact interval

$$\begin{cases} x'(t) \in Ax(t) + F(t, x(t)), & t \in [0, b], \\ x(0) = x_0, \end{cases} \tag{1}$$

and the corresponding inclusion on non-compact interval

$$\begin{cases} x'(t) \in Ax(t) + F(t, x(t)), & t \in \mathbb{R}^+, \\ x(0) = x_0, \end{cases} \tag{2}$$

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where the state $x(\cdot)$ takes values in Banach space X with the norm $|\cdot|$, F is a multimap defined on a subset of $\mathbb{R}^+ \times X$, $A : D(A) \subset X \to X$ is a nondensely defined closed linear operator satisfying the Hille–Yosida condition.

The study of (2) is justified by a partial differential inclusion of parabolic type

$$\begin{cases} \dfrac{\partial}{\partial t} x(t,\xi) \in \Delta x(t,\xi) + F(t,\xi,x(t,\xi)), & t \in \mathbb{R}^+,\ \xi \in \Omega, \\ x(t,\xi) = 0, & t \in \mathbb{R}^+,\ \xi \in \partial\Omega, \\ x(0,\xi) = x_0(\xi), & \xi \in \Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with regular boundary $\partial\Omega$, $x_0 \in C(\Omega;\mathbb{R}^N)$ and $F : \mathbb{R}^+ \times \Omega \to 2^{\mathbb{R}^N}$ is upper semicontinuous with compact convex values.

A strong motivation for investigating this class of inclusions is that a lot of phenomena investigated in hybrid systems with dry friction, processes of controlled heat transfer, obstacle problems and others can be described with the help of various differential inclusions (cf. [13, 15, 21, 28]). The theory of differential inclusions is highly developed and constitutes an important branch of nonlinear analysis, see, e.g., Bressan and Wang [7], Donchev et al. [14], Gabor and Quincampoix [18], Vrabie [25, 26] and the references therein.

The R_δ-property is an important aspect in the study of the topological structure of solution sets for differential inclusions. Recall that a subset D of a metric space is an R_δ-set if there exists a decreasing sequence $\{D_n\}_{n=1}^{\infty}$ of compact and contractible sets such that $D = \bigcap_{n=1}^{\infty} D_n$ (see Definition 2.5 below). This means that an R_δ-set is acyclic (in particular, nonempty, compact and connected) and may not be a singleton but, from the point of view of algebraic topology, it is equivalent to a point, in the sense that it has the same homology groups as one point space. There have been numerous research papers concerning topological structure of solution sets for differential equations or inclusions of various types, see, e.g., Górniewicz and Pruszko [16], De Blasi and Myjak [11], Andres and Pavlačková [3], Zhou et al. [28–31] and references therein.

The topological structure of solution sets of differential inclusions on compact intervals has been investigated intensively by many authors, please see De Blasi and Myjak [12], Bothe [6], Deimling [13], Hu and Papageorgiou [19], Staicu [23] and references therein. Moreover, one can find results on topological structure of solution sets for differential inclusions defined on non-compact intervals (including infinite intervals) from Andres et al. [2], Bakowska and Gabor [5], Chen, Wang and Zhou [10], Gabor and Grudzka [17], Staicu [24], Wang, Ma and Zhou [27] and references therein.

The paper on the existence and controllability results for nondensely defined differential differential inclusions has been investigated by Abada, Benchohra and Hammouche [1]. However, to the best of our knowledge, nothing has been done with the structure of solution sets for nonlinear evolution inclusions with

Hille–Yosida operators. In this paper, we investigate the topological structure of solution sets of (1) and (2) in the case that semigroup is noncompact.

The paper is organized as follows. In Section 2 we recall some notations, definitions, and preliminary facts from multivalued analysis. Section 3 gives the concept of an integral solution for evolution inclusions with Hille–Yosida operators (1). Section 4 is devoted to proving that the solution set for inclusion (1) is nonempty compact. Section 5 is concerned with the R_δ-property on compact intervals, by the inverse limit method, then we proceed to study that the solution set of (2) is a compact R_δ-set. Finally, an example is given to illustrate the obtained theory.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let $(X, |\cdot|)$ be a Banach space. $\mathcal{L}(X)$ stands for the space of all linear bounded operators on Banach space X, with the norm $\|\cdot\|_{\mathcal{L}(X)}$, and $L^1([0,b];X)$ stands for the Banach space consisting of integrable functions from $[0,b]$ to X equipped with the norm

$$\|f\|_{L^1} = \int_0^b |f(t)|dt.$$

We denote by $C([0,b];X)$ the Banach space consisting of continuous functions from $[0,b]$ to X equipped with the norm $\|x\| = \max_{t\in[0,b]} |x(t)|$.

Denote by $\widetilde{C}([0,\infty);X)$ the separated locally convex space consisting of all continuous functions from $[0,\infty)$ to X endowed with the family of seminorms $\{\|\cdot\|_m : \ m \in \mathbb{N}, m > 0\}$, defined by

$$\|x\|_m = \sup_{t\in[0,m]} |x(t)| \quad \text{for each } m \in \mathbb{N},\ m > 0,$$

and a metric

$$d(x_1, x_2) = \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{\|x_1 - x_2\|_m}{1 + \|x_1 - x_2\|_m},$$

$\widetilde{C}([0,\infty);X)$ is a Fréchet space.

A subset K in $L^1([0,b];X)$ is called integrably bounded if there exists $l \in L^1([0,b];\mathbb{R}^+)$ such that

$$|f(t)| \le l(t) \quad \text{a.e. } t \in [0,b]$$

for each $f \in K$. A sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,b];X)$ is said to be semicompact if it is integrably bounded and $\{f_n(t)\} \in K(t)$ for a.e. $t \in [0,b]$, where $K(t) \subset X$, $t \in [0,b]$, is a family of compact sets.

Lemma 2.1 ([21, Proposition 4.2.1]). *Assume that* $\{f_n\} \subset L^1([0,b];X)$ *is semicompact. Then* $\{f_n\}$ *is weakly compact in* $L^1([0,b];X)$.

Let Y and Z be metric spaces. $P(Y)$ stands for the collection of all nonempty subsets of Y. As usual, we denote $P_{cp}(Y) = \{D \in P(Y) : \text{ compact}\}$, $P_{cp,cv}(Y) = \{D \in P(Y) : \text{ compact and convex}\}$, co(D) (resp. $\overline{\text{co}}$(D)) is the convex hull (resp. convex closed hull in D) of a subset D.

A multimap $F : [0,b] \to P_{cp}(X)$ is said to be strongly measurable if there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of step multimaps such that $d_H(F_n(t), F(t)) \to 0$ as $n \to \infty$ for a.e. $t \in [0,b]$, where d_H is the Hausdorff metric on $P_{cp}(X)$.

For the multimap $\varphi : Y \to P(Z)$, if D is a subset of Z, then we denote by $\varphi^{-1}(D) = \{y \in Y : \varphi(y) \cap D \neq \emptyset\}$ the complete preimage of D under φ. φ is called closed if Gra(φ) is closed in $Y \times Z$; quasicompact if $\varphi(D)$ is relatively compact for each compact set $D \subset Y$; upper semicontinuous (shortly, u.s.c.) if $\varphi^{-1}(D)$ is closed for each closed set $D \subset Z$; and weakly semicontinuous (shortly, weakly u.s.c.) if $\varphi^{-1}(D)$ is closed for each weakly closed set $D \subset Z$.

Remark 2.2. (i) Every strongly measurable multimap F admits a strongly measurable selection $f : [0,b] \to X$, i.e., f is strongly measurable and $f(t) \in F(t)$ for a.e. $t \in [0,b]$.

(ii) Let the multimap $F : [0,b] \times X \to P_{cp}(X)$ be such that $F(t,\cdot)$ is u.s.c. for a.e. $t \in \mathbb{R}^+$, and the multimap $F(\cdot,x)$ has a strongly measurable selection for every $x \in X$. Then for every strongly measurable function $x : [0,b] \to X$, there exists a strongly measurable selection $f : [0,b] \to X$ of the multimap $\Phi : [0,b] \to P_{cp}(X)$, $\Phi(t) = F(t,x(t))$.

The following facts will be used.

Lemma 2.3 ([21, Theorem 1.1.12]). *Let* Y *and* Z *be metric spaces and* $\varphi : Y \to P(Z)$ *a closed quasicompact multimap with compact values. Then* φ *is u.s.c.*

X is called an absolute neighborhood retract (ANR-space) if for any metric space Y, closed subset $D \subset Y$ and continuous function $h : D \to X$, there exist a neighborhood $U \supset D$ and a continuous extension $\widetilde{h} : U \to X$ of h.

Definition 2.4. A nonempty subset D of a metric space is said to be contractible if there exist a point $y_0 \in D$ and a continuous function $h : [0,1] \times D \to D$ such that $h(0,y) = y_0$ and $h(1,y) = y$ for every $y \in D$.

Definition 2.5. A subset D of a metric space is called an R_δ-set if there exists a decreasing sequence $\{D_n\}$ of compact and contractible sets such that

$$D = \bigcap_{n=1}^{\infty} D_n.$$

The Hausdorff measure of noncompactness (Hausdorff MNC) $\beta\colon P(X) \to \mathbb{R}^+$ is defined by:

$$\beta(D) = \inf\{r > 0 : D \text{ can be covered by finitely many balls of radius } r\},$$

and it satisfies the following properties:.

monotone: if for all bounded subsets D_1, D_2 of X, $D_1 \subseteq D_2$ implies $\beta(D_1) \leq \beta(D_2)$;

nonsingular: if $\beta(\{x\} \cup D) = \beta(D)$ for every $x \in X$ and every nonempty subset $D \subseteq X$;

regular: $\beta(D) = 0$ if and only if D is relatively compact in X.

Theorem 2.6 ([6, Lemma 5])**.** *Let X be a complete metric space, β denote the Hausdorff MNC in X and let $\emptyset \neq D \subset X$. Then the following statements are equivalent:*

(i) *D is an R_δ-set;*

(ii) *D is an intersection of a decreasing sequence $\{D_n\}$ of closed contractible spaces with $\beta(D_n) \to 0$;*

(iii) *D is compact and absolutely neighborhood contractible, i.e., D is contractible in each neighborhood in $Y \in$ ANR.*

Definition 2.7. A multimap $\varphi : X \to P_{cp}(X)$ is said to be condensing with respect to a MNC β (β-condensing) if for every bounded set $D \subset X$ that is not relatively compact, we have

$$\beta(\varphi(D)) < \beta(D).$$

In subsequent proofs we shall also use the following fixed point result for multimaps.

Theorem 2.8 ([21, Corollary 3.3.1])**.** *Let D be a bounded convex closed subset of a Banach space X, and $\varphi : D \to P_{cp,cv}(D)$ an u.s.c. β-condensing multimap. Then the fixed point set* $\mathrm{Fix}(\varphi) := \{x : x \in \varphi(x)\}$ *is a nonempty compact set.*

3. Statement of the problem

3.1. Nonhomogeneous Cauchy problem. In the following study, we introduce the following hypothesis:

(H_A) the linear operator $A : D(A) \subset X \to X$ satisfies the Hille–Yosida condition, i.e., there exist two constants $\omega \in \mathbb{R}$ and $M > 0$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$\|(\lambda I - A)^{-k}\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^k} \quad \text{for all } \lambda > \omega,\ k \geq 1.$$

It is known that (see [20]) if $\{T(t)\}_{t\geq 0}$ is an integrated semigroup generated by a Hille–Yosida operator A, then $t \mapsto T(t)x$ is differentiable for each $x \in \overline{D(A)}$ and $\{T'(t)\}_{t\geq 0}$ is a C_0-semigroup on $\overline{D(A)}$ generated by the part A_0 of A, which is defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}; \quad A_0 x = Ax \text{ on } D(A_0).$$

(H_T) The C_0-semigroup $\{T'(t)\}_{t\geq 0}$ is norm-continuous, i.e., $t \mapsto T'(t)$ is continuous for $t > 0$.

Let $X_0 = \overline{D(A)}$. Consider Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + f(t), & t \in J = (0, b], \\ x(0) = x_0, \end{cases} \tag{3}$$

where $f \in C([0, b]; X)$ and $x_0 \in X_0$ are given.

Theorem 3.1 ([4,20]). *Let $f \in C([0, b]; X)$ and $x_0 \in X_0$, there exists a unique continuous function $x : [0, b] \to X$ of Cauchy problem* (3) *such that*

(i) $\int_0^t x(s)ds \in D(A)$ *for* $t \in [0, b]$;

(ii) $x(t) = x_0 + A\int_0^t x(s)ds + \int_0^t f(s)ds$.

Moreover, x satisfies the variation of constants formula

$$x(t) = T'(t)x_0 + \frac{d}{dt}\int_0^t T(t-s)f(s)ds, \quad \text{for } t \in [0, b]. \tag{4}$$

Let $\mathcal{J}_\lambda = \lambda(\lambda I - A)^{-1}$, then for all $x \in X_0$, $\lim_{\lambda\to+\infty} \mathcal{J}_\lambda x = x$ (see [20]). Also from the Hille–Yosida condition, it is easy to see that $\lim_{\lambda\to+\infty} |\mathcal{J}_\lambda x| \leq M|x|$. Since

$$\|\mathcal{J}_\lambda\|_{\mathcal{L}(X)} = \|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M\lambda}{\lambda - w},$$

thus $\lim_{\lambda\to+\infty} \|\mathcal{J}_\lambda\|_{\mathcal{L}(X)} \leq M$. Also if x is given by (4), then

$$x(t) = T'(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda f(s)ds \quad \text{for } t \in [0, b].$$

3.2. Integral solutions for inclusion (1). We assume that the multivalued nonlinearity $F : \mathbb{R}^+ \times X \to P_{cp,cv}(X)$ satisfies:

(H_1) $F(t, \cdot)$ is u.s.c. for a.e. $t \in \mathbb{R}^+$, and the multimap $F(\cdot, x)$ has a strongly measurable selection for every $x \in X$;

(H_2) there exists a function $\alpha \in L^1_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq \alpha(t)(1 + |x|) \quad \text{for a.e. } t \in \mathbb{R}^+ \text{ and } x \in X.$$

(H_3) there exists a function $k \in L^1_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ such that

$$\beta(F(t, D)) \leq k(t)\beta(D) \quad \text{for every bounded set } D.$$

Given $x \in C([0,b];X)$, let us denote

$$\mathrm{Sel}_F^b(x) = \{f \in L^1([0,b];X) : f(t) \in F(t, x(t)), \text{ for a.e. } t \in [0,b]\}.$$

Under conditions (H_1) and (H_2), the set $\mathrm{Sel}_F^b(x)$ is always nonempty. We have the following property of weak closedness of Sel_F^b.

Lemma 3.2 ([21, Lemma 5.1.1])**.** *Assume the sequences*

$$\{x_n\} \subset C([0,b];X), \ \{f_n\} \subset L^1([0,b];X),$$

$f_n \in \mathrm{Sel}_F^b(x_n)$ *are such that* $x_n \to x, \ f_n \rightharpoonup g$, *then* $g \in \mathrm{Sel}_F^b(x)$.

Definition 3.3. A continuous function $x : [0,b] \to X$ is said to be an integral solution of differential inclusion (1) if

(i) $\int_0^t x(s)ds \in D(A)$ for $t \in [0,b]$;
(ii) $x(0) = x_0$ and there exists $f(t) \in \mathrm{Sel}_F^b(x)(t)$ satisfying the following integral equation

$$x(t) = T'(t)x_0 + \frac{d}{dt}\int_0^t T(t-s)f(s)ds. \tag{5}$$

We notice also that, if x satisfies (5), then

$$x(t) = T'(t)x_0 + \lim_{\lambda \to +\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda f(s)ds$$

for $t \in [0,b]$.

Remark 3.4. For any $x \in C([0,b];X_0)$, now define a solution multioperator $\mathcal{F}^b : C([0,b];X_0) \to P(C([0,b];X_0))$ as follows

$$\mathcal{F}^b(x)(t) = \left\{T'(t)x_0 + \Gamma(f)(t) : \ f \in \mathrm{Sel}_F^b(x)\right\},$$

where

$$\Gamma(f)(t) = \lim_{\lambda \to +\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda f(s)ds.$$

It is easy to verify that the fixed points of the multioperator $\mathcal{F}^b$ are integral solutions of the inclusion (1).

4. Existence of integral solutions

In this section, we investigate the existence for integral solutions of inclusion (1).

Lemma 4.1. *The operator* Γ *has the following properties:*

(i) *there exists a constant* $c_0 > 0$ *such that*

$$|\Gamma(f)(t) - \Gamma(g)(t)| \leq c_0 \int_0^t |f(s) - g(s)| ds, \quad t \in [0, b]$$

for every $f, g \in L^1([0, b]; X)$*;*

(ii) *for each compact set* $K \subset X$ *and sequence* $\{f_n\} \subset L^1([0, b]; X)$ *such that* $\{f_n(t)\} \subset K$ *for a.e.* $t \in [0, b]$*, the weak convergence* $f_n \rightharpoonup f_0$ *implies the convergence* $\Gamma(f_n) \to \Gamma(f_0)$*.*

Proof. (i) By calculation, we have

$$\begin{aligned} |\Gamma(f)(t) - \Gamma(g)(t)| &\leq \left| \lim_{\lambda \to +\infty} \int_0^t T'(t-s) \mathcal{J}_\lambda (f(s) - g(s)) ds \right| \\ &\leq MM_1 \int_0^t |f(s) - g(s)| ds \\ &\leq c_0 \int_0^t |f(s) - g(s)| ds, \end{aligned}$$

where $c_0 = MM_1$.

(ii) Notice that $\mathcal{J}_\lambda$ is a bounded linear operator and $K \subset X$ is a compact set. Therefore the set $Q(\lambda) \subset X$ defined by

$$Q(\lambda) = \bigcup_{s \in [0,t]} T'(t-s) \mathcal{J}_\lambda K$$

is relatively compact. For every sequence $\{f_n\} \subset L^1([0, b]; X)$ and $\{f_n(t)\} \subset K$ for a.e. $t \in [0, b]$, we have

$$\{\Gamma(f_n)(t)\}_{n=1}^{\infty} \subset \lim_{\lambda \to +\infty} tQ(\lambda)$$

and hence, the sequence $\{\Gamma(f_n)(t)\} \subset X$ is relatively compact for every $t \in [0, b]$.

On the other hand, we have

$$\begin{aligned} &|\Gamma(f_n)(t_2) - \Gamma(f_n)(t_1)| \\ &\leq \left| \lim_{\lambda \to +\infty} \int_{t_1}^{t_2} T'(t-s) \mathcal{J}_\lambda f_n(s) ds \right| + \left| \lim_{\lambda \to +\infty} \int_0^{t_1} [T'(t_2 - s) - T'(t_1 - s)] \mathcal{J}_\lambda f_n(s) ds \right|. \end{aligned}$$

Since $T'(t)$ is strongly continuous and $\{f_n(t)\} \subset K$ for a.e. $t \in [0,b]$, the right-hand side of this inequality tends to zero as $t_2 \to t_1$ uniformly with respect to n. Hence $\{\Gamma(f_n)\}$ is an equicontinuous set. Thus from Arzela-Ascoli theorem, we obtain that the sequence $\{\Gamma(f_n)\} \subset C([0,b];X)$ is relatively compact.

Property (i) ensures that $\Gamma\colon L^1([0,b];X) \to C([0,b];X)$ is a bounded linear operator. Then it is continuous with respect to the topology of weak sequential convergence, that is the weak convergence $f_n \rightharpoonup f_0$ ensuring $\Gamma(f_n) \rightharpoonup \Gamma(f_0)$. Taking into account that $\{\Gamma(f_n)\}$ is relatively compact, we arrive at the conclusion that $\Gamma(f_n) \to \Gamma(f_0)$ in $C([0,b];X)$. □

Similar to the proof of [21, Proposition 4.2.2], we have the following result.

Lemma 4.2. *Let $\{f_n\}$ be integrably bounded, and*

$$\beta(f_n(t)) \le q(t)$$

for a.e. $t \in [0,b]$, where $q \in L^1([0,b];\mathbb{R}^+)$. Then we have

$$\beta(\{\Gamma(f_n)(t)\}_{n=1}^{\infty}) \le 2c_0 \int_0^t q(s)ds$$

for all $t \in [0,b]$, where $c_0 \ge 0$ is the constant in Lemma 4.1(i).

Theorem 4.3. *Let conditions* (H_A), (H_T), (H_1)–(H_3) *be satisfied. Then the inclusion* (1) *has at least one integral solution for each initial value $x_0 \in X_0$.*

Proof. Set

$$\mathcal{M}_0 = \{x \in C([0,b];X_0) : \ |x(t)| \le \psi(t), \ t \in [0,b]\},$$

where $\psi(t)$ is the solution of the integral equation

$$\psi'(t) = MM_1\alpha(t)(1+\psi(t)), \text{ a.e. on } [0,b], \quad \psi(0) = |x_0|.$$

It is clear that $\mathcal{M}_0$ is a closed and convex subset of $C([0,b];X_0)$. We first show that $\mathcal{F}^b(\mathcal{M}_0) \subset \mathcal{M}_0$. Indeed, taking $x \in \mathcal{M}_0$ and $y \in \mathcal{F}^b(x)$, we have

$$\begin{aligned} |y(t)| &\le |T'(t)x_0| + \left| \lim_{\lambda \to +\infty} \int_0^t T'(t-s)\mathcal{J}_\lambda f(s)ds \right| \\ &\le M_1|x_0| + MM_1 \int_0^t \alpha(s)(1+|x(s)|)ds \\ &\le \psi(t). \end{aligned}$$

Thus $y \in \mathcal{M}_0$. Set $\widetilde{\mathcal{M}} = \overline{\mathrm{co}}\mathcal{F}^b(\mathcal{M}_0)$, it is clear that $\widetilde{\mathcal{M}}$ is a closed, bounded and convex set. Moreover, $\mathcal{F}^b(\widetilde{\mathcal{M}}) \subset \widetilde{\mathcal{M}}$.

Claim 1. The multioperator $\mathcal{F}^b$ has closed graph with compact values. Let $\{x_n\} \subset \mathcal{M}_0$ with $x_n \to x$ and $y_n \in \mathcal{F}^b(x_n)$ with $y_n \to y$. We shall prove that $y \in \mathcal{F}^b(x)$. By the definition of $\mathcal{F}^b$, there exist $f_n \in \mathrm{Sel}_F^b(x_n)$ such that

$$y_n(t) = T'(t)x_0 + \Gamma(f_n)(t).$$

We need to prove that there exists $f \in \mathrm{Sel}_F^b(x)$ such that for a.e. $t \in [0, b]$,

$$y(t) = T'(t)x_0 + \Gamma(f)(t).$$

We see that $\{f_n\}$ is integrably bounded by (H_2), and the following inequality holds by (H_3)

$$\beta(\{f_n(t)\}) \leq k(t)\beta(\{x_n(t)\}).$$

Since the sequence $\{x_n\}$ converges in $C([0, b]; X_0)$, we have $\beta(\{f_n(t)\}) = 0$ for a.e. $t \in [0, b]$, then $\{f_n\}$ is a semicompact sequence and then it is also weakly compact in $L^1([0, b]; X)$ by Lemma 2.1. So we can assume, without loss of generality, that $f_n \rightharpoonup f$ in $L^1([0, b]; X)$. By Lemma 4.1(ii), we conclude that

$$y_n(t) = T'(t)x_0 + \Gamma(f_n)(t) \to T'(t)x_0 + \Gamma(f)(t) = y(t).$$

Moreover, from Lemma 3.2, we have that $f \in \mathrm{Sel}_F^b(x)$. It remains to show that, for $x \in \mathcal{M}_0$ and $\{f_n\}$ chosen in $\mathrm{Sel}_F^b(x)$, the sequence $\{\Gamma(f_n)\}$ is relatively compact in $C([0, b]; X_0)$. Conditions (H_2) and (H_3) imply that $\{f_n\}$ is semicompact. Using Lemma 4.1(ii), we obtain that $\{\Gamma(f_n)\}$ is relatively compact in $C([0, b]; X_0)$. Thus $\mathcal{F}^b(x)$ is relatively compact in $C([0, b]; X_0)$, together with the closeness of $\mathcal{F}^b$, then $\mathcal{F}^b(x)$ has compact values.

Claim 2. The multioperator $\mathcal{F}^b$ is u.s.c. In view of Lemma 2.3, it suffices to check that $\mathcal{F}^b$ is a quasicompact multimap. Let Q be a compact set. We prove that $\mathcal{F}^b(Q)$ is a relatively compact subset of $C([0, b]; X_0)$. Assume that $\{y_n\} \subset \mathcal{F}^b(Q)$. Then

$$y_n(t) = T'(t)x_0 + \Gamma(f_n)(t),$$

where $\{f_n\} \in \mathrm{Sel}_F^b(x_n)$, for a certain sequence $\{x_n\} \subset Q$. Conditions (H_2) and (H_3) yield the fact that $\{f_n\}$ is semicompact and then it is a weakly compact sequence in $L^1([0, b]; X)$. Similar arguments as in the previous proof of closeness imply that $\{y_n\}$ is relatively compact in $C([0, b]; X_0)$. Thus, $\{y_n\}$ converges in $C([0, b]; X_0)$, so the multioperator $\mathcal{F}^b$ is u.s.c.

Claim 3. The multioperator $\mathcal{F}^b$ is a condensing multioperator. We first need a MNC constructed suitably for our problem. For a bounded subset $\Omega \subset \mathcal{M}_0$, let $\mathrm{mod}_{\mathrm{C}}(\Omega)$ be the modulus of equicontinuity of the set of functions Ω given by

$$\mathrm{mod}_{\mathrm{C}}(\Omega) = \lim_{\delta \to 0} \sup_{x \in \Omega} \max_{|t_2 - t_1| < \delta} |x(t_2) - x(t_1)|.$$

Given the Hausdorff MNC β, let χ be the real MNC defined on a bounded subset D of $C([0,b];X_0)$ by

$$\chi(D) = \sup_{t\in[0,b]} e^{-Lt}\beta(D(t)).$$

Here, the constant L is chosen such that

$$l := \sup_{t\in[0,b]} \left[2c_0 \int_0^t e^{-L(t-s)}k(s)ds\right] < 1,$$

where k is the function from condition (H_3).

Consider the function $\nu(\Omega) = \max_{D\in\Delta(\Omega)} \big(\chi(D), \mathrm{mod}_{\mathrm{C}}(D)\big)$ in the space of $C([0,b];X_0)$, where $\Delta(\Omega)$ is the collection of all countable subsets of Ω. To show that $\mathcal{F}^b$ is ν-condensing, let $\Omega \subset \mathcal{M}_0$ be a bounded set in $\mathcal{M}_0$ such that

$$\nu(\Omega) \le \nu(\mathcal{F}^b(\Omega)). \tag{6}$$

We will show that Ω is relatively compact. Let $\nu(\mathcal{F}^b(\Omega))$ be achieved on a sequence $\{y_n\} \subset \mathcal{F}^b(\Omega)$, i.e.,

$$\nu(\{y_n\}) = \Big(\chi(\{y_n\}), \mathrm{mod}_{\mathrm{C}}(\{y_n\})\Big).$$

Then $y_n(t) = T'(t)x_0 + \Gamma(f_n)(t)$, $f_n \in \mathrm{Sel}^b_F(x_n)$, where $\{x_n\} \subset \Omega$. Now inequality (6) implies

$$\chi(\{y_n\}) \ge \chi(\{x_n\}). \tag{7}$$

It follows from (H_3) that $\beta(\{f_n(t)\}) \le k(t)\beta(x_n(t))$ for $t \in [0,b]$. Then

$$\beta(\{f_n(t)\}) \le e^{Lt}k(t)\left(\sup_{s\in[0,t]} e^{-Ls}\beta(x_n(t))\right) \le e^{Lt}k(t)\chi(\{x_n\}).$$

Now the application of Lemma 4.2 for Γ yields

$$e^{-Lt}\beta(\{\Gamma(f_n)(t)\}) \le 2c_0 e^{-Lt}\int_0^t e^{Ls}k(s)ds\cdot\chi(\{x_n\}) \le 2c_0\int_0^t e^{-L(t-s)}k(s)ds\cdot\chi(\{x_n\}),$$

for any $t \in [0,b]$. Putting this relation together with (7), we obtain

$$\chi(\{x_n\}) \le \chi(\{y_n\}) = \sup_{t\in[0,b]} e^{-Lt}\beta(y_n(t)) \le l\chi(\{x_n\}).$$

Therefore $\chi(\{x_n\}) = 0$. This implies $\beta(x_n(t)) = 0$.

Using (H_2) and (H_3) again, one gets that $\{f_n\}$ is a semicompact sequence. Then, Lemma 4.1(ii) ensures that $\{\Gamma(f_n)\}$ is relatively compact in $C([0,b];X_0)$. This yields that $\{y_n\}$ is relatively compact in $C([0,b];X_0)$. Hence $\mathrm{mod}_{\mathrm{C}}(\{y_n\}) = 0$. Finally, $\nu(\{y_n\}) = 0$, and so the map $\mathcal{F}^b$ is ν-condensing.

From Theorem 2.8, we deduce that the fixed point set $\mathrm{Fix}(\mathcal{F}^b)$ is a nonempty compact set. □

5. Topological structure of solution sets

In this section, we study the topological structure for solution sets of inclusions (1) and (2). Let $\Theta^b(x_0)$ denote the set of all integral solutions of the inclusion (1).

Lemma 5.1. *Under assumptions in Theorem* 4.3, *there exists a nonempty compact convex subset* $\mathcal{M} \subseteq C([0,b];X_0)$ *such that*

(i) $x(0) = x_0$, *for all* $x \in \mathcal{M}$;

(ii) $T'(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda\overline{\text{co}}F(s,\mathcal{M}(s))ds \subset \mathcal{M}(t)$, *for* $t \in [0,b]$, *where* $\mathcal{M}(t) = \{x(t) : x \in \mathcal{M}\}$.

Proof. Let us construct the decreasing sequence of closed convex sets $\{\mathcal{M}_n\} \subset C([0,b];X_0)$ by the following inductive process. Let

$$\widetilde{\mathcal{M}}_0 = \{x \in C([0,b];X_0) : x(0) = x_0,\ |x(t)| \le N,\ t \in [0,b]\},$$

where $N = \Big(M_1|x_0| + MM_1\|\alpha\|_{L^1}\Big)\exp(MM_1\|\alpha\|_{L^1})$.

Then $\mathcal{M}_n = \overline{\mathcal{N}_n},\ n \ge 1$, where $\mathcal{N}_n \subset C([0,b];X_0)$, $\mathcal{M}_0 = \widetilde{\mathcal{M}}_0$ and

$$\mathcal{N}_n = \left\{ y \in C([0,b];X_0) \;\middle|\; \begin{array}{r} y(t) = T'(t)x_0 + \lim\limits_{\lambda\to+\infty}\displaystyle\int_0^t T'(t-s)\mathcal{J}_\lambda f(s)ds, \\ f \in \text{Sel}^b_{\overline{\text{co}}F}(\cdot,\mathcal{M}_{n-1}(\cdot)) \end{array} \right\}.$$

First of all, let us note that all $\mathcal{M}_n,\ n \ge 1$ are nonempty since $\Theta^b(x_0) \subset \mathcal{M}_n$ for all $n \ge 0$.

We proceed to verify that the sets $\mathcal{M}_n$ are equicontinuous on $[0,b]$. Taking $0 < t_1 < t_2 \le b$, for any $v \in \mathcal{N}_n$, we obtain

$$\begin{aligned} |v(t_2) - v(t_1)| \le \|T'(t_2) - T'(t_1)\|_{\mathcal{L}(X)}|x_0| + \left|\lim_{\lambda\to+\infty}\int_{t_1}^{t_2} T'(t-s)\mathcal{J}_\lambda f(s)ds\right| \\ + \left|\lim_{\lambda\to+\infty}\int_0^{t_1}\big(T'(t_2-s) - T'(t_1-s)\big)\mathcal{J}_\lambda f(s)ds\right|. \end{aligned}$$

The right-hand side tends to zero as $t_2 - t_1 \to 0$ by (H_T). Thus all sets $\mathcal{M}_n$, $n \ge 1$ are equicontinuous.

Using the condition (H_3) we have the following estimation for $0 \le s \le t \le b$:

$$\beta\Big(\overline{\text{co}}F(s,\mathcal{M}_{n-1}(s))\Big) \le k(t)e^{Lt}\Big(\sup_{s\in[0,t]} e^{-Ls}\beta(\mathcal{M}_{n-1}(s))\Big) \le k(t)e^{Lt}\chi(\mathcal{M}_{n-1}).$$

For any $t \in [0, b]$ we have

$$\begin{aligned} e^{-Lt}\beta(\mathcal{N}_n(t)) &= e^{-Lt}\beta\bigg(\lim_{\lambda\to+\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda\overline{\text{co}}F(s,\mathcal{M}_{n-1}(s))ds\bigg) \\ &\leq 2c_0e^{-Lt}\bigg(\int_0^t e^{Ls}k(s)ds\bigg)\chi(\mathcal{M}_{n-1}) \\ &\leq 2c_0\bigg(\int_0^t e^{-L(t-s)}k(s)ds\bigg)\chi(\mathcal{M}_{n-1}). \end{aligned}$$

Therefore, $\chi(\mathcal{N}_n) \leq l\chi(\mathcal{M}_{n-1})$.

Finally, we have $\chi(\mathcal{M}_n) \leq l\chi(\mathcal{M}_{n-1})$ and therefore $\chi(\mathcal{M}_n) \to 0$ $(n \to \infty)$. We obtain a compact set $\mathcal{M} = \bigcap_{n=0}^{\infty}\mathcal{M}_n$, which has the desired properties. □

In the following, let us note that we may assume, without loss of generality, that F satisfies the following estimation:

$(\mathrm{H}_2)'$ $|F(t,x)| \leq \eta(t)$ for $\forall\, x \in X$ and a.e. $t \in [0,b]$, where $\eta \in L^1([0,b];\mathbb{R}^+)$.

In fact, let $\|\Theta^b(x_0)\| \leq N$, B_N be a closed ball in the space X and $\rho : X \to B_N$ be a radial retraction. Then it is easy to see that the multimap $\widetilde{F} : [0,b] \times X \to P_{cp,cv}(X)$, defined by $\widetilde{F}(t,x) = F(t,\rho x)$ satisfies conditions (H_1) and (H_3) (note that ρ is a Lipschitz map), the condition $(\mathrm{H}_2)'$ with $\eta(t) = \alpha(t)(1+N)$. The set $\Theta^b(x_0)$ coincides with the set of all integral solutions of the problem

$$\begin{cases} x'(t) \in Ax(t) + \widetilde{F}(t,x(t)), \quad t \in [0,b], \\ x(0) = x_0. \end{cases}$$

Therefore in what follows we will suppose that the multimap $F : [0,b]\times X \to P_{cp,cv}(X)$ satisfies the conditions (H_1), (H_3) and $(\mathrm{H}_2)'$ instead of (H_2).

Now consider a metric projection $P : [0,b] \times X \to P_{cp,cv}(X)$,

$$P(t,x) = \{y \in \mathcal{M}(t),\ \|x-y\| = \text{dist}(x,\mathcal{M}(t))\},$$

and a multimap $\widehat{F} : [0,b] \times X \to P_{cp,cv}(X)$, defined by

$$\widehat{F}(t,x) = \overline{\text{co}}F(t,P(t,x)).$$

From [21, Lemma 5.3.2], we know the multimap P is closed and u.s.c.

Lemma 5.2 ([21])**.** *The multimap* $\widehat{F}$ *satisfies the conditions* (H_1), $(\mathrm{H}_2)'$ *and* (H_3).

The above result implies that the set $\widehat{\Theta}^b(x_0)$ consisted of all integral solutions of the problem

$$\begin{cases} x'(t) \in Ax(t) + \widehat{F}(t,x), \quad t \in [0,b], \\ x(0) = x_0 \end{cases}$$

is nonempty.

Moreover, the following result holds.

Lemma 5.3. *With the help of Lemma* 5.2*, we have* $\widehat{\Theta}^b(x_0) = \Theta^b(x_0)$.

Proof. In fact, let $x \in \widehat{\Theta}^b(x_0)$. Then

$$\begin{aligned} x(t) &\in T'(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda \widehat{F}(s,x(s))ds \\ &= T'(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda \overline{\mathrm{co}}F(s,P(t,x(s)))ds \\ &\subset T'(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda \overline{\mathrm{co}}F(s,\mathcal{M}(s)))ds \subset \mathcal{M}(t), \end{aligned}$$

hence $P(t,x(t)) = \{x(t)\}$. Then $x(t) = T'(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t T'(t-s)\mathcal{J}_\lambda f(s)ds$, where $f \in \mathrm{Sel}^b_{\widehat{F}}(x) = \mathrm{Sel}^b_F(x)$, and so $x \in \Theta^b(x_0)$.

The inclusion $\Theta^b(x_0) \subset \widehat{\Theta}^b(x_0)$ easily follows from the observation that $\Theta^b(x_0) \subset \mathcal{M}$. □

[21, Lemma 5.3.4] yields the following approximation result.

Lemma 5.4. *Let the hypotheses* (H_1) *and* (H_2) *be satisfied. Then there exists a sequence* $\{\widehat{F}_n\}$ *with* $\widehat{F}_n : [0,b]\times X \to P_{cp,cv}(X)$ *such that*

(i) $\widehat{F}(t,x) \subset \cdots \subset \widehat{F}_{n+1}(t,x) \subset \widehat{F}_n(t,x) \subset \cdots \subset \overline{\mathrm{co}}(F(t,\mathcal{M}(t)))$, $n \geq 1$, *for each* $t \in [0,b]$;

(ii) $\widehat{F}(t,x) = \bigcap_{n=1}^\infty \widehat{F}_n(t,x)$;

(iii) $\widehat{F}_n(t,\cdot) : X \to P_{cp,cv}(X)$ *is continuous for a.e.* $t \in [0,b]$ *with respect to Hausdorff metric for each* $n \geq 1$;

(iv) *for each* $n \geq 1$*, there exists a selection* $g_n : [0,b]\times X \to X$ *of* $\widehat{F}_n$ *such that* $g_n(\cdot,x)$ *is measurable and* $g_n(t,\cdot)$ *is locally Lipschitz.*

Theorem 5.5. *Under the conditions in Theorem* 4.3*, the solution set of problem* (1) *is a compact* R_δ*-set in* $C([0,b];X_0)$.

Proof. Now we consider the differential inclusion:

$$\begin{cases} x'(t) \in Ax(t) + \widehat{F}_n(t,x), & t \in [0,b], \\ x(0) = x_0. \end{cases} \tag{8}$$

Let $\widehat{\Theta}^b_n(x_0)$ denote the solution set of inclusion (8). From Lemma 5.2, it follows that each $\widehat{F}_n$ satisfies the conditions (H_1), (H_2)′ and (H_3), hence each set $\widehat{\Theta}^b_n(x_0)$ is nonempty and compact.

We prove that

$$\widehat{\Theta}^b(x_0) = \bigcap_{n\geq 1}\widehat{\Theta}^b_n(x_0).$$

It is clear that $\widehat{\Theta}^b(x_0) \subset \widehat{\Theta}^b_n(x_0)$ and $\widehat{\Theta}^b(x_0) \subset \bigcap_{n\geq 1} \widehat{\Theta}^b_n(x_0)$.

Let $x \in \bigcap_{n\geq 1} \widehat{\Theta}^b_n(x_0)$, then for each $n \geq 1$, we have

$$x(t) = T'(t)x_0 + \lim_{\lambda\to+\infty} \int_0^t T'(t-s)\mathcal{J}_\lambda g_n(s)ds, \quad \text{for } t \in [0,b].$$

where $g_n \in \mathrm{Sel}^b_{\widehat{F}_n}(x)$. From Lemma 5.4, it follows that $\{g_n\}$ is semicompact and by Lemma 2.1 we may assume, up to subsequence, that $g_n \rightharpoonup f \in L^1([0,b],X)$. Lemma 5.4(ii) implies that $f(t) \in \widehat{F}(t,x(t))$ a.e. on $[0,b]$. Applying Lemma 4.1, we derive that

$$x(t) = T'(t)x_0 + \lim_{\lambda\to+\infty} \int_0^t T'(t-s)\mathcal{J}_\lambda f(s)ds, \quad \text{for } t \in [0,b],$$

which means that $x \in \widehat{\Theta}^b(x_0)$.

We show that the set $\widehat{\Theta}^b_n(x_0)$ is contractible for each $n \geq 1$. In fact, let $x_n \in \widehat{\Theta}^b_n(x_0)$ and for any $\lambda \in [0,1]$, the function $y^\lambda_n(t)$ be a unique solution on $[\lambda b, b]$ of the integral equation

$$y_n(t) = T'(t-\lambda b)x_n(\lambda b) + \lim_{\lambda\to+\infty} \int_{\lambda b}^t T'(t-s)\mathcal{J}_\lambda g_n(s, y_n(s))ds, \tag{9}$$

where g_n is the selection of $\widehat{F}_n$. The functions

$$z^\lambda_n(t) = \begin{cases} x_n(t), & t \in [0,\lambda b], \\ y^\lambda_n(t), & t \in [\lambda b, b] \end{cases}$$

belong to $\widehat{\Theta}^b_n(x_0)$. Define the deformation $h : [0,1] \times \widehat{\Theta}^b_n(x_0) \to \widehat{\Theta}^b_n(x_0)$ by the formula

$$h(\lambda, x_n) = \begin{cases} z^\lambda_n(t), & \lambda \in [0,1), \\ x_n, & \lambda = 1. \end{cases}$$

Since the function g_n is locally Lipschitz in Lemma 5.4(iv), the solutions of the equation (9) depend continuously on (λ, x_n), therefore the definition h is continuous. But $h(0,\cdot) = y^0_n(t)$ and $h(1,\cdot)$ is the identity, hence $\widehat{\Theta}^b_n(x_0)$ is contractible for every $n \geq 1$. Consequently, Theorem 2.6 follows that the solution set of problem (1) is a compact R_δ-set, completing this proof. □

Theorem 5.6. *Under the conditions in Theorem* 4.3*, the solution set of problem* (2) *is a compact R_δ-set in $C(\mathbb{R}^+; X_0)$.*

Proof. Firstly, we introduce the following two inverse systems and their limits. For more details about the inverse system and its limit, we refer the reader to [10] (see also [3, 17]).

For each $p, m > 0$ with $p \geq m$, let us consider a projection

$$\pi_m^p : C([0,p]; X_0) \to C([0,m]; X_0),$$

which is defined by $\pi_m^p(x) = x|_{[0,m]}$, $x \in C([0,p]; X_0)$. Put

$$\mathbb{N}_0 = \{m \in \mathbb{N} \setminus \{0\} : m > 0\}, \quad C_m = \{x \in C([0,m]; X_0) : x(0) = x_0\}.$$

Then it is readily checked that $\{C_m, \pi_m^p, \mathbb{N}_0\}$ is an inverse system and its limit is

$$\{x \in \widetilde{C}([0,\infty); X_0) : x(0) = x_0\} =: C.$$

Consider the sequence of multivalued maps

$$\mathcal{F}^m(x)(t) = \big\{T'(t)x_0 + \Gamma(f)(t),\ t \in [0,m] : f \in L^1([0,m]; X),\ f(t) \in F(t, x(t))\big\}$$

We have the equalities

$$\mathcal{F}^m \pi_m^{m+1}(x)(t) = \big\{T'(t)x_0 + \Gamma(f)(t),\ t \in [0,m] :\ f \in L^1([0,m]; X),\\ f(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,m]\big\},$$

and

$$\pi_m^{m+1} \mathcal{F}^{m+1}(x)(t) = \big\{T'(t)x_0 + \Gamma(f)(t),\ t \in [0,m+1] :\ f \in L^1([0,m+1]; X),\\ f(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,m+1]\big\}.$$

Noticing that

$$\big\{f \in L^1([0,m]; X) : f(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,m]\big\}\\ = \big\{f|_{[0,m]},\ f \in L^1([0,m+1]; X) : f(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,m+1]\big\},$$

one can find that $\mathcal{F}^m \pi_m^{m+1} = \pi_m^{m+1} \mathcal{F}^{m+1}$, so the family $\{\mathrm{id}, \mathcal{F}^m\}$ is the map from the inverse system $\{C_m, \pi_m^p, \mathbb{N}_0\}$ into itself, which enables us to conclude that family $\{\mathrm{id}, \mathcal{F}^m\}$ induces a limit mapping

$$\mathcal{F} : C \to P(C),$$

here, for every $x \in C$,

$$\mathcal{F}(x) = \big\{T'(t)x_0 + \Gamma(f)(t),\ t \in \mathbb{R}^+ :\ f \in L^1(\mathbb{R}^+; X),\\ f(t) \in F(t, x(t)) \text{ for a.e. } t \in \mathbb{R}^+\big\}.$$

Moreover, it follows readily that $\Theta(x_0) := \mathrm{Fix}(\mathcal{F}) = \lim_{\leftarrow} \Theta^m(x_0)$.

For every $m \in \mathbb{N} \setminus \{0\}$, the set of all fixed points of $\mathcal{F}^m$ is denoted by $\mathrm{Fix}(\mathcal{F}^m)$, i.e.,

$$\mathrm{Fix}(\mathcal{F}^m) = \{x \in C_m : x \in \mathcal{F}^m(x)\}.$$

Then we see from Theorem 4.3 that $\mathrm{Fix}(\mathcal{F}^m)(= \Theta^m(x_0))$ are compact R_δ-sets. At the end of this step, applying [3, Proposition 4.1] we obtain that the solution set of problem (2) is a nonempty compact R_δ-set, as claimed. □

6. An example

Consider the following partial differential inclusions of parabolic type

$$\begin{cases} \dfrac{\partial}{\partial t} z(t,\xi) \in \Delta z(t,\xi) + f(\xi, z(t,\xi)) + \displaystyle\sum_{i=1}^{m} a_i(t,\xi) y_i(t), & t \in \mathbb{R}^+,\ \xi \in \Omega, \\ \quad y_i(t) \in \left[\displaystyle\int_\Omega k_{1,i}(\xi) z(t,\xi) d\xi, \int_\Omega k_{2,i}(\xi) z(t,\xi) d\xi \right], & 1 \le i \le m, \\ z(t,\xi) = 0, & t \in \mathbb{R}^+,\ \xi \in \partial\Omega, \\ z(0,\xi) = x_0(\xi), & \xi \in \Omega, \end{cases} \tag{10}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with regular boundary $\partial\Omega$ and $\mathcal{O} \subset \Omega$ be an open subset, $x_0 \in C(\overline{\Omega}; \mathbb{R})$.

We choose $X = C(\overline{\Omega}; \mathbb{R})$ and $X_0 = C_0(\overline{\Omega}, \mathbb{R}) = \{z \in C(\overline{\Omega}, \mathbb{R}) : z = 0 \text{ on } \partial\Omega\}$, endowed with the supnorm, and we consider the operator $A : D(A) \subset X \to X$ defined by

$$\begin{aligned} D(A) &= \{z \in C_0(\overline{\Omega}, \mathbb{R}) \cap H_0^1(\Omega, \mathbb{R}) : \ \Delta z \in C_0(\overline{\Omega}, \mathbb{R})\}, \\ Az &= \Delta z. \end{aligned}$$

Now, we have $\overline{D(A)} = X_0 \neq X$ and

$$(0,\infty) \subset \rho(A), \quad \|R(\lambda, A)\| \le \frac{1}{\lambda}, \text{ for } \lambda > 0.$$

This implies that the operator A satisfies the condition (H_A). Moreover, the operator $T(t)$ generated by A_0 is compact in X_0 with $M = 1$ (see [22]).

In this model, we assume that

(a) $a_i \in C(\overline{\Omega}, \mathbb{R})$, $k_{j,i} \in L^1(\mathcal{O}, \mathbb{R})$, $j = 1, 2$;

(b) $f : \Omega \to \mathbb{R}$ such that $f(\cdot, z)$ is measurable for each $z \in \mathbb{R}$ and there exists $\kappa \in C(\overline{\Omega}, \mathbb{R})$ verifying

$$|f(\xi, z_1) - f(\xi, z_2)| \le \kappa(\xi)|z_1 - z_2|, \quad \forall\ \xi \in \Omega,\ z_1, z_2 \in \mathbb{R}.$$

Let $G(t,z) = f(\xi, z(t,\xi)) + \sum_{i=1}^m a_i(t,\xi) y_i(t)$. From our assumptions on (a) and (b), it follows readily that the multivalued function $G(\cdot,\cdot) : \mathbb{R}^+ \times \overline{\Omega} \to P(\mathbb{R})$ satisfies (H_1) and (H_2).

Then the system (10) can be reformulated as

$$\begin{cases} x'(t) \in Ax(t) + F(t, x(t)), & t \in \mathbb{R}^+, \\ x(0) = x_0, \end{cases}$$

where $x(t)(\xi) = z(t,\xi), F(t, x(t))(\xi) = G(t, z(t,\xi))$.

Thus, all the assumptions in Theorem 4.3 are satisfied, our result can be used to the problem (10), which implies that the solution set of problem (10) is a compact R_δ-set.

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