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Well-Posedness and Orbital Stability of Periodic Traveling Waves for Schamel's Equation

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Abstract. In this paper we will prove results of global well-posedness and orbital stability of periodic traveling-wave solutions related to the Schamel equation. The global solvability will be established by using compactness tools. In order to overcome the lack of smoothness of the nonlinearity, we approximate the original problem by regularizing the nonlinear term through a convenient polynomial near the origin. By following an adaptation of the well-known method introduced by Grillakis, Shatah, and Strauss, the orbital stability will be determined by proving, under certain conditions, that the periodic waves minimize an augmented Hamiltonian.

Keywords. Schamel's equation, well-posedness, periodic traveling waves, orbital stability

Mathematics Subject Classification (2010). 76B25, 35B10, 35Q51, 35Q53

1. Introduction

The Schamel equation

$$u_t + \partial_x \left(u_{xx} + |u|^{\frac{3}{2}} \right) = 0, \tag{1}$$

governs the behavior of weakly nonlinear ion-acoustic solitons which are modified by the presence of trapped electrons and it was first derived by Schamel in [29, 30]. More recently, (1) has appeared in the propagation of ion-acoustic waves in non-thermal unmagnetized collisionless electron-ion plasma, featuring

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a non-Maxwellian-trapped electron distribution modelled by a Kappa distribution combined with a Schamel distribution [33]. It also appears in the propagation of dust acoustic waves in dusty plasma consisting of positively charged warm adiabatic dust, negatively charged cold dust, and nonisothermally distributed electrons [20]. Here $u = u(x,t)$ indicates a real-valued function of the real variables x and t.

By assuming convenient boundary conditions, equation (1) has traveling-wave solutions of the form $u(x,t) = \varphi_\omega(x-\omega t)$, where $\omega > 0$ is a real parameter representing the wave speed. Our particular interest in this paper is to study a special kind of traveling waves having periodic boundary conditions. This means that $\varphi_\omega : \mathbb{R} \to \mathbb{R}$ is a periodic function of its argument. To obtain a positive periodic function φ_ω, we substitute the form of u above into (1) and, after an integration, obtain the nonlinear ordinary differential equation

$$-\varphi_\omega'' + \omega\varphi_\omega - \varphi_\omega^{\frac{3}{2}} + A = 0. \tag{2}$$

Here, A is a constant of integration which is assumed to be nonzero. Multiplying equation (2) by φ_ω' and integrating the result once, we obtain

$$\frac{1}{2}[\varphi_\omega']^2 - \frac{\omega}{2}\varphi_\omega^2 + \frac{2}{5}\varphi_\omega^{\frac{5}{2}} = A\varphi_\omega + B, \tag{3}$$

where B is another constant of integration. In order to construct explicit periodic waves which converge to the corresponding solitary-wave solution with a sech-type profile, in what follows we assume $B = 0$, for all $\omega > 0$. Next, we introduce the variable Λ_ω through the relation $\Lambda_\omega = \varphi_\omega^{\frac{1}{2}}$. With this transformation, (3) can be rewritten in quadrature form:

$$[\Lambda_\omega']^2 = \frac{1}{5}\left[-\Lambda_\omega^3 + \frac{5\omega}{4}\Lambda_\omega^2 + \frac{5}{2}A\right] \equiv \frac{1}{5}F_{\varphi_\omega}(\Lambda_\omega).$$

where $F(t) := F_{\varphi_\omega}(t) = -t^3 + \frac{5\omega}{4}t^2 + \frac{5}{2}A$.

A direct integration (see Section 3) yields an explicit formula for Λ_ω. As a consequence, we obtain an explicit solution of (3) given by

$$\varphi_\omega(\xi) = \left[\eta_2 + (\eta_3 - \eta_2)\,\mathrm{cn}^2\left(\sqrt{\frac{\eta_3 - \eta_1}{20}} \cdot \xi; k\right)\right]^2, \quad k^2 = \frac{\eta_3 - \eta_2}{\eta_3 - \eta_1}, \tag{4}$$

where cn represents the Jacobian elliptic function of *cnoidal* type, $k \in (0,1)$ is the elliptic modulus, and η_i, $i = 1,2,3$ are parameters depending on ω. By recalling that the cnoidal function has fundamental period $4K(k)$, where $K(k)$ denotes the complete elliptic integral of the first kind, and adjusting the parameters η_i, we can find a periodic solution having the form (4) with fundamental period L, where $L > 0$ is any fixed real number. In addition, an application of the implicit function theorem allows us to select an open interval $I \subset \mathbb{R}$ such that for any $\omega \in I$, the function φ_ω given by (4) is a solution of (2), has fundamental period L, and the map $\omega \in I \mapsto \varphi_\omega \in H^n_{per}([0, L])$, $n \in \mathbb{N}$, is smooth (see Section 3 for some details). It should be noted that these ideas have been applied to obtain periodic waves related to several dispersive evolution equations (see, for instance, [3, 7–9], and references therein).

With the periodic traveling waves in hand, their orbital stability is at issue. To begin with, we need to study the initial-value problem associated with (1). Here, we deal with the Cauchy problem

$$\begin{cases} u_t + \partial_x \left(u_{xx} + |u|^{\frac{3}{2}} \right) = 0, \\ \qquad\qquad u(x, 0) = u_0(x), \end{cases} \tag{5}$$

where u_0 belongs to the periodic Sobolev space $H^1_{per}([0, L])$, the *energy space*. Most of our arguments will be based on the approach developed in [16,19], where the authors addressed attention to the Cauchy problem associated, respectively, to the logarithmic Schrödinger equation,

$$iu_t + \Delta u + \log(|u|^2)u = 0,$$

and the logarithmic Korteweg–de Vries equation,

$$u_t + u_{xxx} + \partial_x(u \log |u|) = 0, \tag{6}$$

posed on convenient classes of functions. We point out that results of existence and orbital stability of periodic traveling-wave solutions to (6) were dealt with in [27].

Since the function $x \in \mathbb{R} \mapsto |x|^{\frac{3}{2}}$ is not smooth at the origin, the nonlinearity in (1) brings a set of difficulties concerning the local-in-time solvability of (5) (note that the function $x \in \mathbb{R} \mapsto x \log |x|$ is not smooth either). The idea to circumvent the nonsmoothness is to introduce a family of regularized nonlinearities, in such a way that they converge to the nonlinearity under study (see [16, 18, 19]), and solve a regularized or approximate problem. Provided we can obtain suitable uniform estimates for the approximate solutions, we can use compactness tools in order to obtain the global existence of weak solutions.

As for the existence of solutions, another difficulty coming from the lack of smoothness of the nonlinearity concerns the uniqueness of solutions. In order to establish the uniqueness, we usually need to impose suitable regularity on the solution itself. In our case, we need to assume that the solution u satisfies

$$\partial_x \left[\frac{u}{|u|^{\frac{1}{2}}} \right] \in L^\infty([-T, T], L^\infty_{per}([0, L])), \quad \text{for all } T > 0. \tag{7}$$

By combining energy estimates with Gronwall's inequality we see that (7) is sufficient to deduce the uniqueness of global weak solutions (see Theorem 2.1 below).

Having proved the existence and uniqueness of weak solutions, we are enable to deduce that equation (1) admits three conserved quantities, namely,

$$\mathcal{E}(v) = \frac{1}{2}\int_0^L \left(v_x^2 - \frac{4}{5}\mathrm{sign}(v)|v|^{\frac{5}{2}} \right) dx, \tag{8}$$

$$\mathcal{F}(v) = \frac{1}{2}\int_0^L v^2 \, dx, \tag{9}$$

and

$$\mathcal{M}(v) = \int_0^L v \, dx. \tag{10}$$

These conserved quantities play a crucial role in the stability analysis, which we shall describe next. Our ideas are inspired in the recent work [1], where the authors have established sufficient conditions to determine the orbital stability of periodic traveling-wave solutions related to the regularized Schamel equation

$$u_t - u_{xxt} + \partial_x \left(u + |u|^{\frac{3}{2}} \right) = 0. \tag{11}$$

They adapted the arguments in [21,24] by proving that under certain conditions, the augmented energy functional related to (11) has a minimum restricted to a convenient manifold. We point out that the approach in [1] should provide the orbital stability of periodic traveling waves associated with KdV-type equations of the form

$$u_t + u_{xxx} + \partial_x(f(u)) = 0, \tag{12}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear real-valued function. Traveling waves for (12) are also solutions of the form $u(x,t) = \varphi(x - \omega t)$, but now with φ being a solution of

$$\varphi'' - \omega\varphi + f(\varphi) = A. \tag{13}$$

We next describe the main steps to obtain the orbital stability of the periodic waves in (4). First, we see that all parameters appearing in (4) can be written in terms of the period L and the modulus $k \in (0,1)$; thus, we can write $\omega := \omega(k)$ and obtain that, for each $L > 0$ fixed, φ_ω depends smoothly on k, that is, $\varphi_\omega := \varphi_{\omega(k)}$. After that, as is well-known, one of main ingredients to establish the orbital stability of φ_ω is the knowledge of non-positive portion of the spectrum of the linear operator arising in the linearization of (1) around φ_ω. In our case, such operator reads as

$$\mathcal{L}_k := -\partial_x^2 + \omega(k) - \frac{3}{2}\varphi_{\omega(k)}^{\frac{1}{2}}.$$

In general, the spectral properties of $\mathcal{L}_k$ we need are the following:

(S1) $\ker(\mathcal{L}_k) = [\varphi'_{\omega(k)}]$;

(S2) $\mathcal{L}_k$ has a unique negative eigenvalue, which is simple.

Once established (S1) and (S2), a Vakhitov–Kolokolov-type condition,

(S3) $\left\langle \mathcal{L}_k \left(\frac{\partial \varphi_{\omega(k)}}{\partial k} \right), \frac{\partial \varphi_{\omega(k)}}{\partial k} \right\rangle < 0$,

then implies the orbital stability (see Section 3 for details).

Let us brief describe how one can obtain conditions (S1) and (S2). First of all, we should take note that the operators $\mathcal{L}_k$ have compact resolvent, so that their spectra are purely point spectra consisting of isolated eigenvalues with finite algebraic multiplicities. The task then is to locate the first three eigenvalues. We will present two different approaches for establishing (S1) and (S2). In the first one, we see $\mathcal{L}_k$ as an Hill's operator having the form

$$\mathcal{L}_{Hill} = -\partial_x^2 + g'(\omega, \varphi_\omega)$$

and use the recent theory developed in [26, 27], where the authors presented a new method based on the classical Floquet theorem to establish a characterization of the first three eigenvalues of $\mathcal{L}_{Hill}$ by knowing one of its eigenfunctions. It can be shown that $\mathcal{L}_{Hill}$ is isonertial with respect to the velocity and the period of the traveling waves. In the second approach, we observe that the periodic eigenvalue problem associated with $\mathcal{L}_k$ can be reduced to an equivalent eigenvalue problem associated with the well known Lamé equation. Since the eigenvalues of the Lamé equation are given explicitly, we are able to show (S1) and (S2). For the precise statements we refer the reader to Section 3.

Before we leave this section, let us make some comments relating our work with other ones in the current literature.

In [24] the author established sufficient conditions for obtaining the orbital stability of periodic waves related to (12), by using a combination of arguments contained in [12, 21]. Observe that solutions of (13) also must satisfy (compare with equation (3))

$$\frac{1}{2}[\varphi']^2 - \frac{\omega}{2}\varphi^2 + \widetilde{F}(u) = A\varphi + B, \tag{14}$$

where B is a constant of integration, which can be regarded as the energy, and $\widetilde{F}$ satisfies $\widetilde{F}' = f$ with $\widetilde{F}(0) = 0$. Thus, in the general situation, periodic traveling waves of (12) (if they exist) can be smoothly parametrized by the independent triple of parameters $(\omega, A, B) \in \mathcal{O}$, where $\mathcal{O} \subset \mathbb{R}^3$ is an open set. Interestingly, in [24], the author establishes criteria to determine the orbital stability of such waves depending on the sign of certain determinants, which encode some geometric information about the underlying manifold of periodic solutions. In the applications, the main focus studied by the author was the case $f(u) = u^{p+1}$, where p is a positive integer. Although his theory is valid in a general context, the author applies his results for $p \neq 1, 2$ into two particular situations: waves with sufficiently large period, which means in some sense that periodic waves approach the solitary waves, or waves sufficiently near an equilibrium solution (see [24, Section 5] for the details). The reason is that the question of determining the sign of these determinants brings several difficulties and, in general, can only be done analytically in limiting cases.

We believe our work has some different aspects compared with [24]. Indeed, first of all, it should be noted that our periodic waves have zero-energy; so, in this sense the analysis in [24] is more general inasmuch as the author is considering all possible values of the energy B. However, as commented above, it seems that if we apply his method we will be able to prove the orbital stability only in the two described scenarios (see our Remark 3.1). On the other hand, our analysis allows to obtain the stability of periodic waves with arbitrary period $L > 0$, which is independent on the triple (ω, A, B) as determined in [24].

Moreover, in [14] the authors have presented a result of linear stability (which implies in some cases the orbital stability) for (12), with f a C^2-function, providing a stability index theorem in the periodic context. Such approach relates the number of unstable and potentially unstable eigenvalues of the linearized operator to geometric information about a certain map between the constants of integration in (14) and the conserved quantities of (12). It should be noted that the authors also have applied their theory to the model (1). In order to calculate the stability index they provide numeric calculations indicating that the periodic waves are linearly stable. Hence, our main stability result agrees with the ones in [14] at least for zero-energy solutions.

This paper is organized as follows: In Section 2 is proved the well-posedness of the Cauchy problem (5). Existence of explicit periodic traveling-wave solutions as well as their orbital stability is treated in Section 3.

Notation. For $s \in \mathbb{R}$, the Sobolev space $H^s_{per} = H^s_{per}([0, L])$ is the set of all periodic distributions such that $||f||^2_{H^s_{per}} := L \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s |\widehat{f}(k)|^2 < \infty$, where $\widehat{f}$ is the (periodic) Fourier transform of f. The symbols $\mathrm{sn}(\cdot; k)$, $\mathrm{dn}(\cdot; k)$, and $\mathrm{cn}(\cdot; k)$ represent the Jacobi elliptic functions of *snoidal*, *dnoidal*, and *cnoidal* type, respectively. Recall that $\mathrm{sn}(\cdot; k)$ is an odd function, while $\mathrm{dn}(\cdot; k)$, and $\mathrm{cn}(\cdot; k)$ are even functions. For $k \in (0, 1)$, $K(k)$ and $E(k)$ will denote the complete elliptic integrals of the first and second type, defined respectively by $K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$ and $E(k) = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{(1-t^2)}}\, dt$. For the complete definitions and additional properties of the elliptic functions we refer the reader to [15].

2. Well-posedness results

In this section, we prove the global solvability of the initial-value problem

$$\begin{cases} u_t + u_{xxx} + \partial_x \left(|u|^{\frac{3}{2}}\right) = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{15}$$

by taking the initial data u_0 in the energy space $H^1_{per}([0,L])$. Since we want to establish the orbital stability of the periodic traveling waves in $H^1_{per}([0,L])$, this will be enough to our purposes. Here and in what follows in this section $L > 0$ will be a fixed real number representing the period. As we already said, since the nonlinear term $|u|^{\frac{3}{2}}$ is not C^2 at the origin, a fixed point argument cannot be directly applied. In order to overcome this difficult, we shall regularize the nonlinearity.

Our main theorem in this section reads as follows.

Theorem 2.1. *For any $u_0 \in H^1_{per}([0,L])$ and $T > 0$, there is a global solution $u \in L^\infty([-T,T], H^1_{per}([0,L]))$ associated with the Cauchy problem* (15). *In addition, one has*

$$\mathcal{E}(u(\cdot,t)) \le \mathcal{E}(u_0), \quad \mathcal{F}(u(\cdot,t)) = \mathcal{F}(u_0), \quad \mathcal{M}(u(\cdot,t)) = \mathcal{M}(u_0),$$

a.e. $t \in [-T,T]$. Also, if

$$\frac{\partial}{\partial x}\left[\frac{u}{|u|^{\frac{1}{2}}}\right] \in L^\infty([-T,T], L^\infty_{per}([0,L])), \quad \text{for all } T > 0, \tag{16}$$

then the solution u exists in $C([-T,T], H^1_{per}([0,L]))$, is unique in the interval $[-T,T]$, and satisfies $\mathcal{E}(u(\cdot,t)) = \mathcal{E}(u_0)$ for all $t \in [-T,T]$.

We divide the proof of Theorem 2.1 into four steps. The first one concerns in constructing approximate solutions by regularizing the nonlinearity, in terms of a small parameter $\varepsilon > 0$. In the second step we obtain energy estimates, in order to prove that the approximating solutions are indeed global ones. In the third step, we deduce uniform estimates with respect to the parameter ε, in order to use compactness tools and pass to the limit. Finally, in the fourth step, we establish the uniqueness of solutions under assumption (16).

2.1. Step 1: Approximating solutions. In this subsection we will prove the local existence of approximating solutions. To begin with, we recall the following result, which proves the local well-posedness of the initial-value problem associated to KdV-type equations with smooth nonlinearities.

Theorem 2.2. *Consider the initial-value problem*

$$\begin{cases} u_t + u_{xxx} + f'(u)u_x = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(x,0) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{17}$$

Then, (17) *is locally well-posed provided f is a C^6-function and the initial data v_0 belongs to $H^s_{per}([0,L])$, $s > \frac{1}{2}$. More precisely, there are a time $T = T(\|v_0\|_{H^s_{per}}, f) > 0$, depending only on the norm of the initial data and the nonlinearity, and a unique mild solution of* (17),

$$u \in C([-T,T]; H^s_{per}([0,L])).$$

Proof. See [22, Theorem 1.3]. □

Remark 2.3. For f sufficiently smooth, the local well-posedness of (17) in $H^s_{per}([0,L])$, $s \geq 1$, was established in [13]. In particular, a global well-posedness result holds in $H^1_{per}([0,L])$ as long as the solution remains bounded.

Remark 2.4. Notice that the smoothness of the function f in Theorem 2.2 enables us to establish the identities

$$\mathcal{F}(u(\cdot,t)) = \mathcal{F}(v_0), \quad t \in [-T,T],$$

and

$$\widetilde{\mathcal{E}}(u(\cdot,t)) = \widetilde{\mathcal{E}}(v_0), \quad t \in [-T,T],$$

where $\widetilde{\mathcal{E}}$ is the modified energy, defined as

$$\widetilde{\mathcal{E}}(v) = \frac{1}{2}\int_0^L |v_x|^2\,dx - \int_0^L W(v)\,dx, \quad W(v) := \int_0^v f(s)\,ds.$$

Next, let $0 < \varepsilon < 1$ be fixed. Define the family of regularized nonlinearities

$$f_\varepsilon(v) = \begin{cases} f(v) = |v|^{\frac{3}{2}}, & |v| \geq \varepsilon, \\ p_\varepsilon(v), & |v| \leq \varepsilon, \end{cases}$$

where p_ε is the even polynomial defined by

$$p_\varepsilon(v) = \sum_{n=0}^{6} c_n \varepsilon^{-\frac{1}{2}-2n} v^{2n+2},$$

with the coefficients $c_n \in \mathbb{R}$ determined through the equality $\partial_v^n p_\varepsilon(\varepsilon) = \partial_v^n f(\varepsilon)$, $n = 0,1,\ldots,6$. Thus, the coefficients must satisfy the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 2 & 12 & 30 & 56 & 90 & 132 & 182 \\ 0 & 24 & 120 & 336 & 720 & 1320 & 2184 \\ 0 & 24 & 360 & 1680 & 5040 & 11880 & 24024 \\ 0 & 0 & 720 & 6720 & 30240 & 95040 & 240240 \\ 0 & 0 & 720 & 20160 & 151200 & 665280 & 2162160 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{2} \\ \frac{3}{4} \\ -\frac{3}{8} \\ \frac{9}{16} \\ -\frac{45}{32} \\ \frac{315}{64} \end{pmatrix} \tag{18}$$

It is clear that (18) possess a unique solution. Hence, the polynomial p_ε is well defined and the function f_ε is smooth.

Remark 2.5. Following the same strategy as above, one can smooth the nonlinearity f_ε as regular as necessary. However a regularization with the polynomial of degree 14 near the origin is sufficient to our purposes.

Now, we use Theorem 2.2 to establish the global existence of the approximating solutions.

Lemma 2.6. *Fix $0 < \varepsilon < 1$ and consider the initial-value problem*

$$\begin{cases} u_t^\varepsilon + u_{xxx}^\varepsilon + f_\varepsilon'(u^\varepsilon)u_x^\varepsilon = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}, \\ \qquad\qquad u^\varepsilon(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{19}$$

If $u_0 \in H^1_{per}([0,L])$ then (19) *is globally well-posed. This means that, for any $T > 0$,* (19) *possess a unique solution*

$$u^\varepsilon \in C([-T,T]; H^1_{per}([0,L])).$$

Proof. Since $u_0 \in H^1_{per}([0,L])$ one has from Theorem 2.2 the existence of $T_\varepsilon = T_\varepsilon(\|u_0\|_{H^1_{per}}, f_\varepsilon) > 0$ and a unique $u^\varepsilon \in C([-T_\varepsilon, T_\varepsilon], H^1_{per}([0,L]))$ which is a solution of the Cauchy problem (19). Moreover, as in Remark 2.4, we have

$$\mathcal{F}(u^\varepsilon(\cdot,t)) = \mathcal{F}(u_0), \quad t \in [-T_\varepsilon, T_\varepsilon],$$

and

$$\mathcal{E}_\varepsilon(u^\varepsilon(\cdot,t)) = \mathcal{E}_\varepsilon(u_0), \quad t \in [-T_\varepsilon, T_\varepsilon], \tag{20}$$

where

$$\mathcal{E}_\varepsilon(v) := \frac{1}{2}\int_0^L v_x^2 \, dx - \int_0^L W_\varepsilon(v)\, dx, \quad W_\varepsilon(v) := \int_0^v f_\varepsilon(y)\, dy.$$

In order to complete the proof of Lemma 2.6 we need to show that u^ε can be extended to any time interval $[-T,T]$. To do this, since the local existence time T_ε depends on the initial data itself, it is enough to get an a priori bound for u^ε in $H^1_{per}([0,L])$ (see also Remark 2.3). Such an estimate will be obtained below, taking into account that the energy $\mathcal{E}_\varepsilon$ is conserved.

2.2. Step 2: A priori estimates. First we note that

$$W_\varepsilon(v) = \begin{cases} \dfrac{2}{5}\operatorname{sign}(v)|v|^{\frac{5}{2}}, & \text{if } |v| \geq \varepsilon, \\ \displaystyle\sum_{n=0}^{6} \widetilde{c}_n \varepsilon^{-\frac{1}{2}-2n} v^{3+2n}, & \text{if } |v| \leq \varepsilon, \end{cases}$$

where $\widetilde{c_n}$ are real constants depending only on c_n, $n = 0, 1, \ldots, 6$. Hence, for $t \in [-T_\varepsilon, T_\varepsilon]$,

$$\begin{aligned}&\int_0^L W_\varepsilon(u^\varepsilon(x,t))\,dx\\ &\le \frac{2}{5}\int_{|u^\varepsilon(x,t)|>\varepsilon} |u^\varepsilon(x,t)|^{\frac{5}{2}}\,dx + \sum_{n=0}^{6}|\widetilde{c_n}|\int_{|u^\varepsilon(x,t)|\le\varepsilon} \varepsilon^{-\frac{1}{2}-2n}|u^\varepsilon(x,t)|^{3+2n}\,dx.\end{aligned} \tag{21}$$

It should be clear that the integrations on the right-hand side of (21) are also restricted to the interval $[0, L]$. Notice if $0 < |u^\varepsilon(x,t)| \le \varepsilon$, we have

$$|u^\varepsilon(x,t)|^{-\frac{1}{2}-2n} \ge \varepsilon^{-\frac{1}{2}-2n}, \quad n = 0, \ldots, 6. \tag{22}$$

Thus, from (21), (22) and Sobolev's embedding, we see that, for $t \in [-T_\varepsilon, T_\varepsilon]$,

$$\begin{aligned}-\int_0^L W_\varepsilon(u^\varepsilon(\cdot,t))\,dx &\ge -\frac{2}{5}\int_0^L |u^\varepsilon(x,t)|^{\frac{5}{2}}\,dx - \sum_{n=0}^{6}|\widetilde{c_n}|\int_{|u^\varepsilon(x,t)|\le\varepsilon}|u^\varepsilon(x,t)|^{\frac{5}{2}}\,dx\\ &\ge -b_1\|u^\varepsilon(\cdot,t)\|_{L^\infty_{per}}^{\frac{1}{2}}\|u^\varepsilon(\cdot,t)\|_{L^2_{per}}^2\\ &\ge -b_2\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^{\frac{1}{2}}\|u^\varepsilon(\cdot,t)\|_{L^2_{per}}^2,\end{aligned} \tag{23}$$

where b_1 and b_2 are positive constants independent of $\varepsilon \in (0,1)$. Combining (20) and (23), we deduce

$$\mathcal{E}_\varepsilon(u_0) \ge \frac{1}{2}\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^2 - \frac{1}{2}\|u^\varepsilon(\cdot,t)\|_{L^2_{per}}^2 - b_2\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^{\frac{1}{2}}\|u^\varepsilon(\cdot,t)\|_{L^2_{per}}^2. \tag{24}$$

On the other hand, for any $\delta > 0$, Young's inequality guarantees that

$$\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^{\frac{1}{2}}\|u^\varepsilon(\cdot,t)\|_{L^2_{per}}^2 \le \frac{\delta^4}{4}\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^2 + \frac{3}{4\delta^{\frac{3}{4}}}\|u^\varepsilon(\cdot,t)\|_{L^2_{per}}^{\frac{8}{3}}. \tag{25}$$

Hence, in view of (24) and (25) and by choosing an appropriate small δ, we realize that

$$\mathcal{E}_\varepsilon(u_0) \ge \frac{1}{4}\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^2 - b_3\|u_0\|_{L^2_{per}}^2 - b_4\|u_0\|_{L^2_{per}}^{\frac{8}{3}}, \quad t \in [-T_\varepsilon, T_\varepsilon], \tag{26}$$

where b_3 and b_4 are positive constants independent of $\varepsilon \in (0,1)$. From (26), we finally conclude for all $t \in [-T_\varepsilon, T_\varepsilon]$

$$\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^2 \le 4\mathcal{E}_\varepsilon(u_0) + 8b_3\mathcal{F}(u_0) + 16b_4[\mathcal{F}(u_0)]^{\frac{4}{3}} =: C(u_0,\varepsilon), \tag{27}$$

where $C(u_0,\varepsilon)$ is a positive constant depending only on u_0 and ε. As we have already observed, this is enough to guarantee that solution u^ε belongs to $C([-T,T], H^1_{per}([0,L]))$, for any $T > 0$. Observe that bound in (27) holds as long as the solution exists. In particular, we deduce that

$$\|u^\varepsilon(\cdot,t)\|_{H^1_{per}}^2 \le 4\mathcal{E}_\varepsilon(u_0) + 8b_3\mathcal{F}(u_0) + 16b_4[\mathcal{F}(u_0)]^{\frac{4}{3}}, \quad t \in \mathbb{R}. \tag{28}$$

The proof of Lemma 2.6 is now completed. □

2.3. Step 3: Passage to the limit. Here, we will show uniform estimates with respect to ε in order to obtain a solution of (15) as the limit of the approximating solutions. Let us start with the following lemma.

Lemma 2.7. *Assume* $u_0 \in H^1_{per}([0,L])$. *Then, there exists* $0 < \varepsilon_0 < 1$ *such that, for any* $\varepsilon \in (0,\varepsilon_0)$,

$$\mathcal{E}_\varepsilon(u_0) \le \mathcal{E}(u_0) + 1.$$

Proof. It suffices to prove that

$$\mathcal{E}_\varepsilon(u_0) \overset{\varepsilon\to 0^+}{\longrightarrow} \mathcal{E}(u_0), \tag{29}$$

which we shall establish below.

Claim 1. $W_\varepsilon(u_0(x)) \overset{\varepsilon\to 0^+}{\longrightarrow} W(u_0(x))$, $x \in [0,L]$.

Indeed, let $x \in [0,L]$ be fixed. If $u_0(x) = 0$ there is nothing to prove because

$$W(u_0(x)) = 0 = W_\varepsilon(u_0(x)).$$

If $u_0(x) \neq 0$, there is $a > 0$ such that $|u_0(x)| > a$. So, by choosing $\varepsilon_1 > 0$ such that $0 < \varepsilon_1 < a$, we deduce that for any $\varepsilon \in (0,\varepsilon_1)$,

$$W(u_0(x)) = \frac{2}{5}\text{sign}(u_0(x))|u_0(x)|^{\frac{5}{2}} = W_\varepsilon(u_0(x)).$$

This shows Claim 1.

Claim 2. $W_\varepsilon(u_0(x))$ is uniformly bounded with respect to $0 < \varepsilon < 1$ and $x \in [0,L]$.

In fact, since $u_0 \in H^1_{per}([0,L]) \hookrightarrow C_{per}([0,L])$, there is $b_5 > 0$ such that $|u_0(x)| \le b_5$, for all $x \in [0,L]$. Thus, if $|u_0(x)| \ge \varepsilon$, we get

$$|W_\varepsilon(u_0(x))| = \frac{2}{5}|u_0(x)|^{\frac{5}{2}} \le \frac{2}{5}[b_5]^{\frac{5}{2}}.$$

Also, if $|u_0(x)| \le \varepsilon$, as in (22), it follows that

$$|W_\varepsilon(u_0(x))| \le \sum_{n=0}^{6} |\widetilde{c_n}|\varepsilon^{-\frac{1}{2}-2n}|u_0(x)|^{3+2n} \le \sum_{n=0}^{6} |\widetilde{c_n}| \cdot |u_0(x)|^{\frac{5}{2}} \le \sum_{n=1}^{6} |\widetilde{c_n}| \cdot [b_5]^{\frac{5}{2}},$$

which proves Claim 2.

From Claims 1 and 2, we can apply Lebesgue's dominated convergence theorem to infer

$$\int_0^L W_\varepsilon(u_0(x))\, dx \overset{\varepsilon\to 0^+}{\longrightarrow} \int_0^L W(u_0(x))\, dx. \tag{30}$$

The convergence in (30) is sufficient to pass to the limit in $\mathcal{E}_\varepsilon(u_0)$ and obtain (29). □

From Lemma 2.7 and (28) one then has, if $\varepsilon \in (0, \varepsilon_0)$,

$$\|u^\varepsilon(\cdot, t)\|_{H^1_{per}}^2 \leq 4\mathcal{E}(u_0) + 4 + 8b_3\mathcal{F}(u_0) + 16b_4[\mathcal{F}(u_0)]^{\frac{4}{3}}, \quad t \in \mathbb{R}.$$

Therefore, $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is a bounded sequence in $L^\infty([-T,T], H^1_{per}[0,L]))$, for any $T > 0$. Moreover, it follows from (19) that $\{u^\varepsilon_t\}$ is uniformly bounded in $L^\infty([-T,T], H^{-2}_{per}[0,L]))$. Taking into account standard compactness tools, there is $u \in L^\infty([-T,T], H^1_{per}([0,L]))$ such that, up to a subsequence, we have

$$u^\varepsilon \overset{\varepsilon\to 0^+}{\longrightarrow} u \quad \text{strongly in } L^\infty([-T,T], L^2_{per}([0,L]))$$

and

$$u^\varepsilon(x,t) \overset{\varepsilon\to 0^+}{\longrightarrow} u(x,t), \quad \text{a.e. } (x,t) \in \mathbb{R} \times [-T,T],$$

for any $T > 0$ fixed.

Since $u \in L^\infty([-T,T], L^2_{per}([0,L]))$, we obtain

$$\int_0^L |u_0(x)|^2\,dx = \int_0^L |u^\varepsilon(x,t)|^2\,dx \overset{\varepsilon\to 0^+}{\longrightarrow} \int_0^L |u(x,t)|^2\,dx, \quad \text{a.e. } t \in [-T,T]. \tag{31}$$

Hence, from (31), we conclude $\mathcal{F}(u(\cdot,t)) = \mathcal{F}(u_0)$ a.e. $t \in [-T,T]$.

Next, we shall prove the inequality

$$\mathcal{E}(u(\cdot,t)) \leq \mathcal{E}(u_0), \quad \text{a.e. } t \in [-T,T]. \tag{32}$$

By using the weak lower semicontinuity of the H^1_{per}-norm and Fatou's lemma, we obtain

$$\|u(\cdot,t)\|_{H^1_{per}} \leq \liminf_{\varepsilon\to 0^+} \|u^\varepsilon(\cdot,t)\|_{H^1_{per}}, \quad \text{a.e. } t \in [-T,T]. \tag{33}$$

A similar argument used to determine (30) yields

$$\int_0^L W_\varepsilon(u^\varepsilon(x,t))\,dx \overset{\varepsilon\to 0^+}{\longrightarrow} \int_0^L W(u(x,t))\,dx, \quad \text{a.e. } t \in [-T,T]. \tag{34}$$

From (20), (29), (31), (33), and (34), we conclude that (32) holds.

Finally, following similar arguments as the ones in [16], we deduce that the limit u (which belongs to $L^\infty([-T,T], H^1_{per}([0,L])))$ is indeed a weak solution of the initial value problem (15). This concludes the existence part in Theorem 2.1.

2.4. Step 4: Uniqueness of solutions. We complete the proof of Theorem 2.1 by showing the uniqueness of the solution $u \in L^\infty([-T,T], H^1_{per}([0,L]))$, under the assumption (16). Let us suppose that u and v are two solutions, with the same initial condition u_0. By letting $w := u - v$, we see that $w \in L^\infty([-T,T], H^1_{per}([0,L]))$ is a weak global solution of

$$\begin{cases} w_t + w_{xxx} + \partial_x\left(|u|^{\frac{3}{2}} - |v|^{\frac{3}{2}}\right) = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ w(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{35}$$

By multiplying the differential equation in (35) by w and integrating on $[0,L]$, we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^L w^2\,dx\right] + \int_0^L \frac{\partial}{\partial x}\left[|u|^{\frac{3}{2}} - |v|^{\frac{3}{2}}\right] w\,dx = 0.$$

Next, at least formally, we have $\frac{\partial}{\partial x}\left[|u|^{\frac{3}{2}}\right] = \frac{3}{2}\frac{uu_x}{|u|^{\frac{1}{2}}}$ and $\frac{\partial}{\partial x}\left[\frac{u}{|u|^{\frac{1}{2}}}\right] = \frac{1}{2}\frac{u_x}{|u|^{\frac{1}{2}}}$. So, we can write

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^L w^2\,dx\right] + \frac{3}{2}\int_0^L \left[\frac{uu_x}{|u|^{\frac{1}{2}}} - \frac{vv_x}{|v|^{\frac{1}{2}}}\right] w\,dx = 0. \tag{36}$$

Observe that

$$\begin{aligned} &\int_0^L \left[\frac{uu_x}{|u|^{\frac{1}{2}}} - \frac{vv_x}{|v|^{\frac{1}{2}}}\right] w\,dx \\ &= \int_0^L \left[\frac{uu_x}{|u|^{\frac{1}{2}}} - \frac{vu_x}{|u|^{\frac{1}{2}}} + \frac{vu_x}{|u|^{\frac{1}{2}}} - \frac{vv_x}{|v|^{\frac{1}{2}}}\right] w\,dx \\ &= \int_0^L \left[\frac{u_x}{|u|^{\frac{1}{2}}}\right] w^2\,dx + \int_0^L \left[\frac{vu_x}{|u|^{\frac{1}{2}}} - \frac{vu_x}{|v|^{\frac{1}{2}}} + \frac{vu_x}{|v|^{\frac{1}{2}}} - \frac{vv_x}{|v|^{\frac{1}{2}}}\right] w\,dx \\ &= \int_0^L \left[\frac{u_x}{|u|^{\frac{1}{2}}}\right] w^2\,dx + \int_0^L \left[\frac{1}{|u|^{\frac{1}{2}}} - \frac{1}{|v|^{\frac{1}{2}}}\right] wvu_x\,dx + \int_0^L \left[\frac{v}{|v|^{\frac{1}{2}}}\right] ww_x\,dx \\ &= \int_0^L \left[\frac{u_x}{|u|^{\frac{1}{2}}}\right] w^2\,dx + \int_0^L \left[\frac{1}{|u|^{\frac{1}{2}}} - \frac{1}{|v|^{\frac{1}{2}}}\right] wvu_x\,dx - \frac{1}{2}\int_0^L \left[\frac{\partial}{\partial x}\left[\frac{v}{|v|^{\frac{1}{2}}}\right]\right] w^2\,dx. \end{aligned} \tag{37}$$

We can estimate all terms in (37) in the following way

$$\left|\int_0^L \left[\frac{u_x}{|u|^{\frac{1}{2}}}\right] w^2\,dx\right| \le \left\|\frac{u_x}{|u|^{\frac{1}{2}}}\right\|_{L^\infty_{per}} \|w\|^2_{L^2_{per}}, \tag{38}$$

$$\left|\int_0^L \left[\frac{\partial}{\partial x}\left[\frac{v}{|v|^{\frac{1}{2}}}\right]\right] w^2\,dx\right| \le \left\|\frac{\partial}{\partial x}\left[\frac{v}{|v|^{\frac{1}{2}}}\right]\right\|_{L^\infty_{per}} \|w\|^2_{L^2_{per}}, \tag{39}$$

and

$$\left|\int_0^L \left[\frac{1}{|u|^{\frac{1}{2}}} - \frac{1}{|v|^{\frac{1}{2}}}\right] wvu_x \, dx\right| \leq \int_0^L \frac{|v||u_x||w|\big||v|-|u|\big|}{|u|^{\frac{1}{2}}|v|^{\frac{1}{2}}\left[|u|^{\frac{1}{2}}+|v|^{\frac{1}{2}}\right]} \, dx$$
$$\leq \int_0^L \frac{|u_x|}{|u|^{\frac{1}{2}}}|w||u-v| \, dx \tag{40}$$
$$\leq \left\|\frac{u_x}{|u|^{\frac{1}{2}}}\right\|_{L^\infty_{per}} \|w\|^2_{L^2_{per}},$$

where in the last equality we used integration by parts. In view of (36)–(40),

$$\frac{d}{dt}\left[\|w\|^2_{L^2_{per}}\right] \leq 12\left[\left\|\frac{\partial}{\partial x}\left[\frac{u}{|u|^{\frac{1}{2}}}\right]\right\|_{L^\infty_{per}} + \left\|\frac{\partial}{\partial x}\left[\frac{v}{|v|^{\frac{1}{2}}}\right]\right\|_{L^\infty_{per}}\right]\|w\|^2_{L^2_{per}}. \tag{41}$$

Therefore, from Gronwall's inequality, we conclude that $\|w\|^2_{L^2_{per}} = 0$ and the uniqueness of solutions is proved.

The existence of the conserved quantity $\mathcal{E}$ can be determined by using similar arguments as presented in of [17, Theorem 3.3.9]. Furthermore, we obtain that $u \in C([-T,T], H^1_{per}([0,L]))$ as an application of [19, Lemma 2.4.4]. The proof of Theorem 2.1 is thus completed.

3. Stability of periodic waves

This section is devoted to establish the existence and orbital stability of periodic traveling-wave solutions for (1).

3.1. Existence of periodic traveling waves. As we already discussed in the introduction, by assuming periodic traveling waves of the form $u(x,t) = \varphi_\omega(x-\omega t)$, where ω is a real fixed constant, and making the change of variable $\Lambda_\omega = \varphi_\omega^{\frac{1}{2}}$, we are lead to solve the equation

$$[\Lambda_\omega']^2 = \frac{1}{5}\left[-\Lambda_\omega^3 + \frac{5\omega}{4}\Lambda_\omega^2 + \frac{5}{2}A\right] \equiv \frac{1}{5}F(\Lambda_\omega), \tag{42}$$

where $F(t) = -t^3 + \frac{5\omega}{4}t^2 + \frac{5}{2}A$ is a polynomial function. Let us assume that F has three nonzero real roots $\eta_1 < \eta_2 < \eta_3$ in such a way we can write $F(t) = (t-\eta_1)(t-\eta_2)(\eta_3 - t)$. From (42), we must have

$$\begin{cases} \eta_1 + \eta_2 + \eta_3 = \dfrac{5\omega}{4}, \\ \eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3 = 0, \\ \eta_1\eta_2\eta_3 = \dfrac{5A}{2}. \end{cases} \tag{43}$$

The second equality in (43) and (42) imply that the roots η_i, $i = 1, 2, 3$ must satisfy $\eta_1 < 0 < \eta_2 < \eta_3$, in order to obtain positive solutions. Since Λ_ω must be positive, it must be the case that $\eta_2 \le \Lambda_\omega \le \eta_3$. The Leibniz rule and (42) allow to write

$$\int_{\eta_2}^{\Lambda_\omega(\xi)} \frac{dt}{\sqrt{(\eta_3 - t)(t - \eta_2)(t - \eta_1)}} = \frac{\sqrt{5}}{5}\xi. \tag{44}$$

By defining

$$g = \frac{2}{\sqrt{\eta_3 - \eta_1}} \quad \text{and} \quad k^2 = \frac{\eta_3 - \eta_2}{\eta_3 - \eta_1}, \tag{45}$$

from (44) and the definition of the Jacobi elliptic functions (see, e.g, [15]), we deduce

$$\Lambda_\omega(\xi) = \eta_2 + (\eta_3 - \eta_2)\,\mathrm{cn}^2\left(\frac{\sqrt{5}}{5g} \cdot \xi; k\right).$$

Finally, taking into account that $\Lambda_\omega = \varphi_\omega^{\frac{1}{2}}$, we obtain

$$\varphi_\omega(\xi) = \left[\eta_2 + (\eta_3 - \eta_2)\,\mathrm{cn}^2\left(\sqrt{\frac{\eta_3 - \eta_1}{20}} \cdot \xi; k\right)\right]^2. \tag{46}$$

Next we will be concerned with the construction of periodic waves with an arbitrary period. Note that the function φ_ω in (46) has fundamental period given by

$$T_{\varphi_\omega}(k) = \frac{2\sqrt{20}K(k)}{\sqrt{\eta_3 - \eta_1}}. \tag{47}$$

On the other hand, if we combine the first and second equations in (43), we see that

$$\frac{5\omega}{4}(\eta_2 + \eta_3) = \eta_2^2 + \eta_3^2 + \eta_2\eta_3. \tag{48}$$

It follows from (48) that we must have $\omega > 0$ and

$$\left(\frac{5\omega}{4} - \eta_3\right)(\eta_2 + \eta_3) = \eta_2^2. \tag{49}$$

Identity (49) yields that $\eta_3 < \frac{5\omega}{4}$. Our next step is to prove that

$$\eta_1 < 0 < \eta_2 < \frac{5\omega}{6} < \eta_3 < \frac{5\omega}{4}. \tag{50}$$

Indeed, by solving the quadratic equation (48) for η_3, one sees that η_3 can be written as a function of η_2 (recall that $\omega > 0$ is fixed), namely,

$$\eta_3 \equiv \eta_3(\eta_2) = \frac{1}{2}\left(\frac{5\omega}{4} - \eta_2\right) \pm \frac{1}{2}\sqrt{\Delta_\omega(\eta_2)}, \tag{51}$$

where

$$\Delta_\omega(\eta_2) = \left(\eta_2 - \frac{5\omega}{4}\right)^2 - 4\left(\eta_2^2 - \frac{5\omega}{4}\eta_2\right).$$

Since $\omega > 0$ and $0 < \eta_2 < \eta_3 < \frac{5\omega}{4}$, we obtain $\Delta_\omega(\eta_2) \geq 0$. In addition, from (51) it must be the case that

$$\eta_3(\eta_2) = \frac{1}{2}\left(\frac{5\omega}{4} - \eta_2\right) + \frac{1}{2}\sqrt{\Delta_\omega(\eta_2)}.$$

By observing that $\eta_2 \in \left(0, \frac{5\omega}{4}\right) \mapsto \eta_3(\eta_2)$ is a decreasing function and $\eta_2 = \frac{5\omega}{6}$ is a fixed point of $\eta_3 = \eta_3(\eta_2)$, the inequality $0 < \eta_2 < \eta_3$ implies that $0 < \eta_2 < \frac{5\omega}{6}$. Consequently, $\frac{5\omega}{6} < \eta_3 < \frac{5\omega}{4}$ and (50) is proved.

Note that the expression in (48) is symmetric with respect to η_2 and η_3. Thus, the parameter η_2 can also be written as a function of η_3 in a similar formula as in (51). Taking this result in the first equation of (43), we also obtain η_1 as a function of η_3. In particular, from (45) and (47), the period of φ_ω is expressed as

$$T_{\varphi_\omega}(\eta_3) = \frac{2\sqrt{40}K(k_{\varphi_\omega}(\eta_3))}{\sqrt{3\eta_3 - \frac{5\omega}{4} + \sqrt{\frac{25\omega^2}{16} + \frac{5\omega\eta_3}{2} - 3\eta_3^2}}}, \tag{52}$$

with $k = k_{\varphi_\omega}$ given by

$$k^2 = \frac{3\eta_3 - \frac{5\omega}{4} - \sqrt{\frac{25\omega^2}{16} + \frac{5\omega\eta_3}{2} - 3\eta_3^2}}{3\eta_3 - \frac{5\omega}{4} + \sqrt{\frac{25\omega^2}{16} + \frac{5\omega\eta_3}{2} - 3\eta_3^2}}. \tag{53}$$

By using relations in (48) and (50), the expression in (52) allows to describe the behavior of the period of our traveling waves in (46). In fact, if $\eta_3 \to \frac{5\omega}{6}$, we deduce from (48) that $\eta_2 \to \frac{5\omega}{6}$ and $k \to 0$. Therefore, from (52), we conclude $T_{\varphi_\omega}(\eta_3) \to \frac{4\pi}{\sqrt{\omega}}$, as $\eta_3 \to \frac{5\omega}{6}$. On the other hand, if $\eta_3 \to \frac{5\omega}{4}$, we get $\eta_2 \to 0$. Consequently $k \to 1$ and $T_{\varphi_\omega}(\eta_3) \to +\infty$. Since $\eta_3 \in \left(\frac{5\omega}{6}, \frac{5\omega}{4}\right) \mapsto T_{\varphi_\omega}(\eta_3)$ is an increasing function (this fact may be shown as an application of the implicit function theorem, see, for instance, [1,3,7,9]), we have that $T_{\varphi_\omega}(\eta_3) > \frac{4\pi}{\sqrt{\omega}}$.

Remark 3.1. Two different scenarios appear in the limiting cases above. Indeed, as observed, we have $\eta_2 \to \frac{5\omega}{6}$ and $k \to 0$, as $\eta_3 \to \frac{5\omega}{6}$ ($\omega > 0$ fixed). Hence, from (46), the solution φ_ω degenerates to the equilibrium state $\varphi_{\text{cte}} = \frac{25\omega^2}{36}$. In this limiting case, from (43), it follows that $A = -\frac{25\omega^3}{196}$; hence, φ_{cte} is indeed a constant solution of (2) but it has no positive fundamental period.

On the other hand, we obtained that $\eta_2 \to 0$ and $k \to 1$, as $\eta_3 \to \frac{5\omega}{4}$. Hence, the cnoidal function looses periodicity and it degenerates to the hyperbolic-secant function. Thus, in this limiting case, the solution φ_ω degenerates to

$$\varphi_{\rm sol}(\xi) = \frac{25\omega^2}{16}\text{sech}^4\left(\frac{\sqrt{\omega}}{4}\xi\right),$$

which is the solitary wave associated with the Schamel equation (see also [33]). Notice from (43) that in this scenario, the constant A must vanish. It is clear that the existence of solitary waves to (2) is possible only in this situation.

The above discussion permits the construction of periodic traveling waves with arbitrary period $L>0$. Indeed, fixed $L>0$, choose $\omega>0$ satisfying $\omega > \frac{16\pi^2}{L^2}$. Since the function $\eta_3 \in \left(\frac{5\omega}{6}, \frac{5\omega}{4}\right) \mapsto T_{\varphi_\omega}(\eta_3)$ is continuous and increasing, there is a unique η_3 such that $T_{\varphi_\omega}(\eta_3) = L$. In particular, the traveling wave φ_ω, determined by the parameters η_1, η_2, and η_3 according to (43), has fundamental period L. We summarize our constructions in the next theorem.

Theorem 3.2. *Let $L>0$ be fixed. For any $\omega > \frac{16\pi^2}{L^2}$, equation* (2) *has a periodic solution with fundamental period L. The solution $\varphi = \varphi_\omega$ is given by* (46)*, with the parameters η_1, η_2, and η_3 satisfying* (43)*. In addition, the map*

$$\omega \in \left(\frac{16\pi^2}{L^2}, \infty\right) \mapsto \varphi_\omega \in H^n_{per}([0,L]), \quad n \in \mathbb{N},$$

is smooth.

Proof. The proof is an application of the implicit function theorem. The interested reader will find some details, for instance, in [3, 7, 9]. □

As we will see below, for a fixed $L>0$, it is useful to express the parameters appearing in φ_ω as functions of the parameter k. So, we next present exact formulas for ω, η_1, η_2, η_3, and A in terms of the modulus $k \in (0,1)$. Indeed, from (43), (52) and (53), we get

$$\omega = \frac{64K(k)^2}{L^2}\sqrt{1-k^2+k^4}, \tag{54}$$

$$\eta_1 = \frac{80K(k)^2}{3\ L^2}\left[-2+k^2+\sqrt{1-k^2+k^4}\right],$$

$$\eta_2 = \frac{80K(k)^2}{3L^2}\left[1-2k^2+\sqrt{1-k^2+k^4}\right],$$

$$\eta_3 = \frac{80K(k)^2}{3L^2}\left[1+k^2+\sqrt{1-k^2+k^4}\right],$$

and

$$A = \frac{204800K(k)^6}{27L^6}\left[-2k^6+3k^4+3k^2-2-2(1-k^2+k^4)^{\frac{3}{2}}\right]. \tag{55}$$

Moreover, taking this expressions in (46), we see that our periodic traveling waves are written as

$$\begin{aligned}\varphi_\omega(\xi) &= \frac{6400K(k)^4}{9L^4}\left[1-2k^2+\sqrt{1-k^2+k^4}+3k^2\operatorname{cn}^2\left(\frac{2K(k)\xi}{L};k\right)\right]^2 \\ &= \frac{6400K(k)^4}{9L^4}\left[k^2-2+\sqrt{1-k^2+k^4}+3\operatorname{dn}^2\left(\frac{2K(k)\xi}{L};k\right)\right]^2,\end{aligned} \tag{56}$$

where in the last equality we have used the relation $k^2\operatorname{cn}^2(z;k) = k^2-1+\operatorname{dn}^2(z;k)$.

3.2. Spectral analysis. Let $L>0$ be fixed. Here, we will study the (non-positive) spectrum of the linear operator arising from the linearization of (1) around a periodic traveling wave φ_ω. Consider the family of self-adjoint operators

$$\mathcal{L}_\omega = -\partial_x^2+\omega-\frac{3}{2}\varphi_\omega^{\frac{1}{2}}, \quad \omega > \frac{16\pi^2}{L^2}, \tag{57}$$

defined in $L^2_{per}([0,L])$ with domain $H^2_{per}([0,L])$. Since $\omega=\omega(k)$, $k\in(0,1)$ one can also see $\mathcal{L}_\omega$ depending on the parameter k. So, in what follows we also use $\mathcal{L}_k$ to represent the operator $\mathcal{L}_\omega$ in (57).

We shall give, in a few lines, the relevant description of the spectrum of Hill's type operator as in (57). The interested reader will find all details, for instance, in [25]. The spectrum of $\mathcal{L}_\omega$, say, $\sigma(\mathcal{L}_\omega)$, is formed by an unbounded sequence of real eigenvalues

$$\gamma_0<\gamma_1\leq\gamma_2<\gamma_3\leq\gamma_4<\cdots<\gamma_{2n-1}\leq\gamma_{2n}<\cdots,$$

where equality means that $\gamma_{2n-1}=\gamma_{2n}$ is a double eigenvalue. In addition, $\sigma(\mathcal{L}_\omega)$ is characterized by the number of zeros of the eigenfunctions in the following way: if p is an eigenfunction associated to the eigenvalue γ_{2n-1} or γ_{2n}, then p has exactly $2n$ zeros in the half-open interval $[0,L)$.

Proposition 3.3. *Let $\mathcal{L}_\omega$ be the operator given by* (57)*. Then $\mathcal{L}_\omega$ has zero as a simple eigenvalue associated to the eigenfunction φ_ω'. In addition, it admits a unique negative eigenvalue which is simple.*

Proof. Since, from (2), $\mathcal{L}_\omega\varphi_\omega'=0$, we promptly see that zero is an eigenvalue with associated eigenfunction φ_ω'. Also, since φ_ω' has two zeros in $[0,L)$, it follows that zero must be the second or the third eigenvalue. We will prove that it is the second one. Below we present two different ways of proving this.

The first one uses some numerical computations while the second one gives an explicit representation of the spectrum in terms of k.

First proof. Let $\{\bar{y}, \varphi'_\omega\}$ be a basis formed by smooth solutions of the equation

$$-y'' + \omega y - \frac{3}{2}\varphi_\omega^{\frac{1}{2}} y = 0.$$

From classical Floquet's theory (see, e.g., [25]), there is a real constant θ such that

$$\bar{y}(x+L) = \bar{y}(x) + \theta\varphi'_\omega(x), \quad \text{for all } x \in [0, L]. \tag{58}$$

In some sense, constant θ in (58) measures how far the function $\bar{y}$ is from being periodic. In fact, $\bar{y}$ is periodic if, and only if, $\theta = 0$. Hence, zero is a simple eigenvalue of $\mathcal{L}_\omega$ if, and only if, $\theta \neq 0$. Also, according to [28, Theorem 3.1], $\gamma_1 = 0$ if $\theta < 0$ and $\gamma_2 = 0$ if $\theta > 0$. Thus, what is left is to show that $\theta < 0$.

At this point, it should be noted that from the smoothness of the periodic wave φ_ω we deduce (see [26, 27]) that operator $\mathcal{L}_\omega$ is isoinertial with respect to the period L and the velocity ω. More specifically, this means that the number of negative and zero eigenvalues associated with $\mathcal{L}_\omega$ do not depend on ω and L. Equivalently, this means that the number of negative and zero eigenvalues associated with $\mathcal{L}_k$ do not depend on $k \in (0,1)$ and $L > 0$. Thus, in order to compute the constant θ appearing in (58), one can fix $k_0 \in (0,1)$, $L > 0$ and a convenient function $\bar{y}$.

Let us consider $L = 10$ and $k_0 = 0.5$. So, from (54), we get $\omega_0 \approx 1.639$. Now, let $\bar{y}$ be the unique solution of the Cauchy problem

$$\begin{cases} -\bar{y}'' + \left(\omega_0 - \frac{3}{2}\varphi_{\omega_0}^{\frac{1}{2}}\right)\bar{y} = 0, \\ \bar{y}(0) = -\dfrac{1}{\varphi''_{\omega_0}(0)} \approx 2.3736, \quad \bar{y}'(0) = 0. \end{cases} \tag{59}$$

It is clear that φ'_ω and $\bar{y}$ are indeed linearly independent. By solving (59) numerically, from (58), we see that

$$\theta = \frac{\bar{y}'(L)}{\varphi''_{\omega_0}(0)} \approx \frac{5.54784}{-0.42131} \approx -13.121.$$

The proof is thus completed.

Second proof. It suffices to show that zero is the second eigenvalue of the periodic eigenvalue problem associated with the operator $\mathcal{L}_k$, namely,

$$\begin{cases} -y'' + \left(\omega(k) - \frac{3}{2}\varphi_{\omega(k)}^{\frac{1}{2}}\right) y = \gamma y, \\ y(0) = y(L), \quad y'(0) = y'(L). \end{cases} \tag{60}$$

By making the change of variable $\Lambda(x) = y(\eta x)$, with $\eta = \frac{L}{2K(k)}$, and using that $\mathrm{cn}^2(x;k) + \mathrm{sn}^2(x;k) = 1$, we see that (60) is equivalent to

$$\begin{cases} \Lambda''(x) + \left[d - 5 \cdot 6 \cdot k^2 \,\mathrm{sn}^2\,(x;k)\right] \Lambda(x) = 0, \\ \qquad\qquad \Lambda(0) = \Lambda(2K(k)), \quad \Lambda'(0) = \Lambda'(2K(k)), \end{cases} \tag{61}$$

where

$$d = 10(k^2+1) + \eta^2\gamma - 6\sqrt{k^4 - k^2 + 1}. \tag{62}$$

The differential equation in (61) is the well known Lamé equation. According to [23] the first five eigenvalues and eigenfunctions of (61) can be found explicitly (see also [1, 3, 5, 6] for additional references). In particular if d_i, $i = 0, 1, \dots, 4$ denotes such eigenvalues, they must be the real roots of (see the details in [1]),

$$d^2 - 20(1+k^2)d + 64(1+k^2)^2 + 108k^2 = 0 \tag{63}$$

or

$$d^3 - (20+35k^2)d^2 + (64+576k^2+259k^4)d - 225k^6 - 1860k^4 - 960k^2 = 0, \tag{64}$$

It is not difficult to see that the roots of (63) and (64) are different each other and can be ordered, without lost of generality, in the form $d_0 < d_1 < d_2 < d_3 < d_4$. In particular, $d_1 = 10(1+k^2) - 6\sqrt{k^4 - k^2 + 1}$ and $d_3 = 10(1+k^2) + 6\sqrt{k^4 - k^2 + 1}$ are the roots of (63). From (62), one infers that $\gamma = 0$ if and only if $d = d_1$. Since d_1 is the second eigenvalue of (61) and d and γ are related in a linear way, we deduce that $\gamma = 0$ is the second eigenvalue of (60). The proof is thus completed. □

3.3. A Vakhitov–Kolokolov-type condition. Before going further, to motivate our analysis, let us recall some well-known facts concerning the stability (spectral and orbital) of traveling waves associated with KdV-type equations of the form (12). As observed before, traveling waves must be solutions of (13). Assume for the moment that φ_ω is a solitary-wave solution, which means that φ_ω together with all its derivatives go to zero at infinity. It must be the case that the constant of integration A vanishes. Thus, (13) reduces to

$$-\omega\varphi_\omega + \varphi_\omega'' + f(\varphi_\omega) = 0. \tag{65}$$

By assuming the existence of a smooth curve of solitary waves satisfying (65) and suitable spectral conditions on the linearized operator as those ones in (S1) and (S2), as is well-known in the current literature, the Vakhitov–Kolokolov condition

$$\frac{d}{d\omega}\int \varphi_\omega^2 \, dx > 0, \tag{66}$$

is sufficient to imply the linear (and nonlinear) stability of φ_ω. The condition in (66) can be viewed in an equivalent manner, taking into account the linearized operator. In fact, the linearized operator here reads as $\mathcal{L}=-\partial_x^2+\omega-f'(\varphi_\omega)$. Thus, by taking the derivative with respect to ω in (65) we see that $\mathcal{L}(\partial_\omega\varphi_\omega)=-\varphi_\omega$. Therefore,

$$\frac{1}{2}\frac{d}{d\omega}\int \varphi_\omega^2\,dx = \int \varphi_\omega\partial_\omega\varphi_\omega\,dx = -\int \mathcal{L}(\partial_\omega\varphi_\omega)\partial_\omega\varphi_\omega\,dx = -\langle \mathcal{L}(\partial_\omega\varphi_\omega), \partial_\omega\varphi_\omega\rangle,$$

and (66) is equivalent to

$$\mathcal{I} := \langle \mathcal{L}(\partial_\omega\varphi_\omega), \partial_\omega\varphi_\omega\rangle < 0. \tag{67}$$

Now note if the parameters ω and A are independent each other, we still can take the derivative with respect to ω in (13) and obtain $\mathcal{L}(\partial_\omega\varphi_\omega) = -\varphi_\omega$. Hence, the condition (67) still could be sufficient to imply the linear stability of φ_ω.

In the periodic context, if we assume that the constant of integration A is zero (for all ω) or that it does not depend on ω, the analysis above takes place and (66) (or equivalently (67)) is usually sufficient to imply the linear stability of periodic waves.

However, if A depends on ω or both A and ω depend on a third parameter (which is our case), the analysis above must be slightly modified and it seems that (66) itself is not sufficient to obtain the stability. Hence, instead of the Vakhitov–Kolokolov condition, we will see below that a condition similar to (67) still gives a criterion to obtain the stability (nonlinear) of our periodic traveling waves.

To begin with, let $L > 0$ be fixed. Let φ_ω be the periodic wave obtained in Subsection 3.1. Since φ_ω depends smoothly on ω, and ω and A depend smoothly on $k \in (0,1)$, we can define $\psi := \frac{\partial\varphi_\omega}{\partial k}$ and

$$\mathcal{W}_k(v) = \frac{\partial\omega}{\partial k}\mathcal{F}(v) + \frac{\partial A}{\partial k}\mathcal{M}(v), \tag{68}$$

where $\mathcal{F}$ and $\mathcal{M}$ are the functional in (9) and (10). It is easily seen that

$$\mathcal{W}_k'(\varphi_\omega) = \frac{\partial\omega}{\partial k}\varphi_\omega + \frac{\partial A}{\partial k}. \tag{69}$$

In addition, taking the derivative with respect to k in (2), we infer

$$-\psi'' + \frac{\partial\omega}{\partial k}\varphi_\omega + \omega\psi - \frac{3}{2}\varphi_\omega^{\frac{1}{2}}\psi + \frac{\partial A}{\partial k} = 0.$$

Thus, $\mathcal{L}_\omega(\psi) = -\mathcal{W}_k'(\varphi_\omega)$ and

$$\langle \mathcal{L}(\psi), \psi\rangle = -\langle \mathcal{W}_k'(\varphi_\omega), \psi\rangle\,.$$

From the above calculations, we see that in our context, condition (66) must be replaced by $\langle \mathcal{W}_k'(\varphi_\omega), \psi\rangle > 0$. In what follows, we address our efforts to show that this condition holds.

Proposition 3.4. *For all $k \in (0,1)$, one has* $\langle \mathcal{L}_k(\psi), \psi \rangle = -\langle \mathcal{W}_k'(\varphi_\omega), \psi \rangle < 0$.

Proof. As observed above, we only need to show that $\langle \mathcal{W}_k'(\varphi_\omega), \psi \rangle > 0$. In fact, from (69), we obtain

$$\begin{aligned} \langle \mathcal{W}_k'(\varphi_\omega), \psi \rangle &= \int_0^L \frac{\partial \omega}{\partial k} \varphi_\omega \frac{\partial \varphi_\omega}{\partial k} \, dx + \int_0^L \frac{\partial A}{\partial k} \frac{\partial \varphi_\omega}{\partial k} \, dx \\ &= \frac{1}{2} \frac{\partial \omega}{\partial k} \frac{\partial}{\partial k} \left[\int_0^L \varphi_\omega^2 \, dx \right] + \frac{\partial A}{\partial k} \frac{\partial}{\partial k} \left[\int_0^L \varphi_\omega \, dx \right]. \end{aligned} \tag{70}$$

Next, we derive a convenient expression for the right-hand side of (70). From (3) (with $B = 0$), one has

$$[\varphi_\omega']^2 = \omega \varphi_\omega^2 - \frac{4}{5} [\varphi_\omega]^{\frac{5}{2}} + 2A\varphi_\omega. \tag{71}$$

By integrating (71) over $[0, L]$ and using integration by parts, it follows that

$$-\int_0^L \varphi_\omega \varphi_\omega'' \, dx = \int_0^L \left[\omega \varphi_\omega^2 - \frac{4}{5} [\varphi_\omega]^{\frac{5}{2}} + 2A\varphi_\omega \right] \, dx. \tag{72}$$

Combining (2) and (72), we deduce

$$2\omega \int_0^L \varphi_\omega^2 \, dx = \frac{9}{5} \int_0^L [\varphi_\omega]^{\frac{5}{2}} \, dx - 3A \int_0^L \varphi_\omega \, dx. \tag{73}$$

Letting $\zeta := \frac{\partial \varphi_\omega}{\partial \omega}$ and deriving (73) with respect to ω one has

$$2\int_0^L \varphi_\omega^2 \, dx + 4\omega \int_0^L \varphi_\omega \zeta \, dx = \frac{9}{2} \int_0^L [\varphi_\omega]^{\frac{3}{2}} \zeta \, dx - 3\frac{\partial A}{\partial \omega} \int_0^L \varphi_\omega \, dx - 3A \int_0^L \zeta \, dx. \tag{74}$$

The next step is to derive (2) with the respect to ω, multiply the expression by ζ and integrate the final result over $[0, L]$ to get

$$\int_0^L \left[-\varphi_\omega'' \zeta + \varphi_\omega^2 + \omega \varphi_\omega \zeta - \frac{3}{2} [\varphi_\omega]^{\frac{3}{2}} \zeta + \frac{\partial A}{\partial \omega} \varphi_\omega \right] \, dx = 0. \tag{75}$$

Thus, combining (2) and (75), we deduce

$$\frac{1}{2} \int_0^L [\varphi_\omega]^{\frac{3}{2}} \zeta \, dx = \int_0^L \varphi_\omega^2 \, dx + \frac{\partial A}{\partial \omega} \int_0^L \varphi_\omega \, dx - A \int_0^L \zeta \, dx. \tag{76}$$

Equalities (74) and (76) give

$$\frac{\partial}{\partial \omega} \left(\int_0^L \varphi_\omega^2 \, dx \right) = \frac{7}{2\omega} \|\varphi_\omega\|_{L^2_{per}}^2 + \frac{3}{\omega} \frac{\partial A}{\partial \omega} \int_0^L \varphi_\omega \, dx - \frac{6A}{\omega} \int_0^L \zeta \, dx. \tag{77}$$

Therefore, we combine (70) and (77) to conclude that

$$\begin{aligned}&\langle \mathcal{W}_k'(\varphi_\omega),\psi\rangle\\ &=\frac{7}{4\omega}\left(\frac{\partial\omega}{\partial k}\right)^2\|\varphi_\omega\|^2_{L^2_{per}}+\frac{3}{2\omega}\frac{\partial\omega}{\partial k}\frac{\partial A}{\partial k}\left[\int_0^L\varphi_\omega\,dx\right]+\left[\frac{\partial A}{\partial k}-\frac{3A}{\omega}\frac{\partial\omega}{\partial k}\right]\frac{\partial}{\partial k}\left[\int_0^L\varphi_\omega\,dx\right].\end{aligned} \tag{78}$$

We now pay attention to the expression on the right-hand side of (78). First we see from (54) and (55) that $\omega>0$ and $A<0$, for all $k\in(0,1)$. In addition,

$$\frac{\partial\omega}{\partial k}=\frac{64K(k)\left[(2E(k)-3K(k))\left(1-k^2+k^4\right)+K(k)(1+2k^4)\right]}{L^2k(1-k^2)\sqrt{1-k^2+k^4}}, \tag{79}$$

and

$$\begin{aligned}\frac{\partial A}{\partial k}=&\frac{409600K(k)^5}{9L^6k(1-k^2)}\Big[K(k)\big(2-4k^2+k^4+k^6\big)-E(k)\big(2-3k^2-3k^4+2k^6\big)\\ &+\sqrt{1-k^2+k^4}\Big[K(k)\big(2-3k^2+k^4\big)-2E(k)\big(1-k^2+k^4\big)\Big]\Big].\end{aligned} \tag{80}$$

From the properties of the complete elliptic integrals (see, e.g., [15]), it is not difficult to see that (note that these expressions are well suited to numerical computations)

$$\frac{\partial\omega}{\partial k}>0\quad\text{and}\quad\frac{\partial A}{\partial k}>0\quad\text{for all } k\in(0,1). \tag{81}$$

Next, we need to deduce convenient expressions for $\int_0^L\varphi_\omega$ and its derivative with respect to k. By using (56) and the formulas for the Jacobi elliptic function of cnoidal type contained in [15, p. 193], we get

$$\int_0^L\varphi_\omega\,dx=P\left[Q^2L+2QC_2+C_4\right], \tag{82}$$

where,

$$P=\frac{6400k^4K(k)^4}{L^4},\quad Q=\frac{1-2k^2+\sqrt{1-k^2+k^4}}{3k^2}, \tag{83}$$

$$C_2=\frac{L\left[E(k)-(1-k^2)K(k)\right]}{k^2K(k)}, \tag{84}$$

and

$$C_4=\frac{L\left[(2-3k^2)(1-k^2)K(k)+2(2k^2-1)E(k)\right]}{3k^4K(k)}. \tag{85}$$

Gathering together (82)–(85), we obtain

$$\frac{\partial}{\partial k}\left[\int_0^L\varphi_\omega\,dx\right]=\frac{12800K(k)^2}{9L^3k(1-k^2)\sqrt{1-k^2+k^4}}\cdot h(k), \tag{86}$$

where

$$\begin{aligned} h(k) = 2K(k)\Big[2\big(1-k^2+k^4\big)E(k) - \big(2-3k^2+k^4\big)K(k)\Big]\sqrt{1-k^2+k^4} \\ + \big(5-10k^2+7k^4-2k^6\big)K(k)^2 + 9\big(1-k^2+k^4\big)E(k)^2 \\ - \big(14-21k^2+15k^4-4k^6\big)K(k)E(k), \end{aligned} \tag{87}$$

is a positive function for all $k \in (0,1)$ (this can be easily checked numerically). Therefore, collecting the informations in (78)–(81) and (86)–(87), we finally deduce $\langle \mathcal{W}_k'(\varphi_\omega), \psi\rangle > 0$, for all $k \in (0,1)$, and the proof of the proposition is completed. □

3.4. Orbital stability. Let $L > 0$ be fixed. Consider $\omega \in (\frac{16\pi^2}{L^2}, \infty) \mapsto \varphi_\omega \in H^n_{per}([0,L])$, $n \in \mathbb{N}$ the smooth curve of periodic traveling-wave solutions which solves equation (2) as in Subsection 3.1. In this subsection, we establish the orbital stability of the periodic waves $u(x,t) = \varphi_\omega(x-\omega t)$ in the energy space $\Upsilon := H^1_{per}([0,L])$, provided that the uniqueness of solutions according to Theorem 2.1 is assumed. Our approach will be outlined by stability analysis of Lyapunov type (see also [1,2,4,10,11,21,31,32]).

Before going on, let us recall what we mean by orbital stability.

Definition 3.5. We say that the periodic traveling wave $u(x,t) = \varphi_\omega(x-\omega t)$ is orbitally stable in the space Υ, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in \Upsilon$, with $\|u_0 - \varphi_\omega\|_\Upsilon < \delta$ then the solution of (1), say $u(\cdot,t)$, with $u(\cdot,0) = u_0$ exists for all $t \in \mathbb{R}$, belongs to Υ, and satisfies

$$\inf_{s\in\mathbb{R}} \|u(\cdot,t) - \varphi_\omega(\cdot + s)\|_\Upsilon < \varepsilon$$

for all $t \in \mathbb{R}$. Otherwise, $u(x,t) = \varphi_\omega(x-\omega t)$ is said to be orbitally unstable in Υ.

In what follows, we need to introduce the manifold

$$\Sigma_k = \{v \in \Upsilon;\ \mathcal{W}_k(v) = \mathcal{W}_k(\varphi_\omega)\},$$

where $\mathcal{W}_k$ is the conserved quantity in (68). Moreover, if $u, v \in H^1_{per}([0,L])$, we define

$$\rho(u,v) := \inf_{s\in\mathbb{R}} \|u - v(\cdot + s)\|_\Upsilon.$$

It is not difficult to check that ρ is a pseudo-metric, which measures the distance between u and the orbit generated by v. The next two lemmas are well known in the literature. So, we only indicate where the reader may find their proofs.

Lemma 3.6. *There are $\varepsilon > 0$, a neighbourhood of the orbit generated by φ_ω, say,*

$$U_\varepsilon(\varphi_\omega) := \{v \in \Upsilon;\ \rho(v,\varphi_\omega) < \varepsilon\},$$

and a C^1-function $\tau : U_\varepsilon(\varphi_\omega) \to \mathbb{R}$ such that

$$\langle v(\cdot + \tau(v)), \varphi_\omega' \rangle = 0, \quad \textit{for all } v \in U_\varepsilon(\varphi_\omega).$$

Proof. See [1, 12, 21]. □

By using Propositions 3.3 and 3.4, we can prove the following result.

Lemma 3.7. *Set*

$$\mathcal{P}_k := \{\Phi \in \Upsilon;\ \langle \Phi, \mathcal{W}_k'(\varphi_\omega)\rangle = \langle \Phi, \varphi_\omega'\rangle = 0\}.$$

Then, there exists a constant $C > 0$ such that

$$\langle \mathcal{L}_\omega(\Phi), \Phi\rangle \geq C\|\Phi\|_\Upsilon^2, \quad \textit{for all } \Phi \in \mathcal{P}_k.$$

Proof. See [1, 12]. □

Lemma 3.8. *Consider the functional $\mathcal{G}_k := \mathcal{E} + \omega\mathcal{F} + A\mathcal{M}$. There exist $\varepsilon > 0$ and a constant $C = C(\varepsilon) > 0$ satisfying*

$$\mathcal{G}_k(u) - \mathcal{G}_k(\varphi_\omega) \geq C(\varepsilon)[\rho(u, \varphi_\omega)]^2,$$

for all $u \in U_\varepsilon(\varphi_\omega) \cap \Sigma_k$.

Proof. The proof is essentially contained in [1, 24]. For the sake of clearness, we bring the main steps. Since $\mathcal{G}_k$ is invariant under translations one has

$$\mathcal{G}_k(u) = \mathcal{G}_k(u(\cdot + r)), \quad \text{for all } r \in \mathbb{R}.$$

Thus, it is sufficient to show that

$$\mathcal{G}_k(u(\cdot + \tau(u))) - \mathcal{G}_k(\varphi_\omega) \geq C[\rho(u, \varphi_\omega)]^2,$$

where τ is the smooth function obtained in Lemma 3.6.

By using Lemma 3.6, there is a constant $C_1 \in \mathbb{R}$ such that

$$v := u(\cdot + \tau(u)) - \varphi_\omega = C_1\mathcal{W}_k'(\varphi_\omega) + y, \tag{88}$$

where $y \in \mathcal{B}_k := [\mathcal{W}_k'(\varphi_\omega)]^\perp \cap [\varphi_\omega']^\perp$. We note that $v = u(\cdot + \tau(u)) - \varphi_\omega$ satisfies $\|v\|_\Upsilon < \varepsilon$. We claim that $C_1 = \mathcal{O}(\|v\|_\Upsilon^2)$. Indeed, since $\mathcal{W}_k$ is also invariant under translations we can use Taylor's expansion to obtain

$$\mathcal{W}_k(u) = \mathcal{W}_k(u(\cdot + \tau(u))) = \mathcal{W}_k(\varphi_\omega) + \langle \mathcal{W}_k'(\varphi_\omega), v\rangle + \mathcal{O}(\|v\|_\Upsilon^2). \tag{89}$$

On the other hand, since $y \in \mathcal{B}_k$ one has $\langle \mathcal{W}_k'(\varphi_\omega), y\rangle = 0$ and consequently

$$\langle \mathcal{W}_k'(\varphi_\omega), v\rangle = \langle \mathcal{W}_k'(\varphi_\omega), C_1\mathcal{W}_k'(\varphi_\omega) + y\rangle = \langle \mathcal{W}_k'(\varphi_\omega), C_1\mathcal{W}_k'(\varphi_\omega)\rangle = C_1 N, \tag{90}$$

where N is a constant which depends on the wave speed ω. Since $\mathcal{W}_k(u) = \mathcal{W}_k(\varphi_\omega)$, statements in (89) and (90) allow us to conclude that

$$C_1 = \mathcal{O}(\|v\|_\Upsilon^2). \tag{91}$$

Next, we can apply Taylor's expansion to deduce

$$\mathcal{G}_k(u)=\mathcal{G}_k(u(\cdot+\tau(u)))=\mathcal{G}_k(\varphi_\omega)+\langle\mathcal{G}_k'(\varphi_\omega),v\rangle+\frac{1}{2}\langle\mathcal{G}_k''(\varphi_\omega)v,v\rangle+\mathcal{O}(\|v\|_\Upsilon^2). \quad (92)$$

Since $\mathcal{G}_k'(\varphi_\omega)=0$ and $\mathcal{G}_k''(\varphi_\omega)=\mathcal{L}_\omega$, one has

$$\mathcal{G}_k(u)-\mathcal{G}_k(\varphi_\omega)=\frac{1}{2}\langle\mathcal{L}_\omega(v),v\rangle+\mathcal{O}(\|v\|_\Upsilon^2). \quad (93)$$

By using (88) and (91), we have

$$\langle\mathcal{L}_\omega(v),v\rangle=\langle\mathcal{L}_\omega(y),y\rangle+\mathcal{O}(\|v\|_\Upsilon^2). \quad (94)$$

So, from (93) and (94), $\mathcal{G}_k(u)-\mathcal{G}_k(\varphi_\omega)=\frac{1}{2}\langle\mathcal{L}_\omega(y),y\rangle+\mathcal{O}(\|v\|_\Upsilon^2)$. Next, since $y\in\mathcal{B}_k$, by Lemma 3.7, we have for $C>0$, $\langle\mathcal{L}_\omega(y),y\rangle\geq C\|y\|_\Upsilon^2$. Thus,

$$\mathcal{G}_k(u)-\mathcal{G}_k(\varphi_\omega)\geq\overline{C}\|y\|_\Upsilon^2+\mathcal{O}(\|v\|_\Upsilon^2), \quad (95)$$

where $\overline{C}>0$. Therefore, from (88) we deduce, for $\varepsilon>0$ small enough, that

$$\mathcal{G}_k(u)-\mathcal{G}_k(\varphi_\omega)\geq C(\varepsilon)\|v\|_\Upsilon^2\geq C(\varepsilon)[\rho(u,\varphi_\omega)]^2.$$

This completes the proof. □

Now we are able to proof the main result in this section.

Theorem 3.9. *Assume that the uniqueness of solutions occurs for the initial value problem* (15) *according to Theorem* 2.1. *Then, the periodic wave* φ_ω *is orbitally stable in* Υ.

Proof. The proof follows from Theorem 2.1, Lemma 3.8, and an adaptation of [21, Theorem 3.5] (see also [1, 4]). Assume by contradiction the result is false. Then we can select

$$w_n:=u_n(\cdot,0)\in U_{\frac{1}{n}}(\varphi_\omega)\cap\Upsilon,$$

and $\varepsilon>0$, such that $\|w_n-\varphi_\omega\|_\Upsilon\overset{n\to\infty}{\longrightarrow}0$, with

$$\sup_{t\in\mathbb{R}}\rho(u_n(\cdot,t),\varphi_\omega)\geq\varepsilon,$$

where $u_n(\cdot,t)\in\Upsilon$, is the corresponding solution of (15). Without loss of generality, we can assume that $\varepsilon>0$ is the one in Lemma 3.6. From the continuity of $u_n(\cdot,t)$ at any $t\in\mathbb{R}$, we consider the smallest $|t_n|>0$ satisfying

$$\rho(u_n(\cdot,t_n),\varphi_\omega)=\frac{\varepsilon}{2}. \quad (96)$$

Let us define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(\alpha) = \mathcal{W}_k(\alpha u_n(\cdot, t_n))$. Thus,

$$f_n(\alpha) = \frac{\alpha^2}{2}\frac{\partial \omega}{\partial k}\int_0^L |u_n(\cdot, t_n)|^2 \, dx + \alpha \frac{\partial A}{\partial k}\int_0^L u_n(\cdot, t_n) \, dx =: \alpha^2 g_n + \alpha h_n.$$

We note that $f_n(0) = 0$, $g_n > 0$ and $\mathcal{W}_k(\varphi_\omega) > 0$. Thus, for all $n \in \mathbb{N}$, there exists $\alpha_n > 0$ such that $f_n(\alpha_n) = \mathcal{W}_k(\varphi_\omega)$. In other words, there is a sequence $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ satisfying

$$\mathcal{W}_k(\alpha_n u_n(\cdot, t_n)) = \mathcal{W}_k(\varphi_\omega), \quad \text{for all } n \in \mathbb{N},$$

which means that $(\alpha_n u_n(\cdot, t_n))_{n\in\mathbb{N}} \subset \Sigma_k$.

Define $\mathcal{T}_k(u) := \frac{\partial \omega}{\partial k}\mathcal{F}(u)$ and $\mathcal{R}_k(u) := \frac{\partial A}{\partial k}\mathcal{M}(u)$. Since $\mathcal{F}$ and $\mathcal{M}$ are continuous, one has $\mathcal{T}_k(w_n) \longrightarrow \mathcal{T}_k(\varphi_\omega) =: g$, $\mathcal{R}_k(w_n) \longrightarrow \mathcal{R}_k(\varphi_\omega) =: h$, and $\mathcal{W}_k(w_n) \longrightarrow \mathcal{W}_k(\varphi_\omega)$, as $n \longrightarrow +\infty$. So,

$$\begin{aligned} \varrho_n &:= |\alpha_n^2 \mathcal{T}_k(w_n) + \alpha_n \mathcal{R}_k(w_n) - (\mathcal{T}_k(w_n) + \mathcal{R}_k(w_n))| \\ &= |\mathcal{W}_k(\alpha_n u_n(\cdot, t_n)) - \mathcal{W}_k(w_n)| \\ &= |\mathcal{W}_k(w_n) - \mathcal{W}_k(\varphi_\omega)| \longrightarrow 0, \end{aligned}$$

as $n \longrightarrow +\infty$. On the other hand,

$$0 \leq |\alpha_n^2 \mathcal{T}_k(w_n) + \alpha_n \mathcal{R}_k(w_n) - (g+h)| \leq \varrho_n + |\mathcal{T}_k(w_n) - g| + |\mathcal{R}_k(w_n) - h| \longrightarrow 0,$$

that is,

$$z_n := \alpha_n^2 \mathcal{T}_k(w_n) + \alpha_n \mathcal{R}_k(w_n) \longrightarrow g + h. \tag{97}$$

Statement (97) gives that $(\alpha_n)_{n\in\mathbb{N}}$ is a bounded sequence. Thus, up to a subsequence, there is $\alpha_0 \geq 0$, such that $\alpha_n \longrightarrow \alpha_0$, as $n \longrightarrow +\infty$. In addition, also from (97), we get

$$(1 - \alpha_0)[(1 + \alpha_0)g + h] = 0. \tag{98}$$

We note that $1 + \frac{h}{g} = 1 + \frac{\mathcal{R}_k(\varphi_\omega)}{\mathcal{T}_k(\varphi_\omega)} > 1 > 0$. Since $\alpha_0 \geq 0$, one has from (98) that $\alpha_0 = 1$.

Next, we claim that

$$\rho(u_n(\cdot, t_n), \alpha_n u_n(\cdot, t_n)) \longrightarrow 0, \quad n \longrightarrow +\infty. \tag{99}$$

In fact, since $\rho(u_n(\cdot, t_n), \varphi_\omega) = \frac{\varepsilon}{2}$ there are $r_n \in \mathbb{R}$ and $C_3 > 0$ such that

$$\begin{aligned} \|u_n(\cdot, t_n)\|_\Upsilon &\leq \|u_n(\cdot, t_n) - \varphi_\omega(\cdot + r_n)\|_\Upsilon + \|\varphi_\omega(\cdot + r_n)\|_\Upsilon \\ &< \varepsilon + \|\varphi_\omega(\cdot + r_n)\|_\Upsilon = C_3, \end{aligned}$$

that is, $(\|u_n(\cdot, t_n)\|_\Upsilon)_{n\in\mathbb{N}}$ is a bounded sequence. Therefore, the relation

$$\rho(u_n(\cdot, t_n), \alpha_n u_n(\cdot, t_n)) \leq \|u_n(\cdot, t_n) - \alpha_n u_n(\cdot, t_n)\|_\Upsilon \leq |1 - \alpha_n| \cdot \|u_n(\cdot, t_n)\|_\Upsilon,$$

implies (99). Gathering together (96), (99), and Lemma 3.8, we conclude

$$\begin{aligned}\rho(\alpha_n u_n(\cdot,t_n),\varphi_\omega)^2 &\le \widetilde{C}|\mathcal{G}_k(\alpha_n u_n(\cdot,t_n)) - \mathcal{G}_k(\varphi_\omega)| \\ &\le \widetilde{C}|\mathcal{G}_k(\alpha_n u_n(\cdot,t_n)) - \mathcal{G}_k(u_n(\cdot,t_n))| + \widetilde{C}|\mathcal{G}_k(w_n) - \mathcal{G}_k(\varphi_\omega)|,\end{aligned}$$

with $\widetilde{C} > 0$. Consequently,

$$\rho(\alpha_n u_n(\cdot,t_n),\varphi_\omega) \longrightarrow 0, \quad n \longrightarrow +\infty. \tag{100}$$

In other words, for n large enough, $\alpha_n u_n(\cdot,t_n) \in U_\varepsilon(\varphi_\omega) \cap \Sigma_k$. By combining (99) and (100), we finally obtain for $n \longrightarrow +\infty$

$$\frac{\varepsilon}{2} = \rho(u_n(\cdot,t_n),\varphi_\omega) \le \rho(u_n(\cdot,t_n),\alpha_n u_n(\cdot,t_n)) + \rho(\alpha_n u_n(\cdot,t_n),\varphi_\omega) \longrightarrow 0,$$

which gives a contradiction. The proof of Theorem 3.9 is thus completed. □

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