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Continuity and Convergence Properties of Integral Means of Bojanov–Xu Interpolation

Van Manh Phung

Abstract. We study Bojanov–Xu interpolation whose interpolation points are located on concentric circles in $\mathbb{R}^2$. We prove that the integral means of the interpolation polynomial over a fixed circle or a fixed annulus are continuous functions of the radii of circles. We also give a distribution of the radii such that the integral means are convergent.

Keywords. Bojanov–Xu interpolation, Hermite interpolation, continuity properties, convergence properties

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1. Introduction

Let $\mathscr{P}_d(\mathbb{R}^k)$ be the space of all polynomials of degree at most d in $\mathbb{R}^k$. Since it is a finite dimensional vector space $\dim \mathscr{P}_d(\mathbb{R}^k) = \binom{d+k}{k}$, the convergence of polynomials in $\mathscr{P}_d(\mathbb{R}^k)$ can be regarded as the convergence of any norm in the space.

In one variable, the interpolation by polynomials is a well studied problem. The Lagrange and Hermite interpolation polynomials of functions at given points always exist. When the interpolated function is fixed, the interpolation polynomial is continuous with respect to the interpolation points. Moreover, if an array of interpolation points is suitably distributed, then the sequence of interpolation polynomials of a sufficiently smooth function converges uniformly to the function.

Multivariate polynomial interpolation problems are more difficult. For example, it is not easy to decide whether a given set of $\binom{d+k}{k}$ distinct points in $\mathbb{R}^k$ determines the Lagrange interpolation uniquely. Furthermore, the above-mentioned continuity property of Hermite interpolation is not true in the multivariate case without additional assumptions (see for instance [1, 6, 7]). In [3], the authors studied a bivariate Hermite interpolation problem at equidistant points on concentric circles. They proved that the problem has a unique solution.

V. M. Phung: Department of Mathematics, Hanoi National University of Education, 136 Xuan Thuy Street, Cau Giay, Hanoi, Vietnam; manhpv@hnue.edu.vn

Let n be a positive integer, $d = [\frac{n}{2}] + 1$ and $m = [\frac{n+1}{2}]$, where $[x]$ is the integer part of x. We will denote by Θ_m the set of angles

$$\Theta_m := \left\{ \theta_i : \theta_i = \frac{2i\pi}{2m+1},\ i = 0, 1, \ldots, 2m \right\}.$$

For a well-defined function f, let $\frac{\partial}{\partial r}$ be the normal derivative

$$\frac{\partial f}{\partial r}(x,y) = \frac{\partial f}{\partial x}(x,y)\cos\theta + \frac{\partial f}{\partial y}(x,y)\sin\theta, \quad (x,y) = (r\cos\theta, r\sin\theta).$$

The restriction of f on the ray $\{(r\cos\theta_i, r\sin\theta_i) : r \geq 0\}$ is denoted by f_i for $i = 0, \ldots, 2m$, i.e., $f_i(r) = f(r\cos\theta_i, r\sin\theta_i)$, $r \geq 0$. The circle of radius $r > 0$ centered at the origin is denoted by $S(r)$. We denote by $D^k g$ the derivative of order k of the univariate function g. The following Theorem 1.1 of [3] by Bojanov and Xu will be used throughout this paper. We restate it using the notations presented above.

Theorem 1.1. *Let* $0 < r_1 < r_2 < \cdots < r_\lambda \leq 1$ *and let* $\mu_1, \mu_2, \ldots, \mu_\lambda$ *be positive integers such that*

$$\mu_1 + \mu_2 + \cdots + \mu_\lambda = \left[\frac{n}{2}\right] + 1.$$

Let $\{\mathbf{a}_{li} : \mathbf{a}_{li} = (r_l\cos\theta_i,\, r_l\cos\theta_i),\ 0 \leq i \leq 2m\}$ *be equidistant points on the circle* $S(r_l)$*. Then, for any function* f *such that* $D^{\mu_l - 1} f_i(r_l)$ *exists for* $1 \leq l \leq \lambda$ *and* $0 \leq i \leq 2m$*, there is a unique polynomial* $p \in \mathscr{P}_n(\mathbb{R}^2)$ *such that*

$$\left(\frac{\partial^k p}{\partial r^k}\right)(\mathbf{a}_{lj}) = \left(\frac{\partial^k f}{\partial r^k}\right)(\mathbf{a}_{lj}), \quad 0 \leq k \leq \mu_l - 1,\ 1 \leq l \leq \lambda,\ 0 \leq j \leq 2m.$$

The unique polynomial p in Theorem 1.1 will be denoted by

$$\mathbb{H}[\{(r_1, \mu_1), \ldots, (r_\lambda, \mu_\lambda)\}; f]$$

and be called the Bojanov–Xu interpolation polynomial of f. If $s_1, s_2, \ldots, s_d \in (0,1]$ are not assumed to be distinct, then we can write

$$\{s_1, s_2, \ldots, s_d\} = \{(r_1, \mu_1), \ldots, (r_\lambda, \mu_\lambda)\},$$

where (r_i, μ_i) means that r_i is repeated μ_i times. This convention will be used again in Section 2. Hence, we will write $\mathbb{H}[\{s_1, \ldots, s_d\}; f]$ for $\mathbb{H}[\{(r_1, \mu_1), \ldots, (r_\lambda, \mu_\lambda)\}; f]$. If $s_1, \ldots, s_d$ are distinct, then the interpolation polynomial becomes the bivariate Lagrange interpolation polynomial $\mathbb{L}[\{s_1, \ldots, s_d\}; f]$. It is worth pointing out that an explicit formula or an error formula for $\mathbb{H}[\{s_1, \ldots, s_d\}; f]$ has not been available yet. In addition, we have not known whether the following map

$$(s_1, \ldots, s_d) \longmapsto \mathbb{H}[\{s_1, \ldots, s_d\}; f] \in \mathscr{P}_n(\mathbb{R}^2)$$

is continuous when f is a given smooth function. It is of interest to know whether there are weaker continuity properties of the interpolation polynomials. Moreover, from the numerical analysis point of view, $\mathbb{H}[\{s_1,\dots,s_d\};f]$ (resp. its mean values) is expected to approximate f (resp. the corresponding mean values of f) when the number of circles, say λ, is increasing. But it is difficult to find conditions on $\{s_1,\dots,s_d\}$ such that $\mathbb{H}[\{s_1,\dots,s_d\};f]$ converges uniformly to f. In this paper, we are concerned with the following problem.

Problem. *Let P_f be the integral mean of the Bojanov–Xu interpolation polynomial,*

$$P_f(\{s_1,\dots,s_d\})(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{H}[\{s_1,\dots,s_d\};f](r\cos\theta, r\sin\theta)d\theta, \quad r \geq 0.$$

1. *Describe smoothness conditions on f such that P_f and the double integral of $\mathbb{H}[\cdot;f]$ over an annulus are continuous with respect the radii s_k's.*
2. *Find conditions on the radii s_k's such that the integral means of the Lagrange interpolation polynomials of sufficiently smooth functions over circles or annulus converge to the corresponding integral means of the functions.*

Note that P_f was first introduced in [3, Section 3]. Although a compact formula for the Hermite interpolation is not known, the rotational invariance property of interpolation points make it possible to establish a formula for P_f. More precisely, we can write P_f in term of the mean of $2m+1$ univariate Hermite interpolation polynomials. The continuity and convergence properties of integral means reduce to corresponding properties of univariate interpolation. Theorems 3.2 and 3.3 give answers for the first question. The case where some radii s_k tend to 0 is treated separately. For the second question, we prove in Theorems 4.1 and 4.5 that the convergence property will hold when the Lebesgue constants of the sets $\{s_k^2\}$ grow polynomially in n. The rate of convergence is also studied. Here, to obtain the positive answers for the two problems in the case where some interpolation points are allowed to tend to the origin, we must make an assumption on the smoothness of the interpolated functions at the origin.

Finally, we note that Bojanov and Xu extended their results to bivariate Birkhoff interpolation at points that are located on several concentric circles. Hakopian and Khalaf studied the poisedness of Bojanov–Xu type interpolation on conic sections and obtained some interesting results. For a recent account of the theory of Hermite interpolation, we refer the reader to [4, 9, 10, 12, 13]

2. Univariate Hermite interpolation

Let $t_1,\dots,t_\lambda$ be λ distinct real numbers. Let $\mu_1,\dots,\mu_\lambda$ be λ positive integers and $d = \mu_1+\cdots+\mu_\lambda$. The following theorem is well-known.

Theorem 2.1. *Given a function g for which $D^{\mu_i-1}g(t_i)$ exists for $i=1,\dots,\lambda$. Then there exists a unique $p \in \mathscr{P}_{d-1}(\mathbb{R})$ such that*

$$D^j p(t_i) = D^j g(t_i), \quad 0 \leq j \leq \mu_i - 1,\ 1 \leq i \leq \lambda.$$

The polynomial p in Theorem 2.1 is denoted by $\mathbf{H}[\{(t_1,\mu_1),\dots,(t_\lambda,\mu_\lambda)\};g]$ and is called the Hermite interpolation polynomial. Its formula can be found in [2, Theorem 1.1].

Theorem 2.2. *The Hermite interpolation polynomial is given by*

$$\mathbf{H}[\{(t_1,\mu_1),\dots,(t_\lambda,\mu_\lambda)\};g](t)=\sum_{k=1}^{\lambda}\sum_{i=0}^{\mu_k-1}D^i g(t_k)H_{ki}(t),$$

where

$$H_{ki}(t)=\frac{1}{i!}\frac{w(t)}{(t-t_k)^{\mu_k-i}}\sum_{j=0}^{\mu_k-i-1}\frac{1}{j!}\left(\frac{(t-t_k)^{\mu_k}}{w(t)}\right)^{(j)}\Big|_{t=t_k}(t-t_k)^j$$

and

$$w(t)=\prod_{i=1}^{\lambda}(t-t_i)^{\mu_i}.$$

As Bojanov–Xu interpolation, it is convenient to use interpolation sets whose elements are repeated. More precisely, any set $A=\{s_1,\dots,s_d\}\subset\mathbb{R}$ can be identified with $\{(t_1,\mu_1),\dots,(t_\lambda,\mu_\lambda)\}$, where the t_i's are pairwise distinct. Hence, we can write $\mathbf{H}[A;g]$ instead of $\mathbf{H}[\{(t_1,\mu_1),\cdots,(t_\lambda,\mu_\lambda)\};g]$. In the case where the s_i's are pairwise distinct, the interpolation polynomial becomes the ordinary Lagrange interpolation polynomial and will be denoted by $\mathbf{L}[A;g]$. The univariate Hermite interpolation is continuous with respect to the interpolation points (see for instance [2, Theorem 1.4])

Theorem 2.3. *Let $I\subset\mathbb{R}$ be an interval and $g\in C^{d-1}(I)$. Then the map*

$$(t_1,\dots,t_d)\in I^d\mapsto\mathbf{H}[\{t_1,\dots,t_d\};g]$$

is continuous. In particular, if $t_i\to t_0\in I$ for $i=1,\dots,d$, then $\mathbf{H}[\{t_1,\dots,t_d\};g]\to\mathbf{T}_{t_0}^{d-1}(g)$, the univariate Taylor expansion of g at t_0 to the order $d-1$.

Proposition 2.4. *Let $r_1,\dots,r_\lambda$ be distinct real numbers in $(0,a]$ and $\mu_1,\dots,\mu_\lambda$ positive integers. Let g be a function defined on $(0,a]$ such that $D^{\mu_i-1}g(r_i)$ exists for $i=1,\dots,\lambda$. Let $g^*(r)=g(\sqrt{r})$ and $\hat{g}$ be the even extension of g, i.e., $g(r)=\hat{g}(r)=\hat{g}(-r)$ for $0<r\le a$. Then*

$$\begin{aligned}&\mathbf{H}[(r_1,\mu_1),\dots,(r_\lambda,\mu_\lambda),(-r_1,\mu_1),\dots,(-r_\lambda,\mu_\lambda);\hat{g}](r)\\&=\mathbf{H}[(r_1^2,\mu_1),\dots,(r_\lambda^2,\mu_\lambda);g^*](r^2).\end{aligned}\tag{1}$$

Proof. Set $d=\mu_1+\cdots+\mu_\lambda$. Let us define $P(t)=\mathbf{H}[(t_1^2,\mu_1),\dots,(t_\lambda^2,\mu_\lambda);g^*](t)$ and $Q(t)=P(t^2)$. Then Q is an even polynomial of degree at most $2d-2$. Hence, it suffices to check that

$$Q^{(i)}(t_j)=\hat{g}^{(i)}(t_j),\quad Q^{(i)}(-t_j)=\hat{g}^{(i)}(-t_j),\quad 1\le j\le\lambda,\ 0\le i\le\mu_j-1.$$

Since both Q and $\hat{g}$ are even functions, we need only to prove the equalities for derivatives at t_j. Fix $j \in \{1,\dots,\lambda\}$. For $i=0$, by definition, we have $Q(t_i) = g^*(t_i^2) = g(t_i) = \hat{g}(t_i)$. Next, we consider the case $i > 0$. For simplicity, we set $\varphi(t) = t^2$. By the Faà di Bruno formula [16], we obtain

$$Q^{(i)}(t_j) = (P \circ \varphi)^{(i)}(t_j) = \sum \frac{i!}{k_1! \cdots k_i!} P^{(k)}(\varphi(t_j)) \left(\frac{\varphi'(t_j)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(i)}(t_j)}{i!}\right)^{k_i} \quad (2)$$

where, in the second line, $k = k_1 + \cdots + k_i$ and the sum is over all values of $k_1,\dots,k_i \in \mathbb{N}$ such that $k_1 + 2k_2 + \cdots + ik_i = i$. From the interpolation condition, we have

$$P^{(k)}(\varphi(t_j)) = P^{(k)}(t_j^2) = (g^*)^{(k)}(t_j^2) = (g^*)^{(k)}(\varphi(t_j)).$$

Substituting this into (2), we obtain

$$\begin{aligned} Q^{(i)}(t_j) &= \sum \frac{i!}{k_1! \cdots k_i!} (g^*)^{(k)}(\varphi(t_j)) \left(\frac{\varphi'(t_j)}{1!}\right)^{k_1} \cdots \left(\frac{\varphi^{(i)}(t_j)}{i!}\right)^{k_i} \\ &= (g^* \circ \varphi)^{(i)}(t_j) \\ &= g^{(i)}(t_j). \end{aligned}$$

The proof is complete. □

Let $A = \{s_1,\dots,s_d\}$ be a set of d distinct points in $I = [a,b]$. Let $\Delta(A,I)$ be the norm of the Lagrange operator $\mathbf{L}[A;\cdot] : g \in C(I) \mapsto \mathbf{L}[A;g] \in C(I)$, where $C(I)$ is endowed with the sup norm. It is called the Lebesgue constant of A and can be computed by using the fundamental Lagrange interpolation polynomials,

$$\Delta(A,I) = \sup_{r \in [a,b]} \sum_{i=1}^{d} \left| \prod_{j=1, j\neq i}^{d} \frac{r - s_j}{s_i - s_j} \right|.$$

The Lebesgue constant is important for uniform approximation by interpolation polynomials since it measures the stability of the interpolation process. Indeed, we have

$$\sup_{r \in I} |g(r) - \mathbf{L}[A;g](r)| \leq (1 + \Delta(A,I))\, \mathrm{dist}_I(g, \mathscr{P}_{d-1}), \quad (3)$$

where $\mathrm{dist}_I(g,\mathscr{P}_{d-1}) = \inf\{\sup_I |g-p| : p \in \mathscr{P}_{d-1}(\mathbb{R})\}$. This relation is known as the Lebesgue inequality. By the second Jackson theorem in [15, Theorem 1.5], the term $\mathrm{dist}_I(g,\mathscr{P}_{d-1})$ grows like $o(d^{-M})$ as $d \to \infty$ when $g \in C^M(I)$. More precisely, there exists a constant C_0 depending only on a, b and M such that

$$\mathrm{dist}_I(g,\mathscr{P}_{d-1}) \leq \frac{C_0}{d^M} \omega\left(D^M g; \frac{1}{d}\right), \quad (4)$$

where $\omega(h; \frac{1}{d}) = \sup\{|h(r) - h(s)| : r, s \in [a,b], |r - s| \leq \frac{1}{d}\}$ denotes the modulus of continuity.

Note that the Lebesgue constant is invariant under affine transformations of $\mathbb{R}$. Let $\ell(r) = \alpha r + \beta$ with $\alpha \neq 0$, $J = \ell(I)$ and $B = \ell(A)$. Then it is easy to verify that

$$\Delta(A,I) = \Delta(B,J).$$

Hence, from sets of points in $[-1,1]$ whose Lebesgue constants grow slowly, we can construct analogous sets in $[a,b]$. The Lebesgue constant of a set of d distinct points grows at least like $\log d$. It is well-known that zeros of orthogonal polynomials in $[-1,1]$ usually give the optimal growth. For example, the Lebesgue constants of Chebyshev points $\{\cos\frac{(2k+1)\pi}{2d} : k = 0,\dots,d-1\} \subset [-1,1]$ and Chebyshev–Lobatto points $\{\cos\frac{k\pi}{d} : i = 0,\dots,d\} \subset [-1,1]$ grow like $\log d$. Recently, Calvi and Phung proved in [5] that the Lebesgue constant of the first d points of a $\Re$-Leja sequence grows like $O(d^3 \log d)$. Latter, Chkifa gave some refinements of the estimate. For more details we refer the reader to [5, 8]. Finally, we say that $\Delta(A_d, I)$ with $A_d = \{r_{1d},\dots,r_{dd}\} \subset I$ grows at most like a polynomial of degree N in d if there exists a constant $C > 0$ such that $\Delta(A_d, I) \leq Cd^N$ for $d \geq 1$.

3. Some continuity properties

The notations in Section 1 will be used throughout this section. In particular, $d = [\frac{n}{2}] + 1$ and $m = [\frac{n+1}{2}]$. In addition, we also define

$$f_i^*(r) := f_i(\sqrt{r}) = f\left(\sqrt{r}\cos\frac{2i\pi}{2m+1}, \sqrt{r}\sin\frac{2i\pi}{2m+1}\right), \quad i = 0,\dots,2m.$$

Normally, the Euclidean norm of $\mathbf{x} = (x,y) \in \mathbb{R}^2$ is denoted by $\|\mathbf{x}\| = \sqrt{x^2+y^2}$.

In [3], the authors gave a formula for P_f which contains fundamental interpolation polynomials. The following result provides a useful formula for P_f.

Lemma 3.1. *Under the assumptions of Theorem* 1.1*, we have*

$$P_f\left(\{(r_1,\mu_1),\dots,(r_\lambda,\mu_\lambda)\}\right)(r) = \frac{1}{2m+1}\sum_{i=0}^{2m} \mathbf{H}[\{(r_1^2,\mu_1),\dots,(r_\lambda^2,\mu_\lambda)\}; f_i^*](r^2).$$

Proof. The proof strongly relies on that given in [3, Proof of Theorem 3.1]. For convenience, we repeat some arguments. We write

$$\mathbb{H}[\{(r_1,\mu_1),\dots,(r_\lambda,\mu_\lambda)\}; f](r\cos\theta, r\sin\theta) = \sum_{k=0}^{n} r^k q_k(\theta) := \widetilde{P}(r,\theta),$$

where $q_k(\theta)$ are trigonometric polynomials of degree k given in [3, Relations (2.2) and (2.3)]. Since $\widetilde{P}(r,\theta)$ is a trigonometric polynomial of degree at most $2m$ in θ, we can use the quadrature formula for trigonometric polynomials in [18, Vol. 2, p. 8] to obtain

$$P_f\left(\{(r_1,\mu_1),\ldots,(r_\lambda,\mu_\lambda)\}\right)(r) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{P}(r,\theta)d\theta = \frac{1}{2m+1}\sum_{i=0}^{2m} \widetilde{P}(r,\theta_i). \qquad (5)$$

To shorten notation, we write $P_f(r)$ instead of $P_f(\{(r_1,\mu_1),\ldots,(r_\lambda,\mu_\lambda)\})(r)$. Note that q_{2i-1} only contains polynomials of odd degree. It follows that $\int_0^{2\pi} q_{2i-1}(\theta)d\theta = 0$. Hence, $P_f(r)$ is an even polynomial in r of degree at most $2[\frac{n}{2}]$. By the interpolation conditions for $\mathbb{H}[\cdot]$, for $0 \le k \le \mu_l - 1$, $1 \le l \le \lambda$, relation (5) gives

$$D^k P_f(r_l) = \frac{1}{2m+1}\sum_{i=0}^{2m} \frac{\partial^k \widetilde{P}}{\partial r^k}(r,\theta_i)\Big|_{r=r_l} = \frac{1}{2m+1}\sum_{i=0}^{2m} \frac{\partial^k f}{\partial r^k}(r,\theta_i)\Big|_{r=r_l} = \frac{1}{2m+1}\sum_{i=0}^{2m} D^k f_i(r_l). \quad (6)$$

Let $\hat{f}_i$ be the even extension of f_i, i.e., $\hat{f}_i(r) = \hat{f}_i(-r) = f_i(r)$ for $r \ge 0$. Then $(D^k \hat{f}_i)(-r_l) = (-1)^k D^k f_i(r_l)$ for $1 \le l \le \lambda$. Since $P_f(r)$ is an even polynomial, $(D^k P_f)(-r) = (-1)^k D^k P_f(r)$. From this we conclude from (6) that

$$D^k P_f(-r_l) = \frac{1}{2m+1}\sum_{i=0}^{2m} D^k \hat{f}_i(-r_l), \quad 0 \le k \le \mu_l - 1,\ 1 \le l \le \lambda. \qquad (7)$$

Relations (6) and (7) give $2\left[\frac{n}{2}\right] + 2$ interpolation conditions which determine P_f uniquely. Using the formula for the Hermite interpolation polynomial in Theorem 2.2, we obtain

$$\begin{aligned} P_f(r) &= \frac{1}{2m+1}\sum_{i=0}^{2m} \mathbf{H}[\{(-r_1,\mu_1),\ldots,(-r_\lambda,\mu_\lambda),(r_1,\mu_1),\ldots,(r_\lambda,\mu_\lambda)\};\hat{f}_i](r) \\ &= \frac{1}{2m+1}\sum_{i=0}^{2m} \mathbf{H}[\{(r_1^2,\mu_1),\ldots,(r_\lambda^2,\mu_\lambda)\};f_i^*](r^2), \end{aligned}$$

where we use Proposition 2.4 in the second relation. The proof is complete. □

Theorem 3.2. *Let $0 < \rho_1 < \rho_2 \le 1$ and $A(\rho_1,\rho_2) = \{\mathbf{x} \in \mathbb{R}^2 : \rho_1 \le \|\mathbf{x}\| \le \rho_2\}$. Let f be defined on $A(\rho_1,\rho_2)$ such that $f_i \in C^{d-1}[\rho_1,\rho_2]$ for $i = 0,\ldots,2m$. Then the following two maps are continuous*

$$(s_1,\ldots,s_d) \in [\rho_1,\rho_2]^d \longmapsto P_f(\{s_1,\ldots,s_d\})$$

and

$$(s_1,\ldots,s_d) \in [\rho_1,\rho_2]^d \longmapsto \int_{A(\rho_1,\rho_2)} \mathbb{H}[\{s_1,\ldots,s_d\};f](x,y)dxdy.$$

Proof. Let $\{s_{lk}\}_{k=1}^{\infty}$ be a sequence in $[\rho_1, \rho_2]$ such that $\lim_{k\to\infty} s_{lk} = s_l$ for $l = 1, \ldots, d$. By hypothesis, we have $f_i^* \in C^{d-1}[\rho_1^2, \rho_2^2]$ for $i = 0, \ldots, 2m$. Hence, Theorem 2.3 gives

$$\lim_{k\to\infty} \mathbf{H}[\{s_{1k}^2, \ldots, s_{dk}^2\}; f_i^*] = \mathbf{H}[\{s_1^2, \ldots, s_d^2\}; f_i^*], \quad i = 0, \ldots, 2m.$$

By Lemma 3.1, we have

$$\begin{aligned}\lim_{k\to\infty} P_f(\{s_{1k}, \ldots, s_{dk}\}) &= \frac{1}{2m+1} \lim_{k\to\infty} \sum_{i=0}^{2m} \mathbf{H}[\{s_{1k}^2, \ldots, s_{dk}^2\}; f_i^*] \\ &= \frac{1}{2m+1} \sum_{i=0}^{2m} \mathbf{H}[\{s_1^2, \ldots, s_d^2\}; f_i^*] \\ &= P_f(\{s_1, \ldots, s_d\}),\end{aligned}$$

which proves the first assertion.

Taking the sup norm on $\mathscr{P}_n(\mathbb{R})$, we deduce that

$$P_f(\{s_{1k}, \ldots, s_{dk}\}) \longrightarrow P_f(\{s_1, \ldots, s_d\})$$

uniformly on $[\rho_1, \rho_2]$. It follows that

$$\lim_{k\to\infty} \int_{\rho_1}^{\rho_2} P_f(\{s_{1k}, \ldots, s_{dk}\})(r) r dr = \int_{\rho_1}^{\rho_2} P_f(\{s_1, \ldots, s_d\})(r) r dr.$$

The last relation can be rewritten into

$$\lim_{k\to\infty} \int_{A(\rho_1,\rho_2)} \mathbb{H}[\{s_{1k}, \ldots, s_{dk}\}; f](x,y) dx dy = \int_{A(\rho_1,\rho_2)} \mathbb{H}[\{s_1, \ldots, s_d\}; f](x,y) dx dy,$$

which proves the continuity property of the second map. □

Note that the hypothesis $f_i \in C^{d-1}[0,1]$ implies $f_i^* \in C^{d-1}(0,1] \cap C[0,1]$. The function f_i^* is not differentiable at 0 in general. Hence, the sequence of the Hermite interpolation polynomials $\{\mathbf{H}[\{s_{1k}^2, \ldots, s_{dk}^2\}; f_i^*]\}$ can diverge when some s_l's equal 0 since the assumption on the smoothness of f_i^* is not provided. It follows that, to get the continuity of the map $(s_1, \ldots, s_d) \in [0,1]^d \mapsto P_f(\{s_1, \ldots, s_d\})$ at 0, we must assume that f_i^* is sufficiently smooth at 0.

For $d \in \mathbb{N}$, let us set

$$\mathscr{E}_d = \{g : g^* \in C^d[0,1], g^*(r) := g(\sqrt{r})\}. \tag{8}$$

Clearly, if $h \in C^d[0,1]$, then the function $g(t) := h(t^2)$ belongs to $\mathscr{E}_d$. The class $\mathscr{E}_d$ was studied in [14, 17].

Theorem 3.3. *Let f be a function defined on $D = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1\}$ such that $f_i \in \mathscr{E}_{d-1}$ for $i = 0, \ldots, 2m$. Let $\{s_{lk}\}_{k=1}^{\infty}$ be a sequence in $(0,1]$ such that $\lim_{k\to\infty} s_{lk} = s_l \in [0,1]$ for $l = 1, \ldots, d$. Then the following two limits exist*

$$\lim_{k\to\infty} P_f(\{s_{1k}, \ldots, s_{dk}\}) \quad \textit{and} \quad \lim_{k\to\infty} \int_D \mathbb{H}[\{s_{1k}, \ldots, s_{dk}\}; f](x,y) dx dy. \tag{9}$$

Furthermore, if $s_1 = \cdots = s_d = 0$ *and* f *is* n *times differentiable at* 0*, then*

$$\lim_{k\to\infty} P_f(\{s_{1k},\ldots,s_{dk}\})(r) = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{T}_0^n(f)(r\cos\theta, r\sin\theta)d\theta$$

and

$$\lim_{k\to\infty}\int_D \mathbb{H}[\{s_{1k},\ldots,s_{dk}\};f](x,y)dxdy = \int_D \mathbb{T}_0^n(f)(x,y)dxdy.$$

Here $\mathbb{T}_0^n(f)$ *is the bivariate Taylor expansion of* f *at 0 to the order* n*.*

Proof. By hypothesis, we have $f_i^* \in C^{d-1}[0,1]$ for $i = 0,\ldots,2m$. Analysis similar to that in the proof of Theorem 3.2 shows that two limits in (9) exist. To prove the remaining assertions, we first write

$$\widetilde{Q}(r,\theta) := \mathbb{T}_0^n(f)(r\cos\theta, r\sin\theta) = \sum_{k=0}^{n} r^k p_k(\theta), \quad r \geq 0.$$

As in the proof of Lemma 3.1, we have

$$Q_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \mathbb{T}_0^n(f)(r\cos\theta, r\sin\theta)d\theta = \frac{1}{2m+1}\sum_{i=0}^{2m}\widetilde{Q}(r,\theta_i).$$

Moreover, $Q_f(r)$ is an even polynomial of degree at most $2[\frac{n}{2}]$. By the property of the Taylor polynomial, for $0 \leq k \leq n$ and $0 \leq i \leq 2m$, we have

$$\frac{\partial^k}{\partial r^k}\widetilde{Q}(r,\theta_i)\big|_{r=0} = \frac{\partial^k}{\partial r^k}\mathbb{T}_0^n(f)(r\cos\theta_i, r\sin\theta_i)\big|_{r=0} = \frac{\partial^k}{\partial r^k} f(r\cos\theta_i, r\sin\theta_i)\big|_{r=0} = D^k f_i(0).$$

In fact, from the definition of the normal derivative and the property of the bivariate Taylor polynomial, we have

$$\begin{aligned}
\frac{\partial^k}{\partial r^k}\mathbb{T}_0^n(f)(r\cos\theta_i, r\sin\theta_i)\big|_{r=0} &= \Big(\frac{\partial}{\partial x}\cos\theta_i + \frac{\partial}{\partial y}\sin\theta_i\Big)^k \mathbb{T}_0^n(f)(r\cos\theta_i, r\sin\theta_i)\big|_{r=0} \\
&= \sum_{l=0}^{k}\binom{k}{l}\frac{\partial^k \mathbb{T}_0^n(f)(0,0)}{\partial^{k-l}x\partial^l y}\cos^{k-l}\theta_i \sin^l\theta_i \\
&= \sum_{l=0}^{k}\binom{k}{l}\frac{\partial^k f(0,0)}{\partial^{k-l}x\partial^l y}\cos^{k-l}\theta_i \sin^l\theta_i \\
&= \Big(\frac{\partial}{\partial x}\cos\theta_i + \frac{\partial}{\partial y}\sin\theta_i\Big)^k f(r\cos\theta_i, r\sin\theta_i)\big|_{r=0} \\
&= \frac{\partial^k}{\partial r^k} f(r\cos\theta_i, r\sin\theta_i)\big|_{r=0}.
\end{aligned}$$

It follows that

$$D^k Q_f(0) = \frac{1}{2m+1}\sum_{i=0}^{2m}\frac{\partial^k}{\partial r^k}\widetilde{Q}(r,\theta_i)\big|_{r=0} = \frac{1}{2m+1}\sum_{i=0}^{2m} D^k f_i(0), \quad 0 \leq k \leq n. \tag{10}$$

Since $Q_f(r)$ is a polynomial of degree at most $2[\frac{n}{2}]$, we have

$$Q_f(r) = \mathbf{T}_0^{2[\frac{n}{2}]}\left(Q_f\right)(r) = \frac{1}{2m+1}\sum_{i=0}^{2m}\mathbf{T}_0^{2[\frac{n}{2}]}(f_i)(r). \tag{11}$$

where we use (10) in the second equality. By the hypothesis that $f_i^* \in C^{[\frac{n}{2}]}[0,1]$, we can write

$$f_i^*(r) = \mathbf{T}_0^{[\frac{n}{2}]}(f_i^*)(r) + o(r^{[\frac{n}{2}]}).$$

Consequently $f_i(r) = f_i^*(r^2) = \mathbf{T}_0^{[\frac{n}{2}]}(f_i^*)(r^2) + o(r^{2[\frac{n}{2}]})$. The last relation yields $\mathbf{T}_0^{2[\frac{n}{2}]}(f_i)(r) = \mathbf{T}_0^{[\frac{n}{2}]}(f_i^*)(r^2)$. Now (11) becomes

$$Q_f(r) = \frac{1}{2m+1}\sum_{i=0}^{2m}\mathbf{T}_0^{[\frac{n}{2}]}(f_i^*)(r^2), \tag{12}$$

On the other hand, from Theorem 2.3 and Lemma 3.1, we get

$$\begin{aligned}\lim_{k\to\infty} P_f(\{s_{1k},\dots,s_{dk}\})(r) &= \frac{1}{2m+1}\lim_{k\to\infty}\sum_{i=0}^{2m}\mathbf{H}[\{s_{1k}^2,\dots,s_{dk}^2\};f_i^*](r^2)\\ &= \frac{1}{2m+1}\sum_{i=0}^{2m}\mathbf{T}_0^{d-1}(f_i^*)(r^2).\end{aligned} \tag{13}$$

Combining (12) and (13) we obtain the desired relation,

$$\lim_{k\to\infty} P_f(\{s_{1k},\dots,s_{dk}\})(r) = Q_f(r) = \frac{1}{2\pi}\int_0^{2\pi}\mathbb{T}_0^n(f)(r\cos\theta, r\sin\theta)d\theta.$$

Now we can say that $P_f(\{s_{1k},\dots,s_{dk}\})$ converges to $Q_f(r)$ in the sup norm over $[0,1]$. Hence, in the polar coordinates, we have

$$\lim_{k\to\infty}\int_0^1 P_f(\{s_{1k},\dots,s_{dk}\})(r)rdr = \frac{1}{2\pi}\int_0^1 rdr\int_0^{2\pi}\mathbb{T}_0^n(f)(r\cos\theta, r\sin\theta)d\theta.$$

In other words

$$\lim_{k\to\infty}\int_D \mathbb{H}[\{s_{1k},\dots,s_{dk}\};f](x,y)dxdy = \int_D \mathbb{T}_0^n(f)(x,y)dxdy.$$

The proof is complete. □

The following example shows that the conclusions in Theorem 3.3 does not hold when the hypothesis $f_i \in \mathscr{E}_{d-1}$ is omitted.

Example 3.4. Let $n = 2$. Then $d = 2$ and $m = 1$. For $0 < s_1 < s_2 \leq 1$, $\mathbb{L}[\{s_1, s_2\}; f]$ interpolates f at 6 points lying on two circles $S(s_1)$ and $S(s_2)$ (each circle contains 3 equidistant points). We have

$$P_f(\{s_1, s_2\})(r) = \frac{1}{3}\sum_{i=0}^{2} \mathbf{L}[\{s_1^2, s_2^2\}; f_i^*](r^2)$$

where $f_i(r) = f(r\cos\frac{2i\pi}{3}, r\sin\frac{2i\pi}{3})$ for $i = 0, 1, 2$. Let us choose

$$g(r) = \begin{cases} r^\alpha \sin\dfrac{1}{r^2} & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases} \quad \text{and} \quad f(\mathbf{x}) = g(\|\mathbf{x}\|).$$

Then $f_i = g$ for $i = 0, 1, 2$. Hence

$$P_f(\{s_1, s_2\})(r) = \mathbf{L}[\{s_1^2, s_2^2\}; g^*](r^2) = g(s_1) + \frac{g(s_2) - g(s_1)}{s_2^2 - s_1^2}(r^2 - s_1^2).$$

Assume that $2 < \alpha < 4$. Easy computations show that $g \notin \mathscr{E}_1$ but g is differentiable in $[0, \infty)$. Let us choose $s_{1k} = \frac{1}{\sqrt{\frac{\pi}{2} + 2k\pi}}$ and $s_{2k} = \frac{1}{\sqrt{\pi + 2k\pi}}$ for $k \geq 1$. Then $\lim_{k\to\infty} s_{1k} = \lim_{k\to\infty} s_{2k} = 0$. Evidently, the coefficient of r^2 in $P_f(\{s_{1k}, s_{2k}\})(r)$ does not converge when $k \to \infty$,

$$\lim_{k\to\infty} \frac{g(s_{2k}) - g(s_{1k})}{s_{2k}^2 - s_{1k}^2} = \infty.$$

It follows that $\lim_{k\to\infty} P_f(\{s_{1k}, s_{2k}\})(r)$ does not exist.

4. Some convergence properties

The notations and conventions in Sections 1-3 will be kept throughout this section. As we have said, it is not easy to find radii $\{s_1, \ldots, s_d\}$ such that the Bojanov–Xu interpolation polynomials of smooth functions converge uniformly (or in the norm L^p) when $d \to \infty$. The main difficult in carrying out this construction is that we do not know any compact formulas or error formulas for the interpolation polynomials. It is to be expected that weaker convergence properties hold true. Here, we give a condition on the distribution of the radii $\{s_1, \ldots, s_d\}$ that guarantee the convergence of the means of the interpolation polynomials. Note that we only work with Lagrange interpolation. However, our method can be used for general Bojanov–Xu interpolation.

Theorem 4.1. *Let M and N be positive integers with $M \geq N \geq 1$. Let $0 < \rho_1 < \rho_2 \leq 1$ and $f \in C^M(A(\rho_1, \rho_2))$. Let $A_d = \{s_{1d}, \ldots, s_{dd}\}$ be a set of distinct points in $[\rho_1, \rho_2]$ such that $\Delta(\{s_{1d}^2, \ldots, s_{dd}^2\}, [\rho_1^2, \rho_2^2])$ grows at most like a polynomial of degree N in d. Then*

$$\sup_{r \in [\rho_1, \rho_2]} \left| P_f(A_d)(r) - \frac{1}{2\pi} \int_0^{2\pi} f(r\cos\theta, r\sin\theta) d\theta \right| = o\left(\frac{1}{n^{M-N}}\right)$$

and

$$\left| \int_{A(\rho_3, \rho_4)} \mathbb{L}[A_d; f](x,y) dxdy - \int_{A(\rho_3, \rho_4)} f(x,y) dxdy \right| = o\left(\frac{1}{n^{M-N}}\right),$$

where $\rho_1 \leq \rho_3 < \rho_4 \leq \rho_2$.

Proof. For convenience, we define the following functions

$$F(r,\theta) = f(r\cos\theta, r\sin\theta) \quad \text{and} \quad F^*(r,\theta) = f(\sqrt{r}\cos\theta, \sqrt{r}\sin\theta).$$

They induce two types of modulus of continuity

$$\zeta\left(\frac{1}{n}\right) = \sup\left\{ \left| \frac{\partial^M}{\partial\theta^M} F(r,\theta)\big|_{\theta=\theta_1} - \frac{\partial^M}{\partial\theta^M} F(r,\theta)\big|_{\theta=\theta_2} \right| \;\middle|\; \begin{array}{l} \theta_1, \theta_2 \in [0, 2\pi], \\ |\theta_1 - \theta_2| \leq \frac{1}{n}, \\ \rho_1 \leq r \leq \rho_2 \end{array} \right\}$$

and

$$\eta\left(\frac{1}{n}\right) = \sup\left\{ \left| \frac{\partial^M}{\partial r^M} F^*(r,\theta)\big|_{r=s} - \frac{\partial^M}{\partial r^M} F^*(r,\theta)\big|_{r=t} \right| \;\middle|\; \begin{array}{l} s,t \in [\rho_1^2, \rho_2^2], \\ |s-t| \leq \frac{1}{n}, \\ \theta \in [0, 2\pi] \end{array} \right\}.$$

We note that the moduli of continuity ζ and η can be written in terms of classical modulus of continuity,

$$\zeta\left(\frac{1}{n}\right) = \sup_{\rho_1 \leq r \leq \rho_1} \omega\left(\frac{\partial^M}{\partial\theta^M} F(r,\cdot); \frac{1}{n}\right), \quad \eta\left(\frac{1}{n}\right) = \sup_{0 \leq \theta \leq 2\pi} \omega\left(\frac{\partial^M}{\partial r^M} F^*(\cdot,\theta); \frac{1}{n}\right).$$

Since $f \in C^M(A(\rho_1, \rho_2))$, $\frac{\partial^M}{\partial\theta^M} F(r,\theta)$ is continuous on $[\rho_1, \rho_2] \times [0, 2\pi]$ and hence is uniformly continuous on $[\rho_1, \rho_2] \times [0, 2\pi]$. It follows that $\lim_{n\to\infty} \zeta(\frac{1}{n}) = 0$. Similarly, since $\frac{\partial^M}{\partial r^M} F^*(r,\theta)$ is uniformly continuous on $[\rho_1^2, \rho_2^2] \times [0, 2\pi]$, $\eta(\frac{1}{n})$ tends to 0 when $n \to \infty$.

As $f_i^* \in C^M[\rho_1, \rho_2]$, the Jackson theorem in (4) shows that there exists a constant C_0 depending only on ρ_1, ρ_2 and M such that

$$\text{dist}_I(f_i^*, \mathscr{P}_{d-1}(\mathbb{R})) \leq \frac{C_0}{d^M} \omega\left(D^M f_i^*; \frac{1}{d}\right) \leq \frac{2^{M+1} C_0}{n^M} \omega\left(D^M f_i^*; \frac{1}{n}\right) \leq \frac{2^{M+1} C_0}{n^M} \eta\left(\frac{1}{n}\right),$$

$0 \le i \le 2m$, where $\omega(g; \frac{1}{n})$ denotes the ordinary modulus of continuity, $I = [\rho_1^2, \rho_2^2]$. Combining above estimates with the Lebesgue inequality (3) for f_i^* and the hypothesis on the growth of the Lebesgue constant, we obtain

$$\sup_{r \in [\rho_1, \rho_2]} \left| f_i(r) - \mathbf{L}[\{s_{1d}^2, \ldots, s_{dd}^2\}; f_i^*](r^2) \right| \le \frac{C_1}{n^{M-N}} \eta\left(\frac{1}{n}\right), \quad 0 \le i \le 2m,$$

where C_1 is a constant independent of n. Lemma 3.1 now gives

$$\sup_{r \in [\rho_1, \rho_2]} \left| \frac{1}{2m+1} \sum_{i=0}^{2m} f_i(r) - P_f(A_d)(r) \right| \le \frac{C_1}{n^{M-N}} \eta\left(\frac{1}{n}\right). \tag{14}$$

From [11, Theorem 3, p. 57], for each $\rho_1 \le r \le \rho_2$, we can find a constant $C_2 = C_2(M)$ depending only on M and a trigonometric polynomial T_n of degree at most n (depending on r) such that

$$\sup_{\theta \in [0, 2\pi]} |F(r, \theta) - T_n(\theta)| \le \frac{C_2}{n^M} \omega\left(\frac{\partial^M}{\partial \theta^N} F(r, \theta); \frac{1}{n}\right) \le \frac{C_2}{n^M} \zeta\left(\frac{1}{n}\right).$$

The above fact is known as the first Jackson theorem. It follows that

$$\left| \frac{1}{2m+1} \sum_{i=0}^{2m} F(r, \theta_i) - \frac{1}{2m+1} \sum_{i=0}^{2m} T_n(\theta_i) \right| \le \frac{C_2}{n^M} \zeta\left(\frac{1}{n}\right)$$

and

$$\left| \frac{1}{2\pi} \int_0^{2\pi} F(r, \theta) d\theta - \frac{1}{2\pi} \int_0^{2\pi} T_n(\theta) d\theta \right| \le \frac{C_2}{n^M} \zeta\left(\frac{1}{n}\right).$$

On the other hand, using the quadrature formula for T_n, we obtain $\frac{1}{2\pi} \int_0^{2\pi} T_n(\theta) d\theta = \frac{1}{2m+1} \sum_{i=0}^{2m} T_n(\theta_i)$. From what has already been proved, we deduce that

$$\left| \frac{1}{2m+1} \sum_{i=0}^{2m} F(r, \theta_i) - \frac{1}{2\pi} \int_0^{2\pi} F(r, \theta) d\theta \right| \le \frac{2C_2}{n^M} \zeta\left(\frac{1}{n}\right), \quad \rho_1 \le r \le \rho_2. \tag{15}$$

Combining (14) and (15) along with the setting $F(r, \theta) = f(r\cos\theta, r\sin\theta)$, we obtain

$$\begin{aligned} \sup_{r \in [\rho_1, \rho_2]} \left| P_f(A_d)(r) - \frac{1}{2\pi} \int_0^{2\pi} f(r\cos\theta, r\sin\theta) d\theta \right| &\le \frac{C_1}{n^{M-N}} \eta\left(\frac{1}{n}\right) + \frac{2C_2}{n^M} \zeta\left(\frac{1}{n}\right) \\ &= o\left(\frac{1}{n^{M-N}}\right). \end{aligned}$$

Using polar coordinates and the first assertion, we easily prove the estimate for the double integral. The details are left to the reader. □

Corollary 4.2. *Under the hypotheses of Theorem* 4.1 *except for the Lebesgue constant, if the Lebesgue constant grows like* $\log d$*, then the same estimates in Theorem* 4.1 *hold in which* $o\left(\frac{1}{n^{M-N}}\right)$ *is replaced by* $o\left(\frac{\log n}{n^M}\right)$.

Proof. We keep the notations introduced in the proof of Theorem 4.1. Since the Lebesgue constant $\Delta(\{s_{1d}^2,\dots,s_{dd}^2\},[\rho_1^2,\rho_2^2])$ grows like $\log d$ as $d\to\infty$, Lebesgue's inequality enables us to find $C_3>0$ such that

$$\sup_{r\in[\rho_1,\rho_2]}\left|f_i(r)-\mathbf{L}[\{s_{1d}^2,\dots,s_{dd}^2\};f_i^*](r^2)\right|\le\frac{C_3\log n}{n^M}\eta\left(\frac{1}{n}\right),\quad 0\le i\le 2m,$$

Hence, Lemma 3.1 yields an estimate which is analogous to (14):

$$\sup_{r\in[\rho_1,\rho_2]}\left|\frac{1}{2m+1}\sum_{i=0}^{2m}f_i(r)-P_f(A_d)(r)\right|\le\frac{C_3\log n}{n^M}\eta\left(\frac{1}{n}\right).$$

Combining the last estimate with (15) we obtain

$$\begin{aligned}\sup_{r\in[\rho_1,\rho_2]}\left|P_f(A_d)(r)-\frac{1}{2\pi}\int_0^{2\pi}f(r\cos\theta,r\sin\theta)d\theta\right|&\le\frac{C_3\log n}{n^M}\eta\left(\frac{1}{n}\right)+\frac{2C_2}{n^M}\zeta\left(\frac{1}{n}\right)\\&=o\left(\frac{\log n}{n^M}\right).\end{aligned}$$

The proof is complete. □

Corollary 4.3. *Let* $f\in C(A(\rho_1,\rho_2))$ *and* $A_d=\{s_{1d},\dots,s_{dd}\}\subset[\rho_1,\rho_2]$ *such that*

$$\max_{0\le i\le 2m}\sup_{r\in[\rho_1,\rho_2]}\left|f_i(r)-\mathbf{L}[\{s_{1d}^2,\dots,s_{dd}^2\};f_i^*](r^2)\right|\longrightarrow 0\quad as\quad n\to\infty.$$

Then the same estimates in Theorem 4.1 *hold in which* $o\left(\frac{1}{n^{M-N}}\right)$ *is replaced by* $o(1)$.

Proof. By the hypothesis and Lemma 3.1, we get the following estimate

$$\sup_{r\in[\rho_1,\rho_2]}\left|\frac{1}{2m+1}\sum_{i=0}^{2m}f_i(r)-P_f(A_d)(r)\right|=o(1). \tag{16}$$

Let us set

$$\phi\left(\frac{1}{n}\right)=\sup\left\{\left|F(r,\theta_1)-F(r,\theta_2)\right|:\theta_1,\theta_2\in[0,2\pi],|\theta_1-\theta_2|\le\frac{1}{n},\rho_1\le r\le\rho_2\right\}.$$

Since $f\in C(A(\rho_1,\rho_2))$, we have $\lim_{n\to\infty}\phi(\frac{1}{n})=0$.

By the Jackson theorem [11, Theorem 2, p. 56], for each $\rho_1\le r\le\rho_2$, we can find a constant $C_4>0$ and a trigonometric polynomial S_n of degree at most n (depending on r) such that

$$\sup_{\theta\in[0,2\pi]}|F(r,\theta)-S_n(\theta)|\le C_4\phi\left(\frac{1}{n}\right),$$

It follows that

$$\left|\frac{1}{2m+1}\sum_{i=0}^{2m}F(r,\theta_i)-\frac{1}{2m+1}\sum_{i=0}^{2m}S_n(\theta_i)\right|\leq C_4\phi\left(\frac{1}{n}\right)$$

and

$$\left|\frac{1}{2\pi}\int_0^{2\pi}F(r,\theta)d\theta-\frac{1}{2\pi}\int_0^{2\pi}S_n(\theta)d\theta\right|\leq C_4\phi\left(\frac{1}{n}\right).$$

On the other hand, since S_n is a trigonometric polynomial of degree at most n, we have

$$\frac{1}{2\pi}\int_0^{2\pi}S_n(\theta)d\theta=\frac{1}{2m+1}\sum_{i=0}^{2m}S_n(\theta_i).$$

Combining the above estimates, we obtain

$$\left|\frac{1}{2m+1}\sum_{i=0}^{2m}F(r,\theta_i)-\frac{1}{2\pi}\int_0^{2\pi}F(r,\theta)d\theta\right|\leq 2C_4\phi\left(\frac{1}{n}\right),\quad \rho_1\leq r\leq\rho_2. \tag{17}$$

From (16) and (17), we conclude that

$$\sup_{r\in[\rho_1,\rho_2]}\left|P_f(A_d)(r)-\frac{1}{2\pi}\int_0^{2\pi}f(r\cos\theta,r\sin\theta)d\theta\right|=o(1)+2C_4\phi\left(\frac{1}{n}\right)=o(1),$$

and proof is complete. □

The following result gives the rate of numerical approximation of the double integral of smooth functions.

Corollary 4.4. *Under the assumptions of Theorem* 4.1*, we have*

$$\left|\sum_{l=1}^{[\frac{n}{2}]+1}\Lambda_l\sum_{i=0}^{2m}f\left(s_l\cos\theta_i,s_l\sin\theta_i\right)-\int_{A(\rho_3,\rho_4)}f(x,y)dxdy\right|=o\left(\frac{1}{n^{M-N}}\right),$$

where Λ_l *is given by*

$$\Lambda_l=\frac{4\pi r_l}{2m+1}\int_{\rho_3}^{\rho_4}\frac{rg(r)dr}{(r^2-r_l^2)g'(r_l)},\quad g(r)=\prod_{l=1}^{[\frac{n}{2}]+1}(r^2-r_l^2).$$

Proof. The idea of the proof is inspired from [3]. Using [3, Corollary 3.2], we can write

$$P_f(A_d)(r)=\sum_{l=1}^{[\frac{n}{2}]+1}F_l\left[\ell_l(r)+\ell_l(-r)\right],\quad \ell_l(r)=\prod_{j=1}^{[\frac{n}{2}]+1}\frac{r+r_j}{r_l+r_j}\prod_{j=1,j\neq l}^{[\frac{n}{2}]+1}\frac{r-r_j}{r_l-r_j},$$

where $F_l = \frac{1}{2m+1}\sum_{i=0}^{2m} f(r_l\cos\theta_i, r_l\sin\theta_i)$. It follows that

$$\begin{aligned}\int_{A(\rho_3,\rho_4)} \mathbb{L}[A_d;f](x,y)dxdy &= 2\pi\int_{\rho_3}^{\rho_4} P_f(A_d)(r)rdr \\ &= 2\pi\sum_{l=1}^{[\frac{n}{2}]+1} F_l\int_{\rho_3}^{\rho_4}[\ell_l(r)+\ell_l(-r)]\,rdr.\end{aligned} \tag{18}$$

Since $\ell_l(r) = \frac{g(r)}{(r-r_l)g'(r_l)}$ with $g(r) = \prod_{l=1}^{[\frac{n}{2}]+1}(r^2-r_l^2)$, we have

$$\int_{\rho_3}^{\rho_4}[\ell_l(r)+\ell_l(-r)]\,rdr = \int_{\rho_3}^{\rho_4}\left[\frac{g(r)}{(r-r_l)g'(r_l)}+\frac{g(-r)}{(-r-r_l)g'(r_l)}\right]rdr = \int_{\rho_3}^{\rho_4}\frac{2r_l rg(r)dr}{(r^2-r_l^2)g'(r_l)}.$$

Substituting the last relation into (18), we obtain

$$\int_{A(\rho_3,\rho_4)} \mathbb{L}[A_d;f](x,y)dxdy = 2\pi\sum_{l=1}^{[\frac{n}{2}]+1} F_l\int_{\rho_3}^{\rho_4}\frac{2r_l rg(r)dr}{(r^2-r_l^2)g'(r_l)}. \tag{19}$$

The desired estimate follows directly from (19) and Theorem 4.1, and the proof is complete. □

Suppose that $f \in C^M(D)$ and $F(r,\theta) = f(r\cos\theta, r\sin\theta)$. Let $\zeta_1(\cdot)$ denote the modulus of continuity

$$\zeta_1\left(\frac{1}{n}\right) = \sup\left\{\left|\frac{\partial d^M}{\partial\theta^M}F(r,\theta)\big|_{\theta=\theta_1} - \frac{\partial^M}{\partial\theta^M}F(r,\theta)\big|_{\theta=\theta_2}\right| \;\middle|\; \begin{array}{l}\theta_1,\theta_2\in[0,2\pi],\\ |\theta_1-\theta_2|\le\frac{1}{n},\\ 0\le r\le 1\end{array}\right\}.$$

Clearly, $\lim_{n\to\infty}\zeta_1(\frac{1}{n}) = 0$. If we assume that $\frac{\partial^M}{\partial r^M}f(\sqrt{r}\cos\theta,\sqrt{r}\sin\theta)$ is continuous on $[0,1]\times[0,2\pi]$, then the modulus of continuity defined by

$$\eta_1\left(\frac{1}{n}\right) = \sup\left\{\left|\frac{\partial^M}{\partial r^M}F(\sqrt{r},\theta)\big|_{r=s} - \frac{\partial^M}{\partial r^M}F(\sqrt{r},\theta)\big|_{r=t}\right| \;\middle|\; \begin{array}{l}s,t\in[0,1],\\ |s-t|\le\frac{1}{n},\\ \theta\in[0,2\pi]\end{array}\right\}.$$

has the same asymptotic behaviour, that is $\lim_{n\to\infty}\eta_1(\frac{1}{n}) = 0$. The moduli of continuity ζ_1 and η_1 can be also related to classical modulus of continuity. We can prove the following result in much the same way as Theorem 4.1.

Theorem 4.5. *Let M and N be an positive integers with $M \geq N \geq 1$. Let $f \in C^M(D)$ such that $\frac{\partial^M}{\partial r^M} f(\sqrt{r}\cos\theta, \sqrt{r}\sin\theta)$ exists and is continuous on $[0,1] \times [0,2\pi]$. Let $A_d = \{s_{1d}, \ldots, s_{dd}\}$ be a set of distinct points in $[0,1]$ such that the Lebesgue constant $\Delta(\{s_{1d}^2, \ldots, s_{dd}^2\}, [\rho_1^2, \rho_2^2])$ grows at most like a polynomial of degree N in d. Then*

$$\sup_{r\in[0,1]} \left| P_f(A_d)(r) - \frac{1}{2\pi}\int_0^{2\pi} f(r\cos\theta, r\sin\theta)d\theta \right| = o\left(\frac{1}{n^{M-N}}\right)$$

and

$$\left| \int_{A(\rho_3,\rho_4)} \mathbb{L}[A_d; f](x,y)dxdy - \int_{A(\rho_3,\rho_4)} f(x,y)dxdy \right| = o\left(\frac{1}{n^{M-N}}\right), \quad 0 \leq \rho_3 < \rho_4 \leq 1.$$

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