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On Haar Systems for Groupoids

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Abstract. It is shown that a locally compact groupoid with open range map does not always admit a Haar system. It then is shown how to construct a Haar system if the stability groupoid and the quotient by the stability groupoid both admit one.

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1. Introduction

Topological groupoids occur naturally in encoding hidden symmetries like in fundamental groupoids or holonomy groupoids of foliations, see [7], for instance. In order to construct convolution algebras on groupoids [3, 9], one needs continuous families of invariant measures, so called *Haar systems* [12], see also Section 2. These do not always exist. One known criterion is that a Haar system can only exist if the range map is open ([13, Corollary to Lemma 2], see also [15]).

A second criterion, which has been neglected in the literature, is the possibility of *failing support*, i.e., it is possible that, although the range map is open, the support condition of a Haar system cannot be satisfied, see Proposition 3.2. We conjecture, however, that there should always be a Haar system for a locally compact groupoid with open range map, if the groupoid is second countable.

We show how to construct Haar systems if the stability groupoid and its quotient both admit one.

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2. Locally compact groupoids

Definition 2.1. By a *bundle of groups* we understand a continuous map $\pi : G \to X$ between locally compact Hausdorff spaces together with a group structure on each fibre $G_x = \pi^{-1}(x)$, $x \in X$ such that the following maps are

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continuous:

$$\begin{aligned} \varepsilon : X &\to G \quad \text{identity}, \\ m : G^{(2)} &\to G \quad \text{multiplication}, \\ \iota : G &\to G \quad \text{inverse} \qquad , \end{aligned}$$

where $G^{(2)}$ is the set of all $(x, y) \in G \times G$ with $\pi(x) = \pi(y)$.

Note that this implies that ε is a homeomorphism onto the image, so X carries the subspace topology but also X carries the quotient topology induced by the surjective map π. In all, the topology on X is determined by the one on G.

Definition 2.2. Each fibre G_x, being a locally compact group, carries a Haar measure which is unique up to scaling. A *coherent system* of Haar measures is a family $(\mu_x)_{x\in X}$, where μ_x is a Haar measure on G_x such that for each $\phi \in C_c(G)$ the map

$$x \mapsto \int_{G_x} \phi \, d\mu_x$$

is continuous.

Proposition 2.3. *Let $\pi : G \to X$ be a bundle of groups over a paracompact space X. There exists a coherent system of Haar measures μ_x on G if and only if the map π is open.*

Proof. This is [10, Lemma 1.3]. □

Definition 2.4. Let X be a set. By a *groupoid* over X we mean a category with object class X (so it is a small category) in which each arrow is an isomorphism. We write G for the set of arrows and we use the following notation

$$\begin{aligned} r, s : G \to X &\qquad \text{range and source maps}, \\ \varepsilon : X \to G &\qquad \text{identity}, \\ G^{(2)} \subset G \times G &\qquad \text{set of composable pairs}, \\ m : G^{(2)} \to G &\qquad \text{composition}, \\ \iota : G \to G &\qquad \text{inverse}. \end{aligned}$$

Definition 2.5. A *topological groupoid* is a groupoid G over X together with topologies on G and X such that the structure maps $r, s, \varepsilon, m, \iota$ are continuous. Here $G \times G$ carries the product topology and $G^{(2)} \subset G \times G$ the subset topology. Note that if X is Hausdorff, then $G^{(2)} = \{(\alpha, \beta) \in G \times G : r(\beta) = s(\alpha)\}$ is a closed subset of $G \times G$.

A *locally compact groupoid* is a topological groupoid such that G and X are locally compact Hausdorff spaces.

From now on G is assumed to be a locally compact groupoid. We use the notation

$$\begin{aligned} G_x &= \{g \in G : s(g) = x\}, \\ G^y &= \{g \in G : r(g) = y\}, \\ G_x^y &= G_x \cap G^y. \end{aligned}$$

As X is Hausdorff, all three sets are closed in G.

Note that a bundle of groups is a special case of a groupoid G with $G_x^y = \emptyset$ if $x \neq y$.

Definition 2.6. For a groupoid G the *stability groupoid* is defined to be the subset

$$G' = \big\{g \in G : r(g) = s(g)\big\}.$$

If G is a topological groupoid, then G' is a closed subgroupoid.

Definition 2.7. On a groupoid G we install an equivalence relation

$$g \sim h \quad \Leftrightarrow \quad r(g) = r(h) \text{ and } s(g) = s(h).$$

we write $[g]$ for the equivalence class, i.e., $[g] = G_{s(g)}^{r(g)}$.

Now assume that $(\mu_x^x)_{x \in X}$ is a coherent family of measures on the bundle of groups $G' = \{g \in G : r(g) = s(g)\}$. We then get invariant measures $\mu_{[g]}$ on the classes $[g]$ by setting

$$\int_{[g]} \phi(x)\, d\mu_{[g]}(x) = \int_{G_{s(g)}^{s(g)}} \phi(gx)\, d\mu_{s(g)}^{s(g)}(x).$$

The invariance of the μ_x^x yields the well-definedness of the $\mu_{[g]}$. The uniqueness of a Haar measure implies that $\mu_{[g]}$ is, up to scaling, the unique Radon measure on $[g]$ being right-invariant under $G_{s(g)}^{s(g)}$ or left-invariant under $G_{r(g)}^{r(g)}$.

In the sequel, we shall identify a Radon measure with its positive linear functional, so we write $\mu_{[g]}(\phi)$ for the above integral.

Definition 2.8. We shall need the notion of a topological right-action of a topological groupoid H on a topological space Z. This is given by the following data: first there is a continuous surjection $\rho : Z \to X$, where X is the base set of H. We define

$$Z * H = \big\{(z, h) : \rho(z) = r(h)\big\}.$$

This is a closed subset of $Z \times H$ and we consider it equipped with the corresponding topology. Next the action is given by a map

$$Z * H \to Z,$$
$$(z, h) \mapsto zh,$$

such that $\rho(zh) = s(h)$ and $z \cdot 1 = z$ as well as $z(hh') = (zh)h'$ holds for all $(z, h), (z, hh') \in Z * H$.

Note that the action defines an equivalence relation on Z given by $z \sim zh$ for $h \in H$. We naturally equip Z/H with the quotient topology.

Lemma 2.9. *Assume the locally compact groupoid H acts on a locally compact space Z and that H has open range map. Then the projection $Z \to Z/H$ is open.*

Proof. This is [6, Lemma 2.1]. However, in that paper the assertion was given under a stronger definition of H-actions then the one we use, as it was assumed that the map $\rho : Z \to X$ also be open. Lemma 2.1 and its proof in [6], however, are valid under our weaker assumptions. For the convenience of the reader we shall show this by reproducing the proof here: Let $V \subset Z$ be open, in order to show that its image in Z/H is open, it suffices to show that the union of orbits $VH = \{vh : v \in V, (v, h) \in Z * H\}$ is open in Z. So it suffices to show that any net $z_i \to vh$ with $v \in V$ and $h \in H$ eventually is in VH. But $\rho(z_i)$ converges to $\rho(vh) = s(h)$. As the range map of H is open, so is the source map s, hence the set $s(H)$ is open and we can find a net h_i in H on the same index set, such that $\rho(z_i) = s(h_i)$ for all $i \geq i_0$ for some index i_0. Further, the same applies to open neighborhoods of h, so we can choose the net so that $h_i \to h$. Then $z_i h_i^{-1}$ converges to v and thus is eventually in V and $z_i = z_i h_i^{-1} h_i$ is eventually in VH. □

Definition 2.10. An action of a groupoid H on a space Z is called *free* if $zh = z$ implies that $h = 1_{s(g)}$ and it is called *proper*, if the map $Z * H \to Z \times Z$, $(z, h) \mapsto (zh, z)$ is proper.

For any groupoid G the action of G' on G is easily seen to be free and proper.

Lemma 2.11. *Let G be a locally compact groupoid over a paracompact space X and let $(\mu_x^x)_{x \in X}$ be a coherent system of Haar measures on the groups G_x^x, $x \in X$. Then for every $\phi \in C_c(G)$ the function*

$$\overline{\phi} : g \mapsto \mu_{[g]}(\phi)$$

is continuous.

Proof. Since the G' action is free and proper, this is immediate from [5, Lemma 2.9]. □

3. Haar systems

Definition 3.1. A *Haar system* on the locally compact groupoid G is a family $(\mu^x)_{x\in X}$ of Radon measures on G with

(a) $\mathrm{supp}(\mu^x) = G^x$,
(b) $\int_G \phi(\alpha g)\, d\mu^y(g) = \int_G \phi(g)\, d\mu^x$ for every $\phi \in C_c(G)$ and every $\alpha \in G^x_y$,
(c) $x \mapsto \int_G \phi(g)\, d\mu^x(g)$ is continuous on X for every $\phi \in C_c(G)$.

If a locally compact groupoid G admits a Haar system, then the range map, and so the source map, too, is open, see [13, Corollary to Lemma 2], see also [15].

The question for the converse assertion, asked in [15], is answered in the negative by the following proposition.

Proposition 3.2. *There exists a locally compact, even compact, groupoid G, whose range map is open, but no Haar system exists on G.*

Proof. There are locally compact, even compact, Hausdorff spaces which cannot be the support of any Radon measure. Here are two examples:

- Let X be the unit ball of a Hilbert space of uncountable dimension and equip X with the weak topology. By the Banach-Alaoglu-Theorem, X is a compact Hausdorff space. By [1, Corollary 7.14.59 of volume 2], the set X cannot be the support of any Radon measure.
- (Williams) Let Y be an uncountable set with the discrete topology and let $X = Y \cup \{\infty\}$ be its one-point compactification. Then X cannot be the support of any Radon measure. To see this, let m be a Radon measure on X, then $m(X) < \infty$, as X is compact. Further, $m(Y) = \sum_{y\in Y} m(\{y\})$, as m is regular and the only compact subsets of Y are the finite sets. As $m(Y) < \infty$, the set M of all $y \in Y$ with $m(\{y\}) > 0$ is countable, therefore $M \neq Y$ and m is supported in $M \cup \{\infty\}$.

Let now X be any locally compact Hausdorff space which is not the support of a Radon measure. Let $G = X \times X$ with the product topology and make g a groupoid by setting $(x, y)(y, z) = (x, z)$ and $r(x, y) = x$ as well as $s(x, y) = y$. Then the source map is a homeomorphism between G^x and X, so G^x cannot be the support of any Radon measure, hence no Haar system exists. □

Conjecture 3.3. Every second countable, locally compact groupoid with open range map admits a Haar system.

Definition 3.4. Let G be a groupoid over X. We write $E(G) \subset X \times X$ for the image of the map $g \mapsto (s(g), r(g))$. Then $E(G)$ is an equivalence relation on X.

We say that a groupoid G is a *principal groupoid* if $G^x_x = \{1_x\}$ for every $x \in X$. This means that the groupoid is completely described by its equivalence relation. Note, though, that for topological groupoids the topology on G generally differs from the one on $E(G)$ as a subset of $X \times X$.

Lemma 3.5. *Let G be a groupoid over a set X. Define an equivalence relation on G by*

$$g \sim h \quad \Leftrightarrow \quad r(g) = r(h) \text{ and } s(g) = s(h).$$

Then the set $\overline{G} = G/\sim$ becomes a groupoid, indeed a principal groupoid, by setting $[g][h] = [gh]$ whenever g and h are composable.

Proof. This is easily checked. □

Theorem 3.6. *Let G be a locally compact groupoid over a paracompact space X. Suppose that the stability groupoid G' has open range map.*

(a) *The groupoid $\overline{G}$, when equipped with the quotient topology, is a locally compact groupoid. The quotient map $G \to \overline{G}$ is open.*
(b) *If the range map of G is open, then so is the range map of $\overline{G}$.*
(c) *If $\overline{G}$ admits a Haar system, then G admits a Haar system.*

Proof. Ad (a): By Proposition 2.3, the groupoid G' admits a coherent system of Haar measures $(\mu_x^x)_{x \in X}$. Let $g_0 \in G$ and let $\phi \in C_c^+(G)$ such that $\phi(g_0) > 0$. Let

$$\overline{\phi} : g \mapsto \int_{G_{s(g)}^{r(g)}} \phi(gh)\, d\mu_{s(g)}^{r(g)}(h).$$

By Lemma 2.11 the map $\overline{\phi}$ is continuous. It factors over $\overline{G}$, hence defines a continuous map of compact support on $\overline{G}$. The set $U = \{x \in \overline{G} : \overline{\phi}(x) > 0\}$ is an open neighborhood of $[g_0]$, so $\mathrm{supp}(\overline{\phi})$ is a compact neighborhood of $[g_0]$. Therefore $\overline{G}$ is locally compact.

If $[g] \neq [h]$, then we can find $\phi, \psi \in C_c^+(G)$ such that $\overline{\phi}$ and $\overline{\psi}$ have disjoint supports and $\phi(g), \psi(h) > 0$. Considering the continuous function $\overline{\phi} - \overline{\psi}$ on $\overline{G}$, one sees that $[h]$ and $[g]$ have disjoint neighborhoods, so $\overline{G}$ is a Hausdorff space. Together we infer that $\overline{G}$ is a locally compact groupoid.

The quotient map $p : G \to \overline{G}$ is open by Lemma 2.9.

Ad (b): As the range map of G is open and factors over the range map of $\overline{G}$, the range map of $\overline{G}$ is open as well.

Ad (c): If (m^x) is a Haar system for $\overline{G}$, then

$$\phi \mapsto \int_{\overline{G}} \overline{\phi}(g)\, dm^x(g)$$

defines a Haar system on G. □

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