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One-Sided Operators in Grand Variable Exponent Lebesgue Spaces

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Abstract. The boundedness of one-sided integral operators in grand variable exponent Lebesgue spaces unifying grand Lebesgue spaces and variable exponent Lebesgue spaces are established. The conditions on variable exponent is weaker than the log-Hölder continuity condition.

Keywords. Grand variable exponent Lebesgue spaces, one-sided maximal operator, one-sided Calderón–Zygmund operators, one-sided potentials, boundedness.

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1. Introduction

Our aim is to prove the boundedness of one-sided maximal, singular and potential operators in grand variable exponent Lebesgue space (briefly GVELS). This space introduced in [12] (see also [13]) unifies two non-standard function spaces: a variable exponent Lebesgue space and grand Lebesgue space. We refer also to the recent monograph [15, Section 14.11] for related topics. In [12], the authors established the boundedness of maximal, Calderón–Zygmund and fractional integral operators defined on quasi-metric spaces with doubling measure in GVELS $L^{p(\cdot),\theta}$ (see also [15, Section 14.11]). A variable exponent Lebesgue space $L^{p(\cdot)}$ (briefly VELS) is the special case of the one introduced by W. Orlicz in the 30ies of the last century and subsequently generalized by I. Musielak and

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W. Orlicz. Later H. Nakano [21] specified it. The boundedness of one-sided operators in variable exponent Lebesgue spaces was established in [7] (see also [14, Chapter 5]). It should be emphasized that in the latter paper, the authors derived the boundedness of one-sided operators under the condition which is weaker than the well-known log-Hölder continuity condition. Under the latter condition the operators of Harmonic Analysis such as maximal, Calderón–Zygmund, fractional integral operators are bounded in VELS (see, e.g., the monographs [3, 6] and references cited therein).

The grand Lebesgue space $L^{r)}$ was introduced in the 90ies of the last century by T. Iwaniec and C. Sbordone [10] when they studied integrability problems of the Jacobian under minimal hypothesis. The space $L^{r),\theta}$, $\theta > 0$, introduced by L. Greco, T. Iwaniec and C. Sbordone [9] is related to the investigation of the nonhomogeneous n-harmonic equation $\operatorname{div} A(x, \nabla u) = \mu$. In subsequent years, quite a number of problems of harmonic analysis and the theory of non-linear differential equations were studied in these spaces (see, e.g., the papers [8, 11], the monograph [15] and references cited therein).

The spaces under consideration are non-reflexive, non-separable and non-rearrangement invariant. We introduce a variant of GVELS denoted by $\widetilde{L}^{p(\cdot),\theta,\ell}$ and its one-sided analogs $\widetilde{L}_+^{p(\cdot),\theta,\ell_+}$, $\widetilde{L}_-^{p(\cdot),\theta,\ell_-}$. These classes are narrower than the space $L^{p(\cdot),\theta}$ introduced in [12], and $\widetilde{L}^{p(\cdot),\theta}$ introduced and studied in [15, p. 844]. The third parameter ℓ of $\widetilde{L}^{p(\cdot),\theta,\ell}$ is the least upper bound of the best constants in (one-sided) log-Hölder continuity condition for p.

The main results of this paper are Theorems 4.6–4.8, 5.4–5.6.

Constants (often different constants in one and the same chain of inequalities) will be usually denoted by c or C.

2. Preliminaries

Let $I = (a, b)$ be an open interval and let p be a measurable function on I satisfying the condition

$$1 < p_- \le p_+ < \infty, \tag{2.1}$$

where

$$p_- := \operatorname*{ess\,inf}_I p; \quad p_+ := \operatorname*{ess\,sup}_I p.$$

Further, we denote: $p_-(E) := \operatorname*{ess\,inf}_E p;\quad p_+(E) := \operatorname*{ess\,sup}_E p.$

By $P(I)$ we denote the class of all exponents on I satisfying (2.1).

Definition 2.1. We say that an exponent p belongs to the class $\mathcal{P}_-(I)$ if there exists a non-negative constant c_1 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < x - y \le \frac{1}{2}$, the inequality

$$p(x) \le p(y) + \frac{c_1}{\log\left(\frac{1}{x-y}\right)} \tag{2.2}$$

holds. Further, we say that p belongs to $\mathcal{P}_+(I)$ if there exists a non-negative constant c_2 such that for a.e. $x \in I$ and a.e. $y \in I$ with $0 < y - x \le \frac{1}{2}$, the inequality

$$p(x) \le p(y) + \frac{c_2}{\log\left(\frac{1}{y-x}\right)} \tag{2.3}$$

holds.

The class $\mathcal{P}_-(I)$ (resp. $\mathcal{P}_+(I)$) is strictly larger than the class of exponents satisfying the log-Hölder continuity condition: there is a non-negative constant A such that for all $x, y \in I$, $|x - y| < \frac{1}{2}$,

$$|p(x) - p(y)| \le \frac{A}{-\log|x-y|}. \tag{2.4}$$

We denote the class satisfying condition (2.4) by $\mathcal{P}(I)$.

In particular, it is easy to see that if p is a non-increasing function on I, then condition (2.2) is satisfied, while for non-decreasing p, condition (2.3) holds.

Remark 2.2. Let I be a bounded interval in $\mathbb{R}$ and let p be continuous on I. Then $\mathcal{P}(I) = \mathcal{P}_-^{\log}(I) \cap \mathcal{P}_+^{\log}(I)$.

In the sequel we will use the following notation.

$$I_+(x,h) := [x, x+h] \cap I; \quad I_-(x,h) := [x-h, x] \cap I; \quad I(x,h) := [x-h, x+h] \cap I.$$

Observe that either $I_+(x,h) = \emptyset$ or $|I_+(x,h)| > 0$ because I is an open set. The same conclusion is true for $I_-(x,h)$ and $I(x,h)$.

Let $p(\cdot) \in P(I)$. The Lebesgue space with variable exponent denoted by $L^{p(\cdot)}(I)$ (or by $L^{p(x)}(I)$) is the class of all measurable functions f on I for which

$$S_p(f) := \int_I |f(x)|^{p(x)} dx < \infty.$$

The norm in $L^{p(\cdot)}(I)$ is defined as follows

$$\|f\|_{L^{p(\cdot)}(I)} = \inf\left\{\lambda > 0 : S_p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

It is known that $L^{p(\cdot)}(I)$ is a Banach space (see, e.g., [17]). For other properties of spaces $L^{p(\cdot)}$ we refer to [17, 22, 24].

Further, let $\theta > 0$. We denote by $L^{p(\cdot),\theta}(X)$ the class of all measurable functions $f : I \mapsto \mathbb{R}$ for which the norm

$$\|f\|_{L^{p(\cdot),\theta}(I)} := \sup_{0<\varepsilon<p_- - 1} \varepsilon^{\frac{\theta}{p_- - \varepsilon}} \|f\|_{L^{p(x)-\varepsilon}(I)}$$

is finite.

Together with the space $L^{p(\cdot),\theta}$ it is interesting to consider the space $\mathcal{L}^{p(\cdot),\theta}$ which is defined with respect to the norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\theta}} := \sup_{0<\varepsilon<p_--1} \left\| \varepsilon^{\frac{\theta}{p(x)-\varepsilon}} f \right\|_{L^{p(x)-\varepsilon}(I)}.$$

Lemma 2.3. *The following continuous embedding holds:*

$$\mathcal{L}^{p(\cdot),\theta}(I) \hookrightarrow L^{p(\cdot),\theta}(I).$$

Proof. Since $p_- \le p(x)$, for small positive ε, we have $\varepsilon^{\frac{\theta}{p_--\varepsilon}} \le \varepsilon^{\frac{\theta}{p(x)-\varepsilon}}$. Hence,

$$\varepsilon^{\frac{\theta}{p_--\varepsilon}} \|f\|_{L^{p(x)-\varepsilon}(I)} \le c_p \left\| \varepsilon^{\frac{\theta}{p(x)-\varepsilon}} f \right\|_{L^{p(x)-\varepsilon}(I)}$$

for all $\varepsilon \in (0, p_- - 1)$, where the positive constant c_p depends only on p. Now the result follows. □

It is known (see [12]) that there is a function f and $\theta > 0$ such that $f \in L^{p(\cdot),\theta}(I)$ but $f \notin \mathcal{L}^{p(\cdot),\theta}(I)$.

If $p = p_c = \text{const}$, then $L^{p(\cdot),\theta} = \mathcal{L}^{p(\cdot),\theta}$ and it is the grand Lebesgue space $L^{p_c),\theta}$ introduced in [9]. In the case $p = p_c = \text{const}$ and $\theta = 1$, we have the Iwaniec–Sbordone [10] space $L^{p_c)}$.

Proposition 2.4 ([12, Proposition B])**.** *Let $p \in P(I)$ and let $\theta > 0$. Then*

(a) *The spaces $L^{p(\cdot),\theta}(I)$ and $\mathcal{L}^{p(\cdot),\theta}(I)$ are complete.*

(b) *The closure of $L^{p(\cdot)}(I)$ in $L^{p(\cdot),\theta}(I)$ (resp. in $\mathcal{L}^{p(\cdot),\theta}(I)$) consists of those $f \in L^{p(\cdot),\theta}(I)$ (resp. $f \in \mathcal{L}^{p(\cdot),\theta}(I)$) for which $\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p_--\varepsilon}} \|f(\cdot)\|_{L^{p(\cdot)-\varepsilon}(I)} = 0$ $\left(\text{resp. } \lim_{\varepsilon\to 0} \left\| \varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} f(\cdot) \right\|_{L^{p(\cdot)-\varepsilon}(I)} = 0\right)$.*

The following properties hold for $p \in P(I)$:

$$L^{p(\cdot)}(I) \hookrightarrow L^{p(\cdot),\theta}(I) \hookrightarrow L^{p(\cdot)-\varepsilon}(I), \quad 0 < \varepsilon < p_- - 1;$$
$$L^{p(\cdot)}(I) \hookrightarrow \mathcal{L}^{p(\cdot),\theta}(I) \hookrightarrow L^{p(\cdot)-\varepsilon}(I), \quad 0 < \varepsilon < p_- - 1.$$

The following statement was proved in [7] (see Proposition B) but we have to repeat the proof to observe the estimates of constants which are important for us.

Proposition 2.5. *Let p be a measurable positive function on I satisfying the condition $0 < p_-(I) \le p_+(I) < \infty$. The following conditions are equivalent:*

(a) *Condition (2.2) holds.*

(b) *There exists a positive constant C_1 such that for a.e. $x \in I$ and all r with $0 < r \le \frac{1}{2}$ and $I_-(x,r) \neq \emptyset$ the inequality*

$$r^{p_-(I_-(x,r))-p(x)} \le C_1 \tag{2.5}$$

holds. Moreover,

$$C_1 = \max\Big\{2^{p_+-p_-}, e^{2c_1}\Big\}, \tag{2.6}$$

where c_1 is defined in (2.2).

Proof. Let (2.2) hold. Let us take r so that $0 < r \le \frac{1}{2}$ and $I_-(x,r) \neq \emptyset$. Observe that if

$$S_{r,x} := \frac{1}{2} \operatorname*{ess\,sup}_{y \in I_-(x,r)} (p(x) - p(y)) \le 0,$$

then $p(x) \le p(y)$ for a.e. y, $y \in I_-(x,r)$. Therefore $p(x) \le p_-(I_-(x,r))$ and, consequently, (2.5) holds for such r and x with $C_1 = 2^{p_+-p_-}$. Further, if $S_{r,x} > 0$, then we take x_0, $x_0 \in I_-(x,r)$, so that

$$0 < S_{r,x} \le p(x) - p(x_0).$$

Hence,

$$r^{p_-(I_-(x,r))-p(x)} \le \left(\frac{1}{x-x_0}\right)^{2(p(x)-p(x_0))} \le \left(\frac{1}{x-x_0}\right)^{-\frac{2c}{\log(x-x_0)}} \le e^{2c_1}. \qquad \square$$

Definition 2.6. We say that p satisfies the decay condition at infinity (see [4]) if there is a non-negative constant A_∞ such that

$$|p(x) - p(y)| \le \frac{A_\infty}{\log(e + |x|)}$$

for all $x, y \in I$, $|y| > |x|$. In this case we write $p \in \mathcal{P}_\infty(I)$.

Let us introduce the following maximal operators:

$$\big(\mathcal{M}f\big)(x) = \sup_{h>0} \frac{1}{2h} \int_{I(x,h)} |f(t)|dt,$$

$$\big(\mathcal{M}_-f\big)(x) = \sup_{h>0} \frac{1}{h} \int_{I_-(x,h)} |f(t)|dt,$$

$$\big(\mathcal{M}_+f\big)(x) = \sup_{h>0} \frac{1}{h} \int_{I_+(x,h)} |f(t)|dt,$$

where I is an open set in $\mathbb{R}$ and $x \in I$.

It is known (see, e.g., [20], Proposition 3.2), that if r is a constant such that $1 < r < \infty$, then the following estimate holds for the Hardy–Littlewood maximal operator:

$$\|\mathcal{M}\|_{L^r \to L^r} \le 2(r')^{\frac{1}{r}}. \tag{2.7}$$

Using the pointwise estimate $\mathcal{M}_{\pm} f \le \mathcal{M} f$ and (2.7), we have

$$\|\mathcal{M}_{\pm}\|_{L^r \to L^r} \le 2(r')^{\frac{1}{r}}. \tag{2.8}$$

The boundedness of one-sided maximal, singular and potential operators in variable exponent Lebesgue spaces under the "the one-sided" local log-Hölder continuity condition and decay condition at infinity was established in [7]. For example, for the left maximal operator the following statement holds:

Theorem 2.7. *Let I be an interval in $\mathbb{R}$ and let $p \in P(I)$.*

(a) *If I is a bounded interval and $p \in \mathcal{P}_-(I)$, then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$.*
(b) *If I is $\mathbb{R}$ or $\mathbb{R}_+$ and $p \in \mathcal{P}_-(I) \cap \mathcal{P}_\infty(I)$, then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(\mathbb{R}_+)$.*

The next statement was proved in [7] but without clarification of bounds of norms for operators. We will repeat some arguments of the proof to see the constants there.

Proposition 2.8. *Let I be a bounded interval.*

(a) *if $p \in P(I) \cap \mathcal{P}_-^{\log}(I)$. Then $\mathcal{M}_-$ is bounded in $L^{p(\cdot)}(I)$. Moreover,*

$$\|\mathcal{M}_-\|_{L^{p(\cdot)} \mapsto L^{p(\cdot)}} \le C(p)\bigg(\|\mathcal{M}_-\|_{L^{p_-} \to L^{p_-}} + (b-a)^{\frac{1}{p_-}}\bigg), \tag{2.9}$$

where $C(p) = \widetilde{C}\big(\frac{p}{p_-}\big)$ and $\widetilde{C}(p)$ is defined in (2.11) *(see below).*

(b) *Let $p \in P(I) \cap \mathcal{P}_+^{\log}(I)$. Then $\mathcal{M}_+$ is bounded in $L^{p(\cdot)}(I)$. Moreover,*

$$\|\mathcal{M}_+\|_{L^{p(\cdot)} \to L^{p(\cdot)}} \le C(p)\bigg(\|\mathcal{M}_+\|_{L^{p_-} \to L^{p_-}} + (b-a)^{\frac{1}{p_-}}\bigg),$$

with $C(p) = \widetilde{C}\big(\frac{p}{p_-}\big)$ and $\widetilde{C}(p)$ is defined in (2.11) *(see below) replaced C_1 by C_2, where C_2 is defined as C_1 but taking c_2 for c_1, and c_2 is defined by* (2.3).

Proof. For simplicity let us assume that $I = (0, b)$. First we show that the inequality

$$\big(\mathcal{M}_{-,h} f\big)^{p(x)}(x) \le C(p)\bigg(\frac{1}{h}\int_{I_-(x,h)} |f(t)|^{p(t)} dt + 1\bigg), \quad 0 < h < x, \tag{2.10}$$

holds for all f with $\|f\|_{L^{p(\cdot)}} \le 1$, where

$$\big(\mathcal{M}_{-,h} f\big)(x) := \frac{1}{h}\int_{I_-(x,h)} |f(y)| dy$$

and, with C_1 from (2.6),

$$\widetilde{C}(p) = \max\left\{3^{p_+}, 2^{\frac{p_+}{p_-}} C_1^{\frac{1}{p_-}}\right\}. \tag{2.11}$$

If $h \geq \frac{1}{2}$, then

$$\begin{aligned}
\big(\mathcal{M}_{-,h}f\big)^{p(x)}(x) &= \left(\frac{1}{h}\int_{I_-(x,h)} |f(y)|dy\right)^{p(x)} \\
&\leq \left(\frac{1}{h}\int_{I_-(x,h)\cap\{|f|\geq 1\}} |f(y)|^{p(y)}dy + 1\right)^{p(x)} \\
&\leq \left(\frac{1}{h}\int_{I_-(x,h)} |f(y)|^{p(y)}dy + 1\right)^{p(x)} \\
&\leq (2+1)^{p(x)} \\
&\leq 3^{p_+}
\end{aligned}$$

which proves (2.10) for this case.

Let $h < \frac{1}{2}$. Then using the Hölder inequality we have

$$\begin{aligned}
\big(\mathcal{M}_{-,h}f\big)^{p(x)}(x) &\leq \left(\frac{1}{h}\int_{I_-(x,h)} |f(y)|^{p_-(I_-(x,h))}dy\right)^{\frac{p(x)}{p_-(I_-(x,h))}} \\
&\leq \left(\frac{1}{h}\int_{I_-(x,h)\cap\{|f|\geq 1\}} |f(y)|^{p(y)}dy + 1\right)^{\frac{p(x)}{p_-(I_-(x,h))}} \\
&\leq h^{-\frac{p(x)}{p_-(I_-(x,h))}}\left(\int_{I_-(x,h)} |f(y)|^{p(y)}dy + h\right)^{\frac{p(x)}{p_-(I_-(x,h))}}.
\end{aligned}$$

Since $\int_0^b |f(x)|^{p(\cdot)}dx \leq 1$ and $0 < h < \frac{1}{2}$, we have that $\frac{1}{2}\int_{I_-(x,h)}|f(y)|^{p(y)}dy + \frac{1}{2}h \leq 1$. Consequently, taking into account the last estimate and the condition $p \in \mathcal{P}_-^{\log}(I)$ we find that

$$\begin{aligned}
(\mathcal{M}_{-,h})^{p(x)}(x) &\leq 2^{\frac{p(x)}{p_-(I_-(x,h))}} h^{-\frac{p(x)}{p_-(I_-(x,h))}}\left(\frac{1}{2}\int_{I_-(x,h)} |f(y)|^{p(y)}dy + \frac{1}{2}h\right) \\
&\leq 2^{\frac{p_-}{p_+}-1} h^{\frac{p_-(I_-(x,h))-p(x)}{p_-(I_-(x,h))}}\left(\frac{1}{h}\int_{I_-(x,h)} |f(y)|^{p(y)}dy + 1\right) \\
&\leq 2^{\frac{p_-}{p_+}-1} C_1^{\frac{1}{p_-}}\big(\mathcal{M}_{-,h}(|f|^{p(\cdot)})(x) + 1\big).
\end{aligned}$$

Thus (2.10) has been proved. Inequality (2.10) immediately implies

$$\big(\mathcal{M}_-f\big)^{p(x)}(x) \leq \widetilde{C}(p)\big[\big(\mathcal{M}_-(|f|^{p(\cdot)})\big)(x) + 1\big], \tag{2.12}$$

where $\widetilde{C}(p)$ is defined by (2.11).

Using the fact $\frac{p}{p_-} \in \mathcal{P}_-^{\log}(I)$, inequality (2.12) and the boundedness of $\mathcal{M}_-$ in $L^{p_-}(I)$ we find that

$$\begin{aligned} S_p(\mathcal{M}_- f) &= \int_0^b \big(\mathcal{M}_- f(x)\big)^{p(x)} dx \\ &\le \widetilde{C}\left(\frac{p}{p_-}\right)^{p_-} \left(\int_0^b (\mathcal{M}_-(|f|^{q(\cdot)}(x))^{p_-} dx + b\right) \\ &\le \widetilde{C}\left(\frac{p}{p_-}\right)^{p_-} \left(\|\mathcal{M}_-\|^{p_-}_{L^{p_-}\to L^{p_-}} \int_0^b |f(x)|^{p(x)} dx + b\right) \\ &\le \widetilde{C}\left(\frac{p}{p_-}\right)^{p_-} \left(\|\mathcal{M}_-\|^{p_-}_{L^{p_-}\to L^{p_-}} + b\right) := \bar{C}_p. \end{aligned}$$

Hence, $S_p\Big((\mathcal{M}_f)C_p^{-\frac{1}{p(\cdot)}}\Big) \le 1$. Consequently, $\Big\|(\mathcal{M}_- f)\bar{C}_p^{-\frac{1}{p(\cdot)}}\Big\|^{p_+}_{L^{p(\cdot)}(I)} \le 1$. Finally,

$$\|\mathcal{M}_-\|_{L^{p(\cdot)}(I)\to L^{p(\cdot)}(I)} \le \bar{C}_p^{\frac{1}{p_-}}, \quad \text{where} \quad \bar{C}_p = \left[\widetilde{C}\left(\frac{p}{p_-}\right)\right]^{p_-}\Big(\|\mathcal{M}\|^{p_-}_{L^{p_-}\to L^{p_-}} + b\Big). \qquad \square$$

Locally integrable a.e. positive function w on I will be called a weight.

Definition 2.9. Let I be an interval in $\mathbb{R}$ and let r be a constant, $1 < r < \infty$. We say that a weight $w \in A_r^+(I)$ if

$$\|w\|_{A_r^+(I)} := \sup \frac{1}{c-a}\int_a^b w(t)dt \bigg(\frac{1}{c-a}\int_b^c w^{1-r'}(t)dt\bigg)^{r-1} \le \infty,$$

where the supremum is taken for all $a, b, c \in I$ satisfying the condition $a < b < c$.

We say that $w \in A_1^-(I)$ if there exists $c > 0$ such that $(\mathcal{M}_- w)(x) \le cw(x)$ for a.e. $x \in I$. The best possible constant in the latter inequality is denoted by $\|w\|_{A_1^+(I)}$.

We say that $w \in A_r^-(I)$ if

$$\|w\|_{A_r^-(I)} := \sup \frac{1}{c-a}\int_b^c w(t)dt \bigg(\frac{1}{c-a}\int_a^b w^{1-r'}(t)dt\bigg)^{r-1} \le \infty$$

for all $a, b, c \in I$ satisfying the condition $a < b < c$.

We say that $w \in A_1^+(I)$ if there exists $c > 0$ such that $(\mathcal{M}_+ w)(x) \le cw(x)$ for a.e. $x \in I$. The best possible constant in the latter inequality is denoted by $\|w\|_{A_1^+(I)}$.

It is easy to verify that $A_1^+(I) \subset A_p^+(I)$, $A_1^-(I) \subset A_p^-(I)$, $p > 1$. Moreover, $\|w\|_{A_p^+(I)} \le \|w\|_{A_1^+(I)}$; $\|w\|_{A_p^-(I)} \le \|w\|_{A_1^-(I)}$.

Let ρ be a weight on an interval I, i.e. locally integrable a.e. positive function on I. Suppose that $1 < r < \infty$, where r is a constant. We denote by $L^r(I, \rho)$ the Lebesgue space with weight ρ, which is a space of all measurable functions $f : I \to \mathbb{R}$ for which

$$\|f\|_{L^{p(\cdot)}(I,\rho)} = \left(\int_I \big(|f(x)|\rho(x)\big)^r dx \right)^{\frac{1}{r}} < \infty.$$

Further, we denote $\|f\|_{L^r_\rho(I)} := \|\rho^{\frac{1}{r}} f\|_{L^r(I)}$.

The following statements can be found in [23] for $\mathbb{R}$, and [2] for $\mathbb{R}_+$. They can be obtained for maximal operators defined on a bounded interval I by using, e.g., the techniques of dyadic maximal operators to obtain the Sawyer-type criterion. Then it is possible to pass to the Muckenhoupt-type criterion (see [18] for details).

Theorem 2.10. *Let I be an interval in $\mathbb{R}$. Suppose that r is a constant and that $1 < r < \infty$. Then*

(i) *$\mathcal{M}_+$ is bounded in $L^r(I, w)$ iff $w^r \in A_r^+(I)$. Moreover,*

$$\|\mathcal{M}_-\|_{L^r(I,w)\mapsto L^r(I,w)} \leq C_r \|w^r\|^{\gamma}_{A_r^-(I)}$$

for some positive constants C_r and γ depending only on r.

(ii) *$\mathcal{M}_-$ is bounded in $L^r(I, w)$ iff $w^r \in A_r^-(I)$. Moreover, there are positive constants C_r and γ depending only on r such that*

$$\|\mathcal{M}_+\|_{L^r(I,w)\mapsto L^r(I,w)} \leq C_r \|w^r\|^{\gamma}_{A_r^+(I)}.$$

Remark 2.11 ([16, Theorem 2.1])**.** Let $I := \mathbb{R}$. Suppose that r is a constant and that $1 < r < \infty$. Then the following estimates hold:

$$\|\mathcal{M}_+\|_{L^r(I,w)\mapsto L^r(I,w)} \leq C_r \|w^r\|^{r'-1}_{A_r^+(I)}, \quad \text{resp.} \quad \|\mathcal{M}_-\|_{L^r(I,w)\mapsto L^r(I,w)} \leq C_r \|w^r\|^{r'-1}_{A_r^-(I)}.$$

In these inequalities the exponent $r' - 1$ is best possible.

Definition 2.12. Let I be an interval in $\mathbb{R}$ and let p and q be constants such that $1 < p < \infty$, $1 < q < \infty$. We say that $\mathcal{U} \in A^+_{p,q}(I)$ if

$$\|\mathcal{U}\|_{A^+_{p,q}(I)} := \sup \left(\frac{1}{h} \int_{x-h}^{x} \mathcal{U}^q(t)dt \right)^{\frac{1}{q}} \left(\frac{1}{h} \int_{x}^{x+h} \mathcal{U}^{-p'}(t)dt \right)^{\frac{1}{p'}} < \infty,$$

where the supremum is taken over all $x \in I$ and $h > 0$ with $(x - h, x + h) \subset I$. Further, $\mathcal{U} \in A^-_{p,q}(I)$ if

$$\|\mathcal{U}\|_{A^-_{p,q}(I)} := \sup \left(\frac{1}{h} \int_{x}^{x+h} \mathcal{U}^q(t)dt \right)^{\frac{1}{q}} \left(\frac{1}{h} \int_{x-h}^{x} \mathcal{U}^{-p'}(t)dt \right)^{\frac{1}{p'}} < \infty,$$

where the supremum is taken over all $x \in I$ and $h > 0$ with $(x - h, x + h) \subset I$.

The following statement is known for $I := \mathbb{R}_+$, or for $I := \mathbb{R}$ (see [2]) but is can be derived also for finite interval $I = (a, b)$. It is possible, e.g., by obtaining the one-weight criterion for appropriate one-sided fractional maximal operator defined on I (see [18]) and then passing to the one-sided potentials by using the estimate of weighted norms which are true for one-sided A_∞ weights. We omit the details not to repeat the arguments used for unbounded intervals.

Theorem 2.13. *Let $I := (a, b)$, r and α be constants. Suppose that $0 < \alpha < 1$, $1 < r < \frac{1}{\alpha}$ and $s = \frac{r}{1-\alpha r}$.*

(i) *The Weyl operator $\mathcal{W}^\alpha$ given by*

$$\mathcal{W}^\alpha f(x) = \int_x^b f(t)(t-x)^{\alpha-1}dt, \quad x \in I,$$

is bounded from $L^r(I,\mathcal{U})$ to $L^s(I,\mathcal{U})$ iff $\mathcal{U} \in A^+_{r,s}(I)$. Moreover, there are positive constants $c_{r,\alpha}$ and γ such that

$$\|\mathcal{W}^\alpha\|_{L^r(I,\mathcal{U})\to L^s(I,\mathcal{U})} \le c_{r,\alpha}\|\mathcal{U}\|^\gamma_{A^+_{r,s}(I)}; \tag{2.13}$$

(ii) *the Riemann–Liouville operator*

$$\mathcal{R}^\alpha f(x) = \int_a^x f(t)(x-t)^{\alpha-1}dt, \quad x \in I,$$

is bounded from $L^r(I,\mathcal{U})$ to $L^s(I,\mathcal{U})$ iff $\mathcal{U} \in A^-_{r,s}(I)$. Moreover, there is a positive constants $c_{r,\alpha}$ and γ such that

$$\|\mathcal{W}^\alpha\|_{L^r(I,\mathcal{U})\to L^s(I,\mathcal{U})} \le c_{r,\alpha}\|\mathcal{U}\|^\gamma_{A^+_{r,s}(I)}. \tag{2.14}$$

Remark 2.14. It is known that in the case $I := \mathbb{R}$ the best possible constant γ in (2.13) (or in (2.14)) is equal to $(1-\alpha)\max\{1, \frac{p'}{q}\}$.

3. One-sided extrapolation

The next statement is a modification of the one-sided extrapolation theorem proven in [12] (see [5] for Euclidean spaces). In what follows the following notation is used:

$$\bar{q}(\cdot) := \frac{q(\cdot)}{q_0}, \quad \text{where } 0 < q_0 < \infty;$$
$$B^+_q := \|\mathcal{M}_+\|_{L^{\bar{q}'(\cdot)}(I)\to L^{\bar{q}'(\cdot)}(I)}, \quad B^-_q := \|\mathcal{M}_-\|_{L^{\bar{q}'(\cdot)}(I)\to L^{\bar{q}'(\cdot)}(I)}.$$

In particular, if $p_0 = q_0$, then it is assumed $\bar{p}$ and $B^\pm_p$ for $\bar{q}$ and $B^\pm_q$, respectively.

Proposition 3.1. *Let* $I := (a, b)$ *be an interval in* $\mathbb{R}$ (*bounded or unbounded*). *Let* $\mathcal{F}$ *be a family of pairs of nonnegative functions such that for some* p_0 *and* q_0 *with* $0 < p_0 \le q_0 < \infty$, *the inequality*

$$\left(\int_I f(x)^{q_0} w(x) dx\right)^{\frac{1}{q_0}} \le c_0 \left(\int_I g(x)^{p_0} w(x)^{\frac{p_0}{q_0}} dx\right)^{\frac{1}{p_0}} \tag{3.1}$$

holds for all $(f, g) \in \mathcal{F}$, *where* $w \in A_1^+(I)$ (*resp.* $A_1^-(I)$) *and the positive constant* $c_0 := c_0(\|w\|_{A_1^+(I)})$ (*resp.* $c_0 := c_0(\|w\|_{A_1^-(I)})$) *is independent of* (f, g) *and depends on* $\|w\|_{A_1^+}(I)$ (*resp.* $\|w\|_{A_1^+}(I)$). *Given* $p \in P(I)$ *satisfying the condition* $p_0 < p_-(I) \le p_+(I) < \frac{p_0 q_0}{q_0 - p_0}$, *define a function* q *by*

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}, \quad x \in I. \tag{3.2}$$

If $\mathcal{M}_-$ (*resp.* $\mathcal{M}_+$) *is bounded in* $L^{(\frac{q(\cdot)}{q_0})'}(I)$, *then for all* $(f, g) \in \mathcal{F}$ *such that* $f \in L^{q(\cdot)}(I)$ *the inequality*

$$\|f\|_{L^{q(\cdot)}(I)} \le b_0^- \|g\|_{L^{p(\cdot)}(I)} \quad \Big(\text{resp.} \quad \|f\|_{L^{q(\cdot)}(I)} \le b_0^+ \|g\|_{L^{p(\cdot)}(I)}\Big)$$

holds, where $b_0^- := b_0^-(B_q^-, q)$ (*resp.* $b_0^+ := b_0^+(B_q^+, q)$) *is independent of* (f, g) *and depends on* q *and* B_q^- (*resp. on* q *and* B_q^+). *Moreover, if the mapping* $x \to c_0(x)$ *is non-decreasing on* $(1, \infty)$, *then there exists a small positive constant* δ *such that*

$$\sup_{0<\lambda_- \le \lambda_+ < \delta} b_0^-(B_{q-\lambda}^-, q-\lambda) < \infty \quad \Big(\text{resp.} \sup_{0<\lambda_- \le \lambda_+ < \delta} b_0^+(B_{q-\lambda}^+, q-\lambda) < \infty\Big),$$

where $q - \lambda$ *is defined by* (3.2) *replaced* q *by* $q - \lambda$ *and* $p(\cdot)$ *by* $p(\cdot) - \lambda(\cdot)$ (*here* λ *denotes continuous bounded functions on* I).

Proof. For simplicity let us prove the theorem for $p_0 = q_0$ and $w \in A_1^+(I)$. The proofs for other cases are the same. Thus, assume that (3.1) holds for $p_0 = q_0$ and $w \in A_1^+(I)$. First notice that $\bar{p} \in P(I)$, where $\bar{p}(\cdot) = \frac{p(\cdot)}{p_0}$. Observe that in this case $p(\cdot) = q(\cdot)$ and, consequently, $B_q^- = B_p^-$.

We set:

$$\mathcal{H}\phi(x) = \sum_{k=0}^{+\infty} \frac{\big(\mathcal{M}_-^{(k)}\phi\big)(x)}{2^k (B_p^-)^k},$$

where

$$\mathcal{M}_-^{(k)} = \underbrace{\mathcal{M}_- \circ \mathcal{M}_- \circ \cdots \circ \mathcal{M}_-}_{k}; \quad \mathcal{M}_-^{(0)} = Id.$$

From the definition it follows that

(a) if $\phi \geq 0$, then $\phi(x) \leq \big(\mathcal{H}\phi\big)(x)$;

(b) $\|\mathcal{H}\phi\|_{L^{(\bar{p})'(\cdot)}(I)} \leq 2\|\phi\|_{L^{(\bar{p})'(\cdot)}(I)}$;

(c) $\mathcal{M}_-\big(\mathcal{H}\phi\big)(x) \leq 2B_p^- \,\mathcal{H}\phi(x)$ for every $x \in I$.

The latter inequality implies that $\mathcal{H}\phi \in A_1^+(I)$ with an $A_1^+(I)$ constant independent of ϕ.

Further, by the definition and elementary properties of $L^{p(\cdot)}$ spaces we have

$$\|f\|^{p_0}_{L^{p(\cdot)}(I)} = \|\, |f|^{p_0}\|_{L^{\bar{p}(\cdot)}(I)} \leq \sup \int_I |f(x)|^{p_0} h(x) dx,$$

where the supremum is taken over all nonnegative $h \in L^{(\bar{p})'(\cdot)}(I)$ with the norm $\|h\|_{L^{(\bar{p})'(\cdot)}(I)} = 1$. Let us fix such an h. We will show that

$$\int_I |f|^{p_0} h(x)\; dx \leq c\; \|g\|^{p_0}_{L^{p(\cdot)}(I)},$$

where c is independent of h and $f \in L^{p(\cdot)}(I)$. By (a),(b) and the Hölder inequality for $L^{p(\cdot)}$ spaces we have

$$\begin{aligned}\int_I |f|^{p_0} h(x)\; dx &\leq \int_I |f|^{p_0}\; \mathcal{H}h(x)\; dx \\ &\leq 2\; \|\; |f|^{p_0}\|_{L^{\bar{p}}(I)}\|\mathcal{H}h\|_{L^{(\bar{p})'}(I)} \\ &\leq 2c\; \|\; f\|^{p_0}_{L^{p(\cdot)}(I)}\|h\|_{L^{(\bar{p})'(\cdot)}(I)} \\ &= 2c\; \|\; f\|^{p_0}_{L^{p(\cdot)}(I)} \\ &< \infty.\end{aligned}$$

Using the fact that $A_1^+(I)$ constant of $\mathcal{H}h$ is bounded by $2B_p^-$, applying (3.1) and the Hölder inequality with respect to $\bar{p}$ we find that

$$\begin{aligned}\int_I f^{p_0}(x)h(x)d\mu(x) &\leq \int_I f^{p_0}(x)\mathcal{H}h(x)d\mu(x) \\ &\leq c_0(\|\mathcal{H}h\|_{A_1^+}) \int_I g^{p_0}(x)\mathcal{H}h(x)d\mu(x) \\ &\leq c_0(\|\mathcal{H}h\|_{A_1^+})\Big(\frac{p_0}{p_-} + \frac{p_+ - p_0}{p_+}\Big)\|g^{p_0}\|_{L^{\bar{p}(\cdot)}(I)}\|\mathcal{H}h\|_{L^{(\bar{p})'(\cdot)}(I)} \\ &\leq 2c_0(2B_p^-)\Big(\frac{p_0}{p_-} + \frac{p_+ - p_0}{p_+}\Big)\|g\|^{p_0}_{L^{p(\cdot)}(I)}\|h\|_{L^{(\bar{p})'(\cdot)}(I)} \\ &\leq 2c_0(2B_p^-)\Big(\frac{p_0}{p_-} + \frac{p_+ - p_0}{p_+}\Big)\|g\|^{p_0}_{L^{p(\cdot)}(I)}.\end{aligned}$$

This completes the proof of the statement. □

4. One-sided maximal and Calderón–Zygmund Operators

We begin this section with the following statement.

Proposition 4.1 (Reduction Statement ([12, Proposition 2.10], [15, p. 841])). *Let I be a bounded interval in $\mathbb{R}$ and let $p \in P(I)$. Suppose that $\theta > 0$.*

(a) *Suppose that $\mathcal{F}$ is a family of pairs (f, g) such that*

$$\|f\|_{L^{p(\cdot)-\varepsilon}(I)} \le c_{p,\varepsilon} \|g\|_{L^{p(\cdot)-\varepsilon}(I)},$$

for all small positive ε. If $\sup_{0<\varepsilon\le\sigma} c_{p,\varepsilon} < \infty$ for some positive constant σ, then for all $(f, g) \in \mathcal{F}$,

$$\|f\|_{L^{p(\cdot),\theta}(I)} \le c\|g\|_{L^{p(\cdot),\theta}(I)};$$

(b) *Suppose that $\mathcal{F}$ is a family of pairs (f, g) such that*

$$\|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} f\|_{L^{p(\cdot)-\varepsilon}(I)} \le b_{p,\varepsilon} \|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} g\|_{L^{p(\cdot)-\varepsilon}(I)}$$

for some positive constant $b_{p,\varepsilon}$. If $\sup_{0<\varepsilon<\sigma} b_{p,\varepsilon} < \infty$ for some positive constant σ, then for all $(f, g) \in \mathcal{F}$,

$$\|f\|_{\mathcal{L}^{p(\cdot),\theta}(I)} \le c\|g\|_{\mathcal{L}^{p(\cdot),\theta}(I)},$$

where the positive constant c does not depend on (f, g).

Now we give the definition of the Calderón–Zygmund kernel.

Definition 4.2. Let $I := (-a, a)$, $0 < a \le \infty$. We say that a function k in $L^1_{loc}(I \setminus \{0\})$ is a Calderón–Zygmund kernel if the following properties are satisfied:

(a) There exists a constant A_1 such that

$$\left| \int_{\varepsilon<|x|<N} k(x)dx \right| \le A_1 < \infty$$

for all ε and all N, with $0 < \varepsilon < N < a$, and furthermore

$$\lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<N} k(x)dx$$

exists.

(b) There exists a positive constant A_2 such that

$$\big|k(x)\big| \le \frac{A_2}{|x|}, \quad x \in I \setminus \{0\}.$$

(c) There exists a positive constant A_3 such that for all $x, y \in I$ with $|x| > 2|y| > 0$ the inequality

$$|k(x-y) - k(x)| \le A_3 \frac{|y|}{|x|^2}$$

holds.

It is known (see [1]) that if $a = \infty$, (a)–(c) are satisfied for the kernel k defined on $\mathbb{R}$, then the operators

$$K^* f(x) = \sup_{\varepsilon > 0} \big| K_\varepsilon f(x) \big|, \quad K f(x) = \lim_{\varepsilon \to 0} K_\varepsilon f(x)$$

where

$$K_\varepsilon f(x) = \int_{|x-y|>\varepsilon} k(x-y) f(y) dy,$$

are of weak (1,1) type and are bounded in $L^r(\mathbb{R})$, $1 < r < \infty$. It is clear that $Kf(x) \le K^* f(x)$.

The following example shows the existence of a non-trivial Calderón–Zygmund kernel with a support contained in $(0, a)$.

Example 4.3. The function

$$k(x) = \frac{1}{x} \frac{\sin(\log x)}{\log x} \chi_{(0,a)}(x)$$

is a Calderón–Zygmund kernel (cf. [1]).

There exists also a non-trivial Calderón–Zygmund kernel supported in the interval $(-a, 0)$.

The next results are well-known for the Calderón–Zygmund kernels supported in the interval in the interval $(0, \infty)$ (resp. $(-\infty, 0)$) (see [1]), but the techniques developed in those papers enable us to formulate it for a finite interval.

Theorem 4.4. *Let $I := (0, a)$ be a bounded interval and let r be a constant, $1 < r < \infty$, and let k be a Calderón–Zygmund kernel with support in $(0, 2a)$. Then the condition $w \in A_r^-(I)$ implies the inequality*

$$\int_I \big| K^* f(x) \big|^r w(x) dx \ \le c \int_I \big| f(x) \big|^r w(x) dx, \quad f \in L_w^r(I).$$

Moreover,

$$\|T^*\|_{L^r \mapsto L^r} \le C_r \|w\|^\gamma_{A_r^-(I)}$$

for some positive constants C_r and γ depending only on r.

Theorem 4.5. *Let $I := (0, a)$ be a bounded interval and let r be a constant such that $1 < r < \infty$. Let k be a Calderón–Zygmund kernel with support in $(-2a, 0)$. If $w \in A_r^+(I)$, then it follows that T^* is bounded in $L_w^r(I)$. Moreover,*

$$\|T^*\|_{L^r \mapsto L^r} \leq C_r \|w\|_{A_r^+(I)}^{\gamma}$$

for some positive constant constants C_r and γ depending only on r.

Theorem 4.6. *Let $I := (0, a)$, $0 < a < \infty$ be a bounded interval and let $\theta > 0$. Suppose that $p \in P(I)$.*

(i) *If $p \in \mathcal{P}_-(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_-$ is bounded in $L^{p(\cdot),\theta}(I)$;*

(ii) *If $p \in \mathcal{P}_+(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_+$ is bounded in $L^{p(\cdot),\theta}(I)$;*

Proof. We show only part (i) since part (ii) follows analogously. By Hölder's inequality we can easily see that

$$\|\mathcal{M}_- f\|_{L^{p(\cdot),\theta}(I)} \leq C_{p,\sigma} \sup_{0<\varepsilon<\sigma} \varepsilon^{\frac{\theta}{p_- - \varepsilon}} \|\mathcal{M}_- f\|_{L^{p(\cdot)-\varepsilon}(I)},$$

where σ is a small positive number. Applying Proposition 2.8, estimate (2.8), and taking σ sufficiently small, we find that

$$\|\mathcal{M}_- f\|_{L^{p(\cdot)-\varepsilon}(I)} \leq C(p-\varepsilon)\Big(2[(p-\varepsilon)'_-]^{\frac{1}{(p-\varepsilon)_-}} + |I|\Big)\|f\|_{L^{p(\cdot)-\varepsilon}(I)} := b_{p,\varepsilon}\|f\|_{L^{p(\cdot)-\varepsilon}(I)},$$

where, obviously,

$$\sup_{0<\varepsilon\leq\sigma} b_{p,\varepsilon} < \infty$$

for some sufficiently small positive σ (here $C(p)$ is defined by (2.9)). Now the result follows from Proposition 4.1(a). □

In the next statement by the symbol $D(I)$ is denoted the class of bounded functions defined on I with compact support.

Theorem 4.7. *Let $I := (0, a)$ be a bounded interval and let $\theta > 0$. Suppose that $p \in P(I)$.*

(i) *If $p \in \mathcal{P}_+(I)$, then for the Calderón–Zygmund operator K with kernel supported on $(-2a, 0)$, there is a positive constant c such that for all $f \in D(I)$, the inequality*

$$\|K^* f\|_{L^{p(\cdot),\theta}(I)} \leq c\|f\|_{L^{p(\cdot),\theta}(I)};$$

holds;

(ii) *If $p \in \mathcal{P}_-(I)$, then for the Calderón–Zygmund operator K with kernel supported on $(0, 2a)$, there is a positive constant c such that that for all $f \in D(I)$, the inequality*

$$\|K^* f\|_{L^{p(\cdot),\theta}(I)} \le c\|f\|_{L^{p(\cdot),\theta}(I)};$$

holds.

Proof. (i). Observe that Theorem 4.4 and Proposition 3.1 yield that there is a small positive constant σ such that for all $f \in D(I)$,

$$\|K^* f\|_{L^{p(\cdot)-\varepsilon}(I)} \le c_{p,\varepsilon}\|g\|_{L^{p(\cdot)-\varepsilon}(I)},$$

with $\sup_{0<\varepsilon\le\sigma} c_{p,\varepsilon} < \infty$. Now by using Proposition 4.1 we have the desired result. Part (ii) follows similarly. □

Regarding the space $\mathcal{L}^{p(\cdot),\theta}(I)$ we have the following statement.

Theorem 4.8. *Let I be a bounded interval and let $\theta > 0$. Suppose that $p \in P(I)$.*

(i) *If $p \in \mathcal{P}_-(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_-$ is bounded in $\mathcal{L}^{p(\cdot),\theta}(I)$;*

(ii) *If $p \in \mathcal{P}_+(I)$, then the one-sided Hardy–Littlewood maximal operator $\mathcal{M}_+$ is bounded in $\mathcal{L}^{p(\cdot),\theta}(I)$.*

Proof. We prove (i). First observe that by Hölder's inequality we have that

$$\|\mathcal{M}_- f\|_{\mathcal{L}^{p(\cdot),\theta}(I)} \le C(p,\theta,\sigma) \sup_{0<\varepsilon\le\sigma} \|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} \mathcal{M}_- f(\cdot)\|_{L^{p(\cdot)-\varepsilon}(I)},$$

where σ is a small positive number. Further, let

$$\sup_{0<\varepsilon\le\sigma} \|\varepsilon^{\frac{\theta}{p(\cdot)-\varepsilon}} f(\cdot)\|_{L^{p(\cdot)-\varepsilon}(I)} \le 1.$$

We will show that

$$\varepsilon^\theta \int_I (\mathcal{M}_- f(x))^{p(x)-\varepsilon} dx \le C, \quad \varepsilon \in (0,\sigma],$$

for some positive constant C independent of ε. Let $f \ge 0$. Applying estimates (2.12),(2.8) we find that

$$\begin{aligned}
\varepsilon^\theta \int_I (\mathcal{M}_- f(x))^{p(x)-\varepsilon} dx &\le \widetilde{C}(p_\varepsilon)^{p_- -\varepsilon} \varepsilon^\theta 2^{p_- -\varepsilon-1} \left[\int_I \left[\mathcal{M}_-\left(f^{\frac{p(x)-\varepsilon}{p_- -\varepsilon}}\right)\right]^{p_- -\varepsilon} (x)\, dx + |I| \right] \\
&\le \widetilde{C}(p_\varepsilon)^{p_- -\varepsilon} (p_- - \varepsilon)' 2^{p_- -\varepsilon-1} \varepsilon^\theta \left[\int_I (f(x)^{p(x)-\varepsilon} dx + |I| \right] \\
&\le \widetilde{C}(p_\varepsilon)^{p_- -\varepsilon} (p_- - \varepsilon)' 2^{p_- -\varepsilon} \\
&\le C,
\end{aligned}$$

where $\varepsilon \le \sigma$, $p_\varepsilon := \frac{p-\varepsilon}{p_- -\varepsilon}$ and $\widetilde{C}(p)$ is defined by (2.11). □

5. One-sided fractional integrals

In this section we study the boundedness of one-sided fractional integral operators $\mathcal{W}_\alpha$ and $\mathcal{R}_\alpha$ in GVELSs which are narrower than the space $L^{p(\cdot),\theta}(I)$. To formulate the main result of this section we introduce new classes of exponents related to the classes $\mathcal{P}_-(I)$ and $\mathcal{P}_+(I)$. The class $\widetilde{\mathcal{P}}_-^{\ell_-}(I)$ (resp. $\widetilde{\mathcal{P}}_+^{\ell_+}(I)$) is the class of all non-negative $p \in \mathcal{P}_-(I)$ (resp. $p \in \mathcal{P}_+(I)$) such that $0 \le \ell_- := \sup c_1(p) < \infty$ (resp. $0 \le \ell_+ := \sup c_2(p) < \infty$), where $c_1(p)$ (resp. $c_2(p)$) is the best possible constant in (2.2) (resp. in (2.3)). Analogously, $\mathcal{P}^\ell(I)$ is the class of all $p \in \mathcal{P}(I)$ such that $0 \le \ell := \sup A(p) < \infty$, where $A(p)$ is the best possible constant in (2.4).

Let $p \in P(I)$ and let $\theta > 0$. We introduce new spaces $\widetilde{L}^{p(\cdot),\theta,\ell}(I)$, $\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)$ and $\widetilde{L}_-^{p(\cdot),\theta,\ell_-}(I)$ defined with respect to the norms

$$\|f\|_{\widetilde{L}^{p(\cdot),\theta,\ell}(I)} := \sup\left\{\eta_+^{\frac{\theta}{p_- - \eta_+}} \|f\|_{L^{p(x)-\eta(x)}(I)} : 0 < \eta_- \le \eta_+ < \eta_0,\ p(\cdot) - \eta(\cdot) \in \widetilde{\mathcal{P}}^\ell(I)\right\}$$

$$\|f\|_{\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)} := \sup\left\{\eta_+^{\frac{\theta}{p_- - \eta_+}} \|f\|_{L^{p(x)-\eta(x)}(I)} : 0 < \eta_- \le \eta_+ < \eta_0,\ p(\cdot) - \eta(\cdot) \in \widetilde{\mathcal{P}}_+^{\ell_+}(I)\right\}$$

$$\|f\|_{\widetilde{L}_-^{p(\cdot),\theta,\ell_-}(I)} := \sup\left\{\eta_+^{\frac{\theta}{p_- - \eta_+}} \|f\|_{L^{p(x)-\eta(x)}(I)} : 0 < \eta_- \le \eta_+ < \eta_0,\ p(\cdot) - \eta(\cdot) \in \widetilde{\mathcal{P}}_-^{\ell_-}(I)\right\}$$

where in the definition of these norms η_0 is some positive constant such that $\eta_0 < p_- - 1$ and $\eta(\cdot)$ is a measurable function defined on $(0, \eta_0)$ with the property $0 < \eta_- \le \eta_+ < \eta_0$.

It can be checked that the spaces $\widetilde{L}_-^{p(\cdot),\theta,\ell_-}(I)$ and $\widetilde{L}_+^{p(\cdot),\theta,\ell_-}(I)$ are Banach spaces. Let $p \in \mathcal{P}_+(I)$. Then the closure of $L^{p(\cdot)}(I)$ in $\widetilde{L}^{p(\cdot),\theta}(I)$ consists of those $f \in \widetilde{L}^{p(\cdot),\theta}(I)$ having the following property: for any sequence $\varepsilon^{(n)}(\cdot)$ such that $p - \varepsilon^{(n)} \in \widetilde{\mathcal{P}}_+(I)$ and $\varepsilon_+^{(n)} \to 0$,

$$(\varepsilon_+^{(n)})^{\frac{\theta}{p_- - \varepsilon_+^{(n)}}} \|f(\cdot)\|_{L^{p(\cdot)-\varepsilon^{(n)}(\cdot)}(I)} \to 0.$$

If $p = \text{const}$, then the spaces $\widetilde{L}_-^{p(\cdot),\theta,\ell_-}(I)$ and $\widetilde{L}_+^{p(\cdot),\theta,\ell_-}(I)$ are constant exponent grand Lebesgue spaces.

The next statement is a corollary of Theorem 2.13.

Proposition 5.1. *Let $\mathcal{W}^\alpha$ and $\mathcal{R}^\alpha$ be one-sided operators defined in Theorem* 2.10. *Let p_0 and α be constants such that $1 < p_0 < \infty$ and $0 < \alpha < \frac{1}{p_0}$. We set $q_0 = \frac{p_0}{1-\alpha p_0}$.*

(i) *The operator $\mathcal{W}^\alpha$ is bounded from $L^{p_0}_{w^{\frac{p_0}{q_0}}}(I)$ to $L^{q_0}_w(I)$ iff $w \in A^+_{1+\frac{q_0}{(p_0)'}}$;*

(ii) *The operator $\mathcal{R}^\alpha$ is bounded from $L^{p_0}_{w^{\frac{p_0}{q_0}}}(I)$ to $L^{q_0}_w(I)$ iff $w \in A^-_{1+\frac{q_0}{(p_0)'}}$.*

Proposition 5.2 ([7, Theorem 4.2])**.** *Let $p \in P(I)$. Suppose that α is a constant such that $0 < \alpha < \frac{1}{p_+}$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$.*

(i) *Let $p \in \mathcal{P}_+(I)$. Then there is a positive constant $b_{p,\alpha}$ depending only on p and α such that the following inequality holds*

$$\|\mathcal{W}^\alpha f\|_{L^{q(\cdot)}(I)} \le b_{p,\alpha} \|f\|_{L^{p(\cdot)}(I)}.$$

(ii) *Let $p \in \mathcal{P}_-(I)$. Then there is a positive constant $b_{p,\alpha}$ depending only on p and α such that the following inequality holds*

$$\|\mathcal{R}^\alpha f\|_{L^{q(\cdot)}(I)} \le b_{p,\alpha} \|f\|_{L^{p(\cdot)}(I)}.$$

Proposition 5.3 (Reduction Statement)**.** *Let $p \in P(I)$ and let $\theta > 0$. Suppose that $0 \le \alpha < \frac{1}{p_+}$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Suppose that $\mathcal{F}$ is a family of pairs (f,g) such that*

$$\|f\|_{L^{q(\cdot)-\varepsilon(\cdot)}(I)} \le c_{p,\alpha,\eta} \|g\|_{L^{p(\cdot)-\eta(\cdot)}(I)}$$

for all $\varepsilon(\cdot)$ and $\eta(\cdot)$ satisfying the conditions:

(a) *$1 < \eta_- \le \eta_+ < \sigma$, where σ is a small positive number;*
(b) *$\frac{1}{p(x)-\eta(x)} - \frac{1}{q(x)-\varepsilon(x)} = \alpha$;*
(c) *$p - \eta \in \widetilde{\mathcal{P}}_+^{\ell_+}(I)$ (resp. $p - \eta \in \widetilde{\mathcal{P}}_-^{\ell_-}(I)$).*

If $\sup_{0<\eta_- \le \eta_+ \le \sigma} c_{p,\alpha,\eta} < \infty$ for some positive constant σ, then there exists a positive constant c such that for all $(f,g) \in \mathcal{F}$,

$$\|f\|_{\widetilde{L}_+^{q(\cdot),\frac{\theta q_-}{p_-},\widetilde{\ell}_+}(I)} \le c\|g\|_{\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)} \quad \Big(\textit{resp.} \quad \|f\|_{\widetilde{L}_-^{q(\cdot),\frac{\theta q_-}{p_-},\widetilde{\ell}_-}(I)} \le c\|g\|_{\widetilde{L}_-^{p(\cdot),\theta,\ell_-}(I)}\Big), \quad (5.1)$$

where $\widetilde{\ell}_\pm(I) = \frac{\ell_\pm}{(1-\alpha p_+)^2}$.

Proof. We repeat the arguments of [15, proof of Proposition 14.144, p. 847]. We will prove (5.1).

Observe that it is enough to show that

$$\sup_{0<\varepsilon_- \le \varepsilon_+ < \delta} \varepsilon_+^{\frac{\theta}{p_-}} \|f\|_{L^{q(\cdot)-\varepsilon(\cdot)}(X)} \le C \sup_{0<\eta_- \le \eta_+ < \sigma} \eta_+^{\frac{\theta}{p_-}} \|g\|_{L^{p(\cdot)-\eta(\cdot)}(X)}$$

for some positive numbers σ and δ, where $\theta > 0$.

We take η so that $0 < \eta_- \le \eta_+ < \sigma$. We define $\varepsilon(\cdot)$ so that

$$\frac{1}{p(x)-\eta(x)} - \frac{1}{q(x)-\varepsilon(x)} = \alpha. \quad (5.2)$$

Observe that if $p - \eta \in \widetilde{\mathcal{P}}_+^{\ell_+}(I)$, then by (5.2) we have that $q - \varepsilon \in \widetilde{\mathcal{P}}_+^{\widetilde{\ell}_+}(I)$, where $\widetilde{\ell}_+(I) = \frac{\ell_+}{(1-\alpha p_+)^2}$.

It is easy to see that since the function $t \mapsto \frac{t}{1-\alpha t}$ is increasing on $[0, \frac{1}{\alpha}]$, we have

$$\varepsilon_+ \le q_+ - \frac{p_- - \eta_+}{1-\alpha(p_- - \eta_+)} \sim \eta_+ \quad \text{as } \varepsilon_+ \to 0.$$

In particular, it can be checked that $\frac{\varepsilon_+}{\eta_+} \le \frac{1}{(1-\alpha p_-)^2}$ for sufficiently small ε_+. Hence

$$\varepsilon_+^{\frac{\theta}{p_-}} \|f\|_{L^{q(\cdot)-\varepsilon(\cdot)}(X)} \le c_{p,\alpha,\varepsilon} \varepsilon_+^{\frac{\theta}{p_-}} \|g\|_{L^{p(\cdot)-\eta(\cdot)}(X)} \le (1-\alpha p_-)^{-\frac{2\theta}{p_-}} c_{p,\alpha,\varepsilon} \eta_+^{\frac{\theta}{p_-}} \|g\|_{L^{p(\cdot)-\eta(\cdot)}(X)}.$$

Since $\sup_{0<\varepsilon_- \le \varepsilon_+ < \sigma} c_{p,\alpha,\varepsilon} < \infty$, we have the desired result. □

Finally, we can formulate the statement concerning the fractional integrals.

Theorem 5.4. *Let $p \in P(I)$ and let $\theta > 0$. Suppose that α is a constant such that $0 < \alpha < \frac{1}{p_+}$. We set $q(x) = \frac{p(x)}{1-\alpha p(x)}$. Then*

(i) *$\mathcal{W}^\alpha$ is bounded from $\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)$ to $\widetilde{L}_+^{q(\cdot),\frac{\theta q_-}{p_-},\widetilde{\ell}_+}(I)$;*

(ii) *$\mathcal{R}^\alpha$ is bounded from $\widetilde{L}_-^{p(\cdot),\theta,\widetilde{\ell}_-}(I)$ to $\widetilde{L}_-^{q(\cdot),\frac{\theta q_-}{p_-},\widetilde{\ell}_-}(I)$, where $\widetilde{\ell}_+(I)$ (resp. $\widetilde{\ell}_-(I)$) is defined in Proposition* 5.3.

Proof. (i). Observe that Propositions 3.1 and 5.1 yield that inequality

$$\|\mathcal{W}^\alpha f\|_{L^{q(\cdot)-\varepsilon(\cdot)}(I)} \le c_{p,\alpha,\eta} \|g\|_{L^{p(\cdot)-\eta(\cdot)}(I)}$$

for all $f \in \widetilde{L}^{p(\cdot),\theta,\ell_+}(I)$, $\varepsilon(\cdot)$ and $\eta(\cdot)$ satisfying the conditions (a)–(c) of Proposition 5.3, where σ is a sufficiently small positive number and $\sup_{0<\eta_- \le \eta_+ \le \sigma} c_{p,\alpha,\eta} < \infty$. Now Proposition 5.3 completes the proof.

Part (ii) follows similarly. □

Theorem 5.5. *Let $p \in P(I)$ and let $\theta > 0$. Then*

(i) *$\mathcal{M}_+$ is bounded in $\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)$;*

(ii) *$\mathcal{M}_-$ is bounded in $\widetilde{L}_-^{p(\cdot),\theta,\ell_-}(I)$.*

Proof. (i). This statement follows in the same way as Theorem 4.6 taking into account the bounds of $\mathcal{M}_-$ and $\mathcal{M}_+$ in $L^{p(\cdot)}(I)$. We only need to notice that if $f \in L^{p(\cdot)-\varepsilon(\cdot)}$ with $p(\cdot)-\varepsilon(\cdot) \in \widetilde{\mathcal{P}}_+^{\ell_+}(I)$, then $\mathcal{M}_+ \in L^{p(\cdot)-\varepsilon(\cdot)}$ with $p(\cdot)-\eta(\cdot) \in \widetilde{\mathcal{P}}_+^{\ell_+}(I)$.

Part (ii) follows similarly. □

Theorem 5.6. *Let $I := (0, a)$ be a bounded interval and let $\theta > 0$. Suppose that $p \in P(I)$. Then*

(i) *For the Calderón–Zygmund operator K with kernel supported on $(-2a, 0)$, there is a positive constant c such that for all bounded f defined on I the inequality*

$$\|K^* f\|_{\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)} \le c\|f\|_{\widetilde{L}_+^{p(\cdot),\theta,\ell_+}(I)};$$

holds;

(ii) *For the Calderón–Zygmund operator K with kernel supported on* $(0, 2a)$*, there is a positive constant* c *such that for all bounded* f *defined on* I *the inequality*

$$\|K^* f\|_{\widetilde{L}_{-}^{p(\cdot),\theta,\ell_-}(I)} \le c\|f\|_{\widetilde{L}_{-}^{p(\cdot),\theta,\ell_-}(I)}$$

holds.

Proof. This statement can be obtained in the same way as Theorem 5.4 was proved by using Propositions 5.3, 3.1, and Theorem 4.4. Details are omitted. □

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