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# Averaging of Nonclassical Diffusion Equations with Memory and Singularly Oscillating Forces

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**Abstract.** We consider for $\rho \in [0,1)$ and $\varepsilon > 0$, the following nonclassical diffusion equation with memory and singularly oscillating external force

$$u_t - \Delta u_t - \Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)ds + f(u) = g_0(t) + \varepsilon^{-\rho} g_1\left(\frac{t}{\varepsilon}\right),$$

together with the averaged equation

$$u_t - \Delta u_t - \Delta u - \int_0^\infty \kappa(s)\Delta u(t-s)ds + f(u) = g_0(t)$$

formally corresponding to the limiting case $\varepsilon = 0$. Under suitable assumptions on the nonlinearity and on the external force, we prove the uniform (w.r.t. $\varepsilon$) boundedness as well as the convergence of the uniform attractor $\mathcal{A}^\varepsilon$ of the first equation to the uniform attractor $\mathcal{A}^0$ of the second equation as $\varepsilon \to 0^+$.

**Keywords.** Nonclassical diffusion equation, uniform attractor, memory, singularly oscillating force, boundedeness, convergence

**Mathematics Subject Classification (2010).** Primary 35B41, secondary 45K05, 76R50, 35D30

## 1. Introduction

Let $\rho \in [0,1)$ be a fixed parameter, and let $\Omega$ be a bounded domain in $\mathbb{R}^N (N \geq 3)$ with smooth boundary $\partial\Omega$. For every $\varepsilon \in (0,1]$ and any given $\tau \in \mathbb{R}$, we consider

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for $t > \tau$ the following semilinear nonclassical diffusion equation with memory and a singularly oscillating external force,

$$\begin{cases} u_t - \Delta u_t - \Delta u - \displaystyle\int_0^\infty \kappa(s)\Delta u(t-s)ds + f(u) = g^\varepsilon(t), & x \in \Omega,\ t > \tau \\ u(x,t) = 0, & x \in \partial\Omega,\ t > \tau \\ u(x,\tau) = u_\tau(x), & x \in \Omega \\ u(x,\tau - s) = q_\tau(x,s), & x \in \Omega,\ s > 0 \end{cases} \tag{1}$$

where

$$g^\varepsilon(t) = g_0(t) + \varepsilon^{-\rho} g_1\left(\frac{t}{\varepsilon}\right).$$

Along with (1), we consider the equation

$$\begin{cases} u_t - \Delta u_t - \Delta u - \displaystyle\int_0^\infty \kappa(s)\Delta u(t-s)ds + f(u) = g_0(t), & x \in \Omega,\ t > \tau \\ u(x,t) = 0, & x \in \partial\Omega,\ t > \tau \\ u(x,\tau) = u_\tau(x), & x \in \Omega \\ u(x,\tau - s) = q_\tau(x,s), & x \in \Omega,\ s > 0 \end{cases} \tag{2}$$

without rapid and singular oscillations, which formally corresponds to the case $\varepsilon = 0$ in (1). The speed of energy dissipation for equations (1) and (2) is faster than for the usual nonclassical diffusion equation. The conduction of energy is not only affected by present external forces but also by historic external forces.

In recent years, the existence and long-time behavior of solutions to nonclassical diffusion equations with memory has been addressed by a number of authors (see [4, 10, 15, 16, 28–30]). In the most of existing papers dealing with the memory relaxation, the function $\mu(s) := -\kappa'(s)$ is assumed to satisfy the inequality

$$\mu'(s) + \delta\mu(s) \le 0,$$

which was introduced in the seminal paper [17] and commonly adopted thereafter, and the nonlinearity is assumed to be locally Lipschitz continuous and satisfy a Sobolev growth condition

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \quad |f'(u)| \le C(1 + |u|^{\frac{4}{N-2}}),$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta_D$ in $\Omega$ with the homogeneous Dirichlet boundary condition.

In the case $\kappa \equiv 0$, we obtain the so-called nonclassical diffusion equation

$$u_t - \Delta u_t - \Delta u + f(u) = g_0(t) + \varepsilon^{-\rho} g_1\left(\frac{t}{\varepsilon}\right). \tag{3}$$

The nonclassical diffusion equation arises as a model to describe physical phenomena, such as non-Newtonian flows, soil mechanics and heat conduction theory (see, e.g., [1, 21, 26]). In the past years, the existence and long-time behavior of solutions to nonclassical diffusion equations has been studied extensively, for both autonomous case [18, 23, 24, 27, 31, 32, 34] and non-autonomous case [2,3,5,6,24,30], and even in the case with delays [8,9,25,35]. In [6], the authors proved the uniform boundedness and the upper semicontinuity of uniform attractors for equation (3) with the nonlinearity of Sobolev type and singularly oscillating external forces. We also refer the reader to [11–13, 19, 22, 33] for some other results for partial differential equations with singularly oscillating external forces.

To study problem (1), we assume that the initial datum $u_\tau \in H_0^1(\Omega)$ is given, the nonlinearity $f$ and the external force $g$ satisfy the following conditions:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that

$$f'(u) \geq -\alpha, \tag{4}$$

$$|f'(u)| \leq C\left(1 + |u|^{p-1}\right), \quad 1 \leq p \leq \frac{N+2}{N-2}. \tag{5}$$

We need some dissipation conditions. For $p > 1$, we assume that

$$f(u)u \geq d_0|u|^{p+1} - C_0, \tag{6}$$

while if $p = 1$ (the case of linear growth), in place of (6), we require a weaker condition

$$f(u)u \geq -\beta u^2 - C_1, \quad 0 < \beta < \lambda_1. \tag{7}$$

Here $\alpha, \beta, d_0, C, C_0, C_1$ are positive constants, and $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ in $\Omega$ with the homogeneous Dirichlet boundary condition.

A typical example of such a nonlinearity is

$$f(u) = k|u|^{p-1}u \quad \left(k > 0,\ 1 \leq p \leq \frac{N+2}{N-2}\right), \quad \text{or} \quad f(u) = k \sin u.$$

(H2) The external forces $g_0, g_1 \in L_b^2(\mathbb{R}; L^2(\Omega))$, the space of translation bounded functions in $L_{loc}^2(\mathbb{R}; L^2(\Omega))$, that is,

$$\|g_0\|_{L_b^2}^2 := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_0(y)\|^2 dy = M_0,$$

$$\|g_1\|_{L_b^2}^2 := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_1(y)\|^2 dy = M_1, \tag{8}$$

for some $M_0, M_1 \geq 0$. A straightforward consequence of (8) is

$$\int_t^{t+1} \left\| g_1\left(\frac{y}{\varepsilon}\right) \right\|^2 dy = \varepsilon \int_{\frac{t}{\varepsilon}}^{\frac{t+1}{\varepsilon}} \|g_1(y)\|^2 dy \leq \varepsilon \left(1 + \frac{1}{\varepsilon}\right) M_1 \leq 2M_1,$$

so that

$$\|g_1(\tfrac{\cdot}{\varepsilon})\|_{L_b^2}^2 \leq 2M_1, \quad \forall \varepsilon \in (0,1].$$

Hence

$$\|g^\varepsilon\|_{L_b^2}^2 \leq 2\|g_0\|_{L_b^2}^2 + 2\varepsilon^{-2\rho}\|g_1(\tfrac{\cdot}{\varepsilon})\|_{L_b^2}^2 \leq 2M_0 + 4M_1\varepsilon^{-2\rho}.$$

(H3) The memory kernel $\kappa$ is a nonnegative summable function of total mass $\int_0^\infty \kappa(s)ds = \kappa_0$ having the explicit form

$$\kappa(s) = \int_s^\infty \mu(y)dy,$$

where $\mu \in L^1(\mathbb{R})$ is a nonincreasing (hence nonnegative) piecewise absolutely continuous function allowed to exhibit (infinitely many) jumps. Moreover, we assume the existence of $\delta > 0$ such that

$$\mu'(s) + \delta\mu(s) \leq 0, \tag{9}$$

for almost everywhere $s \geq 0$.
It is noticed that the above condition of the memory term is slightly weaker than the usual condition in [10, 17, 28–30] in the sense that $\mu$ can be weakly singular at the origin. For instance, we can take

$$\mu(s) = ke^{-as}s^{1-b}$$

with $k \geq 0$ and $a > 0, b > 1$.

The paper is organized as follows. In Section 2, for convenience of the reader, we recall some preliminary results which will be used later. Section 3 is devoted to proving the uniform boundedness of the uniform attractors $\mathcal{A}_\epsilon$ with respect to $\varepsilon$. The convergence of the uniform attractors $\mathcal{A}_\epsilon$ as $\epsilon \to 0$ is investigated in the last section.

**Remark 1.1.** It is noticed that all results obtained in the present paper are still true in the cases $N = 1, 2$, with a simpler proof, and we do not need any restriction on the growth exponent $p \geq 1$ of the nonlinearity in (5) due to the Sobolev embbeding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \geq 1$ in these cases. However, for the coherence of the presentation, in what follows we only deal with the case $N \geq 3$.

## 2. Notations and preliminaries

In this section, we recall some notations about function spaces and preliminary results.

As in [7, 17], a new variable which reflects the past history of equation (1) is introduced, that is to be,

$$\eta^t(x,s) = \eta(x,t,s) = \int_0^s u(x,t-r)dr, \quad s \geq 0,$$

then we can check that

$$\partial_t \eta^t(x,s) = u(x,t) - \partial_s \eta^t(x,s), \quad s \geq 0.$$

Since $\mu(s) = -\kappa'(s)$, the first equation of (1) can be transformed into the following system

$$\begin{cases} u_t - \Delta u_t - \Delta u - \displaystyle\int_0^\infty \mu(s)\Delta\eta^t(s)ds + f(u) = g_0(t) + \varepsilon^{-\rho} g_1\left(\frac{t}{\varepsilon}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad \eta_t^t = -\eta_s^t + u. \end{cases}$$

The associated initial-boundary conditions are

$$\begin{cases} u(x,t) = 0, & x \in \partial\Omega,\ t > \tau, \\ \eta^t(x,s) = 0, & (x,s) \in \partial\Omega \times \mathbb{R}^+,\ t > \tau, \\ u(x,\tau) = u_\tau, & x \in \Omega, \\ \eta^\tau(x,s) = \eta_\tau(x,s) := \displaystyle\int_0^s q_\tau(x,\tau - r)dr, & (x,s) \in \Omega \times \mathbb{R}^+. \end{cases}$$

Denoting

$$z(t) = (u(t), \eta^t), \quad z_\tau = (u_\tau, \eta^\tau).$$

Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the $L^2(\Omega)$-inner product and $L^2(\Omega)$-norm, respectively. In view of (9), let $L^2_\mu(\mathbb{R}^+; H^1_0(\Omega))$ be the Hilbert space of functions $\varphi\colon \mathbb{R}^+ \to L^2(\Omega)$ endowed with the inner product

$$\langle \varphi_1, \varphi_2 \rangle_{1,\mu} = \int_0^\infty \mu(s) \langle \nabla\varphi_1(s), \nabla\varphi_2(s) \rangle \, ds,$$

and let $\|\varphi\|_{1,\mu}$ denote the corresponding norm. We introduce the Hilbert space

$$\mathcal{H}_1 = H^1_0(\Omega) \times L^2_\mu(\mathbb{R}^+; H^1_0(\Omega)),$$

which is endowed with the inner product

$$\langle w_1, w_2 \rangle_{\mathcal{H}_1} = \langle \nabla\psi_1, \nabla\psi_2 \rangle + \langle \varphi_1, \varphi_2 \rangle_{1,\mu},$$

where $w_i = (\psi_i, \varphi_i) \in \mathcal{H}_1$ for $i = 1, 2$. The norm induced on $\mathcal{H}_1$ is

$$\|(\psi, \varphi)\|^2_{\mathcal{H}_1} = \|\psi\|^2_{H^1_0(\Omega)} + \int_0^\infty \mu(s)\|\nabla\varphi(s)\|^2 ds.$$

Integrating by parts and using (H3), we have

$$\langle \eta^t, \eta^t_s \rangle_{1,\mu} = \frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\|\nabla\eta^t\|^2 ds = -\frac{1}{2}\int_0^\infty \mu'(s)\|\nabla\eta^t\|^2 ds \geq \frac{\delta}{2}\|\eta^t\|^2_{1,\mu}, \quad (10)$$

for any $\eta^t \in C([\tau, T]; L^2_\mu(\mathbb{R}^+, H^1_0(\Omega)))$.

If $g$ is translation bounded in $L^2_{loc}(\mathbb{R}; L^2(\Omega))$, we denote by $\mathcal{H}_w(g)$ the closure of the set $\{g(\cdot + h)|\, h \in \Omega\}$ in $L^2_{loc}(\mathbb{R}; L^2(\Omega))$ with the weak topology. Under the assumptions (H1)–(H3) above, the following result was proved in [30].

**Theorem 2.1.** *Assume that conditions* (H1)–(H3) *hold. Then for any fixed positive number* $\varepsilon$*, the family of processes* $\{U^\varepsilon_\sigma(t, \tau)\}_{\sigma\in\mathcal{H}_w(g^\varepsilon)}$ *generated by problem* (1) *possesses a uniform attractor* $\mathcal{A}^\varepsilon$ *in the space* $\mathcal{H}_1$*. Moreover,*

$$\mathcal{A}^\varepsilon = \bigcup_{\sigma\in\mathcal{H}_w(g^\varepsilon)} \mathcal{K}^\varepsilon_\sigma(s), \quad \forall s \in \mathbb{R}, \quad (11)$$

*where* $\mathcal{K}^\varepsilon_\sigma(s)$ *is the kernel section at time* $s$ *of the process* $U^\varepsilon_\sigma$*.*

In this paper, we will prove the following facts concerning the family of uniform attractors $\{\mathcal{A}^\varepsilon\}_{\varepsilon\in[0,1]}$ of the processes generated by (1) and (2):

(i) The family $\mathcal{A}^\varepsilon$ is uniformly (w.r.t. $\varepsilon$) bounded in $\mathcal{H}_1$:

$$\sup_{\varepsilon\in(0,1]} \|\mathcal{A}^\varepsilon\|_{\mathcal{H}_1} < \infty;$$

(ii) The uniform attractor $\mathcal{A}^\varepsilon$ converges to $\mathcal{A}^0$ as $\varepsilon \to 0^+$ in the standard Hausdorff semi-distance in $\mathcal{H}_1$:

$$\lim_{\varepsilon\to 0^+} \{\text{dist}_{\mathcal{H}_1}(\mathcal{A}^\varepsilon, \mathcal{A}^0)\} = 0.$$

## 3. Uniform boundedness of the uniform attractors

We now give a sufficient condition to ensure that the family $\mathcal{A}_\varepsilon$ is bounded in $\mathcal{H}_1$ uniformly with respect to $\varepsilon \in (0, 1]$. Such a condition only involves the function $g_1$, which introduces singular oscillations in the external force. To this end, setting $G(t, \tau) = \int_\tau^t g_1(s)ds, t \geq \tau$, we assume that

$$\sup_{t\geq\tau,\tau\in\mathbb{R}} \left( \|G(s, \tau)\|^2_{H^{-1}} + \int_t^{t+1} \|G(s, \tau)\|^2 ds \right) \leq \ell^2. \quad (12)$$

**Remark 3.1.** Condition (12) takes place, for instance, when

$$g_1 \in L^\infty(\mathbb{R}; H^{-1}(\Omega)) \cap L^2_{loc}(\mathbb{R}; L^2(\Omega))$$

is a time periodic function of period $T > 0$ with zero mean, that is,

$$\int_0^T g_1(s)ds = 0.$$

Other examples of quasiperiodic and almost periodic in time functions satisfying (12) can be found in [14].

**Proposition 3.2.** *Assume that $g_1 \in L^2_b(\mathbb{R}; L^2(\Omega))$ satisfies (12). Then, the solution $(v, \eta_1^t)$ to the problem*

$$\begin{cases} v_t - \Delta v_t - \Delta v - \displaystyle\int_0^\infty \mu(s)\Delta\eta_1^t(s)ds = g_1\left(\frac{t}{\varepsilon}\right), \\ \partial_t\eta_1^t = -\partial_s\eta_1^t + v, \\ v|_{\partial\Omega} = 0, \ \eta_1^t|_{\partial\Omega} = 0, \\ (v(\tau), \eta_1^\tau) = (0, 0), \end{cases} \tag{13}$$

*with $\varepsilon \in (0, 1]$, satisfies the inequality*

$$\|v(t)\|^2_{H^1_0(\Omega)} + \|\eta_1^t\|^2_{1,\mu} \le C\ell^2\varepsilon^2, \quad \forall t \ge \tau, \tag{14}$$

*where $C$ is a constant independent of $g_1$.*

*Proof.* Without loss of generality, we may assume $\tau = 0$. Denoting

$$V(t) = \int_0^t v(y)dy \quad \text{and} \quad \overline{\eta_1}^t = \int_0^t \eta_1^y(s)dy.$$

Integrating (13) in time from 0 to $t$, we see that the function $V(t)$ solves the problem

$$V_t - \Delta V_t - \Delta V - \int_0^\infty \mu(s)\Delta\overline{\eta_1}^t(s)ds = G_\varepsilon(t), \quad V|_{\partial\Omega} = 0, \quad V|_{t=0} = 0, \tag{15}$$

where

$$G_\varepsilon(t) = \int_0^t g_1\left(\frac{s}{\varepsilon}\right)ds = \varepsilon\int_0^{\frac{t}{\varepsilon}} g_1(s)ds = \varepsilon G\left(\frac{t}{\varepsilon}, 0\right).$$

It follows from (12) that

$$\sup_{t\ge 0}\|G_\varepsilon(t)\|_{H^{-1}} \le \ell\varepsilon, \tag{16}$$

and

$$\sup_{t\geq 0}\int_t^{t+1}\|G_\varepsilon(s)\|^2ds \leq 2\ell^2\varepsilon^2.$$

Indeed, (16) is straightforward, whereas

$$\int_t^{t+1}\|G_\varepsilon(s)\|^2ds=\varepsilon^3\int_{\frac{t}{\varepsilon}}^{\frac{t+1}{\varepsilon}}\|G(s,0)\|^2ds\leq\varepsilon^3\left(1+\frac{1}{\varepsilon}\right)\sup_{t\geq 0}\left(\int_t^{t+1}\|G(s,0)\|^2ds\right)\leq 2\ell^2\varepsilon^2.$$

Multiplying (15) by $V$ in $L^2(\Omega)$, then using Young's inequality and (10), we get

$$\frac{d}{dt}y(t)+\alpha_1 y(t)\leq\frac{2}{\lambda_1}\|G_\varepsilon(t)\|^2,$$

where $y(t)=\|V\|^2+\|\nabla V\|^2+\|\overline{\eta_1}^t\|^2_{1,\mu},\ 0<\alpha_1<\min\{1,\delta,\frac{\lambda_1}{2}\}$. Hence by the Gronwall inequality, we deduce that

$$\|V\|^2+\|\nabla V\|^2+\|\overline{\eta_1}^t\|^2_{1,\mu}\leq C\int_0^t e^{-\alpha_1(t-s)}\|G_\varepsilon(s)\|^2ds,$$

where we have used the fact that $V(0)=0$ and $\overline{\eta_1}^0=0$. Since

$$\begin{aligned}
&\int_0^t e^{-\alpha_1(t-s)}\|G_\varepsilon(s)\|^2ds\\
&=\int_{t-1}^t e^{-\alpha_1(t-s)}\|G_\varepsilon(s)\|^2ds+\int_{t-2}^{t-1}e^{-\alpha_1(t-s)}\|G_\varepsilon(s)\|^2ds+\cdots\\
&\leq\int_{t-1}^t\|G_\varepsilon(s)\|^2ds+e^{-\alpha_1}\int_{t-2}^{t-1}\|G_\varepsilon(s)\|^2ds+\cdots\\
&\leq\frac{1}{1-e^{-\alpha_1}}\sup_{t\geq 0}\int_t^{t+1}\|G_\varepsilon(s)\|^2ds\\
&\leq C\ell^2\varepsilon^2,
\end{aligned}\tag{17}$$

so

$$\|V\|^2+\|\nabla V\|^2+\|\overline{\eta_1}^t\|^2_{1,\mu}\leq C\ell^2\varepsilon^2.\tag{18}$$

On the other hand, multiplying (15) by $V_t$, we obtain

$$\|V_t\|^2+\|\nabla V_t\|^2\leq|\langle G_\varepsilon(t),V_t\rangle_{H^{-1},H_0^1}|+|\langle\nabla V,\nabla V_t\rangle|+\left|\int_0^\infty\mu(s)\langle\nabla\overline{\eta_1}^t,\nabla V_t\rangle ds\right|.$$

Applying the Hölder and Cauchy inequalities, we have

$$\begin{aligned}
\|V_t\|^2+\|\nabla V_t\|^2\leq{}& C(\varepsilon_0)\|G_\varepsilon(t)\|^2_{H^{-1}}+\varepsilon_0\|V_t\|^2_{H_0^1}+\varepsilon_0\|\nabla V_t\|^2+C(\varepsilon_0)\|\nabla V\|^2\\
&+C(\varepsilon_0)\|\overline{\eta_1}^t\|^2_{1,\mu}+\varepsilon_0\int_0^\infty\mu(s)ds\|\nabla V_t\|^2.
\end{aligned}$$

Choosing $\varepsilon_0$ small enough, then using (16), (18), and noting that $\mu \in L^1(\mathbb{R}^+)$, we deduce that

$$\|V_t\|^2 + \|\nabla V_t\|^2 \leq C\ell^2\varepsilon^2, \quad \text{i.e.,} \quad \|v\|^2 + \|\nabla v\|^2 \leq C\ell^2\varepsilon^2. \tag{19}$$

Multiplying the second equation in (13) by $\eta_1^t$ in $L^2_\mu(\mathbb{R}^+; H_0^1(\Omega))$, then using (10) and the Cauchy inequality, we get

$$\frac{d}{dt}\|\eta_1^t\|^2_{1,\mu} + \frac{\delta}{2}\|\eta_1^t\|^2_{1,\mu} \leq \frac{2}{\delta}\|v\|^2_{H_0^1(\Omega)} \leq C\ell^2\varepsilon^2.$$

Hence using the Gronwall inequality we obtain

$$\|\eta_1^t\|^2_{1,\mu} \leq C\ell^2\varepsilon^2. \tag{20}$$

Combining (19) and (20), we get (14) as desired. □

**Theorem 3.3.** *Let conditions* (H1)–(H3) *and* (12) *hold. Then the uniform attractors $\mathcal{A}^\varepsilon$ are uniformly (w.r.t. $\varepsilon$) bounded in $\mathcal{H}_1$, that is,*

$$\sup_{\varepsilon \in (0,1]} \|\mathcal{A}^\varepsilon\|_{\mathcal{H}_1} < \infty.$$

*Proof.* Let $z(t) = (u, \eta^t)$ be the solution to (1) with the initial datum $z_\tau \in \mathcal{H}_1$. For $\varepsilon > 0$, we consider the problem

$$\begin{cases} v_t - \Delta v_t - \Delta v - \displaystyle\int_0^\infty \mu(s)\Delta\eta_1^t(s)ds = \varepsilon^{-\rho} g_1\left(\frac{t}{\varepsilon}\right), \\ \partial_t \eta_1^t = -\partial_s \eta_1^t + v, \\ v|_{\partial\Omega} = 0, \ \eta_1^t|_{\partial\Omega} = 0, \\ (v(\tau), \eta_1^\tau) = (0, 0). \end{cases} \tag{21}$$

Proposition 3.2 provides the estimate

$$\|v(t)\|^2_{H_0^1(\Omega)} + \|\eta_1^t\|^2_{1,\mu} \leq C\ell^2\varepsilon^{2(1-\rho)}, \quad \forall t \geq \tau. \tag{22}$$

Then, the function $(w(t), \eta_2^t) = z(t) - -(v(t), \eta_1^t)$ clearly satisfies the problem

$$\begin{cases} w_t - \Delta w_t - \Delta w - \displaystyle\int_0^\infty \mu(s)\Delta\eta_2^t(s)ds + f(w) = -(f(u) - f(w)) + g_0(t), \\ \partial_t \eta_2^t = -\partial_s \eta_2^t + w, \\ w|_{\partial\Omega} = 0, \ \eta_2^t|_{\partial\Omega} = 0, \\ w|_{t=\tau} = u_\tau, \ \eta_2^\tau = \eta_\tau. \end{cases}$$

Multiplying the first equation by $w$, then using (10) and the second equation, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|_{1,\mu}^2\right)+\|\nabla w\|^2+\frac{\delta}{2}\|\eta_2^t\|_{1,\mu}^2+\int_\Omega f(w)wdx$$
$$=-\int_\Omega (f(u)-f(w))\,wdx+\int_\Omega g_0(t)wdx.$$

We consider two cases:

**Case 1:** $p>1$**.** Using the dissipation condition (6) and the Cauchy inequality, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|_{1,\mu}^2\right)+\|\nabla w\|^2+\frac{\delta}{2}\|\eta_2^t\|_{1,\mu}^2+d_0\|w\|_{L^{p+1}}^{p+1}-C$$
$$\le \varepsilon_0\|w\|^2+\int_\Omega |f(u)-f(w)||w|dx+C(\varepsilon_0)\|g_0(t)\|^2.$$

We estimate the second term on the right-hand side as follows

$$\begin{aligned}
&\int_\Omega |f(u)-f(w)||w|dx\\
&\le C\int_\Omega (1+|w|^{p-1}+|v|^{p-1})|v||w|dx\\
&\le C\|w\|\|v\|+C\|w\|_{L^{p+1}}^p\|v\|_{L^{p+1}}+C\|v\|_{L^{p+1}}^p\|w\|_{L^{p+1}}\\
&\le \varepsilon_0\|w\|^2+C(\varepsilon_0)\|v\|^2+\varepsilon_0\|w\|_{L^{p+1}}^{p+1}+C(\varepsilon_0)\|v\|_{L^{p+1}}^{p+1}\\
&\le \varepsilon_0\|w\|^2+\varepsilon_0\|w\|_{L^{p+1}}^{p+1}+C(\varepsilon_0)\left(\|v\|^2+\|v\|_{H_0^1}^{p+1}\right),
\end{aligned}$$

where we have used the Hölder and Young inequalities, and the embeddding $H_0^1(\Omega)\hookrightarrow L^{p+1}(\Omega)$ due to the condition $p\le\frac{N+2}{N-2}$. Therefore,

$$\begin{aligned}
&\frac{d}{dt}(\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|_{1,\mu}^2)+\|\nabla w\|^2+(\lambda_1-2\varepsilon_0)\|w\|^2+\delta\|\eta_2^t\|_{1,\mu}^2\\
&+2(d_0-\varepsilon_0)\|w\|_{L^{p+1}}^{p+1}\\
&\le C\left(1+\|g_0(t)\|^2+\|v\|_{H_0^1}^2+\|v\|_{H_0^1}^{p+1}\right).
\end{aligned}\tag{23}$$

**Case 2:** $p=1$**.** Using the dissipation condition (7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|_{1,\mu}^2\right)+\|\nabla w\|^2+\frac{\delta}{2}\|\eta_2^t\|_{1,\mu}^2\\
&\le \beta\|w\|^2+C_1|\Omega|+\varepsilon_0\|w\|^2+C\int_\Omega |v||w|dx+C(\varepsilon_0)\|g_0(t)\|^2\\
&\le (\beta+2\varepsilon_0)\|w\|^2+C_1|\Omega|+C(\varepsilon_0)\|v\|^2+C(\varepsilon_0)\|g_0(t)\|^2,
\end{aligned}$$

where we have used the fact that $p-1=0$ in the second line of the above estimate of $\int_\Omega |f(u)-f(w)||w|dx$ and the Hölder inequality. Hence

$$\begin{aligned}
&\frac{d}{dt}\left(\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|^2_{1,\mu}\right)\\
&+2\varepsilon_0\|\nabla w\|^2+2(\lambda_1(1-\varepsilon_0)-\beta-2\varepsilon_0)\|w\|^2+\delta\|\eta_2^t\|^2_{1,\mu}\\
&\le C\left(1+\|g_0(t)\|^2+\|v\|^2_{H_0^1}\right).
\end{aligned}\tag{24}$$

From (23) and (24), by choosing $\varepsilon_0$ small enough and using (22), in both cases we have for some $\alpha_2>0$ and for all $t\ge\tau$,

$$\frac{d}{dt}y(t)+\alpha_2 y(t)\le\ C\left(1+\|g_0(t)\|^2+\ell^2\varepsilon^{2(1-\rho)}+\ell^{p+1}\varepsilon^{(p+1)(1-\rho)}\right),$$

where $y(t)=\|w\|^2+\|\nabla w\|^2+\|\eta_2^t\|^2_{1,\mu}$. Hence, by the Gronwall inequality, we obtain

$$y(t)\le Ce^{-\alpha_2(t-\tau)}y(\tau)+C\left(1+M_0^2+\ell^2\varepsilon^{2(1-\rho)}+\ell^{p+1}\varepsilon^{(p+1)(1-\rho)}\right),$$

where we have used the fact that (see (17) for a similar proof)

$$\int_\tau^t e^{-\alpha_2(t-s)}\|g_0(s)\|^2ds\le\frac{1}{1-e^{-\alpha_2}}\|g_0\|^2_{L_b^2}.$$

Recalling that $z(t)=(w,\eta_1^t)+(v,\eta_2^t)$ and using (22) once again, we have for all $t\ge\tau$,

$$\|z(t)\|^2\le Ce^{-\alpha_2(t-\tau)}\|z_\tau\|^2+C\left(1+M_0^2+\ell^2\varepsilon^{2(1-\rho)}+\ell^{p+1}\varepsilon^{(p+1)(1-\rho)}\right).\tag{25}$$

Hence, the processes $\{U_\varepsilon(t,\tau)\}$ have a bounded absorbing set $B^*$, which is independent of $\varepsilon$ (because $\rho<1$). Since $\mathcal{A}^\varepsilon\subset B^*$, the proof is complete. □

## 4. Convergence of the uniform attractors

The main result of this section is to establish the upper semicontinuity of the uniform attractors $\mathcal{A}^\varepsilon$ at $\varepsilon=0$.

**Theorem 4.1.** *Let* (H1)–(H3) *and* (12) *hold. Then, for every $\rho\in[0,1)$, the uniform attractor $\mathcal{A}^\varepsilon$ converges to $\mathcal{A}^0$ with respect to the Hausdorff semidistance in $\mathcal{H}_1$ as $\varepsilon\to 0^+$, i.e.,*

$$\lim_{\varepsilon\to0^+}\{\mathrm{dist}_{\mathcal{H}_1}(\mathcal{A}^\varepsilon,\mathcal{A}^0)\}=0.$$

In order to prove this theorem, we make a comparison between some particular solutions to (1) corresponding to $\varepsilon > 0$ and $\varepsilon = 0$, respectively, starting from the same initial data. We denote

$$u^\varepsilon(t) = U_\varepsilon(t,\tau)u_\tau,$$

with $u_\tau$ belonging to the absorbing set $B^*$ found in the previous section. From (25), we have the uniform bound:

$$\|u^\varepsilon(t)\|^2_{H^1_0} \leq R_1^2 \quad \text{for some } R_1 > 0. \tag{26}$$

In particular, for $\varepsilon = 0$, since $u_\tau \in B^*$, we get

$$\|u^0(t)\|^2_{H^1_0} \leq R_0^2, \tag{27}$$

for some $R_0 > 0$.

On the other hand, to prove the convergence of the uniform attractors, we actually need consider whole family of equations

$$\widehat{u}_t - \Delta\widehat{u}_t - \Delta\widehat{u} + f(\widehat{u}) - \int_0^\infty \mu(s)\Delta\widehat{\eta}^t(s)ds = \widehat{g}^\varepsilon(t), \tag{28}$$

with the external force $\widehat{g} = \widehat{g}^\varepsilon \in \mathcal{H}_w(g^\varepsilon)$. To this end, we observe that every function $\widehat{g}_1 \in \mathcal{H}_w(g_1)$ fulfills the inequality (12).

For any $\varepsilon \in [0,1]$, we denote

$$\widehat{u^\varepsilon}(t) = U_{\widehat{g}^\varepsilon}(t,\tau)u_\tau,$$

where $u_\tau$ belongs to the absorbing set $B^*$ found in the previous section. Then

$$\widehat{z}^\varepsilon(t) = (\widehat{u}^\varepsilon(t), \widehat{\eta}^t_\varepsilon) = U_{\widehat{g}^\varepsilon}(t,\tau)\widehat{z}_\tau,$$

is the solution to (28) with the external force $\widehat{g}^\varepsilon = \widehat{g}_0 + \varepsilon^{-\rho}\widehat{g}_1(\frac{\cdot}{\varepsilon}) \in \mathcal{H}_w(g^\varepsilon)$. Due to Theorem 3.3, along with the estimate of Theorem 2.1 to handle the case $\varepsilon = 0$, we have the uniform bound

$$\sup_{\varepsilon\in[0,1]} \|\widehat{z}^\varepsilon(t)\|_{\mathcal{H}_1} \leq C, \quad \forall t \geq \tau.$$

Next, we define the deviation

$$\overline{z}(t) = \widehat{z}^\varepsilon(t) - \widehat{z}^0(t) = (r(t), \zeta^t).$$

**Lemma 4.2.** *For every $\varepsilon \in (0,1]$, we have the estimate*

$$\|\overline{z}(t)\|^2 \leq \left(C\ell^2\varepsilon^{2(1-\rho)} + R_3\ell\varepsilon^{1-\rho}\right)e^{C(t-\tau)}, \quad \forall t \geq \tau,$$

*for some positive constant $C$ independent of $\varepsilon, \tau, \widehat{g}^\varepsilon$.*

*Proof.* Let $(v(t), \eta_1^t)$ be the solution to the auxiliary problem (21) with null initial datum $(v_\tau, \eta_1^\tau) = (0,0)$.

The difference $(w(t), \eta_2^t) = \overline{z}(t) - (v(t), \eta_1^t)$ clearly satisfies the equations

$$w_t - \Delta w_t - \Delta w - \int_0^\infty \mu(s)\Delta\eta_2^t(s)ds + f(u^\varepsilon) - f(u^0) = 0,$$
$$\partial_t \eta_2^t = -\partial_s \eta_2^t + w,$$

with initial condition $(w(\tau), \eta_2^\tau) = (0,0)$. Taking the scalar product the first equation by $w$, we obtain

$$\frac{d}{dt}\left(\|w\|^2 + \|\nabla w\|^2 + \|\eta_2^t\|_{1,\mu}^2\right) + 2\|\nabla w\|^2 + \delta\|\eta_2^t\|_{1,\mu}^2 + 2(f(u^\varepsilon) - f(u^0), w + v)$$
$$\leq 2\left|(f(u^\varepsilon) - f(u^0), v)\right|,$$

thus

$$\frac{d}{dt}\left(\|w\|^2 + \|\nabla w\|^2 + \|\eta_2^t\|_{1,\mu}^2\right) + 2\|\nabla w\|^2 + \delta\|\eta_2^t\|_{1,\mu}^2 + 2\int_\Omega f'(\xi)(w+v)^2 dx$$
$$\leq 2\int_\Omega (|f(u^\varepsilon)| + |f(u^0)|)|v|dx.$$

Exploiting conditions (4) and (5), we readily obtain

$$\frac{d}{dt}\left(\|w\|^2 + \|\nabla w\|^2 + \|\eta_2^t\|_{1,\mu}^2\right) \leq 2\alpha\|w+v\|^2 + C\int_\Omega (1 + |u^\varepsilon|^p + |u^0|^p)|v|dx,$$

where we have used the fact that $|f(u)| \leq C(1 + |u|^p)$. Using the Hölder inequality, we get

$$\frac{d}{dt}\left(\|w\|^2 + \|\nabla w\|^2 + \|\eta_2^t\|_{1,\mu}^2\right)$$
$$\leq C\alpha\|w\|^2 + C\alpha\|v\|^2 + C\|v\|^2 + \left(\|u^\varepsilon\|_{L^{p+1}}^p + \|u^0\|_{L^{p+1}}^p\right)\|v\|_{L^{p+1}}.$$

Exploiting the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, (22), (26) and (27), we have

$$\frac{d}{dt}y(t) \leq Cy(t) + C\ell^2\varepsilon^{2(1-\rho)} + R_3\ell\varepsilon^{1-\rho},$$

where

$$y(t) = \|w\|^2 + \|\nabla w\|^2 + \|\eta_2^t\|_{1,\mu}^2 \quad \text{and} \quad R_3 = C\left(R_0^p + R_1^p\right).$$

Since $(w(\tau), \eta_2^\tau) = (0,0)$, the Gronwall inequality leads to

$$\|w\|^2 + \|\nabla w\|^2 + \|\eta_2^t\|_{1,\mu}^2 \leq \left(C\ell^2\varepsilon^{2(1-\rho)} + R_3\ell\varepsilon^{1-\rho}\right)e^{C(t-\tau)}, \quad \forall t \geq \tau.$$

The desired conclusion follows then by comparison. □

*Proof of Theorem* 4.1. For $\varepsilon > 0$, let $z^\varepsilon \in \mathcal{A}^\varepsilon$. Thus, in view of (11), there exists a complete bounded trajectory $\widehat{z}^\varepsilon(t)$ of (28), with the external force

$$\widehat{g}^\varepsilon = \widehat{g}_0 + \varepsilon^{-\rho}\widehat{g}_1(\tfrac{\cdot}{\varepsilon}) \in \mathcal{H}_w(g^\varepsilon), \quad \text{where} \quad \widehat{g}_0 \in \mathcal{H}_w(g_0), \quad \widehat{g}_1 \in \mathcal{H}_w(g_1),$$

such that $\widehat{z}^\varepsilon(0) = z^\varepsilon$.

By Lemma 4.2 with $t = 0$,

$$\|z^\varepsilon - U_{\widehat{g}_0}(0,\tau)\widehat{z}^\varepsilon(\tau)\|_{\mathcal{H}_1} \le \left(C\ell\varepsilon^{1-\rho} + R_3^{\frac{1}{2}}\ell^{\frac{1}{2}}\varepsilon^{\frac{1-\rho}{2}}\right)e^{C\tau}, \quad \forall \tau \le 0.$$

On the other hand, it is known (see, e.g., [14]) that the set $\mathcal{A}^0$ attracts $U_{\widehat{g}_0}(t,\tau)B^*$, uniformly not only with respect to $\tau \in \mathbb{R}$, but also with respect to $\widehat{g}_0 \in \mathcal{H}_w(g^0)$. Then, for every $\delta > 0$, there is $\tau = \tau(\delta) \le 0$ independent of $\varepsilon$ such that

$$\text{dist}_{\mathcal{H}_1}\left(U_{\widehat{g}_0}(0,\tau)\widehat{z}^\varepsilon(\tau), \mathcal{A}^0\right) \le \delta.$$

Using the triangle inequality we get

$$\text{dist}_{\mathcal{H}_1}\left(z^\varepsilon, \mathcal{A}^0\right) \le \left(C\ell\varepsilon^{1-\rho} + R_3^{\frac{1}{2}}\ell^{\frac{1}{2}}\varepsilon^{\frac{1-\rho}{2}}\right)e^{C\tau} + \delta.$$

Because $z^\varepsilon \in \mathcal{A}^\varepsilon$ is arbitrary, we reach the conclusion

$$\limsup_{\varepsilon\to 0^+}\{\text{dist}_{\mathcal{H}_1}(\mathcal{A}^\varepsilon, \mathcal{A}^0)\} \le \delta.$$

Letting $\delta \to 0$ we complete the proof. □

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