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Energy Decay of Solutions to a Wave Equation with a Dynamic Boundary Dissipation of Fractional Derivative Type

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Abstract. We consider the wave equation with a dynamic boundary control condition of fractional derivative type. We study stability of the system using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov.

Keywords. Wave equation, dynamic boundary dissipation of fractional derivative type, frequency domain method, polynomial stability

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1. Introduction

In this paper we investigate the existence and decay properties of solutions for the initial boundary value problem of the wave equation of the type

$$\varphi_{tt}(x,t) - \varphi_{xx}(x,t) = 0 \quad \text{in } (0,L)\times(0,+\infty) \tag{P}$$

where $(x,t) \in (0,L)\times(0,+\infty)$. This system is subject to the boundary conditions

$$\begin{aligned} \varphi(0,t) &= 0 && \text{in } (0,+\infty)\\ m\varphi_{tt}(L,t) + \varphi_x(L,t) &= -\gamma \partial_t^{\alpha,\eta}\varphi(L,t) && \text{in } (0,+\infty) \end{aligned}$$

where $m > 0$ and $\gamma > 0$. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo fractional derivative of order α, $0 < \alpha < 1$, with respect to the time variable (see Choi and MacCamy [13] and E. Blanc, G. Chiavassa, and B. Lombard [7]). It is defined as follows

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{dw}{ds}(s)\,ds, \quad \eta \geq 0.$$

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The system is finally completed with initial conditions

$$\varphi(x,0)=\varphi_0(x), \quad \varphi_t(x,0)=\varphi_1(x)$$

where the initial data (φ_0,φ_1) belong to a suitable function space.

The problem (P) describes the motion of a pinched vibration cable with tip mass $m>0$.

The problem of global existence and computing decay rates for the initial boundary value problem

$$\begin{cases} u_{tt}-\Delta u=0 & \text{on } \Omega\times(0,+\infty),\\ u=0 & \text{on } \Gamma_0\times(0,+\infty),\\ \dfrac{\partial u}{\partial\nu}+m\cdot\nu\sigma(t)g(u_t)=0 & \text{on } \Gamma_1\times(0,+\infty),\\ u(x,0)=\varphi_0(x), \quad u_t(x,0)=\varphi_1(x) & \text{on } \Omega, \end{cases}$$

has attracted a lot of attention in recent years. The bibliography of works in the direction is truly long (see [1,11,12,15,20,21]) and many energy estimates have been derived for arbitrary growing feedbacks (polynomial, exponential or logarithmic decay). The decay rate of the energy (when t goes to infinity) depends on the function σ and on the function H which represents the growth at the origin of g.

In [30], B. Mbodje studies the decay rate of the energy of the wave equation with a boundary fractional derivative control as in this paper. Using energy methods, he proves strong asymptotic stability under the condition $\eta=0$ and a polynomial type decay rate $E(t)\le\frac{C}{t}$ if $\eta\neq 0$.

In [23], Z. H. Luo, B. Z. Guo and O. Morgul studied the decay rate of the energy of the wave equation with a dynamic boundary condition with linear feedback control instead of linear feedback control of fractional derivative type. Using frequency domain method, they proved that the system is only asymptotically stable but not exponentially stable.

The boundary feedback under the consideration here are of fractional type and are described by the fractional derivatives

$$\partial_t^{\alpha,\eta}w(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds$$

The order of our derivatives is between 0 and 1. Very little is known in the literature. In addition to being nonlocal, fractional derivatives involve singular and nonintegrable kernels ($t^{-\alpha}$, $0<\alpha<1$). This makes the problem more delicate.

It has been shown (see [31]) that, as ∂_t, the fractional derivative ∂_t^α forces the system to become dissipative and the solution to approach the equilibrium state. Therefore, when applied on the boundary, we can consider them as controllers which help to reduce the vibrations.

In the recent years, fractional calculus has been applied successfully in various areas to modify many existing models of physical processes such as heat conduction, diffusion, viscoelasticity, wave propagation, electronics etc. Caputo and Mainardi [10] have established the relation between fractional derivative and theory of viscoelasticity. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definition of fractional derivative appeared in [17, 19]. One can refer to Podlubny [34] (see also [36]) for a survey of applications of fractional calculus.

With the rapid development of polymer science and plastic industry, the theoretical study and application in viscoelastic material has become an important task for solid mechanics (see [3–5, 25]). The theory of viscoelasticity and the solution for some boundary value problem of viscoelasticity were investigated by Llioushin and Pobedria [22]. In our case, the fractional dissipations may describe an active boundary viscoelastic damper designed for the purpose of reducing the vibrations (see [30, 31]).

In [30], by redescribing the fractional derivative term by means of a suitable diffusion equation, the original model is transformed into an augmented system which can be more easily tackled by the energy method. This concept of diffusive representation or realization in the sense of systems theory was introduced by Staffans [37] and Desch and Miller [14] in the aim of transforming fractional operators into classical input output dynamic systems (see also [32, 33].

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) with a dynamic boundary control of fractional derivative type.

The organization of this paper is as follows. In Section 2, we show that the above system can be replaced by an augmented one obtained by coupling the wave equation with a suitable diffusion equation (as in [30]). In Section 3, we introduce our functional analytic setting with a view of tackling the problem later on. Sections 2 and 3 are closely related to the Sections 4 and 5 of reference [30] by Mbodje. In Section 4, existence and uniqueness of strong and weak solutions of the system are proved, using the Hille–Yosida theorem. In Section 5, we show the lack of exponential stability by spectral analysis. In Section 6, we study asymptotic stability of the above model, once formulated as a first-order system. Since no compactness property can be found a priori, thus forbidding the use of LaSalle's invariance principle, then a refined analysis of the spectrum of the generator of the semigroup is carried out. The main results are Theorems 6.1, 6.5 and 6.6. In Theorem 6.6, we show a polynomial type decay rate depending on parameter α. The proof heavily relies on multiplier method and Borichev–Tomilov theorem and will be proved in three steps.

Finally, Section 7 is devoted to conclusions on the problems treated in this paper and future works, including some possible generalizations and interesting open questions.

2. Augmented model

This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.1 (see [30]). *Let μ be the function:*

$$\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}, \quad -\infty<\xi<+\infty,\ 0<\alpha<1. \tag{1}$$

Then the relationship between the "input" U and the "output" O of the system

$$\partial_t\phi(\xi,t) + (\xi^2+\eta)\phi(\xi,t) - U(t)\mu(\xi) = 0, \quad -\infty<\xi<+\infty,\ \eta\geq 0,\ t>0, \tag{2}$$

$$\phi(\xi,0) = 0, \tag{3}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)\,d\xi \tag{4}$$

is given by

$$O = I^{1-\alpha,\eta}U = D^{\alpha,\eta}U \tag{5}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau.$$

Proof. From (2) and (3), we have

$$\phi(\xi,t) = \int_0^t \mu(\xi)e^{-(\xi^2+\eta)(t-\tau)}U(\tau)\,d\tau. \tag{6}$$

Hence, by using (4), we get

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)e^{-\eta t}\int_0^t\left[2\int_0^{+\infty}|\xi|^{2\alpha-1}e^{-\xi^2(t-s)}\,d\xi\right]e^{\eta\tau}U(\tau)\,d\tau. \tag{7}$$

Thus,

$$\begin{aligned} O(t) &= (\pi)^{-1}\sin(\alpha\pi)e^{-\eta t}\int_0^t\left[(t-s)^{-\alpha}\Gamma(\alpha)\right]e^{\eta\tau}U(\tau)\,d\tau \\ &= (\pi)^{-1}\sin(\alpha\pi)\int_0^t\left[(t-\tau)^{-\alpha}\Gamma(\alpha)\right]e^{-\eta(t-\tau)}U(\tau)\,d\tau, \end{aligned} \tag{8}$$

which completes the proof. Indeed, we know that

$$(\pi)^{-1}\sin(\alpha\pi) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}. \qquad \square$$

Lemma 2.2. *If* $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty, -\eta]$ *then*

$$\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\, d\xi = \frac{\pi}{\sin \alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

Proof. Let us set

$$f_\lambda(\xi) = \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}.$$

We have

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\mathrm{Re}\lambda+\eta+\xi^2} & \text{or} \\ \dfrac{\mu^2(\xi)}{|\mathrm{Im}\lambda|+\eta+\xi^2} \end{cases}$$

Then the function f_λ is integrable. Moreover

$$\left|\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\right| \leq \begin{cases} \dfrac{\mu^2(\xi)}{\eta_0+\eta+\xi^2} & \text{for all} \quad \mathrm{Re}\lambda \geq \eta_0 > -\eta \\ \dfrac{\mu^2(\xi)}{\tilde{\eta}_0+\xi^2} & \text{for all } |\mathrm{Im}\lambda| \geq \tilde{\eta}_0 > 0 \end{cases}$$

From [38, Theorem 1.16.1], the function $f_\lambda : D_\eta \to \mathbb{C}$ is holomorphic. For a real number $\lambda > -\eta$, we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\, d\xi &= \int_{-\infty}^{+\infty} \frac{|\xi|^{2\alpha-1}}{\lambda+\eta+\xi^2}\, d\xi \\
&= \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda+\eta+x}\, dx \\
&= (\lambda+\eta)^{\alpha-1}\int_1^{+\infty} y^{-1}(y-1)^{\alpha-1}\, dy \quad (\text{with } y = \frac{x}{\lambda+\eta}+1) \\
&= (\lambda+\eta)^{\alpha-1}\int_0^1 z^{-\alpha}(1-z)^{\alpha-1}\, dz \quad (\text{with } z = \frac{1}{y}) \\
&= (\lambda+\eta)^{\alpha-1} B(1-\alpha, \alpha) \\
&= (\lambda+\eta)^{\alpha-1}\Gamma(1-\alpha)\Gamma(\alpha) \\
&= (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin \pi\alpha}.
\end{aligned}$$

Both holomorphic functions f_λ and $\lambda \mapsto (\lambda+\eta)^{\alpha-1}\frac{\pi}{\sin \pi\alpha}$ coincide on the half line $]-\eta, +\infty[$, hence on D_η following the principe of isolated zeroes. $\square$

We are now in a position to reformulate system (P). Indeed, by using Theorem 2.1, system (P) is equivalent to the following:

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0, \\ \partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - \varphi_t(L, t)\mu(\xi) = 0, \\ \varphi(0, t) = 0, \\ m\varphi_{tt}(L, t) + \varphi_x(L, t) = -\zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi, \quad \zeta = (\pi)^{-1}\sin(\alpha\pi)\gamma, \\ \varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x). \end{cases} \tag{P'}$$

For the solution of problem (P'), we define the energy functional

$$E(t) = \frac{1}{2}\|\varphi_t\|_2^2 + \frac{1}{2}\|\varphi_x\|_2^2 + \frac{m}{2}|\varphi_t(L, t)|^2 + \frac{\zeta}{2}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2\, d\xi. \tag{9}$$

Lemma 2.3. *Let* (φ, ϕ) *be a solution of the problem* (P')*. Then, the energy functional defined by* (9) *satisfies*

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2\, d\xi \le 0. \tag{10}$$

Proof. Multiplying the first equation in (P') by $\overline{\varphi}_t$, integrating over $(0, L)$ and using integration by parts, we get $\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_2^2 - \mathrm{Re}\int_0^L \varphi_{xx}\overline{\varphi}_t dx = 0$. Then

$$\frac{d}{dt}\left(\frac{1}{2}\|\varphi_t\|_2^2 + \frac{1}{2}\|\varphi_x\|_2^2 + \frac{m}{2}|\varphi_t(L, t)|^2\right) + \zeta \mathrm{Re}\overline{\varphi}_t(L, t)\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t)\, d\xi = 0. \tag{11}$$

Multiplying the second equation in (P') by $\zeta\overline{\phi}_t$ and integrating over $(-\infty, +\infty)$, to obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\|\phi\|_2^2 + \zeta\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2\, d\xi - \zeta \mathrm{Re}\varphi_t(L, t)\int_{-\infty}^{+\infty} \mu(\xi)\overline{\phi}(\xi, t)\, d\xi = 0. \tag{12}$$

From (9), (11) and (12) we get

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2\, d\xi.$$

This completes the proof of the lemma. □

3. Functional analytic setting

Let us introduce the semigroup representation of the (P'). We consider the following condition of the right end contour of wave

$$v(t) = \varphi_t(L, t), \quad \text{for } t > 0 \tag{13}$$

were v solve the equation

$$mv_t(t) + \varphi_x(L,t) + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,t)\, d\xi = 0. \tag{14}$$

Let $U = (\varphi, \varphi_t, \phi, v)^T$ and rewrite (P') as

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = (\varphi_0, \varphi_1, \phi_0, v_0), \end{cases} \tag{15}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} \varphi \\ u \\ \phi \\ v \end{pmatrix} = \begin{pmatrix} u \\ \varphi_{xx} \\ -(\xi^2+\eta)\phi + u(L)\mu(\xi) \\ -\frac{1}{m}\varphi_x(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi \end{pmatrix}. \tag{16}$$

We consider the following space

$$H_L^1(0,L) = \{\varphi \in H^1(0,L),\ \varphi(0) = 0\}$$

and and the Hilbert space

$$\mathcal{H} = H_L^1(0,L) \times L^2(0,L) \times L^2(-\infty,+\infty) \times \mathbb{C}$$

equipped with the inner product

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \int_0^L (u\overline{\tilde{u}} + \varphi_x\overline{\tilde{\varphi}}_x)\, dx + \zeta\int_{-\infty}^{+\infty} \phi\overline{\tilde{\phi}}\, d\xi + mv\overline{\tilde{v}}.$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ (\varphi, u, \phi, v)^T \in \mathcal{H} \;\middle|\; \begin{array}{l} \varphi \in H^2(0,L) \cap H_L^1(0,L) \\ u \in H_L^1(0,L),\ v \in \mathbb{C} \\ -(\xi^2+\eta)\phi + u(L)\mu(\xi) \in L^2(-\infty,+\infty) \\ u(L) = v,\ |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\} \tag{17}$$

4. Global existence

In this section we will give well-posedness results for problem (P') using semi-group theory. We show that the operator $\mathcal{A}$ generates a C_0- semigroup in $\mathcal{H}$. We prove that $\mathcal{A}$ is a maximal dissipative operator. For this purpose we need the following two lemmas.

Lemma 4.1. *The operator $\mathcal{A}$ is dissipative and satisfies, for any $U \in D(\mathcal{A})$,*

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2 \, d\xi. \tag{18}$$

Proof. For any $U = (\varphi, u, \phi, v)^T \in D(A)$, Using (15), (10) and the fact that $E(t) = \frac{1}{2}\|U\|_{\mathcal{H}}^2$, estimate (18) easily follows. □

Lemma 4.2. *The operator $\lambda I - \mathcal{A}$ is surjective for all $\lambda > 0$.*

Proof. We need to show that for all $F = (f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, there exists $U = (\varphi, u, \phi, v)^T \in D(\mathcal{A})$ such that

$$\lambda U - \mathcal{A}U = F. \tag{19}$$

Then, in terms of components, the above equation reads

$$\begin{cases} \lambda\varphi - u = f_1, \\ \lambda u - \varphi_{xx} = f_2, \\ \lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ \lambda v + \dfrac{1}{m}\varphi_x(L) + \dfrac{\zeta}{m} \displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi) \, d\xi = f_4. \end{cases} \tag{20}$$

Suppose φ is found with the appropriate regularity. Then, $(20)_1$ yields

$$u = \lambda\varphi - f_1. \tag{21}$$

It is clear that $u \in H_L^1(0, L)$. Furthermore, by (20) we can find ϕ as

$$\phi = \frac{f_3(\xi) + \mu(\xi)u(L)}{\xi^2 + \eta + \lambda}. \tag{22}$$

By using (20) and (21) the function φ satisfying the following equation

$$\lambda^2\varphi - \varphi_{xx} = f_2 + \lambda f_1. \tag{23}$$

Solving (23) is equivalent to finding $\varphi \in H^2 \cap H_L^1(0, L)$ such that

$$\int_0^L (\lambda^2\varphi\overline{w} - \varphi_{xx}\overline{w}) \, dx = \int_0^L (f_2 + \lambda f_1)\overline{w} \, dx, \tag{24}$$

for all $w \in H_L^1(0, L)$. Using integration by parts in (24) and taking into account (22), we obtain

$$\begin{cases} \displaystyle\int_0^L (\lambda^2\varphi\overline{w} + \varphi_x\overline{w}_x) \, dx + (\lambda m + \tilde{\zeta})u(L)\overline{w}(L) \\ = \displaystyle\int_0^L (f_2 + \lambda f_1)\overline{w} \, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2 + \eta + \lambda} f_3(\xi) \, d\xi \; \overline{w}(L) + m f_4\overline{w}(L) \end{cases} \tag{25}$$

where $\tilde{\zeta} = \zeta \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{\xi^2+\eta+\lambda}\, d\xi$. Using again (21), we deduce that

$$u(L) = \lambda\varphi(L) - f_1(L). \tag{26}$$

Inserting (26) into (25), we get

$$\begin{cases} \displaystyle\int_0^L (\lambda^2\varphi\overline{w} + \varphi_x\overline{w}_x)\, dx + \lambda(\lambda m + \tilde{\zeta})\varphi(L)\overline{w}(L) \\ \displaystyle= \int_0^L (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi\ \overline{w}(L) \\ \quad + (\lambda m + \tilde{\zeta}) f_1(L)\overline{w}(L) + m f_4 \overline{w}(L). \end{cases} \tag{27}$$

Consequently, problem (27) is equivalent to the problem

$$a(\varphi, w) = L(w) \tag{28}$$

where the bilinear form $a : H_L^1(0,L) \times H_L^1(0,L) \to \mathbb{C}$ and the linear form $L : H_L^1(0,L) \to \mathbb{C}$ are defined by

$$a(\varphi, w) = \int_0^L (\lambda^2\varphi\overline{w} + \varphi_x\overline{w}_x)\, dx + \lambda(\lambda m + \tilde{\zeta})\varphi(L)\overline{w}(L)$$

and

$$\begin{aligned} L(w) = {} & \int_0^L (f_2 + \lambda f_1)\overline{w}\, dx - \zeta \int_{-\infty}^{+\infty} \frac{\mu(\xi)}{\xi^2+\eta+\lambda} f_3(\xi)\, d\xi\ \overline{w}(L) \\ & + (\lambda m + \tilde{\zeta}) f_1(L)\overline{w}(L) + m f_4 \overline{w}(L). \end{aligned}$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax–Milgram theorem, we deduce that for all $w \in H_L^1(0,L)$ problem (28) admits a unique solution $\varphi \in H_L^1(0,L)$. Applying the classical elliptic regularity, it follows from (27) that $\varphi \in H^2(0,L)$. Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. □

Consequently, using Hille–Yosida Theorem, we have the following existence and uniqueness result.

Theorem 4.3. *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C^0(\mathbb{R}_+, \mathcal{H})$, of problem* (15). *Moreover if $U_0 \in D(\mathcal{A})$, then*

$$U \in C^0(\mathbb{R}_+, D(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

5. Lack of exponential stability

In order to state and prove our stability results, we need some Theorems.

Theorem 5.1 ([18, 35])**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R} \quad \text{and} \quad \overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Theorem 5.2 ([8])**.** *Let $S(t) = e^{\mathcal{A}t}$ be a bounded C_0-semigroup on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{|\beta|\geq 1} \frac{1}{\beta^\delta} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < M$$

for some $\delta > 0$, then there exist c such that

$$\|e^{\mathcal{A}t} U_0\|^2 \leq \frac{c}{t^{\frac{2}{\delta}}} \|U_0\|^2_{D(\mathcal{A})}.$$

Theorem 5.3 ([2, 24])**.** *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0. semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*

(i) *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

(ii) *The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set, then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e., $\|S(t)z\|_H \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.*

Our main result is the following

Theorem 5.4. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. We will examine two cases.

Case 1. $\eta = 0$: We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin x, 0, 0, 0)^T \in \mathcal{H}$, and denoting by $(\varphi, u, \phi, v)^T$ the image of $(\sin x, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\alpha-5}{2}} \sin L$. But, then $\phi \notin L^2(-\infty, +\infty)$, since $\alpha \in]0, 1[$. And so $(\varphi, u, \phi, v)^T \notin D(\mathcal{A})$.

Case 2. $\eta \neq 0$: We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the wave system (P) from being exponentially stable. Indeed We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (\varphi, u, \phi, v)^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} \lambda\varphi - u = 0, \\ \lambda u - \varphi_{xx} = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = 0, \\ \lambda v + \dfrac{1}{m}\varphi_x(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0 \end{cases} \tag{29}$$

From $(29)_1$, $(29)_2$ for such λ, we find

$$\lambda^2\varphi - \varphi_{xx} = 0. \tag{30}$$

Since $v = u(L)$, using $(29)_3$ and $(29)_4$, we get

$$\begin{cases} \varphi(0) = 0, \\ \qquad \left(\lambda + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \dfrac{\mu^2(\xi)}{\xi^2+\lambda+\eta}\,d\xi\right) u(L) + \dfrac{1}{m}\varphi_x(L) \\ = \left(\lambda + \dfrac{\gamma}{m}(\lambda+\eta)^{\alpha-1}\right)\lambda\varphi(L) + \dfrac{1}{m}\varphi_x(L) \\ = 0. \end{cases} \tag{31}$$

The matrix of the system determining is not singular. Set $X = (\varphi, \varphi_x)^T$

$$\frac{d}{dx}X = \tilde{\mathcal{B}}X \quad \text{where} \quad \tilde{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}. \tag{32}$$

The characteristic polynomial of $\tilde{\mathcal{B}}$ is $s^2 - \lambda^2 = 0$. We find the roots

$$t_1(\lambda) = \lambda, \quad t_2(\lambda) = -\lambda.$$

Here and below, for simplicity we denote $t_i(\lambda)$ by t_i. The solution φ is given by

$$\varphi(x) = \sum_{i=1}^{2} c_i e^{t_i x}. \tag{33}$$

Thus the boundary conditions may be written as the following system:

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 \\ h(t_1)e^{t_1 L} & h(t_2)e^{t_2 L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{34}$$

where we have set

$$h(r) = \frac{1}{m}r + \lambda^2 + \frac{\gamma}{m}\lambda(\lambda+\eta)^{\alpha-1}.$$

Hence a non-trivial solution φ exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = \det M(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Our purpose in the sequel is to prove, thanks to Rouché's Theorem, that there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \le \mathcal{R}(\lambda) \le 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i}, i = 1, 2$ remains bounded.

Lemma 5.5. *There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbb{Z}^*,|k|\geq N} \subset \sigma(\mathcal{A}) \tag{35}$$

where

$$\lambda_k = i\left(\frac{k\pi}{L}+\frac{1}{mk\pi}\right)+\frac{\tilde{\alpha}}{k^{3-\alpha}}+\frac{\beta}{k^{(3-\alpha)}}+o\left(\frac{1}{k^{3-\alpha}}\right), \quad k\geq N,\ \tilde{\alpha}\in i\mathbb{R},\ \beta\in\mathbb{R},\ \beta<0,$$
$$\lambda_k = \overline{\lambda_{-k}} \quad \textit{if } k\leq -N.$$

Moreover for all $|k| \geq N$, the eigenvalues λ_k are simple.

Proof. **Step 1.**

$$\begin{aligned} f(\lambda) &= e^{t_2}h(t_2) - e^{t_1}h(t_1) \\ &= -e^{-\lambda L}h(-\lambda)\left(e^{2\lambda L} + \frac{-\frac{\lambda}{m}+\lambda^2+\frac{\gamma}{m}\lambda(\lambda+\eta)^{\alpha-1}}{\frac{\lambda}{m}+\lambda^2+\frac{\gamma}{m}\lambda(\lambda+\eta)^{\alpha-1}}\right) \\ &= -e^{-\lambda L}h(-\lambda)\left(e^{2\lambda L} - 1 - \frac{2}{1+m\lambda+\gamma(\lambda+\eta)^{\alpha-1}}\right). \end{aligned} \tag{36}$$

We set

$$\begin{aligned} \tilde{f}(\lambda) &= e^{2\lambda L} - 1 - \frac{2}{1+m\lambda+\gamma(\lambda+\eta)^{\alpha-1}} \\ &= f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^2} + + \frac{f_3(\lambda)}{\lambda^{3-\alpha}} + o\left(\frac{1}{\lambda^{3-\alpha}}\right) \end{aligned} \tag{37}$$

where

$$f_0(\lambda) = e^{2\lambda L} - 1, \tag{38}$$

$$f_1(\lambda) = \frac{2}{m}, \tag{39}$$

$$f_2(\lambda) = -\frac{2}{m^2}, \tag{40}$$

$$f_3(\lambda) = -\frac{2\gamma}{m^2}. \tag{41}$$

Note that f_0, f_1, f_2 and f_3 remain bounded in the strip $-\alpha_0 \leq \mathcal{R}(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (38), f_0 has one familie of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \ \Leftrightarrow \ e^{2\lambda L} = 1.$$

Hence

$$2\lambda L = i2k\pi, \quad \text{i.e., } \lambda_k^0 = \frac{ik\pi}{L}, \quad k \in \mathbb{Z}.$$

Now with the help of Rouché's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Changing in (37) the unknown λ by $u = 2\lambda L$ then (37) becomes

$$\tilde{f}(u) = (e^u - 1) + O\left(\frac{1}{u}\right) = f_0(u) + O\left(\frac{1}{u}\right).$$

The roots of f_0 are $u_k = \frac{ik}{L}\pi, k \in \mathbb{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u - 1| \geq Cr$ for r small enough. This allows to apply Rouché's Theorem. Consequently, there exists a subsequence of roots of $\tilde{f}$ which tends to the roots u_k of f_0. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\frac{ik}{L}\pi$ of f_0. Finally for $|k| \geq N, \lambda_k$ is simple since λ_k^0 is.

Step 3. From Step 2, we can write

$$\lambda_k = i\frac{1}{L}k\pi + \varepsilon_k. \tag{42}$$

Using (42), we get

$$e^{2\lambda_k L} = 1 + 2L\varepsilon_k + 2L^2\varepsilon_k^2 + o(\varepsilon_k^2). \tag{43}$$

Substituting (43) into (37), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = 2L\varepsilon_k + 2L^2\varepsilon_k^2 + \frac{\frac{2}{m}}{\frac{k\pi i}{L} + \varepsilon_k} - \frac{\frac{2}{m^2}}{(\frac{k\pi i}{L} + \varepsilon_k)^2} + o(\varepsilon_k^2) = 2L\varepsilon_k + \frac{\frac{2L}{m}}{k\pi i} + o(\varepsilon_k) + o\left(\frac{1}{k}\right) = 0$$

and hence

$$\varepsilon_k = \frac{i}{mk\pi}.$$

Step 4. From Step 3, we can write

$$\lambda_k = i\frac{1}{L}k\pi + \frac{1}{mk\pi}i + \varepsilon_k \tag{44}$$

Using (44), we get

$$e^{2\lambda_k L} = 1 + \left(\frac{2L}{mk\pi}i + 2L\varepsilon_k\right) + \frac{1}{2}\left(\frac{2L}{mk\pi}i + 2L\varepsilon_k\right)^2 + o(\varepsilon_k^3) \tag{45}$$

Substituting (45) into (37), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\begin{aligned}
\tilde{f}(\lambda_k) &= \left(\frac{2L}{mk\pi}i + 2L\varepsilon_k\right) + \frac{1}{2}\left(\frac{2L}{mk\pi}i + 2L\varepsilon_k\right)^2 + \frac{\frac{2}{m}}{\frac{k\pi i}{L} + \frac{1}{mk\pi}i + \varepsilon_k} \\
&\quad - \frac{\frac{2}{m^2}}{(\frac{k\pi i}{L} + \frac{1}{mk\pi}i + \varepsilon_k)^2} - \frac{\frac{2\gamma}{m^2}}{(\frac{k\pi i}{L} + \frac{1}{mk\pi}i + \varepsilon_k)^{(3-\alpha)}} + O(\varepsilon_k^3) + O\left(\frac{1}{k^3}\right) \\
&= 2L\varepsilon_k - \frac{2\gamma}{m^2}\left(\frac{L}{k\pi i}\right)^{3-\alpha} + o(\varepsilon_k^3) + o\left(\frac{1}{k^3}\right) \\
&= 0
\end{aligned} \tag{46}$$

$$\varepsilon_k = \frac{\gamma}{m^2 L^{\alpha-2}(k\pi i)^{3-\alpha}} + o\left(\frac{1}{k^{3-\alpha}}\right)$$
$$= \begin{cases} -\dfrac{\gamma}{m^2 L^{\alpha-2}(k\pi)^{3-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} - i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{k^{3-\alpha}}\right) & \text{for } k \succeq 0 \\ -\dfrac{\gamma}{m^2 L^{\alpha-2}(-k\pi)^{3-\alpha}}\left(\cos(1-\alpha)\dfrac{\pi}{2} - i\sin(1-\alpha)\dfrac{\pi}{2}\right) + o\left(\dfrac{1}{k^{3-\alpha}}\right) & \text{for } k \preceq 0 \end{cases}$$

From this equation we obtain $|k|^{3-\alpha}\mathcal{R}\lambda_k \sim \beta$ in that case, with

$$\beta = -\frac{\gamma}{m^2 L^{\alpha-2}\pi^{3-\alpha}}\cos(1-\alpha)\frac{\pi}{2}.$$

The operator $\mathcal{A}$ has a non exponential decaying branch of eigenvalues. Thus the proof is complete. □

6. Asymptotic stability

Because of the unboundedness of the ξ-domain for the diffusive equation, the resolvent of $\mathcal{A}$ is not compact, and a major difficulty arises in the use of LaSalle's invariance principle to prove asymptotic stability. A refined analysis of the spectrum of generator of the semigroup can be performed, which allows for the use of the stability results of [2, 24]. A direct application of this result on the pseudo-differentially damped linearized pendulum, can be found in [28].

6.1. Strong stability of the system. In this part, we use a general criteria of Lemma 5.3 to show the strong stability of the C_0-semigroup $e^{t\mathcal{A}}$ associated to the wave system (P') in the absence of the compactness of the resolvent of $\mathcal{A}$. Our main result is the following theorem:

Theorem 6.1. *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of* (15) *satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 6.1, we need the following two lemmas.

Lemma 6.2. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof. We make a distinction between $i\lambda = 0$ and $i\lambda \neq 0$.

Step 1. Solving for $\mathcal{A}U = 0$ leads to $U = 0$, thanks to the boundary conditions in (17). Hence, $i\lambda = 0$ is not is not an eigenvalue of $\mathcal{A}$.

Step 2. We will argue by contradiction. Let us suppose that there $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $U \neq 0$, such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda\varphi - u = 0, \\ i\lambda u - \varphi_{xx} = 0, \\ i\lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = 0, \\ i\lambda v + \dfrac{1}{m}\varphi_x(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0. \end{cases} \tag{47}$$

Then, from (18) we have

$$\phi \equiv 0. \tag{48}$$

From $(47)_3$, we have

$$u(L) = 0. \tag{49}$$

Hence, from $(47)_1$ and $(47)_4$ we obtain

$$\varphi(L) = 0 \quad \text{and} \quad \varphi_x(L) = 0. \tag{50}$$

From $(47)_1$ and $(47)_2$, we have

$$-\lambda^2\varphi - \varphi_{xx} = 0. \tag{51}$$

Consider $X = (\varphi, \varphi_x)$. Then we can rewrite (50) and (51) as the initial value problem

$$\begin{array}{l} \dfrac{d}{dx}X = \mathcal{B}X \\ X(0) = 0, \end{array} \quad \text{where} \quad \mathcal{B} = \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{pmatrix} \tag{52}$$

By the Picard Theorem for ordinary differential equations the system (52) has a unique solution $X = 0$. Therefore $\varphi = 0$. It follows from (47), that $u = 0$ and $v = 0$, i.e., $U = 0$. Consequently, $\mathcal{A}$ does not have purely imaginary eigenvalues, so the condition (i) of Theorem 5.3 holds. $\square$

The condition (ii) of Theorem 5.3 will be satisfied if we show that $\sigma(\mathcal{A}) \cap \{i\mathbb{R}\}$ is at most a countable set. We have the following lemma.

Lemma 6.3. 1. *If $\lambda \neq 0$, the operator $\lambda I - \mathcal{A}$ is surjective.*
2. *If $\lambda = 0$ and $\eta = 0$, the operator $\lambda I - \mathcal{A}$ is surjective. where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.*

Proof. **Case 1:** $\lambda \neq 0$ **.** Let $F = (f_1, f_2, f_3, f_3)^T \in \mathcal{H}$ be given, and let $X = (\varphi, u, \phi, v)^T \in D(\mathcal{A})$ be such that

$$(i\lambda I - \mathcal{A})X = F. \tag{53}$$

Equivalently, we have

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \varphi_{xx} = f_2, \\ i\lambda\phi + (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3, \\ i\lambda v + \dfrac{1}{m}\varphi_x(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi = f_4. \end{cases} \tag{54}$$

We divide the proof into three steps, as follows:

Step 1. With the first two equations of (54), we get

$$\lambda^2\varphi + \varphi_{xx} = -(f_2 + i\lambda f_1).$$

As $\varphi(0) = 0$, then

$$\varphi(x) = c_1 \sin\lambda x - \frac{1}{\lambda}\int_0^x (f_2(\sigma) + i\lambda f_1(\sigma))\sin\lambda(x-\sigma)\,d\sigma \tag{55}$$

and hence

$$\varphi_x(x) = c_1\lambda\cos\lambda x - \int_0^x (f_2(\sigma) + i\lambda f_1(\sigma))\cos\lambda(x-\sigma)\,d\sigma. \tag{56}$$

Step 2. With the third equation of (54), we get

$$\phi(\xi) = \frac{u(L)\mu(\xi) + f_3(\xi)}{i\lambda + \xi^2 + \eta}. \tag{57}$$

Inserting (57) in the last equation of (54), we get

$$\left(i\lambda + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda+\xi^2+\eta}\,d\xi\right)u(L) + \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi = f_4. \tag{58}$$

Since $\frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{i\lambda+\xi^2+\eta}\,d\xi = \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}$ and $u(L) = i\lambda\varphi(L) - f_1(L)$, we deduce that

$$\begin{aligned} &i\lambda\left(i\lambda + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}\right)\varphi(L) + \frac{1}{m}\varphi_x(L) \\ &= \left(i\lambda + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}\right)f_1(L) + f_4 - \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi \end{aligned} \tag{59}$$

Step 3. Using (55), (56) we can rewrite (59) as an equation in the unknown c_1

$$\begin{aligned} &\lambda c_1\left[iI\sin\lambda L + \frac{1}{m}\cos\lambda L\right] \\ &= J + If_1(L) + \frac{1}{m}\int_0^L (f_2(\sigma) + i\lambda f_1(\sigma))\cos\lambda(L-\sigma)\,d\sigma \\ &\quad + iI\int_0^L (f_2(\sigma) + i\lambda f_1(\sigma))\sin\lambda(L-\sigma)\,d\sigma \end{aligned} \tag{60}$$

where

$$I = i\lambda + \frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}, \quad J = f_4 - \frac{\zeta}{m}\int_{-\infty}^{+\infty} \frac{\mu(\xi)f_3(\xi)}{i\lambda+\xi^2+\eta}\,d\xi.$$

We set

$$\begin{aligned} g(\lambda) &= iI\sin\lambda L + \frac{1}{m}\cos\lambda L \\ &= -\lambda\sin\lambda L + \frac{1}{m}\cos\lambda L + i\frac{\gamma}{m}(i\lambda+\eta)^{\alpha-1}\sin\lambda L \\ &= -\lambda\sin\lambda L + \frac{1}{m}\cos\lambda L + \frac{\gamma}{m}(\lambda^2+\eta^2)^{\frac{\alpha-1}{2}}\sin(1-\alpha)\theta\sin\lambda L \\ &\quad + i\frac{\gamma}{m}(\lambda^2+\eta^2)^{\frac{\alpha-1}{2}}\cos(1-\alpha)\theta\sin\lambda L \end{aligned}$$

where $\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ such that

$$\cos\theta = \frac{\eta}{\sqrt{\lambda^2+\eta^2}}, \quad \sin\theta = \frac{\lambda}{\sqrt{\lambda^2+\eta^2}}$$

It is clear that

$$g(\lambda) \neq 0 \quad \forall\lambda\in\mathbb{R}.$$

Hence $i\lambda - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.

Case 2: $\lambda = 0$ **and** $\eta \neq 0$ **.** The system (54) is reduced to the following

$$\begin{cases} u = -f_1, \\ \varphi_{xx} = -f_2, \\ (\xi^2+\eta)\phi - u(L)\mu(\xi) = f_3, \\ \dfrac{1}{m}\varphi_x(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi = f_4. \end{cases} \tag{61}$$

With the second equation of (54) we get

$$\varphi(x) = -\int_0^x\int_0^s f_2(r)\,dr\,ds + Cx.$$

From $(61)_1, (61)_3$ and $(61)_4$, we have

$$-\frac{\gamma}{m}\eta^{\alpha-1}f_1(L) + \frac{1}{m}\varphi_x(L) = f_4 - \frac{\zeta}{m}\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{\xi^2+\eta}\,d\xi.$$

We find $C = \int_0^L f_2(r)\,dr + \gamma\eta^{\alpha-1}f_1(L) + mf_4 - \zeta\int_{-\infty}^{+\infty}\frac{\mu(\xi)f_3(\xi)}{\xi^2+\eta}\,d\xi$. Hence $\mathcal{A}$ is surjective. The proof is thus complete. □

Proof of Theorem 6.1. By Lemma 6.2, the operator $\mathcal{A}$ has no pure imaginary eigenvalues and by Lemma 6.3 $R(i\lambda-\mathcal{A}) = \mathcal{H}$ for all $\lambda \in \mathbb{R}^*$ and $R(i\lambda-\mathcal{A}) = \mathcal{H}$ for $\lambda = 0$ and for all $\eta > 0$. Therefore, the closed graph theorem of Banach implies that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. □

6.2. Residual spectrum of $\mathcal{A}$.

Lemma 6.4. *Let $\mathcal{A}$ be defined by* (16). *Then*

$$\mathcal{A}^* \begin{pmatrix} \varphi \\ u \\ \phi \\ v \end{pmatrix} = \begin{pmatrix} -u \\ -\varphi_{xx} \\ -(\xi^2+\eta)\phi - u(L)\mu(\xi) \\ \frac{1}{m}\varphi_x(L) + \frac{\zeta}{m}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi \end{pmatrix} \tag{62}$$

with domain

$$D(\mathcal{A}^*) = \left\{ (\varphi,u,\phi,v)^T \in \mathcal{H} \;\middle|\; \begin{array}{l} \varphi \in H^2(0,L)\cap H_L^1(0,L), \\ u \in H_L^1(0,L), \quad v \in \mathbb{C}, \\ -(\xi^2+\eta)\phi - u(L)\mu(\xi) \in L^2(-\infty,+\infty) \\ u(L) = v, \quad |\xi|\phi \in L^2(-\infty,+\infty) \end{array} \right\} \tag{63}$$

Proof. Let $U = (\varphi,u,\phi,v)^T$ and $V = (\tilde{\varphi},\tilde{u},\tilde{\phi},\tilde{v})^T$. We have

$$\langle \mathcal{A}U, V\rangle_{\mathcal{H}} = \langle U, \mathcal{A}^*V\rangle_{\mathcal{H}}.$$

$$\begin{aligned} \langle \mathcal{A}U, V\rangle_{\mathcal{H}} &= \int_0^L u_x\overline{\tilde{\varphi}}_x\,dx + \int_0^L \overline{\tilde{u}}\varphi_{xx}\,dx + \zeta\int_{-\infty}^{+\infty}[-(\xi^2+\eta)\phi + u(L)\mu(\xi)]\overline{\tilde{\phi}}\,d\xi \\ &\quad + m\left(-\frac{1}{m}\varphi_x(L) - \frac{\zeta}{m}\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\right)\overline{\tilde{v}} \\ &= -\int_0^L u\overline{\tilde{\varphi}}_{xx}\,dx - \int_0^L \overline{\tilde{u}}_x\varphi_x\,dx + \varphi_x(L)\overline{\tilde{u}}(L) + \overline{\tilde{\varphi}}_x(L)u(L) \\ &\quad - \zeta\int_{-\infty}^{+\infty}\phi[(\xi^2+\eta)\overline{\tilde{\phi}}]\,d\xi + \zeta u(L)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\tilde{\phi}}\,d\xi - \varphi_x(L)\overline{\tilde{v}} \\ &\quad - \zeta\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi)\,d\xi\overline{\tilde{v}} \end{aligned}$$

As $v = u(L)$ and if we set $\tilde{v} = \tilde{u}(L)$, we find

$$\begin{aligned} \langle \mathcal{A}U, V\rangle_{\mathcal{H}} &= -\int_0^L u\overline{\tilde{\varphi}}_{xx}\,dx - \int_0^L \overline{\tilde{u}}_x\varphi_x\,dx - \zeta\int_{-\infty}^{+\infty}\phi[(\xi^2+\eta)\overline{\tilde{\phi}} + \mu(\xi)\overline{\tilde{u}}(L)]\,d\xi \\ &\quad + v\left(\overline{\tilde{\varphi}}_x(L) + \zeta\int_{-\infty}^{+\infty}\mu(\xi)\overline{\tilde{\phi}}\,d\xi\right). \end{aligned}$$ □

Theorem 6.5. $\sigma_r(\mathcal{A}) = \emptyset$, *where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof. Since $\lambda \in \sigma_r(\mathcal{A}), \overline{\lambda} \in \sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (62), the eigenvalue problem $\mathcal{A}^*Z = \lambda Z$ for $\lambda \in \mathbb{C}$ and $0 \neq Z = (\varphi, u, \phi, v) \in D(\mathcal{A}^*)$ we have

$$\begin{cases} \lambda\varphi + u = 0, \\ \lambda u + \varphi_{xx} = 0, \\ \lambda\phi + (\xi^2 + \eta)\phi + u(L)\mu(\xi) = 0, \\ \lambda v - \dfrac{1}{m}\varphi_x(L) - \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = 0 \end{cases} \tag{64}$$

From $(64)_1$ and $(64)_2$, we find

$$\lambda^2\varphi - \varphi_{xx} = 0, \tag{65}$$

As $v = u(L) = -\lambda\varphi(L)$, we deduce from $(64)_3$ and $(64)_4$ that

$$\left(\lambda + \frac{\gamma}{m}(\lambda + \eta)^{\alpha - 1}\, d\xi\right) \lambda\varphi(L) + \frac{1}{m}\varphi_x(L) = 0 \tag{66}$$

with the following conditions

$$\varphi(0) = 0. \tag{67}$$

System (65)–(67) is the same as (30) and (31). Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete. □

6.3. Polynomial stability (for $\eta \neq 0$). In this part, we prove that the system (P') is polynomially stable when $\eta > 0$. Note that in [27], an early example of such refined decay estimate had been proved for Webster-Lokshin model with constant coefficients in the case $\alpha = \frac{1}{2}$ and inferred for other values of α by using a modal decomposition on a Riesz basis and the asymptotic of the eigenfunctions of the ∂_t^α operator.

Theorem 6.6. *The semigroup $S_{\mathcal{A}}(t)_{t \geq 0}$ is polynomially stable and*

$$\|S_{\mathcal{A}}(t)U_0\|_{\mathcal{H}} \leq \frac{1}{t^{\frac{1}{4-2\alpha}}}\|U_0\|_{D(\mathcal{A})}.$$

Proof. An early example of such refined decay estimate had been proved for the case $\alpha = \frac{1}{2}$ and inferred for other values of α in [27]. We will need to study the resolvent equation $(i\lambda - \mathcal{A})U = F$, for $\lambda \in \mathbb{R}$, namely

$$\begin{cases} i\lambda\varphi - u = f_1, \\ i\lambda u - \varphi_{xx} = f_2, \\ i\lambda\phi + (\xi^2 + \eta)\phi - u(L)\mu(\xi) = f_3, \\ i\lambda v + \dfrac{1}{m}\varphi_x(L) + \dfrac{\zeta}{m}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi)\, d\xi = f_4, \end{cases} \tag{68}$$

where $F = (f_1, f_2, f_3, f_4)^T$.

Step 1. Taking inner product in $\mathcal{H}$ with U and using (18) we get

$$|\mathrm{Re}\langle \mathcal{A}U, U\rangle| \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{69}$$

This implies that

$$\zeta \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\varphi(\xi, t)|^2 \, d\xi \leq \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{70}$$

and, applying $(68)_1$, we obtain

$$||\lambda||\varphi(L)| - |f_1(L)||^2 \leq |u(L)|^2.$$

We deduce that $|\lambda|^2|\varphi(L)|^2 \leq c|f_1(L)|^2 + c|u(L)|^2$. Moreover, from $(68)_4$, we have

$$\varphi_x(L) = -im\lambda u(L) - \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi) \, d\xi + mf_4.$$

Then

$$\begin{aligned} |\varphi_x(L)|^2 &\leq 2m^2|\lambda|^2|u(L)|^2 + 2m^2 f_4^2 + 2\zeta^2 \left| \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi) \, d\xi \right|^2 \\ &\leq 2m^2|\lambda|^2|u(L)|^2 + 2m^2 f_4^2 \\ &\quad + 2\zeta^2 \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1}|\mu(\xi)|^2 \, d\xi \right) \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi)|^2 \, d\xi \\ &\leq 2m^2|\lambda|^2|u(L)|^2 + c\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + c'\|F\|_{\mathcal{H}}^2. \end{aligned} \tag{71}$$

From $(68)_3$, we obtain

$$u(L)\mu(\xi) = (i\lambda + \xi^2 + \eta)\phi - f_3(\xi). \tag{72}$$

By multiplying $(72)_1$ by $(i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)$, we get

$$(i\lambda + \xi^2 + \eta)^{-1}u(L)\mu^2(\xi) = \mu(\xi)\phi - (i\lambda + \xi^2 + \eta)^{-1}\mu(\xi)f_3(\xi). \tag{73}$$

Hence, by taking absolute values of both sides of (73), integrating over the interval $]-\infty, +\infty[$ with respect to the variable ξ and applying Cauchy–Schwartz inequality, we obtain

$$\mathcal{S}|u(L)| \leq \mathcal{U} \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi|^2 \, d\xi \right)^{\frac{1}{2}} + \mathcal{V} \left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \tag{74}$$

where

$$\mathcal{S} = \int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-1} |\mu(\xi)|^2 \, d\xi = (|\lambda| + \eta)^{\alpha-1},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta)^{-1} |\mu(\xi)|^2 \, d\xi \right)^{\frac{1}{2}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|\lambda| + \xi^2 + \eta)^{-2} |\mu(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}$$

$$= \left((1-\alpha) \frac{\pi}{\sin \alpha\pi} (|\lambda| + \eta)^{\alpha-2} \right)^{\frac{1}{2}}.$$

Thus, by using again the inequality $2PQ \leq P^2 + Q^2, P \geq 0, Q \geq 0$, we get

$$\mathcal{S}^2 |u(L)|^2 \leq 2\mathcal{U}^2 \left(\int_{-\infty}^{+\infty} (\xi^2 + \eta) |\phi|^2 \, d\xi \right) + 2\mathcal{V}^2 \left(\int_{-\infty}^{+\infty} |f_3(\xi)|^2 \, d\xi \right). \tag{75}$$

We deduce that

$$|u(L)|^2 \leq c|\lambda|^{2-2\alpha} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + c\|F\|^2_{\mathcal{H}}. \tag{76}$$

Step 2. Now, we use the classical multiplier method. Let us introduce the following notation

$$\mathcal{I}_\varphi(\alpha) = |u(\alpha)|^2 + |\varphi_x(\alpha)|^2, \quad \mathcal{E}_\varphi(L) = \int_0^L q(s) \mathcal{I}_\varphi(s) \, ds.$$

Lemma 6.7. *Let $q \in H^1(0, L)$. We have that*

$$\mathcal{E}_\varphi(L) = [q\mathcal{I}_\varphi]_0^L + R \tag{77}$$

where R satisfies

$$|R| \leq C\mathcal{E}_\varphi(L) + \|q^{\frac{1}{2}} F\|^2_{\mathcal{H}}.$$

for a positive constant C.

Proof. To get (77), let us multiply the equation $(68)_2$ by $q\overline{\varphi}_x$, integrating on $(0, L)$ we obtain $i\lambda \int_0^L uq\overline{\varphi}_x \, dx - \int_0^L \varphi_{xx} q\overline{\varphi}_x \, dx = \int_0^L f_2 q\overline{\varphi}_x \, dx$ or

$$-\int_0^L uq(\overline{i\lambda\varphi_x}) \, dx - \int_0^L q\varphi_{xx}\overline{\varphi}_x \, dx = \int_0^L f_2 q\overline{\varphi}_x \, dx.$$

Since $i\lambda\varphi_x = u_x + f_{1x}$ taking the real part in the above equality results in

$$-\tfrac{1}{2} \int_0^L q \tfrac{d}{dx} |u|^2 \, dx - \tfrac{1}{2} \int_0^L q \tfrac{d}{dx} |\varphi_x|^2 \, dx = \mathrm{Re} \int_0^L f_2 q\overline{\varphi}_x \, dx + \mathrm{Re} \int_0^L uq\overline{f}_{1x} \, dx.$$

Performing an integration by parts we get $\int_0^L q'(s)[|u(s)|^2 + |\varphi_x(s)|^2]\,ds = [q\mathcal{I}_\varphi]_0^L + R$ where

$$R = 2\mathrm{Re}\int_0^L f_2 q\overline{\varphi}_x\,dx + 2\mathrm{Re}\int_0^L uq\overline{f}_{1x}\,dx.$$

If we take $q(x) = \int_0^x e^{ns}\,ds = \frac{e^{nx}-1}{n}$ (Here n will be chosen large enough) in Lemma 6.7 we arrive at

$$\mathcal{E}_\varphi(L) = q(L)\mathcal{I}_\varphi(L) + R. \tag{78}$$

Also, we have

$$\begin{aligned}|R| &\leq \int_0^L q(x)(|u(s)|^2 + |\varphi_x(s)|^2)\,ds + \int_0^L q(x)(|f_2(s)|^2 + |f_{1x}(s)|^2)\,ds \\ &\leq C\frac{e^{Ln}}{n}\|F\|_{\mathcal{H}}^2 + \frac{c'}{n}\int_0^L q'(s)[|u(s)|^2 + |\varphi_x(s)|^2]\,ds\end{aligned} \tag{79}$$

Using inequalities (78) and (79) we conclude that there exists a positive constant C such that

$$\int_0^L \mathcal{I}_\varphi(s)\,ds \leq C\mathcal{I}_\varphi(L) + C'\|F\|_{\mathcal{H}}^2. \tag{80}$$

provided n is large enough. □

Step 3. Since that

$$\int_{-\infty}^{+\infty} |\phi(\xi)|^2\,d\xi \leq C\int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi)|^2\,d\xi \leq C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}.$$

Substitution of inequalities (71)and (76) into (80) we get that

$$\|U\|_{\mathcal{H}}^2 \leq C(|\lambda|^{4-2\alpha} + |\lambda|^{2-2\alpha} + 1)\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C'(|\lambda|^2+1)\|F\|_{\mathcal{H}}^2.$$

So we have

$$\|U\|_{\mathcal{H}} \leq C|\lambda|^{4-2\alpha}\|F\|_{\mathcal{H}} \quad \text{as } |\lambda| \to \infty.$$

Then, using Theorem 5.2 with $\delta = 4 - 2\alpha$ one has conclusion of Theorem. The proof is now complete. □

7. Conclusions and future works

7.1. Conclusions. We have studied the dynamic boundary stabilization of the wave system with dissipation law of fractional derivative type. Using a spectral analysis we have proved a non-uniform stability. Using Arendt-Batty Theorem, we have proved the strong asymptotic stability. If $\eta > 0$, using a frequency domain approach, we prove some polynomial energy decay rate depending on parameter α.

7.2. Future works. In Theorems 4.3, 6.1, 6.5, 6.6, our approach can be generalized to multi-dimensional spaces. But it is difficult to use spectral analysis to generalize Theorem 5.4. Instead we can show the lack of exponential stability by proving that the second condition in Theorem 5.1 does not hold.

We can extend (paper in preparation) the results of this paper to more general measure density instead of (1). Indeed we can consider $\int_{-\infty}^{\infty} \frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\, d\xi$ as Stieltjes function. By the help of Abelian/Tauberian theorem of Karamata, we obtain many interesting cases that is resolvent growth slower or faster. We use a general Borichev–Tomilov theorem (see [6]).

It seems to be interesting to study a global decaying solutions of hyperbolic systems (strong and weakly) under control of fractional derivative type. We think that the interaction of the hyperbolicity (order of multiplicity) and the number of dissipative terms have an effect on the result.

It seems to be interesting to develop some energy methods to treat nonlinear evolution under control of fractional derivative type. The problem of global existence and energy decay for the following wave equation of Kirchhoff type is open

$$\begin{cases} \varphi_{tt}(x,t) - M(\|\varphi_x\|^2_{L^2(0,L)})\varphi_{xx}(x,t) = 0 & \text{in } (0,L)\times(0,+\infty) \\ \varphi(0,t) = 0 & \text{in } (0,+\infty) \\ M(\|\varphi_x\|^2_{L^2(0,L)})\varphi_x(L,t) = -\gamma\partial_t^{\alpha,\eta}\varphi(L,t) & \text{in } (0,+\infty) \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & \text{in } (0,L). \end{cases}$$

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