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On the Global Existence of Strong Solution to the 3D Damped Boussinesq Equations with Zero Thermal Diffusion

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Abstract. In this note we establish the regularity and uniqueness for the three-dimensional incompressible damped Boussinesq equations with zero thermal diffusion.

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1. Introduction

The three-dimensional (3D) incompressible damped Boussinesq equations with zero thermal diffusion read as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \Delta u + |u|^{\beta-1}u + \nabla p = \theta e_3, \quad x\in\mathbb{R}^3,\ t>0, \\ \partial_t\theta + (u\cdot\nabla)\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0)=u_0(x), \quad \theta(x,0)=\theta_0(x), \end{cases} \tag{1}$$

where $u = u(x,t)\in\mathbb{R}^3$ is the velocity, $p = p(x,t)\in\mathbb{R}$ is the scalar pressure, $\theta=\theta(x,t)\in\mathbb{R}$ is the temperature and $e_3 = (0,\, 0,\, 1)^{\mathrm{T}}$. Here the term $|u|^{\beta-1}u$ with $\beta\geq 1$ is called as damping term. When the damping term is absent, the system (1) reduces to the classical 3D Boussinesq equations with zero thermal diffusion (cf. [4,5,13,14]). The Boussinesq system is one of the most commonly used models, since it shares vortex stretching effect similar to the 3D incompressible flow (see [11]). Moreover, the Boussinesq system has important roles in the atmospheric sciences [10]. Although the local existence and uniqueness of smooth solutions for 3D Boussinesq equations with zero thermal diffusion

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were well-known (see, e.g., [2,3,11]), whether the unique local smooth solution can exist globally is an outstanding challenging open problem. Up to now, we know the blow-up (regularity) criteria of the 3D Boussinesq equations with zero thermal diffusion (cf. [4,5,13]).

For the case $\theta = 0$, the system (1) is reduced to the 3D incompressible damped Navier–Stokes equations which were studied first by Cai and Jiu [1]. The authors in [1] proved that the corresponding system admits global weak solutions for any $\beta \geq 1$, and global strong solution for any $\beta \geq \frac{7}{2}$. Moreover, the strong solution is unique for any $\frac{7}{2} \leq \beta \leq 5$. Subsequently, a considerable works are devoted to the Navier–Stokes equations with damping term, we refer the readers to the interesting works [6–8,12,15,16].

Before stating our main result, we give the definition of the strong solution to the system (1).

Definition 1.1. A measurable function pair $(u(x,t), \theta(x,t))$ is said to be a strong solution to the system (1) if it satisfies that for any given $T > 0$

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)), \quad \nabla u \in L^1(0,T;L^\infty(\mathbb{R}^3)),$$

$$\theta \in L^\infty(0,T;L^2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)).$$

Our purpose in this paper is to establish the regularity and uniqueness for the system (1) with $\beta \geq 3$. More precisely, the main result can be stated as follows.

Theorem 1.2. *Suppose $\beta \geq 3$, $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $\theta_0 \in L^2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$. Then there exists a unique global strong solution of the system* (1) *satisfying for any given $T > 0$*

$$u \in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^2(\mathbb{R}^3)), \quad \nabla u \in L^1(0,T;L^\infty(\mathbb{R}^3)),$$
$$|u|^{\frac{\beta-1}{2}} \nabla u \in L^2(0,T;L^2(\mathbb{R}^3)), \quad \nabla |u|^{\frac{\beta+1}{2}} \in L^2(0,T;L^2(\mathbb{R}^3)),$$
$$\theta \in L^\infty(0,T;L^2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)).$$

2. The proof of Theorem 1.2

This section is devoted to the proof of the Theorem 1.2. The local strong solution can be obtained like the classical Navier–Stokes equations, we only need to establish *a priori* estimates. Throughout the paper, C represents a real positive constant which may be different in each occurrence.

Multiplying $(1)_2$ by $|\theta|^{p-2}\theta$, integrating the resulting equation over $\mathbb{R}^3$ and using the incompressible condition, we have $\frac{d}{dt}\|\theta(t)\|_{L^p} = 0$, for all $p \in [2,\infty)$.

For the endpoint case $p = \infty$, it follows from the maximum principle that $\frac{d}{dt}\|\theta(t)\|_{L^\infty} = 0$. We therefore obtain

$$\frac{d}{dt}\|\theta(t)\|_{L^p} = 0, \quad \forall p \in [2, \infty].$$

Integrating the above equality from 0 to t yields

$$\|\theta(t)\|_{L^p} = \|\theta_0\|_{L^p} \le \|\theta_0\|_{L^2} + \|\theta_0\|_{L^\infty}, \quad \forall p \in [2, \infty]. \tag{2}$$

Testing $(1)_1$ by u gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|u\|_{L^{\beta+1}}^{\beta+1} \le \|u\|_{L^2}\|\theta\|_{L^2},$$

which along with (2) implies that

$$\|u(t)\|_{L^2}^2 + \int_0^t (\|\nabla u(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^{\beta+1}}^{\beta+1})\, d\tau \le C(t, u_0, \theta_0). \tag{3}$$

Multiplying the equation of $(1)_1$ by Δu, integration by parts and using the incompressible condition, it leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2 + \frac{4(\beta-1)}{(\beta+1)^2}\|\nabla |u|^{\frac{\beta+1}{2}}\|_{L^2}^2 \\
&= -\int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u\, dx + \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u\, dx.
\end{aligned}$$

According to the Young inequality, one obtains

$$-\int_{\mathbb{R}^3} \theta e_3 \cdot \Delta u\, dx \le \|\Delta u\|_{L^2}\|\theta\|_{L^2} \le \frac{1}{4}\|\Delta u\|_{L^2}^2 + C\|\theta\|_{L^2}^2.$$

The estimate [16, (2.1)] yields for $\beta \ge 3$ that

$$\begin{aligned}
\int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u\, dx &\le \frac{1}{2}\|\Delta u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 |u|^2\, dx \\
&\le \frac{1}{2}\|\Delta u\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 (|u|^{\beta-1} + 1)\frac{|u|^2}{|u|^{\beta-1} + 1}\, dx \\
&\le \frac{1}{2}\|\Delta u\|_{L^2}^2 + \frac{1}{2}\Big\||u|^{\frac{\beta-1}{2}}|\nabla u|\Big\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2,
\end{aligned}$$

where we have applied the following fact due to $\beta \ge 3$

$$\frac{|u|^2}{|u|^{\beta-1} + 1} \le 1.$$

Therefore, we conclude

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2+\|\Delta u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla |u|^{\frac{\beta+1}{2}}\|_{L^2}^2\leq C\|\nabla u\|_{L^2}^2+C\|\theta\|_{L^2}^2.$$

We get by the Gronwall inequality

$$\|\nabla u(t)\|_{L^2}^2+\int_0^t\Big(\|\Delta u\|_{L^2}^2+\||u|^{\frac{\beta-1}{2}}\nabla u\|_{L^2}^2+\|\nabla |u|^{\frac{\beta+1}{2}}\|_{L^2}^2\Big)(\tau)\,d\tau\leq C(t,u_0,\theta_0), \quad (4)$$

which together with the simple embedding yields

$$\int_0^t \|u(\tau)\|_{L^{3(\beta+1)}}^{\beta+1}\,d\tau\leq \int_0^t \|\nabla |u|^{\frac{\beta+1}{2}}(\tau)\|_{L^2}^2\,d\tau\leq C(t,u_0,\theta_0). \quad (5)$$

Come back to the first equation of (1), namely

$$\partial_t u-\Delta u+\nabla P=g:=-(u\cdot\nabla)u-|u|^{\beta-1}u+\theta e_3. \quad (6)$$

The simple interpolation inequality allows us to deduce

$$\begin{aligned}\|(u\cdot\nabla)u\|_{L^{\frac{3(\beta+1)}{\beta}}}&\leq C\|u\|_{L^{\frac{6(\beta+1)}{\beta-1}}}\|\nabla u\|_{L^6}\\&\leq C\|u\|_{L^6}^{\frac{\beta}{\beta+1}}\|\Delta u\|_{L^2}^{\frac{1}{\beta+1}}\|\Delta u\|_{L^2}\\&\leq C\|\nabla u\|_{L^2}^{\frac{\beta}{\beta+1}}\|\Delta u\|_{L^2}^{\frac{\beta+2}{\beta+1}},\end{aligned}$$

$$\||u|^{\beta-1}u\|_{L^{\frac{3(\beta+1)}{\beta}}}\leq C\|u\|_{L^{3(\beta+1)}}^{\beta},$$

$$\|\theta e_3\|_{L^{\frac{3(\beta+1)}{\beta}}}\leq C\|\theta_0\|_{L^{\frac{3(\beta+1)}{\beta}}}.$$

Due to $\beta\geq 3$, (3), (4) and (5), we deduce

$$\|g\|_{L_t^{\frac{\beta+1}{\beta}}L_x^{\frac{3(\beta+1)}{\beta}}}\leq C(t,u_0,\theta_0).$$

According to the incompressible condition, the equation (6) can be rewritten as

$$\partial_t u-\Delta u=(I+\mathcal{R}_i\mathcal{R}_j)g, \quad (7)$$

where $\mathcal{R}_i$ is the classical Riesz operator and I is the identity operator. Applying operator Δ to equality (7), we have

$$\partial_t\Delta u-\Delta\Delta u=\Delta(I+\mathcal{R}_i\mathcal{R}_j)g,$$

Thanks to the Duhamel Principle, Δu can be solved by

$$\Delta u(x,t) = e^{t\Delta}\Delta u_0(x) + \int_0^t e^{(t-s)\Delta}\Delta(I+\mathcal{R}_i\mathcal{R}_j)g(x,s)\,ds.$$

Now we recall the following maximal $L_t^q L_x^p$ regularity for the heat kernel (see [9])

$$\left\|\int_0^t e^{(t-s)\Delta}\Delta g(s,x)\,ds\right\|_{L_t^q L_x^p} \le C\|g\|_{L_t^q L_x^p}$$

for any $(p,q)\in(1,\infty)^2$ and $T\in(0,\infty]$. For any $1<q<\min\left\{\frac{4(\beta+1)}{3\beta+5},\ \frac{\beta+1}{\beta}\right\}$, we have

$$\begin{aligned}
&\|\Delta u\|_{L_t^q L_x^{\frac{3(\beta+1)}{\beta}}} \\
&\le \|e^{t\Delta}\Delta u_0\|_{L_t^q L_x^{\frac{3(\beta+1)}{\beta}}} + \left\|\int_0^t e^{(t-s)\Delta}\Delta(I+\mathcal{R}_i\mathcal{R}_j)g(x,s)\,ds\right\|_{L_t^q L_x^{\frac{3(\beta+1)}{\beta}}} \\
&\le C(t,u_0,\theta_0) + C\big\|(I+\mathcal{R}_i\mathcal{R}_j)g(x,s)\big\|_{L_t^q L_x^{\frac{3(\beta+1)}{\beta}}} \\
&\le C(t,u_0,\theta_0) + C\|g\|_{L_t^q L_x^{\frac{3(\beta+1)}{\beta}}} \\
&\le C(t,u_0,\theta_0),
\end{aligned} \tag{8}$$

where we have used the following estimate due to the property of the heat operator

$$\begin{aligned}
\|e^{t\Delta}\Delta u_0\|^q_{L_t^q L_x^{\frac{3(\beta+1)}{\beta}}} &= \int_0^t \|e^{t\Delta}\Delta u_0(\tau)\|^q_{L_x^{\frac{3(\beta+1)}{\beta}}}\,d\tau \\
&\le C\int_0^t \tau^{-\frac{q}{2}-\frac{3q}{2}\left(\frac{1}{2}-\frac{\beta}{3(\beta+1)}\right)}\|\nabla u_0\|^q_{L^2}\,d\tau \\
&\le Ct^{1-\frac{(3\beta+5)q}{4(\beta+1)}}\|\nabla u_0\|^q_{L^2} \\
&\le C(t,u_0,\theta_0).
\end{aligned}$$

By means of the bounds (3) and (8), it follows that

$$u\in L^q\big(0,T;W^{2,\frac{3(\beta+1)}{\beta}}(\mathbb{R}^3)\big),\quad 1<q<\min\left\{\frac{4(\beta+1)}{3\beta+5},\ \frac{\beta+1}{\beta}\right\}.$$

Thus, we have

$$\nabla u\in L^1\big(0,T;L^\infty(\mathbb{R}^3)\big).$$

Applying ∇ to the temperature θ equation, one has

$$\partial_t\nabla\theta + (u\cdot\nabla)\nabla\theta = -(\nabla u\cdot\nabla)\theta.$$

Multiplying the above equation by $|\nabla\theta|^{p-2}\nabla\theta$, integrating the resulting equation and using the divergence-free condition, we arrive at

$$\frac{1}{p}\frac{d}{dt}\|\nabla\theta(t)\|_{L^p}^p = -\int_{\mathbb{R}^3}(\nabla u\cdot\nabla)\theta\,|\nabla\theta|^{p-2}\nabla\theta\,dx \le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p}^p.$$

This also implies that $\frac{d}{dt}\|\nabla\theta(t)\|_{L^p} \le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^p}$. By letting $p\to\infty$, we infer

$$\frac{d}{dt}\|\nabla\theta\|_{L^\infty} \le \|\nabla u\|_{L^\infty}\|\nabla\theta\|_{L^\infty}.$$

According to the Gronwall inequality, we obtain

$$\|\nabla\theta(t)\|_{L^\infty} \le \|\nabla\theta_0\|_{L^\infty}\exp\left[\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\right] \le C(t,u_0,\theta_0).$$

The uniqueness to the strong solution can be proved as follows. Let (u,θ,π) and $(\overline{u},\overline{\theta},\overline{\pi})$ are two solutions to system (1) with the initial datum satisfying $u(x,0)=\overline{u}(x,0)$, $\theta(x,0)=\overline{\theta}(x,0)$. Taking the difference and taking the inner product, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|u-\overline{u}\|_{L^2}^2+\|\theta-\overline{\theta}\|_{L^2}^2\big)+\|\nabla(u-\overline{u})\|_{L^2}^2+\int_{\mathbb{R}^3}(|u|^{\beta-1}u-|\overline{u}|^{\beta-1}\overline{u})(u-\overline{u})\,dx\\
&\le \int_{\mathbb{R}^3}|u-\overline{u}|^2\,|\nabla\overline{u}|\,dx+\int_{\mathbb{R}^3}|u-\overline{u}||\theta-\overline{\theta}|\,|\nabla\overline{\theta}|\,dx+\int_{\mathbb{R}^3}|u-\overline{\theta}|\,|\theta-\overline{\theta}|\,dx\\
&\le \|\nabla\overline{u}\|_{L^\infty}\|u-\overline{u}\|_{L^2}^2+\|\nabla\overline{\theta}\|_{L^\infty}\|u-\overline{u}\|_{L^2}\|\theta-\overline{\theta}\|_{L^2}+\|u-\overline{u}\|_{L^2}\|\theta-\overline{\theta}\|_{L^2}\\
&\le C(1+\|\nabla\overline{u}\|_{L^\infty}+\|\nabla\overline{\theta}\|_{L^\infty})(\|u-\overline{u}\|_{L^2}^2+\|\theta-\overline{\theta}\|_{L^2}^2).
\end{aligned}$$

Thanks to the Hölder inequality, we obtain for $\beta\ge 1$

$$\begin{aligned}
&\int_{\mathbb{R}^3}(|u|^{\beta-1}u-|\overline{u}|^{\beta-1}\overline{u})(u-\overline{u})\,dx\\
&=\int_{\mathbb{R}^3}|u|^{\beta+1}\,dx-\int_{\mathbb{R}^3}|\overline{u}|^{\beta-1}\overline{u}u\,dx-\int_{\mathbb{R}^3}|u|^{\beta-1}u\overline{u}\,dx+\int_{\mathbb{R}^3}|\overline{u}|^{\beta+1}\,dx\\
&\ge \|u\|_{L^{\beta+1}}^{\beta+1}-\|\overline{u}\|_{L^{\beta+1}}^{\beta}\|u\|_{L^{\beta+1}}-\|u\|_{L^{\beta+1}}^{\beta}\|\overline{u}\|_{L^{\beta+1}}+\|\overline{u}\|_{L^{\beta+1}}^{\beta+1}\\
&=(\|u\|_{L^{\beta+1}}^{\beta}-\|\overline{u}\|_{L^{\beta+1}}^{\beta})(\|u\|_{L^{\beta+1}}-\|\overline{u}\|_{L^{\beta+1}})\\
&\ge 0.
\end{aligned}$$

We thus get

$$\frac{d}{dt}(\|u-\overline{u}\|_{L^2}^2+\|\theta-\overline{\theta}\|_{L^2}^2)\le C(1+\|\nabla\overline{u}\|_{L^\infty}+\|\nabla\overline{\theta}\|_{L^\infty})(\|u-\overline{u}\|_{L^2}^2+\|\theta-\overline{\theta}\|_{L^2}^2).$$

It follows from the Gronwall inequality that

$$\begin{aligned}
\|u-\overline{u}\|_{L^2}^2+\|\theta-\overline{\theta}\|_{L^2}^2 &\le (\|u(x,0)-\overline{u}(x,0)\|_{L^2}^2+\|\theta(x,0)-\overline{\theta}(x,0)\|_{L^2}^2)\\
&\quad\times\exp\left(C\int_0^t(1+\|\nabla\overline{u}(\tau)\|_{L^\infty}+\|\nabla\overline{\theta}(\tau)\|_{L^\infty})\,d\tau\right).
\end{aligned}$$

Therefore, the uniqueness follows from the fact $u(x,0)=\overline{u}(x,0)$, $\theta(x,0)=\overline{\theta}(x,0)$. This completes the proof of Theorem 1.2.

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