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Stabilization of a Drude/Vacuum Model

Serge Nicaise

Abstract. We analyze the stability of a dispersive medium immersed in vacuum (with Silver–Müller boundary condition in the exterior boundary) or vice versa. The dispersive medium model corresponds to the coupling between Maxwell's system and a first order ordinary differential equation (of parabolic type). For a dispersive medium coupled with vacuum, the ordinary differential equation will be set in a subset of the full domain. We show that this model is well-posed and is strongly stable in a closed subspace of the energy space. We further identify some sufficient conditions that guarantee the exponential or polynomial decay of the associated energy in this subspace.

Keywords. Dispersive media, stabilization

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1. Introduction

Dispersion is a well-known phenomenon that is characterized by the fact that all frequencies of a polychromatic wave do not travel at the same speed through the medium. One example is the Maxwell–Drude (or cold plasma) model [10, 18–20, 28, 29] given by

$$\begin{cases} \varepsilon_0\varepsilon_\infty E_t - \operatorname{curl} H = -J & \text{in } Q_m = \Omega_m \times (0,+\infty), \\ \mu_0 H_t + \operatorname{curl} E = 0 & \text{in } Q_m, \\ J_t + \gamma J = \varepsilon_0\omega_p^2 E & \text{in } Q_m, \end{cases} \tag{1.1}$$

where E (resp. H) is the electric (resp. magnetic) field and J is the dipolar current vector. The different positive parameters are ε_0 the permittivity of the free space, μ_0 the permeability of the free space, ϵ_∞ the permittivity at infinite frequency, ω_p the plasma frequency of the electrons and γ the electron-neutral collision frequency. Here and below $E_t = \frac{\partial E}{\partial t}$ is the partial derivative of E with

S. Nicaise: Université de Valenciennes, EA 4015 - LAMAV - Laboratoire de Mathématiques et leurs Applications de Valenciennes, FR CNRS 2956, F-59313 Valenciennes, France; Serge.Nicaise@univ-valenciennes.fr

respect to the time t, while in this introduction n denotes the unit outer normal vector along the boundary. Note that by renormalisation, we can assume that $\epsilon_0 = \mu_0 = 1$.

In practical applications, the dispersive medium corresponds to a piece of metal (gold or silver for instance). In this paper we are interested in two particular situations: first the case when a "small" piece of metal is immersed in the whole free space, the resulting system being a coupling between the Maxwell–Drude equation in Ω_m with Maxwell's system in $\mathbb{R}^3 \setminus \Omega_m$; second, the case when a "large" piece of metal contains vacuum. In both cases, to reduce the problem to a bounded (or relatively small) domain (for computational purposes for instance), two commonly used strategies can be mentioned: either use absorbing boundary conditions or use a perfectly matched layer. Here we restrict ourselves to the first strategy. In the first case we replace the problem set in the whole space to a bounded domain Ω containing Ω_m with absorbing boundary conditions on the exterior boundary Γ, see [28, §6.2]. Therefore the final problem is the next one

$$\begin{cases} \varepsilon E_t - \operatorname{curl} H = -1_{\Omega_m} \tilde{J} & \text{in } Q = \Omega \times (0, +\infty), \\ H_t + \operatorname{curl} E = 0 & \text{in } Q, \\ J_t + \gamma J = \omega_p^2 E & \text{in } Q_m, \\ E_T - H \times n = \mathbf{0} & \text{on } \Gamma \times (0, +\infty), \\ E(x,0) = E_0(x), \quad H(x,0) = H_0(x) & \text{in } \Omega, \\ J(x,0) = J_0(x) & \text{in } \Omega_m, \end{cases} \tag{1.2}$$

where $\tilde{J}$ is any extension of J outside Ω_m, 1_{Ω_m} is the characteristic function of Ω_m (equal to 1 in Ω_m and 0 elsewhere),

$$\varepsilon = \begin{cases} \varepsilon_\infty & \text{in } \Omega_m, \\ 1 & \text{in } \Omega_e = \Omega \setminus \bar{\Omega}_m, \end{cases}$$

and $E_T = n \times (E \times n)$ is the tangential component of E along $\Gamma \times (0, +\infty)$. The boundary condition on $\Gamma \times (0, +\infty)$ corresponds to the so-called Silver–Müller boundary condition.

In the second situation, Ω_e corresponds to vacuum, while we truncate the original domain Ω_m into a smallest one (still called Ω_m for simplicity) and again imposed Silver–Müller boundary condition on the boundary Γ of the truncated domain. At the end, we arrive at the same problem (1.2), but contrary to the first case, Ω_m here surrounds Ω_e.

Before going on, we notice that the original problem (1.2) implies some hidden constraints on the divergence of H, E and J. Namely, the second equation in (1.2) yields

$$(\operatorname{div} H)_t = 0 \quad \text{in } Q,$$

therefore if we assume the divergence free property of H at $t = 0$, it will remain valid for all $t > 0$. Similarly the first equation in (1.2) implies that

$$(\operatorname{div} E)_t = 0 \quad \text{in } Q_e = \Omega_e \times (0, +\infty),$$

consequently the divergence free property of E in Ω_e at $t = 0$ will guarantee the same property at all $t > 0$. Finally the first and third equations in (1.2) imply that

$$\begin{aligned} \epsilon_\infty (\operatorname{div} E)_t + \operatorname{div} J = 0 \qquad & \text{in } Q_m, \\ (\operatorname{div} J)_t + \gamma \operatorname{div} J = \omega_p^2 \operatorname{div} E \quad & \text{in } Q_m, \end{aligned}$$

or in matrix form

$$D_t = MD \quad \text{in } Q_m,$$

where $D = (\operatorname{div} E, \operatorname{div} J)^\top$ and M is the 2×2 matrix

$$M = \begin{pmatrix} 0 & -\epsilon_\infty^{-1} \\ \omega_p^2 & -\gamma \end{pmatrix}.$$

Hence for all $t \geq 0$, one has $D(t, \cdot) = e^{tM} D(t = 0, \cdot)$ and consequently if E and J are divergence free in Ω_e at $t = 0$, they will remain divergence free forever.

The model (1.1) is the simplest one among Maxwell's equations for dispersive media, see [28] and it was shown in [24, Theorem 4.12] that it is polynomially stable, namely that its energy decays like t^{-1} for sufficiently smooth initial data. Mathematically it corresponds to the coupling between Maxwell's system in E and H with a first order ordinary differential equation of parabolic type in J, and this last equation is responsible of this decay. On the other hand, the full Maxwell system in Ω with the Silver–Müller boundary condition (corresponding to (1.2) with $\Omega_m = \emptyset$) is exponentially stable under some geometrical conditions on Ω, see [15, 16, 26]. Therefore the natural question to raise is whether the system (1.2) is stable or not, and if yes, determine the decay rate of its energy. The main goal of this paper is to answer to these questions. The strong stability (into a closed subspace of the energy space) is obtained with the help of Arendt–Batty/Lyubich–Vũ theorem [2, 22]. Here the main difficulty relies on the non-compactness of the resolvent of the associated operator, but eliminating some variables and perturbating the system, we can fall into a compact perturbation argument. Our decay results are based on a frequency domain approach [7, 13, 27] and use the exponential decay of the full Maxwell system with the Silver–Müller boundary condition.

The paper is organized as follows: in Section 2 we introduce some notations, some function spaces and prove some equivalences of norms. Section 3 is devoted to the well-posedness of the problem, that is proved using semi-group theory. The strong stability of the system is analyzed in Section 4, and finally Section 5 is devoted to the exponential or polynomial decay of the energy under appropriate sufficient conditions.

In the whole paper, the notation $A \lesssim B$ is used for the estimate $A \leq C\,B$, where C is a generic constant that does not depend on A and B. The notation $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold.

2. Preliminaries

In the whole paper Ω will be a non empty bounded and simply connected domain of $\mathbb{R}^3$ with a connected and Lipschitz boundary Γ. As suggested in the introduction, we consider two cases, illustrated in Figures 1 and 2 respectively:

Case A: We fix Ω_m a non empty open subset of Ω with a Lipschitz and connected boundary Σ such that $\bar{\Omega}_m \subset \Omega$ and set $\Omega_e = \Omega \setminus \bar{\Omega}_m$, whose boundary is made of two connected components Σ and Γ.

Case B: We fix Ω_e a non empty open subset of Ω with a Lipschitz and connected boundary Σ such that $\bar{\Omega}_e \subset \Omega$ and set $\Omega_m = \Omega \setminus \bar{\Omega}_e$, whose boundary is made of two connected components Σ and Γ.

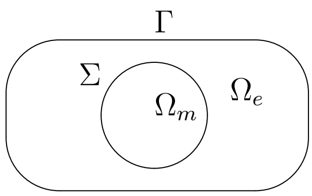

Figure 1: Illustration of the Case A

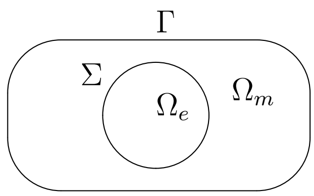

Figure 2: Illustration of the Case B

In both cases, for a function u from Ω to $\mathbb{C}$, we will denote by u_m (resp. u_e), its restriction to Ω_m (resp. Ω_e).

For a subset $\mathcal{O}$ of Ω or of its boundary and a real number s, $H^s(\mathcal{O})$ is the usual Sobolev space defined in $\mathcal{O}$ and for shortness we denote $\mathbf{H}^s(\mathcal{O}) = H^s(\mathcal{O})^3$ and $\mathbf{L}^2(\mathcal{O}) = L^2(\mathcal{O})^3$. Their norm (resp. semi-norm) will be denoted by $\|\cdot\|_{s,\mathcal{O}}$ (resp. $|\cdot|_{s,\mathcal{O}}$); for $s = 0$, we drop the index 0 and for $\mathcal{O} = \Omega$, we also drop the index Ω. The duality pairing between $H^{-\frac{1}{2}}(\Sigma)$ and $H^{\frac{1}{2}}(\Sigma)$ will be denoted by $\langle \cdot\,; \cdot \rangle_\Sigma$. As usual $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with a zero trace on the boundary. The unit outer normal vector along $\partial\Omega$ (resp. $\partial\Omega_m$, $\partial\Omega_e$) defined almost everywhere will be denoted by n (resp. n_m, n_e).

We recall that

$$\begin{aligned}
\mathbf{H}(\mathrm{div}, \Omega) &= \{U \in \mathbf{L}^2(\Omega) : \mathrm{div}\, U \in L^2(\Omega)\},\\
\mathbf{H}(\mathrm{curl}, \Omega) &= \{U \in \mathbf{L}^2(\Omega) : \mathrm{curl}\, U \in \mathbf{L}^2(\Omega)\},\\
\mathbf{H}_0(\mathrm{curl}, \Omega) &= \{U \in \mathbf{H}(\mathrm{curl}, \Omega) : U \times n = 0 \text{ on } \partial\Omega\},\\
X_N(\Omega) &= \mathbf{H}_0(\mathrm{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega),\\
W &= \{U \in \mathbf{H}(\mathrm{curl}, \Omega) : U \times n \in \mathbf{L}^2(\Gamma)\},\\
W_0 &= \{U \in \mathbf{H}(\mathrm{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega) : U \times n \in \mathbf{L}^2(\Gamma)\},
\end{aligned}$$

are Hilbert spaces with their natural norm, in particular

$$\|U\|_W^2 = \|U\|^2 + \|\operatorname{curl} U\|^2 + \|U \times n\|_\Gamma^2, \quad \forall U \in W,$$
$$\|U\|_{W_0}^2 = \|U\|^2 + \|\operatorname{curl} U\|^2 + \|\operatorname{div} U\|^2 + \|U \times n\|_\Gamma^2, \quad \forall U \in W_0.$$

Recall that the main theorem of [6] states that $C^\infty(\bar{\Omega})^3$ is dense in W and by a density argument (see [25, Lemma 2.2]) we deduce that the next Green formula holds:

$$\int_\Omega \left(\operatorname{curl} E \cdot \bar{H} - E \cdot \operatorname{curl} \bar{H}\right) dx = \int_\Gamma E_T \cdot \bar{H} \times n \, d\sigma, \quad \forall E, H \in W. \tag{2.1}$$

Furthermore, according to [8, Theorem 2], W_0 is embedded into $\mathbf{H}^{\frac{1}{2}}(\Omega)$. From our assumptions on Ω, we deduce the

Lemma 2.1. *The semi-norm*

$$|U|_{W_0} = (\|\operatorname{curl} U\|^2 + \|\operatorname{div} U\|^2 + \|U \times n\|_\Gamma^2)^{\frac{1}{2}}, \quad \forall U \in W_0,$$

is a norm in W_0 equivalent to the natural one. In other words, we have

$$\|U\| \lesssim |U|_{W_0}, \quad \forall U \in W_0.$$

Proof. The proof is based on a contradiction argument, the compact embedding of W_0 into $\mathbf{L}^2(\Omega)$ and the fact that the sole element $U \in W_0$ such that $|U|_{W_0} = 0$ is $U = \mathbf{0}$ as the boundary of Ω is connected (see [1, Proposition 3.18]). □

In Case B, the boundary of Ω_e is connected, and the space

$$K_N(\Omega_e) = \{U \in X_N(\Omega_e) : \operatorname{curl} U = \mathbf{0} \text{ and } \operatorname{div} U = 0 \quad \text{in } \Omega_e\}$$

is then reduced to $\{\mathbf{0}\}$ (see [1, Proposition 3.18]). On the contrary in Case A, as the boundary of Ω_e is not connected, the space $K_N(\Omega_e)$ is not reduced to $\{\mathbf{0}\}$ (see again [1, Proposition 3.18]), but is one-dimensional and spanned by ∇q_0, where $q_0 \in H^1(\Omega_e)$ satisfies

$$\begin{cases} \Delta q_0 = 0 & \text{in } \Omega_e, \\ q_0 = 1 & \text{in } \Sigma, \\ q_0 = 0 & \text{in } \Gamma. \end{cases}$$

As Green's formula yields

$$\langle \nabla q_0 \cdot n_e; 1 \rangle_\Sigma = \langle \nabla q_0 \cdot n_e; q_0 \rangle_\Sigma = \int_{\Omega_e} |\nabla q_0|^2 \, dx, \tag{2.2}$$

and since q_0 is not identically equal to 0, we deduce that

$$\langle \nabla q_0 \cdot n_e; 1\rangle_\Sigma > 0. \tag{2.3}$$

For Σ smooth enough, this property can be alternatively obtained using Hopf Lemma.

Below we also extend the function q_0 by 1 in Ω_m and denote this extension by φ_0, namely

$$\varphi_0 = \begin{cases} 1 & \text{in } \Omega_m, \\ q_0 & \text{in } \Omega_e. \end{cases} \tag{2.4}$$

Clearly φ_0 belongs to $H_0^1(\Omega)$.

For further uses, let us also introduce the spaces

$$\tilde{Y}_\tau = \{U \in \mathbf{H}(\mathrm{curl}, \Omega) | \operatorname{div} U_e \in L^2(\Omega_e), \operatorname{div} U_m \in L^2(\Omega_m) \text{ and } U \times n \in \mathbf{L}^2(\Gamma)\},$$
$$Y_\tau = \{U \in \tilde{Y}_\tau | \langle U_e \cdot n; 1\rangle_\Sigma = 0\},$$

that are Hilbert spaces with norm

$$\|U\|_{Y_\tau}^2 = \|U\|^2 + \|\operatorname{curl} U\|^2 + \|\operatorname{div} U_m\|_{\Omega_m}^2 + \|\operatorname{div} U_e\|_{\Omega_e}^2 + \|U \times n\|_\Gamma^2, \quad \forall U \in \tilde{Y}_\tau.$$

Note that $Y_\tau = \tilde{Y}_\tau$ in Case B, and that the space Y_τ differs from the space $Y(\Omega)$ defined in [9, p. 811] by the boundary conditions on Γ. Nevertheless, the next Lemma gives a result similar to the one from [9, Lemma 2.2].

Lemma 2.2. *The semi-norm*

$$|U|_{Y_\tau} = \left(\|U_m\|_{\Omega_m}^2 + \|\operatorname{curl} U\|^2 + \|\operatorname{div} U_m\|_{\Omega_m}^2 + \|\operatorname{div} U_e\|_{\Omega_e}^2 + \|U \times n\|_\Gamma^2\right)^{\frac{1}{2}}, \quad \forall U \in Y_\tau,$$

is a norm in Y_τ equivalent to the natural one. In other words, we have

$$\|U_e\|_{\Omega_e} \lesssim |U|_{Y_\tau}, \quad \forall U \in Y_\tau. \tag{2.5}$$

Proof. Fix $U \in Y_\tau$. As it is not divergence free in Ω, we subtract from it $\nabla\varphi$, with $\varphi \in H_0^1(\Omega)$ solution of

$$\int_\Omega \nabla\varphi \cdot \nabla\chi \, dx = \int_\Omega U \cdot \nabla\chi \, dx, \quad \forall \chi \in H_0^1(\Omega).$$

As a consequence, $V = U - \nabla\varphi$ is divergence free in Ω, hence belongs to W_0 and owing to Lemma 2.1 we have

$$\|V\| \lesssim |V|_{W_0} \lesssim \|\operatorname{curl} U\| + \|U \times n\|_\Gamma. \tag{2.6}$$

Furthermore, in Case A, according to [9, Lemma 2.1], we have the estimate

$$|\varphi|_{1,\Omega_e} \lesssim \|\operatorname{div} U_e\|_{\Omega_e} + \|\varphi\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} + |\langle V_e \cdot n; 1\rangle_\Sigma|\,, \tag{2.7}$$

where we recall that $\|\varphi\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} = \inf_{c\in\mathbb{C}} \|\varphi - c\|_{H^{\frac{1}{2}}(\Sigma)}$. On the contrary, in Case B, we clearly have

$$|\varphi|_{1,\Omega_e} \lesssim \|\operatorname{div} U_e\|_{\Omega_e} + \|\varphi\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}},$$

so that (2.7) still holds, since $\langle V_e \cdot n; 1\rangle_\Sigma = \int_{\Omega_e} \operatorname{div} V_e \, dx = 0$. As

$$\begin{aligned}\|U_e\|_{\Omega_e} &\lesssim \|V_e\|_{\Omega_e} + |\varphi|_{1,\Omega_e} \\ &\lesssim \|\operatorname{curl} U\| + \|U\times n\|_\Gamma + \|\operatorname{div} U_e\|_{\Omega_e} + \|\varphi\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} + |\langle V_e\cdot n;1\rangle_\Sigma|\,,\end{aligned} \tag{2.8}$$

it remains to estimate the two last terms of this right-hand side. For the last term, as mentioned before it is zero in Case B, while in Case A, as V is divergence free, we have

$$\langle V_e \cdot n; 1\rangle_\Sigma = \langle V_e \cdot n; q_0\rangle_\Sigma = \int_{\Omega_e} V_e \cdot \nabla q_0 \, dx,$$

and by Cauchy–Schwarz's inequality, we deduce that $|\langle V_e \cdot n; 1\rangle_\Sigma| \lesssim \|V_e\|_{\Omega_e}$. Thanks to (2.6), we obtain

$$|\langle V_e \cdot n; 1\rangle_\Sigma| \lesssim \|\operatorname{curl} U\| + \|U \times n\|_\Gamma. \tag{2.9}$$

For the term $\|\varphi\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}}$, by the trace theorem in $\mathbf{H}(\mathrm{curl}, \Omega_m)$ (see [11, Theorem I.2.11]), we have $\|\varphi\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} \lesssim \|\nabla_T\varphi\|_{H^{-\frac{1}{2}}(\Sigma)} \lesssim \|U \times n\|_{H^{-\frac{1}{2}}(\Sigma)} + \|V\times n\|_{H^{-\frac{1}{2}}(\Sigma)} \lesssim \|U\|_{\mathbf{H}(\mathrm{curl},\Omega_m)} + \|V\|_{\mathbf{H}(\mathrm{curl},\Omega_m)}$. Again by (2.6) and the fact that $\operatorname{curl} V = \operatorname{curl} U$, we obtain

$$\|\varphi\|_{H^{\frac{1}{2}}(\Sigma)/\mathbb{C}} \lesssim \|U_m\|_{\Omega_m} + \|\operatorname{curl} U\| + \|U \times n\|_\Gamma. \tag{2.10}$$

The two estimates (2.9) and (2.10) in (2.8) directly lead to (2.5). □

3. Well-posedness of the system

To prove an existence result for system (1.2), we re-write it in the following framework:

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{3.1}$$

where U is the vectorial unknown

$$U = \begin{pmatrix} E \\ H \\ J \end{pmatrix},$$

where $E, H \in \mathbf{L}^2(\Omega)$ and $J \in \mathbf{L}^2(\Omega_m)$, and for smooth enough E, H and J

$$\mathcal{A}U = \begin{pmatrix} \epsilon^{-1}(\operatorname{curl} H - 1_{\Omega_m}\tilde{J}) \\ -\operatorname{curl} E \\ -\gamma J + \omega_p^2 E \end{pmatrix}. \tag{3.2}$$

The existence of a solution to (3.1) is obtained by using semi-group theory in the appropriate Hilbert setting described here below (see for instance [17, 23]). The considerations on the divergence properties on E, H and J from the introduction suggest introducing

$$J(\Omega) = \{U \in \mathbf{L}^2(\Omega) | \operatorname{div} U = 0\},$$
$$J_m(\Omega) = \{U \in \mathbf{L}^2(\Omega) | \operatorname{div} U = 0 \text{ in } \Omega_m \cup \Omega_e\},$$

and the Hilbert space

$$\mathcal{H} = J_m(\Omega) \times J(\Omega) \times J(\Omega_m),$$

with the inner product

$$\big((E, H, J), (E', H', J')\big)_{\mathcal{H}} := \int_\Omega (\epsilon E \cdot \bar{E}' + H \cdot \bar{H}')\, dx + \int_{\Omega_m} \omega_p^{-2} J \cdot \bar{J}'\, dx,$$

$\forall\, (E, H, J), (E', H', J') \in \mathcal{H}$.

We now define the operator $\mathcal{A}$ as follows:

$$D(\mathcal{A}) := \left\{ (E, H, P) \in \mathcal{H} \,\middle|\, \begin{array}{l} E, H \in W \text{ satisfying the Silver–Müller} \\ \text{boundary condition } E_T - H \times n = \mathbf{0} \text{ on } \Gamma. \end{array} \right\} \tag{3.3}$$

and for all $U = (E, H, P) \in D(\mathcal{A})$, $\mathcal{A}U$ is given by (3.2).

Let us check that $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.

Theorem 3.1. *The operator $\mathcal{A}$ defined by* (3.2) *with domain* (3.3) *generates a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ on $\mathcal{H}$. Therefore for all $U_0 \in \mathcal{H}$, problem* (3.1) *has a weak solution $U \in C([0, \infty), H)$ given by $U(t) = T(t)U_0$, for all $t \geq 0$. If moreover $U_0 \in D(\mathcal{A}^k)$, with $k \in \mathbb{N}^*$, problem* (3.1) *has a strong solution $U \in C([0, \infty), D(\mathcal{A}^k)) \cap C^1([0, \infty), D(\mathcal{A}^{k-1}))$.*

Proof. It suffices to show that $\mathcal{A}$ is a maximal dissipative operator, then by Lumer–Phillips' theorem it generates a C_0-semigroup of contractions $(T(t))_{t\geq 0}$ on $\mathcal{H}$.

Let us first show the dissipativeness. Let $U = (E, H, J)^\top \in D(\mathcal{A})$ be fixed. Then by the definition of $\mathcal{A}$, we have

$$(\mathcal{A}U, U)_{\mathcal{H}} = \int_\Omega \Big((\operatorname{curl} H - 1_{\Omega_m}\tilde{J}) \cdot \bar{E} - \operatorname{curl} E \cdot \bar{H} \Big)\, dx + \int_{\Omega_m} \omega_p^{-2}(-\gamma J + \omega_p^2 E) \cdot \bar{J}\, dx.$$

Taking the real part of this identity, we find

$$\Re(\mathcal{A}U,U)_{\mathcal{H}} = \Re\int_{\Omega}\left(\operatorname{curl} H\cdot\bar{E} - \operatorname{curl} E\cdot\bar{H}\right)\,dx - \omega_p^{-2}\gamma\int_{\Omega_m}|J|^2\,dx.$$

Using Green's formula (2.1) and the Silver–Müller boundary condition $E_T - H\times n = \mathbf{0}$ on Γ, the previous identity becomes

$$\Re(\mathcal{A}U,U)_{\mathcal{H}} = -\omega_p^{-2}\gamma\int_{\Omega_m}|J|^2\,dx - \int_{\Gamma}|E\times n|^2\,d\sigma \le 0. \tag{3.4}$$

This shows that $\mathcal{A}$ is dissipative.

Let us go on with the maximality. Let $\lambda > 0$ be fixed. For $(F,G,R)^\top\in\mathcal{H}$, we look for $U=(E,H,J)^\top\in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = (F,G,R)^\top. \tag{3.5}$$

According to (3.2) this is equivalent to

$$\epsilon\lambda E - \operatorname{curl} H + 1_{\Omega_m}\tilde{J} = \epsilon F, \tag{3.6}$$

$$\lambda H + \operatorname{curl} E = G, \tag{3.7}$$

$$\lambda J + \gamma J - \omega_p^2 E = R. \tag{3.8}$$

Assume for the moment that U exists. Then the first and second equations allow to eliminate J and E since they are equivalent to

$$J = \frac{1}{\lambda+\gamma}(\omega_p^2 E + R) \quad \text{in } \Omega_m, \tag{3.9}$$

$$E = \alpha(\lambda)\operatorname{curl} H + \alpha(\lambda)\left(\epsilon F - 1_{\Omega_m}\frac{1}{\lambda+\gamma}\tilde{R}\right) \quad \text{in } \Omega, \tag{3.10}$$

where

$$\alpha(\lambda) = \begin{cases} \left(\epsilon_\infty\lambda + \frac{\omega_p^2}{\lambda+\gamma}\right)^{-1} & \text{in } \Omega_m,\\ \lambda^{-1} & \text{in } \Omega_e. \end{cases} \tag{3.11}$$

Thus multiplying (3.7) by a test function $\bar{H}'\in W$ and using Green's formula (2.1), we obtain

$$\int_{\Omega}\left(\lambda H\cdot\bar{H}' + E\cdot\operatorname{curl}\bar{H}'\right)\,dx + \int_{\Gamma}E_T\cdot(\bar{H}'\times n)\,d\sigma = \int_{\Omega}G\cdot\bar{H}'\,dx.$$

Using the Silver–Müller boundary condition and the expression (3.10), we arrive at

$$\int_{\Omega}\left(\lambda H\cdot\bar{H}' + \alpha(\lambda)\operatorname{curl} H\cdot\operatorname{curl}\bar{H}'\right)\,dx + \int_{\Gamma}(H\times n)\cdot(\bar{H}'\times n)\,d\sigma = L_\lambda(H'), \tag{3.12}$$

for all $H' \in W$, where we have set

$$L_\lambda(H') = \int_\Omega \left(G \cdot \bar{H}' - \alpha(\lambda) \left(\epsilon F - 1_{\Omega_m} \frac{1}{\lambda + \gamma} \tilde{R} \right) \cdot \operatorname{curl} \bar{H}' \right) dx. \tag{3.13}$$

As H is divergence free, we can add to the left-hand side of (3.12) the term

$$\int_\Omega \operatorname{div} H \operatorname{div} \bar{H}' \, dx,$$

that is zero and find an augmented formulation:

$$a_\lambda(H, H') = L_\lambda(H'), \quad \forall\, H' \in W_0, \tag{3.14}$$

where for all $H, H' \in W_0$, we have set

$$\begin{aligned} a_\lambda(H, H') = & \int_\Omega \left(\lambda H \cdot \bar{H}' + \alpha(\lambda) \operatorname{curl} H \cdot \operatorname{curl} \bar{H}' + \operatorname{div} H \operatorname{div} \bar{H}' \right) dx \\ & + \int_\Gamma (H \times n) \cdot (\bar{H}' \times n) \, d\sigma. \end{aligned} \tag{3.15}$$

Clearly L_λ is a continuous and linear form on W_0, while a_λ is a continuous, sesquilinear and coercive form on W_0. Hence by Lax-Milgram lemma, problem (3.14) has a unique solution $H \in W_0$. As G is divergence free in Ω, H will be also divergence free. Indeed for any $h \in L^2(\Omega)$, as test-function in (3.14), we take $H' = \nabla\psi$, with $\psi \in H_0^1(\Omega)$ the unique variational solution of

$$\Delta\psi - \lambda\psi = h \quad \text{in } \Omega.$$

With such a choice, we find $\int_\Omega \left(\lambda H \cdot \nabla\bar{\psi} + \operatorname{div} H \Delta\bar{\psi} \right) dx = \int_\Omega G \cdot \nabla\bar{\psi} \, dx$. As G is divergence free, by Green's formula, the previous identity becomes

$$\int_\Omega \operatorname{div} H \left(-\lambda\bar{\psi} + \Delta\bar{\psi} \right) dx = \int_\Omega \operatorname{div} H \, \bar{h} \, dx = 0,$$

and since h is arbitrary in $L^2(\Omega)$, we deduce that $\operatorname{div} H = 0$ in Ω. Now since any element H' of W can be written as

$$H' = H_0 + \nabla\psi,$$

with $\psi \in H_0^1(\Omega)$ and $H_0 \in W_0$, (3.14) remains valid for test-functions in W, namely we have

$$a_\lambda(H, H') = L_\lambda(H'), \quad \forall\, H' \in W. \tag{3.16}$$

We would like to come back to problem (3.5), with H in hand. We thus define E by (3.10) that clearly belongs to $\mathbf{L}^2(\Omega)$. Similarly we define J by (3.9) that belongs to $\mathbf{L}^2(\Omega_m)$. First we notice that using (3.10) in (3.16), we get equivalently

$$\int_\Omega \left(\lambda H\cdot\bar{H}'+E\cdot\operatorname{curl}\bar{H}'\right) dx+\int_\Gamma (H\times n)\cdot(\bar{H}'\times n)\, d\sigma=\int_\Omega G\cdot\bar{H}'dx, \quad \forall\, H'\in W. \tag{3.17}$$

Hence taking test-functions $H' \in \mathcal{D}(\Omega)^3$, we find that

$$\lambda H + \operatorname{curl} H = G \quad \text{in } \mathcal{D}'(\Omega)^3.$$

This implies that $\operatorname{curl} H$ belongs to $\mathbf{L}^2(\Omega)$ and that (3.5) holds. With this regularity in hand, in (3.17), taking test-functions $H' \in \mathbf{H}^1(\Omega)$, using Green's formula (I.2.22) of [11] and the previous identity, we find that

$$\int_\Gamma (H \times n) \cdot (\bar{H}' \times n)\, d\sigma - \langle E; H' \times n\rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma)-\mathbf{H}^{\frac{1}{2}}(\Gamma)} = 0, \quad \forall\, H' \in \mathbf{H}^1(\Omega).$$

This implies that

$$H \times n = E_T \quad \text{in } \mathbf{H}^{-\frac{1}{2}}(\Gamma),$$

and, as $E \times n$ belongs to $\mathbf{L}^2(\Gamma)$, $H \times n$ as well and the Silver–Müller boundary condition holds.

The surjectivity of $\lambda I - \mathcal{A}$ finally holds because (3.6) and (3.8) yield

$$\begin{aligned} \epsilon\lambda \operatorname{div} E &= \epsilon \operatorname{div} F = 0 \quad \text{in } \Omega_e,\\ \epsilon\lambda \operatorname{div} E + \operatorname{div} J &= \epsilon \operatorname{div} F = 0 \quad \text{in } \Omega_m,\\ (\lambda+\gamma)\operatorname{div} J - \omega_p^2 \operatorname{div} E &= \quad \operatorname{div} R = 0 \quad \text{in } \Omega_m. \end{aligned}$$

Hence E is clearly divergence free in Ω_e. On the other hand the second and third identity can be written in the matrix form

$$\begin{pmatrix} \epsilon\lambda & 1 \\ -\omega_p^2 & \lambda+\gamma \end{pmatrix} \begin{pmatrix} \operatorname{div} E \\ \operatorname{div} J \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega_m.$$

That leads to $\operatorname{div} E = \operatorname{div} J = 0$ in Ω_e since the determinant of the above 2×2 matrix is equal to $\epsilon\lambda(\lambda+\gamma) + \omega_p^2$ and is clearly positive. □

4. Strong stability

One simple way to prove the strong stability of (3.1) is to use the following theorem due to Arendt–Batty and Lyubich–Vũ (see [2, 22]).

Theorem 4.1 (Arendt–Batty/Lyubich–Vũ)**.** *Let X be a reflexive Banach space and $(T(t))_{t\geq 0}$ be a C_0-semigroup generated by A on X. Assume that $(T(t))_{t\geq 0}$ is bounded and that no eigenvalues of A lie on the imaginary axis. If $\sigma(A) \cap i\mathbb{R}$ is countable, then $(T(t))_{t\geq 0}$ is stable.*

Since the resolvent of our operator is not compact, we have to analyze the full spectrum on the imaginary axis. This is done in the next Lemmas.

Lemma 4.2. *For all* $\xi \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$, *we have*

$$\ker(i\xi - \mathcal{A}) = \{0\}.$$

Furthermore in Case A, 0 *is an eigenvalue of* $\mathcal{A}$ *whose associated eigenvector is* $(\nabla\varphi_0, 0, 0)^\top$, *where* φ_0 *is defined by* (2.4), *otherwise* $\ker \mathcal{A} = \{0\}$.

Proof. Let $\xi \in \mathbb{R}$ and $U = (E, H, J)^\top \in D(\mathcal{A})$ be such that $(i\xi - \mathcal{A})U = 0$, or equivalently

$$\epsilon\imath\xi E - \operatorname{curl} H + 1_{\Omega_m}\tilde{J} = \mathbf{0}, \tag{4.1}$$
$$\imath\xi H + \operatorname{curl} E = \mathbf{0}, \tag{4.2}$$
$$(i\xi + \gamma)J - \omega_p^2 E = \mathbf{0}. \tag{4.3}$$

By the dissipativeness of $\mathcal{A}$ (identity (3.4)), we get

$$J = \mathbf{0} \quad \text{in } \Omega_m, \tag{4.4}$$
$$E \times n = \mathbf{0} \quad \text{on } \Gamma. \tag{4.5}$$

By (4.3) and the Silver–Müller boundary condition, we find

$$E = \mathbf{0} \quad \text{in } \Omega_m, \tag{4.6}$$
$$H \times n = \mathbf{0} \quad \text{on } \Gamma. \tag{4.7}$$

We remark that (4.1) is now equivalent to

$$\imath\xi E - \operatorname{curl} H = \mathbf{0}, \tag{4.8}$$

since, by (4.6), $\epsilon E = E$ in Ω and recalling (4.4).

Now for $\xi \neq 0$, by (4.8), $E = \frac{1}{i\xi} \operatorname{curl} H$, and by (4.2), we arrive at

$$-\xi^2 H + \operatorname{curl}\operatorname{curl} H = \mathbf{0} \quad \text{in } \Omega.$$

As (4.2) and (4.6) lead to $H = \mathbf{0}$ in Ω_m, by Holmgren's theorem, we conclude that $H = \mathbf{0}$ in Ω and hence $E = \mathbf{0}$ in Ω.

In the case $\xi = 0$, owing to (4.4), the identities (4.1) and (4.2) reduce to

$$\operatorname{curl} E = \operatorname{curl} H = \mathbf{0}.$$

And by (4.5) and (4.7) (recalling that Ω is simply connected), there exist $\varphi, \psi \in H_0^1(\Omega)$ such that

$$E = \nabla\varphi, \quad H = \nabla\psi.$$

By the divergence free property of H, we directly get $\psi = 0$. For φ, by (4.6), there exists a constant $c \in \mathbb{C}$ such that

$$\varphi = c \quad \text{in } \Omega_m.$$

And by the divergence free property of E in Ω_e, we deduce that $\varphi = c\varphi_0$ in Case A, otherwise $\varphi = 0$. □

Lemma 4.3. *For all $\xi \in \mathbb{R}^*$, $i\xi - \mathcal{A}$ is surjective.*

Proof. For any $\xi \in \mathbb{R}^*$ and any $(F, G, R)^\top \in \mathcal{H}$, we look for a unique solution $U = (E, H, J)^\top \in D(\mathcal{A})$ to

$$(i\xi - \mathcal{A})U = (F, G, R)^\top,$$

or equivalently to (compare with (3.6)–(3.8))

$$\epsilon\imath\xi E - \operatorname{curl} H + 1_{\Omega_m}\tilde{J} = \epsilon F, \tag{4.9}$$

$$\imath\xi H + \operatorname{curl} E = G, \tag{4.10}$$

$$\imath\xi J + \gamma J - \omega_p^2 E = R. \tag{4.11}$$

Following the arguments of the proof of Theorem 3.1, if a solution $U \in D(\mathcal{A})$ of (4.9)–(4.11) exists, then H belongs to W_0 and is solution of

$$a_{\imath\xi}(H, H') = L_{\imath\xi}(H'), \quad \forall\, H' \in W_0, \tag{4.12}$$

where a_λ and L_λ have been defined in (3.15) and (3.13). Dropping the term $\imath\xi H \cdot \bar{H}'$ in $a_{\imath\xi}$, we get the sesquilinear form

$$a_{0,\imath\xi}(H, H') = \int_\Omega \left(\alpha(\imath\xi) \operatorname{curl} H \cdot \operatorname{curl} \bar{H}' + \operatorname{div} H \operatorname{div} \bar{H}'\right) dx + \int_\Gamma (H \times n) \cdot (\bar{H}' \times n)\, d\sigma.$$

Multiplying $a_{0,\imath\xi}(H, H')$ by $\imath\xi e^{i\theta}$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we get the form

$$b(H, H') = \imath\xi e^{i\theta} a_{0,\imath\xi}(H, H'),$$

and check that it is coercive for an appropriated choice of θ. Owing to Lemma 2.1, it suffices to see that there exists $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$\Re b(H, H) \geq C(\omega)|H|^2_{W_0}, \quad \forall\, H \in W_0,$$

for some positive constant $C(\omega)$ (that may depend on ω). To obtain this estimate, we only need to impose that

$$\Re\left(\imath\xi e^{i\theta}\right) > 0 \quad \text{and} \quad \Re\left(\imath\xi e^{i\theta}\alpha(\imath\xi)\right) > 0,$$

that, in view of the definition (3.11) of $\alpha(\imath\xi)$, is equivalent to (as $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$)

$$\Re\left(\imath\xi e^{i\theta}\right) > 0 \quad \text{and} \quad \Re\left(\frac{\imath\xi e^{i\theta}}{\epsilon\imath\xi + \frac{\omega_p^2}{\imath\xi+\gamma}}\right) > 0.$$

After some computations, these two conditions are equivalent to

$$\xi \sin\theta < 0, \quad \omega_p^2\xi\gamma\sin\theta < \xi^2(2\epsilon\gamma^2 - \omega_p^2)\cos\theta.$$

In the case $\xi > 0$, they yield the constraints

$$-\frac{\pi}{2} < \theta < \min\{0, \arctan t_0\},$$

with $t_0 = \frac{\xi(2\epsilon\gamma^2-\omega_p^2)}{\omega_p^2\gamma}$; while in the case $\xi < 0$, they yield the constraints

$$\max\{0, \arctan t_0\} < \theta < \frac{\pi}{2}.$$

In both cases, such a θ exists.

Now we define the bounded operators $\mathcal{B}$, $\mathcal{A}_{0,\imath\xi}$ and $\mathcal{A}_{\imath\xi}$ from W_0 into W_0' as follows:

$$\left.\begin{aligned} \langle\mathcal{B}H, H'\rangle &= b(H, H'),\\ \langle\mathcal{A}_{0,\imath\xi}H, H'\rangle &= a_{0,\imath\xi}(H, H'),\\ \langle\mathcal{A}_{\imath\xi}H, H'\rangle &= a_{\imath\xi}(H, H'), \end{aligned}\right\} \quad \forall\, H, H' \in W_0.$$

Coming back to the coerciveness of b, by Lax–Milgram Lemma, we deduce that $\mathcal{B}$ is an isomorphism. Since $\imath\xi e^{i\theta}$ is different from 0, we get equivalently that $\mathcal{A}_{0,\imath\xi}$ is an isomorphism. But we notice that

$$\mathcal{A}_{\imath\xi} - \mathcal{A}_{0,\imath\xi} = \imath\xi\mathbb{I},$$

where $\mathbb{I}$ is the identity operator. As W_0 is compactly embedded into $\mathbf{L}^2(\Omega)$, we deduce that $\mathcal{A}_{\imath\xi} - \mathcal{A}_{0,\imath\xi}$ is a compact operator and consequently, $\mathcal{A}_{\imath\xi}$ is a Fredholm operator of index 0. Hence it will be an isomorphism if and only if it is injective. Let us then finally show that

$$\ker\mathcal{A}_{\imath\xi} = \{\mathbf{0}\}.$$

Indeed let $H \in \ker\mathcal{A}_{\imath\xi}$. Then it satisfies

$$a_{\imath\xi}(H, H') = 0, \qquad \forall\, H' \in W_0.$$

Then the arguments of Theorem 3.1 show that if we define E (resp. J) by (3.10) (resp. (3.9)) with $\lambda = \imath\xi$ and $(F, G, R) = \mathbf{0}$, then the triple (E, H, J) belongs to $D(\mathcal{A})$ and to $\ker(\imath\xi - \mathcal{A})$. By Lemma 4.2, we conclude that $H = \mathbf{0}$.

Once $\mathcal{A}_{\imath\xi}$ is an isomorphism from W_0 into W_0', problem (4.12) has a unique solution $H \in W_0$, and again the arguments of Theorem 3.1 allow to show that if we define E (resp. J) by (3.10) (resp. (3.9)) with $\lambda = \imath\xi$, then the triple (E, H, J) belongs to $D(\mathcal{A})$ and satisfies (4.9)–(4.11). $\square$

To characterize the range of $\mathcal{A}$, we introduce

$$\mathcal{H}_0 := \{(E, H, J)^\top \in \mathcal{H} : \langle E_e \cdot n_e; 1\rangle_\Sigma = 0\}.$$

Note that $\mathcal{H}_0 = \mathcal{H}$ in Case B, otherwise $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$ of codimension 1.

Lemma 4.4. *The range $R(\mathcal{A})$ of $\mathcal{A}$ is equal to $\mathcal{H}_0$.*

Proof. Let us first notice that $R(\mathcal{A})$ is included in $\mathcal{H}_0$ in Case A. Indeed, for $U = (E, H, J)^\top \in D(\mathcal{A})$, $\mathcal{A}U$ belongs to $\mathcal{H}_0$ if and only if

$$\langle(\operatorname{curl} H - 1_{\Omega_m}\tilde{J})_e \cdot n_e; 1\rangle_\Sigma = 0,$$

or equivalently $\langle(\operatorname{curl} H)_e \cdot n_e; 1\rangle_\Sigma = 0$. As $\operatorname{curl} H$ belongs to $H(\operatorname{div}, \Omega)$, we deduce that

$$\langle(\operatorname{curl} H)_e \cdot n_e; 1\rangle_\Sigma = \langle(\operatorname{curl} H)_m \cdot n_e; 1\rangle_\Sigma = -\int_{\Omega_m} \operatorname{div}(\operatorname{curl} H)\, dx = 0,$$

this last identity following from Green's formula and recalling that n_e is pointing outside Ω_e.

Now for any $(F, G, R)^\top \in \mathcal{H}_0$, we look for a solution $U = (E, H, J)^\top \in D(\mathcal{A})$ to $-\mathcal{A}U = (F, G, R)^\top$ or equivalently to (compare with (3.6)–(3.8))

$$\begin{aligned} -\operatorname{curl} H + 1_{\Omega_m}\tilde{J} &= \epsilon F, \\ \operatorname{curl} E &= G, \\ \gamma J - \omega_p^2 E &= R. \end{aligned}$$

Since neither E nor H can be eliminated, we perturb this problem into

$$-\operatorname{curl} H + 1_{\Omega_m}\tilde{J} = \epsilon F, \tag{4.13}$$

$$H + \operatorname{curl} E = G, \tag{4.14}$$

$$\gamma J - \omega_p^2 E = R. \tag{4.15}$$

In other words, we solve $\mathcal{B}U = (F, G, R)^\top$, where $\mathcal{B}$ is defined by $D(\mathcal{B}) = D(\mathcal{A})$ and

$$\mathcal{B}U = -\mathcal{A}U + \mathcal{R}U, \quad \forall\, U \in D(\mathcal{A}),$$

with

$$\mathcal{R}(E, H, J)^\top = (\mathbf{0}, H, \mathbf{0})^\top.$$

If we show that $\mathcal{B}$ is an isomorphism from $D(\mathcal{A}) \cap \mathcal{H}_0$ into $\mathcal{H}_0$, then $\mathbb{I} - \mathcal{R}\mathcal{B}^{-1}$ is a Fredholm operator from $\mathcal{H}_0$ into itself of index 0, since $\mathcal{R}\mathcal{B}^{-1}$ is a compact operator from $\mathcal{H}_0$ into itself because W_0 is compactly embedded into $\mathbf{L}^2(\Omega)$. Hence $\mathbb{I} - \mathcal{R}\mathcal{B}^{-1}$ is an isomorphism if and only if it is injective. Now we remark that $V \in \mathcal{H}_0$ belongs to $\ker(\mathbb{I} - \mathcal{R}\mathcal{B}^{-1})$ if and only if $U = \mathcal{B}^{-1}V$ belongs to $\ker(\mathcal{B} - \mathcal{R}) = \ker \mathcal{A}$. By Lemma 4.2, we deduce that $U = \mathbf{0}$ in Case B, while in Case A, we deduce that $U = c(\nabla\varphi_0, 0, 0)^\top$, for some $c \in \mathbb{C}$. As U is also in $\mathcal{H}_0$, due to (2.3) we deduce that $c = 0$. Consequently $\mathbb{I} - \mathcal{R}\mathcal{B}^{-1}$ is an isomorphism from $\mathcal{H}_0$ into itself, in other words, for all $(F, G, R)^\top \in \mathcal{H}_0$, there exists a unique solution V to

$$(\mathbb{I} - \mathcal{R}\mathcal{B}^{-1})V = (F, G, R)^\top,$$

and therefore $U = \mathcal{B}^{-1}V$ belongs to $D(\mathcal{A})$ and satisfies

$$-\mathcal{A}U = (\mathcal{B} - \mathcal{R})U = (F, G, R)^\top,$$

which proves that $R(\mathcal{A}) = \mathcal{H}_0$.

It then remains to analyze the operator $\mathcal{B}$. For system (4.13)–(4.15), assuming that a solution (E, H, J) exists, we first eliminate J by the relation

$$J = \frac{1}{\gamma}(\omega_p^2 E + R), \tag{4.16}$$

to obtain

$$-\operatorname{curl} H + 1_{\Omega_m}\frac{\omega_p^2}{\gamma}\tilde{E} = \epsilon F - 1_{\Omega_m}\frac{1}{\gamma}\tilde{R}.$$

Multiplying this identity by $\bar{E}' \in Y_\tau$, integrating in Ω and using Green's formula (2.1), we get

$$-\int_\Omega H \cdot \operatorname{curl} \bar{E}'\,dx + \int_\Gamma (H \times n) \cdot \bar{E}'_T\,d\sigma + \int_{\Omega_m} \frac{\omega_p^2}{\gamma} E \cdot \bar{E}'\,dx = \int_\Omega (\epsilon F - 1_{\Omega_m}\frac{1}{\gamma}\tilde{R}) \cdot E'\,dx$$

Using (4.14) and the Silver–Müller boundary condition, we arrive at

$$\begin{aligned}\int_\Omega \operatorname{curl} E \cdot \operatorname{curl} \bar{E}'\,dx + \int_\Gamma E_T \cdot \bar{E}'_T\,d\sigma + \int_{\Omega_m} \frac{\omega_p^2}{\gamma} E \cdot \bar{E}'\,dx \\ = \int_\Omega \left((\epsilon F - 1_{\Omega_m}\frac{1}{\gamma}\tilde{R}) \cdot E' + G \cdot \operatorname{curl} \bar{E}' \right) dx\end{aligned}$$

As E is divergence free in Ω_m and Ω_e, we can add the terms $\int_{\Omega_m} \operatorname{div} E_m \operatorname{div} \bar{E}'_m\,dx + \int_{\Omega_e} \operatorname{div} E_e \operatorname{div} \bar{E}'_e\,dx$, and find

$$b(E, E') = L(E'), \quad \forall\, E' \in Y_\tau, \tag{4.17}$$

where for all $E, E' \in Y_\tau$, we set

$$\begin{aligned}b(E, E') &= \int_\Omega \operatorname{curl} E \cdot \operatorname{curl} \bar{E}'\,dx + \int_\Gamma E_T \cdot \bar{E}'_T\,d\sigma + \int_{\Omega_m} \frac{\omega_p^2}{\gamma} E \cdot \bar{E}'\,dx \\ &\quad + \int_{\Omega_m} \operatorname{div} E_m \operatorname{div} \bar{E}'_m\,dx + \int_{\Omega_e} \operatorname{div} E_e \operatorname{div} \bar{E}'_e\,dx, \\ L(E') &= \int_\Omega \left((\epsilon F - 1_{\Omega_m}\frac{1}{\gamma}\tilde{R}) \cdot E' + G \cdot \operatorname{curl} \bar{E}' \right) dx.\end{aligned}$$

Since $B(E, E) \gtrsim |E|^2_{Y_\tau}$, by Lemma 2.2, the sesquilinear form b is coercive in Y_τ, and by Lax–Milgram Lemma, problem (4.17) has a unique solution $E \in Y_\tau$. As before, from this solution we have to come back to the original problem (4.13)–(4.15). But first we have to check that E is divergence free in Ω_m and Ω_e. To do so, we first notice that, in Case A, (4.17) remains valid for any test-functions $E' \in \tilde{Y}_\tau$; in other words,

$$b(E, E') = L(E'), \quad \forall\, E' \in \tilde{Y}_\tau. \tag{4.18}$$

Indeed in Case A, any $E' \in \tilde{Y}_\tau$ can be splitted up as

$$E' = F + \delta \nabla \varphi_0,$$

where $\delta \in \mathbb{C}$ is chosen such that $\langle F_e \cdot n; 1\rangle_\Sigma = 0$, which yields

$$\delta = \frac{\langle E'_e \cdot n; 1\rangle_\Sigma}{\langle \nabla \varphi_0 \cdot n; 1\rangle_\Sigma}.$$

As $\nabla\varphi_0$ belongs to $\tilde{Y}_\tau$, we deduce that F is in Y_τ. On one hand, as $\nabla\varphi_0$ is zero on Ω_m, one sees that $b(E, \nabla\varphi_0) = 0$, on the other hand,

$$L(\nabla\varphi_0) = \epsilon \int_{\Omega_e} F \cdot \nabla\varphi_0 \, dx = -\epsilon \int_{\Omega_e} \operatorname{div} F \varphi_0 \, dx + \langle F_e \cdot n; 1\rangle_\Sigma = 0,$$

recalling that $(F, G, R)^\top$ belongs to $\mathcal{H}_0$. The first identity implies that $b(E, E') = b(E, F)$, hence by (4.17) with the test function $F \in Y_\tau$ and then the second identity we obtain

$$b(E, E') = b(E, F) = L(F) = L(E'),$$

and prove (4.18).

Now we chose different test functions in (4.18):

1) For an arbitrary $h \in L^2(\Omega_e)$, in (4.18), take $E' \in \tilde{Y}_\tau$ defined by

$$E' = \begin{cases} 0 & \text{in } \Omega_m, \\ \nabla\varphi_e & \text{in } \Omega_e, \end{cases}$$

where $\varphi_e \in H^1_0(\Omega_e)$ is the unique solution of

$$\Delta\varphi_e = h \quad \text{in } \Omega_e.$$

Then (4.18) reduces to

$$\int_{\Omega_e} \operatorname{div} E_e h \, dx = \int_{\Omega_e} F \cdot \nabla\varphi_e \, dx = -\int_{\Omega_e} \operatorname{div} F\varphi_e \, dx + \langle F_e \cdot n; \varphi_e\rangle_\Sigma = 0.$$

As h was arbitrary, we find that

$$\operatorname{div} E_e = 0 \quad \text{in } \Omega_e. \tag{4.19}$$

2) For an arbitrary $h \in L^2(\Omega_m)$, in (4.18), take $E' \in \tilde{Y}_\tau$ defined by

$$E' = \begin{cases} \nabla\varphi_m & \text{in } \Omega_m, \\ 0 & \text{in } \Omega_e, \end{cases}$$

where $\varphi_m \in H_0^1(\Omega_m)$ is the unique solution of

$$\Delta\varphi_m - \frac{\omega_p^2}{\gamma}\varphi_m = h \quad \text{in } \Omega_m.$$

Then (4.18) reduces to

$$\int_{\Omega_m} \frac{\omega_p^2}{\gamma} E \cdot \nabla\bar{\varphi}_m \, dx + \int_{\Omega_m} \operatorname{div} E_m \Delta\bar{\varphi}_m \, dx = \int_{\Omega_m} (\epsilon_\infty F - \frac{1}{\gamma} R) \cdot \nabla\bar{\varphi}_m \, dx.$$

Recalling that F and R are divergence free in Ω_m and using Green's formula, we get $\int_{\Omega_m} \operatorname{div} E_m h \, dx = 0$, and as h was arbitrary, we find that

$$\operatorname{div} E_m = 0 \quad \text{in } \Omega_m. \tag{4.20}$$

Using the two properties (4.19) and (4.20) in (4.18) and defining H by (see (4.14))

$$H = G - \operatorname{curl} E,$$

that is clearly divergence free in Ω (as G is) and J by (4.16), the identity (4.18) becomes

$$\int_\Omega (-H \cdot \operatorname{curl}\bar{E}' + 1_{\Omega_m} \tilde{J} \cdot \bar{E}') \, dx + \int_\Gamma E_T \cdot \bar{E}'_T \, d\sigma = \int_\Omega \epsilon F \cdot \bar{E}' \, dx, \quad \forall\, E' \in \tilde{Y}_\tau. \tag{4.21}$$

First by taking test functions $E' \in \mathcal{D}(\Omega)^3$, we find that

$$-\operatorname{curl} H + 1_{\Omega_m} \tilde{J} = \epsilon F \quad \text{in } \mathcal{D}'(\Omega)^3.$$

And since $1_{\Omega_m} \tilde{J}, \epsilon F$ are in $\mathbf{L}^2(\Omega)$, we deduce that H belongs to $\mathbf{H}(\operatorname{curl}, \Omega)$. This last identity and the definitions of H and J show that (4.13)–(4.15) hold. The only missing properties to have $(E, H, J) \in D(\mathcal{A})$ are $H \times n \in \mathbf{L}^2(\Gamma)$ and the Silver–Müller boundary condition. To prove that properties, we take in (4.21) test functions $E' \in \mathbf{H}^1(\Omega)$, by Green's formula [11, Theorem I.2.11], and the previous identity we find that

$$-\langle (H \times n); \bar{E}' \rangle_{\mathbf{H}^{-\frac{1}{2}}(\Gamma) - \mathbf{H}^{\frac{1}{2}}(\Gamma)} + \int_\Gamma E_T \cdot \bar{E}' \, d\sigma = 0.$$

This proves that $(H \times n) = E_T$ in $\mathbf{H}^{-\frac{1}{2}}(\Gamma)$, since E' is arbitrary in $\mathbf{H}^{\frac{1}{2}}(\Gamma)$. This proves the Silver–Müller boundary condition and as E_T is in $\mathbf{L}^2(\Gamma)$, $H \times n$ as well. The proof of the Lemma is complete. □

This result suggests to introduce the operator $\mathcal{A}_0$ from $\mathcal{H}_0$ into itself defined by $\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}) \cap \mathcal{H}_0$ and

$$\mathcal{A}_0 U = \mathcal{A} U, \quad \forall\, U \in \mathcal{D}(\mathcal{A}_0).$$

Corollary 4.5. *The operator $\mathcal{A}_0$ is strongly stable.*

Proof. The three previous Lemmas yield

$$\sigma(\mathcal{A}_0) \cap i\mathbb{R} = \emptyset.$$

As $\mathcal{A}_0$ is also dissipative in $\mathcal{H}_0$, we then conclude by Theorem 4.1. □

5. Stability results

Our stability results are based on a frequency domain approach, namely for the exponential decay of the energy we use the following result (see [27] or [13]):

Lemma 5.1. *A C_0-semigroup $(e^{t\mathcal{L}})_{t\geq 0}$ of contractions on a Hilbert space H is exponentially stable, i.e., satisfies*

$$\|e^{t\mathcal{L}} U_0\| \leq C\, e^{-\omega t} \|U_0\|_H, \quad \forall\, U_0 \in H,\ \forall\, t \geq 0,$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{L}) \supset \big\{ i\beta \ \big|\ \beta \in \mathbb{R} \big\} \equiv i\mathbb{R}, \tag{5.1}$$

and

$$\sup_{\beta \in \mathbb{R}} \|(i\beta - \mathcal{L})^{-1}\| < \infty, \tag{5.2}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

On the contrary the polynomial decay of the energy is based on the following result stated in [7, Theorem 2.4] (see also [4, 5, 21] for weaker variants).

Lemma 5.2. *A C_0-semigroup $(e^{t\mathcal{L}})_{t\geq 0}$ of contractions on a Hilbert space satisfies*

$$\|e^{t\mathcal{L}} U_0\| \leq C\, t^{-\frac{1}{\ell}} \|U_0\|_{\mathcal{D}(\mathcal{L})}, \quad \forall\, U_0 \in \mathcal{D}(\mathcal{L}),\ \forall\, t > 1,$$

as well as

$$\|e^{t\mathcal{L}} U_0\| \leq C\, t^{-1} \|U_0\|_{\mathcal{D}(\mathcal{L}^\ell)}, \quad \forall\, U_0 \in \mathcal{D}(\mathcal{L}^\ell),\ \forall\, t > 1,$$

for some constant $C > 0$ and for some positive integer ℓ if (5.1) *holds and if*

$$\limsup_{|\beta| \to \infty} \frac{1}{\beta^\ell} \|(i\beta - \mathcal{L})^{-1}\| < \infty. \tag{5.3}$$

Since condition (5.1) was already treated in the previous section, it remains to analyze the behaviour of the resolvent on the imaginary axis. Let us start with the exponential decay.

Lemma 5.3. *Suppose that the Maxwell system with Silver–Müller boundary condition:*

$$\begin{cases} \epsilon\partial_t E = \operatorname{curl} H, \ \partial_t H = -\operatorname{curl} E & \text{in } \Omega \times (0, +\infty), \\ \operatorname{div}(\epsilon E) = \operatorname{div} H = 0 & \text{in } \Omega \times (0, +\infty), \\ E_T - H \times n = \mathbf{0} & \text{on } \Gamma \times (0, +\infty), \end{cases} \tag{5.4}$$

is exponentially stable. Then the resolvent of the operator of $\mathcal{A}_0$ satisfies condition (5.2).

Proof. We use a contradiction argument, i.e., we suppose that (5.2) is false. Then there exist a sequence of real numbers $\xi_n \to +\infty$ and a sequence of vectors $U_n = (E_n, H_n, J_n)^\top$ in $D(\mathcal{A}_0)$ with

$$\|U_n\|_{\mathcal{H}} = 1, \tag{5.5}$$

such that

$$\|(i\xi_n - \mathcal{A})U_n\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty. \tag{5.6}$$

By (3.2), this directly implies that

$$\imath\xi_n\epsilon E_n - \operatorname{curl} H_n + 1_{\Omega_m}\tilde{J}_n \to 0 \quad \text{in } \mathbf{L}^2(\Omega), \qquad \text{as } n \to \infty, \tag{5.7}$$

$$\imath\xi_n H_n + \operatorname{curl} E_n \to 0 \quad \text{in } \mathbf{L}^2(\Omega), \qquad \text{as } n \to \infty, \tag{5.8}$$

$$(\imath\xi_n + \gamma)J_n - \omega_p^2 E_n \to 0 \quad \text{in } \mathbf{L}^2(\Omega_m), \quad \text{as } n \to \infty. \tag{5.9}$$

By the dissipativeness of $\mathcal{A}$ (see (3.4)), we further have

$$\omega_p^{-2}\gamma\int_{\Omega_m} |J_n|^2\, dx + \int_\Gamma |E_n \times n|^2\, d\sigma = \Re((\imath\xi_n - \mathcal{A})U_n, U_n)_{\mathcal{H}} \le \|(\imath\xi_n - \mathcal{A})U_n\|_{\mathcal{H}}, \tag{5.10}$$

and therefore by (5.6)

$$J_n \to 0 \quad \text{in } \mathbf{L}^2(\Omega_m), \quad \text{as } n \to \infty, \tag{5.11}$$

$$E_n \times n \to 0 \quad \text{in } \mathbf{L}^2(\Gamma), \qquad \text{as } n \to \infty. \tag{5.12}$$

In order to use the exponential stability of (5.4), we need to correct E_n since they do not satisfy $\operatorname{div}(\epsilon E_n) = 0$ on the whole Ω. Therefore we consider $\varphi_n \in H_0^1(\Omega)$ such that

$$\int_\Omega \epsilon\nabla\varphi_n \cdot \nabla\psi\, dx = \int_\Omega \epsilon E_n \cdot \nabla\psi\, dx, \quad \forall\, \psi \in H_0^1(\Omega), \tag{5.13}$$

and set

$$\tilde{E}_n = E_n - \nabla\varphi_n,$$

that belongs to W and satisfies

$$\operatorname{div}(\epsilon\tilde{E}_n) = 0 \quad \text{in } \Omega. \tag{5.14}$$

By setting

$$F_n = \imath\xi_n\epsilon E_n - \operatorname{curl} H_n + 1_{\Omega_m}\tilde{J}_n,$$

the identity (5.13) implies that

$$\begin{aligned}\int_\Omega \epsilon|\nabla\varphi_n|^2\,dx &= \frac{1}{\imath\xi_n}\int_\Omega (F_n + \operatorname{curl} H_n - 1_{\Omega_m}\tilde{J}_n)\cdot\nabla\varphi_n\,dx \\ &= \frac{1}{\imath\xi_n}\int_\Omega (F_n - 1_{\Omega_m}\tilde{J}_n)\cdot\nabla\varphi_n\,dx,\end{aligned}$$

since by Green's formula (2.1) $\int_\Omega \operatorname{curl} H_n \cdot \nabla\varphi_n\,dx = 0$. Hence by Cauchy–Schwarz's inequality we find that

$$\|\nabla\varphi_n\| \lesssim \frac{1}{\xi_n}(\|F_n\| + \|J_n\|_{\Omega_m}).$$

By (5.7) and (5.11), we conclude that

$$\|\imath\xi_n\nabla\varphi_n\| \to 0 \quad \text{as } n \to \infty. \tag{5.15}$$

At this stage we remark that the pair $(\tilde{E}_n, H_n)$ satisfies the Silver–Müller boundary condition on Γ, the condition (5.14), $\operatorname{div} H_n = 0$ in Ω and Maxwell's equations (compare with (5.7) and (5.8))

$$\begin{cases} \imath\xi_n\epsilon\tilde{E}_n - \operatorname{curl} H_n = \tilde{F}_n = F_n - 1_{\Omega_m}\tilde{J}_n - \imath\xi_n\epsilon\nabla\varphi_n, \\ \imath\xi_n H_n + \operatorname{curl}\tilde{E}_n = G_n = \imath\xi_n H_n + \operatorname{curl} E_n. \end{cases} \tag{5.16}$$

With the help of (5.8), (5.11) and (5.15), we have

$$\tilde{F}_n, G_n \to 0 \quad \text{in } \mathbf{L}^2(\Omega), \quad \text{as } n \to \infty. \tag{5.17}$$

As by assumption, system (5.4) is exponentially stable, applying Lemma 5.1 to this system, its resolvent is uniformly bounded on the imaginary axis. In other words, there exists a positive constant C independent of n such that the solution $(\tilde{E}_n, H_n)$ of (5.16) satisfies

$$\|\tilde{E}_n\| + \|H_n\| \le C(\|\tilde{F}_n\| + \|G_n\|).$$

The property (5.17) then yields $\|\tilde{E}_n\| + \|H_n\| \to 0$. By (5.15), we conclude that

$$\|E_n\| + \|H_n\| \to 0, \quad \text{as } n \to \infty,$$

and recalling (5.11), we arrive at a contradiction with (5.5). □

Remark 5.4. Due to [14, §5], system (5.4) is exponentially stable in Case B and if Ω_e and Ω are strictly star-shaped with respect to a point since for any metals (experimentally $\epsilon_\infty = 3.2629$ for gold and $\epsilon_\infty = 3.7362$ for silver, see [28, Appendix A])

$$\epsilon_\infty > 1.$$

This is physically reasonable since the speed of propagation is fast inside Ω_e and slow outside. In the opposite case, the exponential decay does not hold for physical models even if Ω_m and Ω are strictly star-shaped, since from [14, §5], we need that $\epsilon_\infty \leq 1$ which is never physically satisfied; the reason is that for light rays in the slow metal that approach the vaccum boundary, there is total reflection for rays that are incident close to tangency.

This remark motivates us to find weaker assumptions that yield a polynomial decay.

Lemma 5.5. *Assume that Maxwell's system in Ω with constant coefficient and the Silver–Müller boundary condition on $\partial\Omega$ is exponentially stable. Then the resolvent of the operator of $\mathcal{A}_0$ satisfies condition* (5.3) *with $\ell = 4$.*

Proof. We again use a contradiction argument, i.e., we suppose that (5.3) is false with $\ell \in \mathbb{N}$. Then there exist a sequence of real numbers $\xi_n \to +\infty$ and a sequence of vectors $U_n = (E_n, H_n, J_n)^\top$ in $D(\mathcal{A})$ satisfying (5.5) and

$$\xi_n^\ell \left\|(i\xi_n - \mathcal{A})U_n\right\|_{\mathcal{H}} \to 0 \quad \text{as } n \to \infty. \tag{5.18}$$

As before this directly implies that

$$F_n = \xi_n^\ell(\imath\xi_n\epsilon E_n - \operatorname{curl} H_n + 1_{\Omega_m}\tilde{J}_n) \to 0 \quad \text{in } \mathbf{L}^2(\Omega), \qquad \text{as } n \to \infty, \tag{5.19}$$

$$\xi_n^\ell(\imath\xi_n H_n + \operatorname{curl} E_n) \to 0 \quad \text{in } \mathbf{L}^2(\Omega), \qquad \text{as } n \to \infty, \tag{5.20}$$

$$\xi_n^\ell((\imath\xi_n + \gamma)J_n - \omega_p^2 E_n) \to 0 \quad \text{in } \mathbf{L}^2(\Omega_m), \quad \text{as } n \to \infty, \tag{5.21}$$

By (5.10) and (5.18), we deduce that

$$\xi_n^{\ell/2} J_n \to 0 \quad \text{in } \mathbf{L}^2(\Omega_m), \quad \text{as } n \to \infty, \tag{5.22}$$

$$\xi_n^{\ell/2}(E_n \times n) \to 0 \quad \text{in } \mathbf{L}^2(\Gamma), \qquad \text{as } n \to \infty. \tag{5.23}$$

The first property and (5.21) directly imply that

$$\xi_n^{\frac{\ell}{2}-1} E_n \to 0 \quad \text{in } \mathbf{L}^2(\Omega_m), \quad \text{as } n \to \infty. \tag{5.24}$$

By using the trace estimate [11, (I.2.16)]:

$$\|U \cdot n_m\|_{H^{-\frac{1}{2}}(\Sigma)} \lesssim \|U\|_{\Omega_m} + \|\operatorname{div} U\|_{\Omega_m}, \quad \forall U \in H(\operatorname{div}, \Omega_m),$$

and recalling that E_n is divergence free in Ω_m, we get that

$$\xi_n^{\frac{\ell}{2}-1}(E_n)_{|\Omega_m}\cdot n_m \to 0 \quad \text{in } H^{-\frac{1}{2}}(\Sigma), \quad \text{as } n\to\infty. \tag{5.25}$$

On the other hand, Green's formula [11, (I.2.17)] leads to

$$\langle[\epsilon E_n\cdot n];\varphi\rangle_\Sigma = \int_\Omega \epsilon E_n\cdot\nabla\varphi\,dx, \quad \forall\,\varphi\in H_0^1(\Omega),$$

where $[\epsilon E_n\cdot n] = (E_n)_{|\Omega_m}\cdot n_m + (E_n)_{|\Omega_e}\cdot n_e$ means the jump of $\epsilon E_n\cdot n$ through Σ. By using (5.19), we get

$$\langle[\epsilon E_n\cdot n];\varphi\rangle_\Sigma = (\imath\xi_n)^{-1}\int_\Omega(\xi_n^{-\ell}F_n + \operatorname{curl} H_n - 1_{\Omega_m}\tilde{J}_n)\cdot\nabla\varphi\,dx, \quad \forall\,\varphi\in H_0^1(\Omega).$$

Hence by Green's formula (2.1), we obtain

$$\langle[\epsilon E_n\cdot n];\varphi\rangle_\Sigma = (\imath\xi_n)^{-1}\int_\Omega(\xi_n^{-\ell}F_n - 1_{\Omega_m}\tilde{J}_n)\cdot\nabla\varphi\,dx, \quad \forall\,\varphi\in H_0^1(\Omega),$$

and by Cauchy–Schwarz's inequality and the definition of the norm of $H^{-\frac{1}{2}}(\Sigma)$, we get

$$\|[\epsilon E_n\cdot n]\|_{H^{-\frac{1}{2}}(\Sigma)} \lesssim \xi_n^{-\ell-1}\|F_n\| + \xi_n^{-1}\|J_n\|_{\Omega_m}.$$

Owing to (5.19) and (5.22), we deduce that $\xi_n^{\frac{\ell}{2}+1}[\epsilon E_n\cdot n]\to 0$ in $H^{-\frac{1}{2}}(\Sigma)$, as $n\to\infty$. This property combined with (5.25) yields

$$\xi_n^{\frac{\ell}{2}-1}[E_n\cdot n]\to 0 \quad \text{in } H^{-\frac{1}{2}}(\Sigma), \quad \text{as } n\to\infty. \tag{5.26}$$

As in the previous Lemma, we now correct E_n by $\varphi_n\in H_0^1(\Omega)$ solution of

$$\int_\Omega \nabla\varphi_n\cdot\nabla\psi\,dx = \int_\Omega E_n\cdot\nabla\psi\,dx = \langle[E_n\cdot n];\psi\rangle_\Sigma, \quad \forall\,\psi\in H_0^1(\Omega),$$

and set

$$\tilde{E}_n = E_n - \nabla\varphi_n.$$

that belongs to W and satisfies

$$\operatorname{div}\tilde{E}_n = 0 \quad \text{in } \Omega. \tag{5.27}$$

Owing to the property (5.26), φ_n clearly satisfies

$$\xi_n^{\frac{\ell}{2}-1}\varphi_n\to 0 \quad \text{in } H^1(\Omega), \quad \text{as } n\to\infty. \tag{5.28}$$

As before, the pair $(\tilde{E}_n, H_n)$ satisfies the Silver–Müller boundary condition on Γ, the condition (5.27), $\operatorname{div} H_n = 0$ in Ω and Maxwell's equations with contant coefficient

$$\begin{cases} \imath\xi_n \tilde{E}_n - \operatorname{curl} H_n = \tilde{F}_n, \\ \imath\xi_n H_n + \operatorname{curl} \tilde{E}_n = G_n = \imath\xi_n H_n + \operatorname{curl} E_n, \end{cases} \tag{5.29}$$

where $\tilde{F}_n = \xi_n^{-\ell} F_n - 1_{\Omega_m} \tilde{J}_n - \imath\xi_n \epsilon \nabla\varphi_n + \imath\xi_n 1_{\Omega_m}(1-\epsilon_\infty)\tilde{E}_n$.

Let us now check that the property (5.17) holds as long as $\ell \geq 4$. First by (5.19), (5.20) and (5.22), we directly deduce that

$$\xi_n^{-\ell}\|F_n\| + \|G_n\| + \|1_{\Omega_m}\tilde{J}_n\| \to 0, \quad \text{as } n \to \infty,$$

for any $\ell \geq 0$. Now for the term $\imath\xi_n\epsilon\nabla\varphi_n$, we notice that

$$\|\imath\xi_n\epsilon\nabla\varphi_n\| = \xi_n\|\epsilon\nabla\varphi_n\| \leq C\xi_n^{\frac{\ell}{2}-1}\|\nabla\varphi_n\|,$$

for n large enough and a positive constant C independent of n as soon as $\ell \geq 4$. Consequently under this hypothesis, by (5.28), we deduce that $\|\imath\xi_n\epsilon\nabla\varphi_n\| \to 0$, as $n \to \infty$. The same argument and (5.24) leads to

$$\|\imath\xi_n 1_{\Omega_m}(1-\epsilon_\infty)\tilde{E}_n\| \to 0, \quad \text{as } n \to \infty,$$

for $\ell \geq 4$. Alltogether, we have shown (5.17) for $\ell \geq 4$.

Applying Lemma 5.1 to system (5.4), we deduce that its resolvent is uniformly bounded on the imaginary axis. In other words, there exists a positive constant C independent of n such that the solution $(\tilde{E}_n, H_n)$ of (5.29) satisfies

$$\|\tilde{E}_n\| + \|H_n\| \leq C(\|\tilde{F}_n\| + \|G_n\|).$$

By (5.17), we obtain

$$\|\tilde{E}_n\| + \|H_n\| \to 0, \quad \text{as } n \to \infty,$$

and by (5.22) and (5.28), we arrive at a contradiction with (5.5). □

Remark 5.6. Different sufficient conditions on Ω and its boundary that guarantee that Maxwell's system in Ω with constant coefficient and the Silver–Müller boundary condition on $\partial\Omega$ is exponentially stable are available in the literature. Let us mention [15, 16], where the sufficient assumptions are that Ω is a strictly star-shaped domain with a boundary of class C^1 (piecewise smooth is sufficient using the results from [12]), and [26], where the boundary of Ω has to be smooth (of class C^∞, that automatically satisfies the geometric control condition (G.C.C.) from [3]).

The previous lemmas allow to check the hypotheses of Lemma 5.1 or Lemma 5.2 and then lead to the next stability results.

Theorem 5.7. 1. *Under the assumption of Lemma* 5.5*, problem* (3.1) *is polynomially stable in $\mathcal{H}_0$, in other words, there exists a positive constant C such that in Case A, one has*

$$\|U(t) - \kappa(\nabla\varphi_0, 0, 0)^\top\|^2_{\mathcal{H}} \leq C\, t^{-\frac{1}{2}} \|U_0\|^2_{\mathcal{D}(\mathcal{A})}, \quad \forall\, t > 1, \tag{5.30}$$

for all $U_0 = (E_0, H_0, J_0)^\top \in \mathcal{D}(\mathcal{A})$, with

$$\kappa = \left(\int_{\Omega_e} |\nabla q_0|^2\, dx\right)^{-1} \langle (E_0)_{|\Omega_e} \cdot n\,; 1\rangle_\Sigma, \tag{5.31}$$

otherwise

$$\|U(t)\|^2_{\mathcal{H}} \leq C\, t^{-\frac{1}{2}} \|U_0\|^2_{\mathcal{D}(\mathcal{A})}, \quad \forall\, t > 1, \tag{5.32}$$

for all $U_0 = (E_0, H_0, J_0)^\top \in \mathcal{D}(\mathcal{A})$.

2. *On the contrary, under the assumption of Lemma* 5.3*, problem* (3.1) *is exponentially stable in $\mathcal{H}_0$, i.e., there exist two positive constants C and ω such that in Case A, one has*

$$\|U(t) - \kappa(\nabla\varphi_0, 0, 0)^\top\|^2_{\mathcal{H}} \leq C\, e^{-\omega t} E(0), \quad \forall\, t \geq 0,$$

for all $U_0 \in \mathcal{H}$, otherwise

$$\|U(t)\|^2_{\mathcal{H}} \leq C\, e^{-\omega t} E(0), \quad \forall\, t \geq 0,$$

for all $U_0 \in \mathcal{H}$.

Proof. By the proof of Lemma 4.5, we know that $\mathcal{A}_0$ satisfies (5.1). For point 1, we notice that Lemmas 5.5 and 5.2 imply that for all $\tilde{U}_0 \in \mathcal{D}(\mathcal{A}_0)$, the solution $\tilde{U}$ of

$$\begin{cases} \tilde{U}_t = \mathcal{A}_0 \tilde{U}, \\ \tilde{U}(0) = \tilde{U}_0, \end{cases} \tag{5.33}$$

satisfies

$$\|\tilde{U}(t)\|^2_{\mathcal{H}} \leq C\, t^{-\frac{1}{2}} \|\tilde{U}_0\|^2_{\mathcal{D}(\mathcal{A}_0)}, \quad \forall\, t > 1. \tag{5.34}$$

This yields (5.32) in Case B. On the contrary in Case A, it is readily checked that for any $U_0 \in \mathcal{D}(\mathcal{A})$, the solution U of (3.1) can be written as

$$U(t) = \tilde{U}(t) + \kappa(\nabla\varphi_0, 0, 0)^\top,$$

once $U_0 = \tilde{U}_0 + \kappa(\nabla\varphi_0, 0, 0)^\top$, with κ chosen such that $\tilde{U}_0$ belongs to $\mathcal{H}_0$, or equivalently such that $\langle (E_0 - \kappa\nabla\varphi_0)_{|\Omega_e} \cdot n\,; 1\rangle_\Sigma = 0$. By (2.2), we get the expression (5.31) for κ. As $(\nabla\varphi_0, 0, 0)^\top$ belongs to $\mathcal{D}(\mathcal{A})$, $\tilde{U}_0$ is indeed in $\mathcal{D}(\mathcal{A}_0)$. Hence applying the estimate (5.34), we find (5.30) since

$$|\kappa| \lesssim \left|\langle (E_0)_{|\Omega_e} \cdot n; 1\rangle_\Sigma\right| = \left|\int_{\Omega_e} E_0 \cdot \nabla q_0 \, dx\right| \lesssim \|U_0\|_{\mathcal{H}},$$

and therefore

$$\|\tilde{U}_0\|_{\mathcal{H}} \leq \|U_0\|_{\mathcal{H}}, \tag{5.35}$$

as well as

$$\|\tilde{U}_0\|_{\mathcal{D}(\mathcal{A}_0)} = \|\tilde{U}_0\|_{\mathcal{H}} + \|\mathcal{A}\tilde{U}_0\|_{\mathcal{H}} = \|\tilde{U}_0\|_{\mathcal{H}} + \|\mathcal{A}U_0\|_{\mathcal{H}} \lesssim \|U_0\|_{\mathcal{D}(\mathcal{A})}.$$

Point 2 is proved similarly by using (5.35) and the exponential decay of the solution of (5.33). □

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