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Legendre Forms in Reflexive Banach Spaces

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Abstract. Legendre forms are used in the literature for second-order sufficient optimality conditions of optimization problems in (reflexive) Banach spaces. We show that if a Legendre form exists on a reflexive Banach space, then this space is already isomorphic to a Hilbert space.

Keywords. Legendre form, second-order optimality conditions, Hilbertizable space, quadratic form, coercive bilinear form

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1. Introduction

A Legendre form is a quadratic form $Q : X \to \mathbb{R}$ (where X is a normed vector space) that is sequentially weakly lower semi-continuous and has the property that $x_n \to x$ whenever $x_n \rightharpoonup x$ and $Q(x_n) \to Q(x)$, see Definition 2.2[1].

Legendre forms have their origin in the calculus of variations. They are discussed in a Hilbert space setting in [5,6]. However, the definition of a Legendre form naturally extends to Banach spaces. Most notably, in [1] they are defined in arbitrary Banach spaces and used in reflexive Banach spaces. Legendre forms are useful in reflexive Banach spaces for second-order sufficient optimality conditions. For instance, if the second derivative corresponds to a quadratic form $Q(x)$ that is a Legendre form, then it suffices to show that $Q(x) > 0$ for all $x \in C \setminus \{0\}$ for some closed convex cone $C \subset X$ (instead of the usual coercivity of the second derivative), see [1, Lemma 3.75]. The condition that a certain quadratic form is a Legendre form also plays an important role in many other theorems in [1] that are formulated in the setting of a reflexive Banach space.

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[1]The term "Legendre form" is different from the object of the same name that appears in the area of elliptic curves.

It is well known that if a coercive quadratic form exists on a Banach space X, then X is already Hilbertizable, i.e., isomorphic to a Hilbert space (see Proposition 4.7). As a consequence, coercive quadratic forms do not exist on Banach spaces that are not Hilbertizable.

The question arises whether Legendre forms (which can be interpreted as a generalization of coercive quadratic forms) suffer from the same problem or if there exist Legendre forms on Banach spaces that are not Hilbertizable.

For reflexive Banach spaces, we are able to prove the following theorem, which answers this question.

Theorem 1.1. *Let Q be a Legendre form on a reflexive Banach space X. Then X is Hilbertizable.*

This theorem will be proven in Section 4.

An important consequence of this result is that one should not attempt to apply theorems in which the existence of a Legendre form is assumed to reflexive Banach spaces that are not Hilbertizable. An example for such spaces would be $L^p(\Omega)$ where $p \in (1, \infty) \setminus \{2\}$ and $\Omega \subset \mathbb{R}^d$ is a measurable set. Some of these theorems that are formulated in a reflexive Banach space and in which a Legendre form appears in the conditions are [1, Theorem 3.128, Theorem 5.5], [9, Theorem 5.7]. The reflexivity in these and other theorems is usually used to obtain the weak sequential compactness of the closed unit ball, which plays an important role in the proofs. Therefore the question whether there exist Legendre forms in non-reflexive Banach spaces is less relevant for applications.

In this paper, we will proceed as follows. We start with introducing some notation and giving precise definitions in Section 2. Then in Section 3 we will give a brief overview of the established results regarding Legendre forms in Hilbert spaces. In Section 4 we provide a proof of Theorem 1.1 and some intermediate results. Finally, in Section 5 we address extended Legendre forms, discuss the situation in non-reflexive Banach spaces, and give a conclusion.

2. Definitions and notation

We start with defining terminology that is related to quadratic forms.

Definition 2.1. Let X be a normed vector space. We call a function $Q : X \to \mathbb{R}$ a *quadratic form* if there exists a bilinear form $B : X \times X \to \mathbb{R}$ such that

$$Q(x) = B(x, x) \quad \forall\, x \in X.$$

For a quadratic form Q we say that two subsets $Y_1, Y_2 \subset X$ are *Q-orthogonal*, denoted as $Y_1 \perp^Q Y_2$, if

$$Q(y_1 + y_2) = Q(y_1) + Q(y_2) \quad \forall\, y_1 \in Y_1,\, y_2 \in Y_2.$$

We call Q *coercive* (or *elliptic*) if it is continuous and

$$Q(x) \geq \gamma \|x\|^2 \quad \forall\, x \in X$$

for a constant $\gamma > 0$.

We say that Q is *positive* (or *negative*) if $Q(x) > 0$ (or $Q(x) < 0$) for all $x \in X \setminus \{0\}$.

Note that a quadratic form does not need to be continuous. However, due to Lemma 4.6 all quadratic forms that we use in Sections 3 and 4 turn out to be continuous.

In the case that Q is a continuous quadratic form, it is always possible to uniquely choose a continuous linear operator $A : X \to X^*$ in such a way, that

$$Q(x) = \langle Ax,\, x\rangle_{X^* \times X} \quad \text{and} \quad \langle Ax,\, y\rangle_{X^* \times X} = \langle Ay,\, x\rangle_{X^* \times X} \quad \forall\, x, y \in X \qquad (1)$$

holds. Hence, we set the convention that whenever there is a continuous quadratic form Q, we denote by A the unique operator that is given by (1). We note that if X is reflexive, we know that A is self-adjoint. Using the operator A it, it is easy to check that the equivalence

$$Y_1 \perp^Q Y_2 \quad \Leftrightarrow \quad \langle Ay_1,\, y_2\rangle_{X^* \times X} = 0 \quad \forall\, y_1 \in Y_1,\, y_2 \in Y_2$$

holds. This gives us an alternative description of Q-orthogonality. We will also use the notation $y_1 \perp^Q Y_2$ for $\{y_1\} \perp^Q Y_2$ and in the same spirit we abbreviate $\{y_1\} \perp^Q \{y_2\}$ with $y_1 \perp^Q y_2$.

It can be shown that if a function $Q : X \to \mathbb{R}$ is a quadratic form then the parallelogram law

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \quad \forall\, x, y \in X$$

holds. The converse is true if we assume Q to be continuous. In this case, the operator $A : X \to X^*$ that satisfies (1) can be recovered by the polarization identity

$$\langle Ax,\, y\rangle_{X^* \times X} = \frac{1}{4}(Q(x + y) - Q(x - y)).$$

Having discussed quadratic forms, we give the definition of a Legendre form.

Definition 2.2. Let X be a normed vector space. We call a quadratic form $Q : X \to \mathbb{R}$ a *Legendre form* if Q is sequentially weakly lower semi-continuous and if $x_k \rightharpoonup x$ and $Q(x_k) \to Q(x)$ imply $x_k \to x$ for all sequences $\{x_k\}_{k \in \mathbb{N}}$ in X.

Note that as a consequence of Lemma 4.6 all Legendre forms are continuous if X is a Banach space. In this case, it is possible to replace x with 0 in the above definition, i.e., if a quadratic form Q on a Banach space is sequentially weakly lower semi-continuous and $(x_k \rightharpoonup 0 \wedge Q(x_k) \to 0) \Rightarrow x_k \to 0$ holds for all sequences $\{x_k\}_{k \in \mathbb{N}} \subset X$, then Q is already a Legendre form.

Finally, we mention that for linear subspaces Y_1, Y_2, Z of a vector space X we will use the notation $Z = Y_1 \dot{+} Y_2$ if $Z = Y_1 + Y_2$ and $Y_1 \cap Y_2 = \{0\}$.

3. Legendre forms in Hilbert spaces

In this section we give some results from the literature that discuss Legendre forms in Hilbert spaces. Note that with the use of Theorem 1.1 these results extend to reflexive Banach spaces.

The following theorem is due to [5, Theorem 11.6] and yields a good characterization of Legendre forms in Hilbert spaces. Moreover, it is useful for constructing examples of Legendre forms.

Theorem 3.1. *A quadratic form Q on a Hilbert space X is a Legendre form if and only if it can be expressed as*

$$Q(x) = Q_1(x) - Q_2(x)$$

where Q_1 is a coercive quadratic form and Q_2 is a sequentially weakly continuous quadratic form.

It can be shown that a quadratic form $Q(x) = \langle Ax, x\rangle$ in a Hilbert space is weakly sequentially continuous if and only if A is compact, see [6, Theorem 1 in 6.2.3]. A consequence is that in a finite dimensional space all quadratic forms are Legendre forms. We note that if $A : X \to X^*$ is a compact operator, the quadratic form $Q(x) = \langle Ax, x\rangle$ does not need to be weakly continuous. For instance, the compact operator $A : \ell^2 \to \ell^2$ given by $Ae_n = \frac{1}{n}e_n$ for all $n \in \mathbb{N}$ yields a sequentially weakly continuous quadratic form Q that is not weakly continuous. Indeed, $Q(x) = 1$ for all $x \in S := \{\sqrt{n}e_n : n \in \mathbb{N}\}$ but $Q(0) = 0$ and 0 is in the weak closure (but not in the weak sequential closure) of S.

A simple combination of [5, Theorem 7.1, Theorem 11.3] yields a statement that can give us an idea how to prove Theorem 1.1.

Theorem 3.2. *Let Q be a Legendre form on a Hilbert space X. Then there is an orthogonal and Q-orthogonal decomposition*

$$X = Y_+ \dotplus Y_0 \dotplus Y_-$$

with closed subspaces $Y_+, Y_0, Y_- \subset X$ such that Q is coercive on Y_+, $-Q$ is coercive on Y_-, and $Q = 0$ on Y_0. Moreover, Y_0 and Y_- are finite-dimensional.

It should be noted that it is not possible to simply prove this result (without orthogonality) in reflexive Banach spaces in a way that is analogous to the original proof in Hilbert spaces. This is because the proof of [5, Theorem 7.1] uses that a quadratic form can be expressed as a difference of two nonnegative quadratic forms. However, this is not possible in the Banach spaces ℓ^p where $p \in (1, 2)$, see [7, Corollary 1.7].

4. Legendre forms in reflexive Banach spaces

4.1. Basic results from functional analysis. In this section we will recall some established results from the literature of functional analysis. We will need those in Sections 4.2 and 4.3. Additionally, we provide a result that states that lower semi-continuous quadratic forms are continuous, which yields a very basic property of Legendre forms.

Lemma 4.1. *A reflexive Banach space is finite dimensional if and only if every weakly convergent sequence is norm-convergent.*

This result follows from [8, Theorem 1.22] and the fact that every bounded sequence has a weakly convergent subsequence.

The next result which states that finite-dimensional subspaces can be complemented can be found in [8, Lemma 4.21].

Lemma 4.2. *Let X be a Banach space with a closed subspace Y. If* $\dim Y < \infty$ *or* $\dim(X/Y) < \infty$ *then there is a closed subspace $Z \subset X$ such that $X = Y \dot{+} Z$.*

The following lemma will be useful when we want to generalize the property "Hilbertizable" (which means that there exists a scalar product such that the induced norm is equivalent to the original norm) to a larger space.

Lemma 4.3. *Let $X = Y \dot{+} Z$ be a Banach space with closed subspaces Y, Z and* $\dim Z < \infty$. *If Y is Hilbertizable, then X is Hilbertizable.*

Proof. Because Z is finite dimensional, we know that Z is also Hilbertizable.

As the product of two Hilbertizable Banach spaces, $Y \times Z$ (which is algebraically isomorphic to X) can be equipped with an inner product. As a consequence of the open mapping theorem, it follows that $Y \times Z$ equipped with the norm $\|(y, z)\| := \|y\| + \|z\|$ is even topologically isomorphic to X, i.e., the norms are equivalent. □

When we obtain intermediate results that are formulated in a reflexive Banach space, we often intend to apply these results to a closed subspace of a reflexive Banach space. Similarly, we will often apply results where a Legendre form is relevant to closed subspaces of a Banach space, where the Legendre form is defined on the whole Banach space. The following two lemmas guarantee that this is possible.

Lemma 4.4. *A closed subspace of a reflexive Banach space is itself reflexive.*

For this result, see [4, Corollary V.4.3].

Lemma 4.5. *Let Q be a Legendre form on a normed vector space X and let Y be a linear subspace of X. Then Q is also a Legendre form on Y.*

Proof. Every sequence $\{y_n\}_{n\in\mathbb{N}}$ in Y that converges in the weak topology on Y to $y \in Y$, also converges in the weak topology on X. The claim follows. □

The previous two lemmas will be used frequently without always referencing them.

Finally, we give a result that shows that quadratic forms are continuous if they are lower semi-continuous. A similar result is known for convex functions, see [1, Proposition 2.111]. However, a quadratic form does not need to be convex.

Lemma 4.6. *Let Q be a quadratic form on a Banach space X. If Q is lower semi-continuous then it is continuous.*

Proof. Our first goal is to find $x \in X, \varepsilon > 0$ such that Q is bounded on $B_{2\varepsilon}(x)$. In order to do this, consider the sets $C_n := \{y \in X : Q(y) \leq n\}$ for $n \in \mathbb{N}$. Because Q is lower semi-continuous, these sets are closed. Since $X = \bigcup_{n\in\mathbb{N}} C_n$, by Baire's theorem one of the sets C_n has to contain a nonempty open set. Therefore Q is bounded from above on some nonempty open set. Since Q is lower semi-continuous, it is also locally bounded from below, i.e., for a given $x \in X$ we find $\delta > 0, \widetilde{C} > -\infty$ such that $Q(y) > \widetilde{C}$ for all $y \in B_\delta(x)$.

Thus we know that there exist $x \in X, \varepsilon > 0, C > 0$ such that $|Q(y)| < C$ whenever $\|y - x\| < 2\varepsilon$. Using the parallelogram law, we have

$$|Q(z)| = \frac{1}{2}|Q(x+z) + Q(x-z) - 2Q(x)| \leq 2C \quad \forall\, z \in B_{2\varepsilon}(0).$$

Now it follows from the definition of a quadratic form that there exists a bilinear form $B : X \times X \to \mathbb{R}$ such that $Q(x) = B(x, x)$ for all $x \in X$. Without loss of generality we can assume that B is symmetric. Moreover, if $y_1, y_2 \in B_\varepsilon(0)$, we have $|B(y_1, y_2)| = \frac{1}{4}|Q(y_1 + y_2) - Q(y_1 - y_2)| \leq C$. Thus B is bounded in a neighborhood of 0 and therefore continuous. The claim follows. □

Because sequentially weakly lower semi-continuous functions are lower semi-continuous, an important consequence of this lemma is that on a Banach space every Legendre form is continuous.

4.2. Proving Hilbertizability under various assumptions. In this section, we will prepare the proof of Theorem 1.1. We do this by proving results which are similar to Theorem 1.1 but need additional assumptions. We start with the statement that X is Hilbertizable if Q is coercive, see Proposition 4.7. We then relax this assumption to positive quadratic forms in Proposition 4.8 and then nonnegative quadratic forms in Proposition 4.10. In Section 4.3 we will then provide a proof of Theorem 1.1 with the additional assumption that the operator A is injective, see Proposition 4.14.

We start with formulating the following well-known result:

Proposition 4.7. *Let Q be a coercive quadratic form on a Banach space X. Then X is Hilbertizable.*

The proof uses that $Q(\cdot)^{\frac{1}{2}}$ defines a norm on X which is equivalent to the original norm, see, e.g., [1, p. 195].

In order to generalize the result to positive Legendre forms, it suffices to show that a positive Legendre form is already coercive. This is done in the proof of the next result.

Proposition 4.8. *Let Q be a positive Legendre form on a reflexive Banach space X. Then X is Hilbertizable.*

Proof. Assume that Q is not coercive. Then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $Q(x_n) \to 0$ and $\|x_n\| = 1$ for all $n \in \mathbb{N}$.

Because $\{x_n\}_{n\in\mathbb{N}}$ is bounded in a reflexive Banach space, there is a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Without loss of generality, $x_n \rightharpoonup x$ for some $x \in X$. We have $0 \le Q(x) \le \liminf Q(x_n) = 0$. It follows that $Q(x) = 0$ and therefore $x_n \rightharpoonup x = 0$. Applying the definition of a Legendre form yields $x_n \to 0$ which is a contradiction to $\|x_n\| = 1$.

Thus, Q is coercive. Due to Proposition 4.7 it follows that X is Hilbertizable. □

So far we have investigated positive Legendre forms. Now we briefly turn our attention to nonpositive Legendre forms and provide a lemma that will be helpful later on.

Lemma 4.9. *Let Q be a Legendre form on a reflexive Banach space X. If $Q \le 0$ on X, then X is finite-dimensional.*

Proof. Because $-Q \ge 0$ is a quadratic form, it is convex, see, e.g., [1, Proposition 3.71]. Thus $-Q$ is sequentially weakly lower semi-continuous. As a consequence, Q is sequentially weakly continuous.

Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in X such that $x_n \rightharpoonup x$ for some $x \in X$. From $x_n \rightharpoonup x$ it follows that $Q(x_n) \to Q(x)$. Consequently, $\|x_n - x\| \to 0$.

Thus every sequence that converges weakly in X converges strongly in X. Applying Lemma 4.1 yields the desired result. □

With the help of Lemma 4.9 we are now able to extend the result of Proposition 4.8 to nonnegative Legendre forms.

Proposition 4.10. *Let Q be a Legendre form on a reflexive Banach space X. If $Q \ge 0$ on X, then X is Hilbertizable.*

Proof. We define $Y_0 := \{x \in X : Q(x) = 0\}$. In [1, Proposition 3.72] it is shown that Y_0 is a linear subspace of X. We observe that Y_0 is also a closed linear subspace. Therefore we can apply Lemma 4.9 to Y_0, which yields that

$\dim Y_0 < \infty$. This allows us to apply Lemma 4.2, which states that we can find a closed subspace Y_+ such that $X = Y_+ \dot{+} Y_0$. Because $Q > 0$ on $Y_+ \setminus \{0\}$, we know by Proposition 4.8 that Y_+ is Hilbertizable. Applying Lemma 4.3 yields the result. □

4.3. Completion of the proof. Our next goal is to prove the statement of Theorem 1.1 with the additional assumption that $\ker A = \{0\}$, i.e., A is injective. The idea for our proof will be to find closed subspaces Y_-, Y_{+0} such that $X = Y_{+0} \dot{+} Y_-$ where Q is negative on Y_- and $Q \geq 0$ on Y_{+0}. The next three lemmas will be a preparation for this, and we start with proving the existence of a maximal closed subspace Y_-.

Lemma 4.11. *Let Q be a Legendre form on a reflexive Banach space X. Then there exists a maximal closed subspace Y_- (w.r.t. set inclusions) such that Q is negative on Y_-.*

Proof. Suppose there is no maximal closed subspace with the desired property. Then we can construct a sequence of linear subspaces $(Z_i)_{i\in\mathbb{N}}$ such that $Z_i \subsetneq Z_{i+1}$ and Q is negative on Z_i for all $i \in \mathbb{N}$. If we define the $Z := \overline{\bigcup_{i\in\mathbb{N}} Z_i}$ we have that Z is an infinite-dimensional closed subspace with $Q \leq 0$ on Z. By Lemma 4.9 this is a contradiction. □

The following lemma allows us to show that a subspace Y_- is not maximal under certain conditions.

Lemma 4.12. *Let Q be a quadratic form on a finite dimensional vector space $X := Y_- \dot{+} \operatorname{lin}(y) \dot{+} \operatorname{lin}(x)$. We assume that Q is negative on Y_-, $Q(y) = 0$, $y \perp^Q Y_-$, and $\langle Ay, x\rangle \neq 0$.*

Then there is a linear subspace $\tilde{Y}_- \supsetneq Y_-$ such that Q is negative on $\tilde{Y}_-$.

Proof. Because Y_- is finite dimensional, we can find a Q-orthogonal basis $(y_i)_{i=1}^d$ of Y_- with the property that $Q(y_i) = -1$ for all $i \in \{1, \ldots, d\}$. We define

$$z_\alpha := x + \alpha y + \sum_{i=1}^{d} \langle Ax, y_i\rangle y_i$$

for $\alpha \in \mathbb{R}$. Then it can be shown that $z_\alpha \perp^Q Y_-$ for all $\alpha \in \mathbb{R}$. Indeed, $\langle Ay_j, z_\alpha\rangle = \langle Ay_j, x\rangle + \alpha\langle Ay_j, y\rangle + \sum_{i=1}^d \langle Ax, y_i\rangle\langle Ay_j, y_i\rangle = \langle Ay_j, x\rangle + \langle Ax, y_j\rangle\langle Ay_j, y_j\rangle = 0$ is true for every basis vector y_j of Y_-. Now we will calculate $Q(z_\alpha)$. We have

$$\begin{aligned} Q(z_\alpha) = \langle Az_\alpha, x + \alpha y\rangle &= Q(x + \alpha y) + \sum_{i=1}^{d} \langle Ax, y_i\rangle\langle Ay_i, x + \alpha y\rangle \\ &= Q(x) + 2\alpha\langle Ax, y\rangle + \sum_{i=1}^{d} \langle Ax, y_i\rangle^2. \end{aligned}$$

Due to $\langle Ax, y\rangle \neq 0$ we can choose $\alpha \in \mathbb{R}$ such that $Q(z_\alpha) < 0$. Finally, if we set $\tilde{Y}_- := Y_- \dotplus \operatorname{lin}(z_\alpha)$ it is easy to check $Q < 0$ on $\tilde{Y}_- \setminus \{0\}$ by using $z_\alpha \perp^Q Y_-$. □

If a subspace $Y_{+0} \dotplus Y_-$ is not equal to the full space X, the next lemma enables us to find an element outside of $Y_{+0} \dotplus Y_-$ that is Q-orthogonal to that linear subspace.

Lemma 4.13. *Let Q be a Legendre form on a reflexive Banach space X. We assume that A is injective. Let $Y := Y_{+0} \dotplus Y_-$ be a subspace with closed subspaces $Y_{+0}, Y_- \subset X$. Furthermore, we assume that Y_- is a maximal closed subspace (w.r.t. to set inclusions) with the property that Q is negative on Y_-.*

If $X \neq Y$, then there exists $z \in X \setminus Y$ such that $z \perp^Q Y$.

Proof. Consider the set $Y^{\perp^Q} := \{x \in X : x \perp^Q Y\}$. First, we make the assumption that $Y^{\perp^Q} \subset Y$ holds.

Let $x \in X \setminus Y$ be given. Note that because Y_- is finite dimensional according to Lemma 4.9 and the sum of a closed subspace and a finite dimensional subspace is closed (see, e.g., [8, Theorem 1.42]) we know that Y is closed. Therefore we can invoke Hahn-Banach to find $x^* \in X^*$ such that $\langle x^*, x\rangle = 1$ and $\langle x^*, y\rangle = 0$ for all $y \in Y$.

Assume that there exists $y \in X$ such that $Ay = x^*$. Then $y \in Y^{\perp^Q}$ and thus $y \in Y$. Therefore, $Q(y) = \langle Ay, y\rangle = 0$ and $y \perp^Q Y_-$. Since also $\langle Ay, x\rangle \neq 0$ is true, we can apply Lemma 4.12 to $Y_- \dotplus \operatorname{lin}(x) \dotplus \operatorname{lin}(y)$. This yields a contradiction to the maximality of Y_-.

Thus we know that $x^* \notin \operatorname{range}(A)$. However, using that A is injective and self-adjoint, we also know that $\operatorname{range}(A)$ is dense in X^*. Thus, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $Ax_n \to x^*$. We claim that $\|x_n\| \to \infty$ has to hold. Indeed, if $\{x_n\}_{n\in\mathbb{N}}$ has a bounded subsequence, it must also have a weakly convergent subsequence. A weak limit $z \in X$ would have to satisfy $Az = x^*$, which is a contradiction to $x^* \notin \operatorname{range}(A)$.

We define the sequence $\{y_n\}_{n\in\mathbb{N}}$ via $y_n := \frac{1}{\|x_n\|} x_n$. Note that $Ay_n \to 0$ and $\|y_n\| = 1$ for all $n \in \mathbb{N}$. Because $\{y_n\}_{n\in\mathbb{N}}$ is bounded, it has a weakly convergent subsequence. Hence, without loss of generality, $y_n \rightharpoonup z$ for some $z \in X$. Therefore, $Ay_n \to 0$ implies $Az = 0$, hence $z = 0$. We also know that $Q(y_n) = \langle Ay_n, y_n\rangle \to 0$. Using the definition of a Legendre form, we have $y_n \to 0$, which is a contradiction to $\|y_n\| = 1$.

Thus our initial assumption $Y^{\perp^Q} \subset Y$ is false, and any $z \in Y^{\perp^Q} \setminus Y$ satisfies $z \perp^Q Y$. □

Now we are able to prove Hilbertizability if A is injective.

Proposition 4.14. *Let Q be a Legendre form on a reflexive Banach space X. If A is injective then X is Hilbertizable.*

Proof. First, we use Lemma 4.11 to choose a maximal closed subspace Y_- such that Q is negative on Y_-.

We will apply Zorn's Lemma to the collection of linear subspaces

$$\mathcal{F} := \{Z \subset X : Z \text{ closed subspace, } Q \geq 0 \text{ on } Z, Z \perp^Q Y_-\}$$

with the ordering "$\subset$". Let us check the requirements for Zorn's Lemma. Due to $\{0\} \in \mathcal{F}$ it is clear that $\mathcal{F}$ is nonempty. Let $(Z_i)_{i\in I}$ be a given (totally ordered) chain of sets $Z_i \in \mathcal{F}$ for an arbitrary index set I. Then $\tilde{Z} := \overline{\bigcup_{i\in I} Z_i}$ is an upper bound on that chain, and it is easy to see that $\tilde{Z} \in \mathcal{F}$. Thus the requirements for Zorn's Lemma are satisfied and $\mathcal{F}$ contains a maximal element.

We denote a maximal element of $\mathcal{F}$ by Z and assume that $X \neq (Z \dot{+} Y_-)$. We can apply Lemma 4.13 and find $x \in X \setminus (Z \dot{+} Y_-)$ such that $x \perp^Q (Z \dot{+} Y_-)$. Consider the case $Q(x) \geq 0$. Then $Q \geq 0$ on the closed subspace $Z \dot{+} \operatorname{lin}(x)$, which is a contradiction to the maximality of Z. On the other hand, if $Q(x) < 0$, then $Q < 0$ holds on $(Y_- \dot{+} \operatorname{lin}(x)) \setminus \{0\}$, which is a contradiction to the maximality of Y_-. Thus our assumption is wrong and $X = (Z \dot{+} Y_-)$.

Finally, we can combine Proposition 4.10 and Lemma 4.9 with Lemma 4.3, which completes the proof. □

Now it remains to generalize the result of Proposition 4.14 to arbitrary Legendre forms Q and dropping the condition that A is injective.

Proof of Theorem 1.1. Using Lemma 4.9, it follows that $\ker A$ is finite dimensional. By Lemma 4.2 there exists a closed subspace V of X with $X = V \dot{+} \ker A$.

Consider the restriction of Q to V and the corresponding self-adjoint operator $A_V : V \to V^*$. We will show that $\ker A_V = \{0\}$. Indeed, let $z \in \ker A_V$ be given. Then for each $v \in V, x_0 \in \ker A$ we have

$$0 = \langle A_V z, v\rangle_{V^*\times V} = \langle Az, v\rangle_{X^*\times X} + \langle Ax_0, z\rangle_{X^*\times X} = \langle Az, v + x_0\rangle_{X^*\times X}.$$

Thus $\langle Az, x\rangle = 0$ for all $x \in X = V \dot{+} \ker A$ and therefore $z \in \ker A \cap V = \{0\}$.

Now, we can apply Proposition 4.14 in the subspace V, hence V is Hilbertizable. Thus we can apply Lemma 4.3 to $X = V \dot{+} \ker A$, which completes the proof. □

We note that our proof yields a Q-orthogonal decomposition $X = Z \dot{+} Y_-$ such that $Q \geq 0$ on Z and Q is negative on Y_-, which has similarities with Theorem 3.2.

5. Further remarks and conclusion

5.1. Extended Legendre forms. In [1, Definition 3.73] the concept of an extended Legendre form is introduced.

Definition 5.1. An *extended Legendre form* is a continuous and sequentially weakly lower semi-continuous function $Q : X \to \mathbb{R}$ that has the properties $Q(tx) = t^2 Q(x)$ for all $x \in X$, $t > 0$ and

$$(Q(x_n) \to Q(x) \wedge x_n \rightharpoonup x) \quad \Rightarrow \quad x_n \to x$$

for all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X.

Note that an extended Legendre form does not have to be a quadratic form. The question arises whether Theorem 1.1 holds for extended Legendre forms. The following example shows that this is not the case.

Example 5.2. Let $p \in (1, \infty) \setminus \{2\}$ be given. Then the function

$$Q : L^p(0,1) \to \mathbb{R}, \quad Q(x) = \|x\|^2_{L^p(0,1)}$$

is an extended Legendre form on a reflexive Banach space.

Proof. Clearly, Q is continuous, sequentially weakly lower semi-continuous, and has the property $Q(tx) = t^2 Q(x)$ for $x \in L^p(0,1), t \in \mathbb{R}$.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $L^p(0,1)$ such that $x_n \rightharpoonup x \in L^p(0,1)$ and $\|x_n\|^2_{L^p(0,1)} \to \|x\|^2_{L^p(0,1)}$. Then according to [2, Proposition 3.32] it follows that $x_n \to x$ (we can apply this proposition because $L^p(0,1)$ is uniformly convex, see [3]). □

5.2. Legendre forms in non-reflexive Banach spaces. In this subsection we will give a counterexample to show that Theorem 1.1 does not hold for non-reflexive Banach spaces. This will be done using the space ℓ^1.

Example 5.3. There exists a Legendre form on the Banach space ℓ^1, although ℓ^1 is not Hilbertizable.

Proof. It is clear that ℓ^1 is not Hilbertizable. According to [4, Proposition V.5.2] every weakly convergent sequence in ℓ^1 converges in norm. As a consequence, every quadratic form on ℓ^1 is a Legendre form. It remains to show the existence of a quadratic form on ℓ^1. This is indeed the case, the simplest example being $Q(x) = 0$. □

However, as already mentioned in the introduction, Legendre forms are rarely relevant in non-reflexive Banach spaces such as ℓ^1, because there can be bounded sequences without a weakly convergent subsequence.

In the case, that a Banach space X has a separable predual space, the sequential version of the Banach-Alaoglu theorem guarantees that every bounded sequence has a weakly-$*$ convergent subsequence. In these Banach spaces it would be more reasonable to redefine Legendre forms using weakly-$*$ convergent sequences instead of weakly convergent sequences in Definition 2.2. It is an open question, whether one can show a result similar to Theorem 1.1 for these adapted Legendre forms.

5.3. Conclusion. We were able to show that if a Legendre form is defined on a reflexive Banach space, this space is already isomorphic to a Hilbert space. As an example, Legendre forms cannot exist on ℓ^p or $L^p(\Omega)$, where $p \in (1, \infty) \setminus \{2\}$ and $\Omega \subset \mathbb{R}^d$ is an open subset. As a simple consequence, the characterization of Legendre forms in Hilbert spaces in Theorem 3.1 also holds in reflexive Banach spaces.

Our results for Legendre forms in reflexive Banach spaces do not generalize to non-reflexive Banach spaces or to extended Legendre forms.

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