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Coefficient Groups Inducing Nonbranched Optimal Transport

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Abstract. In this work we consider an optimal transport problem with coefficients in a normed Abelian group G, and extract a purely intrinsic condition on G that guarantees that the optimal transport (or the corresponding minimum filling) is not branching. The condition turns out to be equivalent to the nonbranching of minimum fillings in geodesic metric spaces. We completely characterize discrete normed groups and finite-dimensional normed vector spaces of coefficients that induce nonbranching optimal transport plans. We also provide a complete classification of normed groups for which the optimal transport plans, besides being nonbranching, have acyclic support. This seems to initiate new geometric classifications of certain normed groups. In the nonbranching case we also provide a global version of calibration, i.e. a generalization of Monge–Kantorovich duality.

Keywords. Optimal transport, Abelian group, rectifiable chain, minimal filling, calibration

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1. Introduction

1.1. Basic setting and motivation. The present work can be considered as an attempt to do an *ab initio* study of transportation problems, interpreted in a very broad sense. We consider n points $x_1, \dots, x_n$ in a space X, and associated coefficients $g_1, \dots, g_n$ in a space G.

These points and coefficients may be interpreted as locations and quantifications of some entities. Then, informally speaking, we want to study the properties of the "lowest cost 1-dimensional transport" for the quantities g_i between the sources x_i, under minimal assumptions on X and G. For more geometric motivations to the same problems see also the introductions of [6, 8, 14, 15].

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It is a natural assumption to require X to be a *geodesic metric space*, meaning that for any $x, y \in X$ there exists a curve in X of length equal to their distance $d(x, y)$. As we want to be able to implement a "lossless transport" condition for the quantity modelled by G we have to be able to combine together different quantities g_i, and thus the space G has to be a group. Because at a crossing of our transport system the order in which we sum contributions from the different branches is irrelevant, we require G to be an Abelian group. Moreover to compare different coefficients we consider a norm $|\cdot|_G : G \to \mathbb{R}_{\geq 0}$ compatible with the group operation, i.e. we consider a *normed Abelian group* $(G, |\cdot|_G)$. Following [15], the axioms for $|\cdot|_G$ are

(1) $|g + h|_G \leq |g|_G + |h|_G$ for all $g, h \in G$,
(2) $|g|_G = |-g|_G$ for all $g \in G$,
(3) $|g|_G = 0$ if and only if $g = 0_G$.

We often write $|\cdot|$ and 0 rather than $|\cdot|_G$ and 0_G, in case the group and norm are clear from the context.

The optimal transport problem which we consider is the following: Given points $x_1, \ldots, x_n \in X$ and coefficients $g_1, \ldots, g_n \in G$, we split each g_i, $i = 1, \ldots, n$, into parts g_{ij}, $j = 1, \ldots, n$, and then interpret g_{ij} to be the quantity moving from x_i to x_j (g_{ij} is then assumed to be equal to $-g_{ji}$). Among all such decompositions we seek the one minimizing the transport cost

$$\sum_{i<j} |g_{ij}| d_X(x_i, x_j).$$

Definition 1.1 (Optimal transport plans)**.** Let $g_1, \ldots, g_n$ be elements in a normed Abelian group G such that $g_1 + \cdots + g_n = 0$ and assume that an n-point metric space is given by $(\{x_1, \ldots, x_n\}, d)$. Then we set

$$OT\left(\sum_{i=1}^{n} g_i [\![x_i]\!]\right) := \inf \sum_{1 \leq i < j \leq n} |g_{ij}| d(x_i, x_j), \tag{1.1}$$

where the infimum is taken over all g_{ij}, $i, j = 1, \ldots, n$, with $g_{ij} = -g_{ji}$, $g_{ii} = 0$ and $g_i = \sum_{j=1}^{n} g_{ij}$. We say that G has *optimal transport plans* if for any choice of g_i and x_i as above, the infimum in (1.1) is achieved.

Note that if G has no elements g of torsion 2, i.e. there is no $g \neq 0$ with $g + g = 0$, then the condition $g_{ii} = 0$ is implied by $g_{ij} = -g_{ji}$, and thus becomes redundant.

It is clear that in case G is proper, i.e. the closed balls $\mathbf{B}(0, r)$ are compact for all $r > 0$, then G has optimal transport plans. As such, this is a rather weak condition on the group.

Example 1.2. For the case $G = \mathbb{R}$ with the Archimedean norm, we obtain the usual notion of optimal transport: due to the condition $g_1 + \cdots + g_n = 0$, up to reordering we may suppose that there exists $1 \leq k \leq n$ such that $g_1 \geq \cdots \geq g_k \geq 0 \geq g_{k+1} \geq \cdots \geq g_n$, and then the problem (1.1) becomes equivalent to that of transporting at minimal cost (where the transport cost is equal to the distance) the masses $|g_1|, \ldots, |g_k|$ situated at points $x_1, \ldots, x_k$ to masses $|g_{k+1}|, \ldots, |g_n|$ situated at the points $x_{k+1}, \ldots, x_n$.

1.2. Groups with nonbranching optimal transport plans. We next introduce the notion of branching, expressed in terms of group coefficients only. We point out the first basic example of branched transport as considered first by Gilbert in 1967 [7] and more recently formalized by Xia [16] appears for the case $G = \mathbb{R}$ with the norm $|x|_G := |x|^\alpha$, for $\alpha \in]0, 1[$. Then we have the strict subadditivity $|a + b|^\alpha < |a|^\alpha + |b|^\alpha$ for $a, b > 0$, which is the fundamental reason why branching for optimal transport occurs (see the discussion in [1]). The condition from Definition 1.3 below is precisely preventing this to occur, in the general case.

Definition 1.3 (Nonbranching optimal transport plans)**.** Assume that $(G, |\cdot|)$ has optimal transport plans. We say that $(G, |\cdot|)$ has *nonbranching optimal transport plans* if for any finite collection $g_1, \ldots, g_n \in G$ with $\sum_{i=1}^n g_i = 0$ there are $g_{ij} \in G$, $i, j = 1, \ldots n$, with

$$\begin{cases} g_{ij} = -g_{ji} & \text{for all } i, j = 0, \ldots, n, \\ g_{ii} = 0 & \text{for all } \quad i = 1, \ldots, n, \\ g_i = \displaystyle\sum_{j=1}^n g_{ij} & \text{for all } \quad i = 1, \ldots, n, \\ |g_i| = \displaystyle\sum_{j=1}^n |g_{ij}| & \text{for all } \quad i = 1, \ldots, n. \end{cases} \tag{NBP}$$

Once we know that a group G has nonbranching optimal transport plans, the next regularity condition to require is that the graph encoding how G-mass is transported along transport plans, does not have cycles. This kind of requirement turns out to generate interesting geometric conditions on G, and is the content of the next definition:

Definition 1.4 (Acyclic nonbranching optimal transport plans)**.** We say that the normed Abelian group $(G, |\cdot|)$ has *acyclic nonbranching optimal transport plans* if for any finite collection $g_1, \ldots, g_n \in G$ with $\sum_{i=1}^n g_i = 0$ there are $g_{ij} \in G$, $i, j = 1, \ldots n$, as in equations (NBP) such that the graph with vertices $V := \{g_i \,:\, 1 \leq i \leq n\}$ and edges $\{\{g_i, g_j\} : g_i, g_j \in V \,:\, g_{ij} \neq 0\}$ doesn't contain cycles.

In Theorem 1.9 we classify all complete groups G that have acyclic nonbranching optimal transport plans. All such groups turn out to be proper, and thus have optimal transport plans as a consequence of our classification. For this reason we do not put this assumption in the definition of groups with acyclic nonbranching optimal transport plans. We will note in Section 4 that a first necessary condition on G for having acyclic nonbranching optimal transport plans, perhaps geometrically appealing in its own right, is the following:

Definition 1.5 (collinear zero mean triples)**.** Let $(G, |\cdot|)$ be a normed Abelian group. We say that $a, b, c \in G$ form a *zero mean triple* if $a + b + c = 0$. We say that a nontrivial triple is *collinear* if

$$\text{one of} \quad |a| + |b| = |c|, \quad |a| + |c| = |b|, \quad |b| + |c| = |a| \quad \text{holds.} \tag{1.2}$$

We say that $(G, |\cdot|)$ has *collinear zero mean triples* if

$$\text{all zero mean triples are collinear.} \tag{CZT}$$

1.3. Branched transport is a minimal filling problem. In fact, the minimization problems that are considered under the denomination of "branched optimal transport" are usually not formulated in the form of an optimal transport problem in which marginals are fixed and one minimizes over transport plans, but rather they are formulated exactly as a minimal filling problem. This link to the minimization among G-chains is also pointed out in [16]. Motivated by this fact, we introduce the nonbranching property defined in terms of minimal fillings.

The spaces of rectifiable and flat k-dimensional chains in a metric space X with coefficients in G were defined by Fleming [6] for $X = \mathbb{R}^n$ and extended by De Pauw and Hardt [3] to arbitrary metric spaces. A 0-dimensional rectifiable chain in $\mathscr{R}_0(X; G)$ with finite support is simply a finite union of points $p_1, \ldots, p_n$ in X to each of which a coefficient g_i in G is associated. Such a chain T is denoted by

$$T = \sum_{i=1}^{n} g_i [\![p_i]\!].$$

If $\gamma : [0, 1] \to X$ is a Lipschitz path and $g \in G$, then a 1-dimensional Lipschitz G-chain is given by $\gamma_\#(g[\![0, 1]\!]) \in \mathscr{L}_1(X; G)$ and its mass is $\mathbf{M}(S) = |g| \operatorname{length}(\gamma)$ in case γ is injective. See [3] for the precise definition of mass in this context. Moreover, any Lipschitz chain in $\mathscr{L}_1(X; G)$ is the finite sum of chains of this type. A rectifiable chain $S \in \mathscr{R}_1(X; G)$ is induced by a G-valued orientation $\mathbf{g} : A \to G$ on an oriented 1-rectifiable Borel set $A \subset X$ such that the mass $\mathbf{M}(S) = \int_A |\mathbf{g}| \, d\mathscr{H}^1$ is finite, see [3, §3] for more details. As a subset of rectifiable chains, a polyhedral chain $S \in \mathscr{P}_1(X; G)$ in a normed space X is given by

$S = \sum_{\sigma \in K^{(1)}} g_\sigma [\![\sigma]\!]$, where $g_\sigma \in G$ and $K \subset X$ is a finite, oriented, 1-dimensional simplicial complex, see [4, p. 1052].

The *filling problem* for $T \in \mathscr{R}_0(X;G)$ is the following minimization problem

$$\mathrm{Fill}_{G,X}(T) := \inf\big\{\mathbf{M}(C) \,:\, C \in \mathscr{R}_1(X;G),\, \partial C = T\big\},$$

and as usual if T is not a boundary, then the filling length is infinite. It should be noted that in a Lipschitz path connected metric space X, the chain $\sum_{i=1}^n g_i [\![p_i]\!]$ is a boundary of elements in $\mathscr{L}_1(X;G)$ if and only if $\sum_{i=1}^n g_i = 0$. This can be proved by induction for Lipschitz chains using the identity [3, Theorem 4.2.1]:

$$\partial\gamma_\#(g[\![0,1]\!]) = g[\![\gamma(1)]\!] - g[\![\gamma(0)]\!].$$

For general rectifiable chains this follows by approximation, [3, Theorem 4.3.4].

First we state a definition that turns out to be equivalent to Definition 1.3, see Theorem 1.7.

Definition 1.6. We say that $(G, |\cdot|)$ has *nonbranching minimal fillings* if for all $g_1, \dots, g_n \in G$ such that $g_1 + \dots + g_n = 0$ and all $x_1, \dots, x_n \in X$ in a geodesic metric space X there is a $S \in \mathscr{L}_1(X;G)$ with

(1) $\partial S = T := \sum_{i=1}^n g_i [\![x_i]\!]$,
(2) $\mathrm{spt}(S) \subset \bigcup_{1 \le i < j \le n} [x_i, x_j]$, where $[x_i, x_j]$ is a geodesic segment connecting x_i with x_j in X,
(3) $\mathbf{M}(S) = \mathrm{Fill}_{G,X}(T)$.

Note that if the open geodesic segments (x_i, x_j) in the definition above are pairwise disjoint, then the constancy theorem [4, Theorem 6.4] implies that

$$S = \sum_{1 \le i < j \le n} g_{ij} [\![x_j, x_i]\!],$$

for some $g_{ij} \in G$, $i, j = 1, \dots, n$, where we set $g_{ii} = 0$ and $g_{ji} = -g_{ij}$ for $j > i$. If we further assume that $x_i \neq x_j$ for $i \neq j$ the condition $\partial S = T$ implies that $g_i = \sum_{j=1}^n g_{ij}$ for all $i = 1, \dots, n$.

1.4. Main results. As mentioned above, the two conditions of nonbranching (i.e. the one based on transport plans and the one based on fillings) are equivalent. We may interpret this by saying that that subadditivity phenomenon highlighted at the beginning of Subsection 1.2 is robust enough to pass to the case of general normed Abelian groups.

Theorem 1.7. *Let G be a normed Abelian group. The following are equivalent:*

(1) *G has nonbranching optimal transport plans.*
(2) *G has nonbranching minimal fillings.*

Our next step is to classify discrete groups that have nonbranching optimal transport or minimal fillings. We note here that this classification is only using the metric properties of our groups via equation (NBP), and does not require the property of G of having optimal transport plans from Definition 1.1.

First of all we note if A and B have nonbranching optimal transport plans, so does $A \oplus B$ with norm $|(a,b)| = \lambda|a|_A + \mu|b|_B$, where $\lambda, \mu > 0$ are arbitrary, see Lemma 3.1 for a more general statement that treats arbitrary direct sums. This suggests that groups with nonbranching optimal transport plans are ℓ_1-sums of elementary building blocks. Our next main result proves this, and completely classifies discrete normed Abelian groups which have nonbranching optimal transport plans (or minimal fillings, which is equivalent by Theorem 1.7).

Note that beyond discrete groups the class of normed Abelian groups is very large, and in particular contains the class of Banach spaces as a special subclass. Keeping this in mind, we also give a complete classification for the case where $(G, |\cdot|)$ is a finite-dimensional normed vector space, and we leave a more general classification of non-discrete groups with nonbranching optimal transport to a future work.

Theorem 1.8. *If G is a normed Abelian group with nonbranching optimal transport plans, then the following two classification statements hold:*

(1) *If G is discrete, then G is isometrically isomorphic to the direct sum $(\bigoplus_{i\in I}\mathbb{Z}) \oplus (\bigoplus_{j\in J}\mathbb{Z}_2)$ for some I, J with norm*

$$\left|\sum_{i\in I} m_i + \sum_{j\in J} n_j\right| = \sum_{i\in I} |m_i|\mu_i + \sum_{j\in J} |n_j|\lambda_j,$$

for some coefficients μ_i, λ_j with $\inf_{i,j}\{\mu_i, \lambda_i\} > 0$.

(2) *If $(G, |\cdot|)$ can be endowed with a multiplication by real scalars such that it becomes a finite dimensional normed vector space, then G is isometrically isomorphic (as a normed vector space) to ℓ_1^n for some $n \geq 1$, where ℓ_1^n is the vector space $\mathbb{R}^n$ with norm $\|x\| = \sum_{i=1}^n |x_i|$.*

Our next result is a complete classification of groups with acyclic nonbranching transport plans. In this case, within the class of all normed Abelian groups, we find that only four groups satisfy the condition:

Theorem 1.9 (Classification of groups with acyclic nonbranching optimal transport plans)**.** *The following ones are the only complete normed Abelian groups that have acyclic nonbranching optimal transport plans, up to rescaling of their norm by a constant factor:*

- $\mathbb{R}$ *with the Archimedean norm,*
- $\mathbb{Z}$ *with the Archimedean norm,*
- $\mathbb{Z}_2$*,*
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *with norm satisfying* $|(1,0)| = 1, |(0,1)| = \alpha, |(1,1)| = 1 + \alpha$ *for any choice of* $\alpha \geq 1$.

The above theorem is based on the complete classification of groups with collinear zero mean triples:

Proposition 1.10 (Classification of groups with collinear zero mean triples)**.** *The following ones are the only complete normed Abelian groups that have* (CZT)*, up to rescaling of their norm by a constant factor:*

- $\mathbb{R}$ *with the Archimedean norm,*
- $\mathbb{Z}$ *with the Archimedean norm,*
- $\mathbb{Z}_2$*,*
- $\mathbb{Z}_4$ *with norm satisfying* $|1| = 1, |2| = 2$*,*
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *with norm satisfying* $|(1,0)| = 1, |(0,1)| = \alpha, |(1,1)| = 1 + \alpha$ *for any choice of* $\alpha \geq 1$.

We then note that the groups with nonbranching optimal trasport plans as extracted in Theorem 1.8 are endowed with a version of a global nonlinear duality, or, in more geometric terms, they have *calibrations*. This result is based on the corresponding result on the existence of calibrations for the minimum filling problem with coefficients in $\mathbb{Z}_2$ as obtained in [11] and on Kantorovich duality, for the cases of coefficients in $\mathbb{R}$ or $\mathbb{Z}$.

Proposition 1.11. *Let* $G = \mathbb{R}^k \oplus \mathbb{Z}^l \oplus \mathbb{Z}_2^m$ *with the* ℓ_1*-norm as in Theorem* 1.8*, i.e.*

$$|(a_1, \ldots, a_k, b_1, \ldots, b_l, c_1, \ldots, c_m)| = \sum_{h=1}^{k} \lambda_h |a_h| + \sum_{i=1}^{l} \mu_i |b_i| + \sum_{j=1}^{m} \nu_j |c_j|,$$

for real numbers $\lambda_h, \mu_i, \nu_j > 0$. *Consider a chain* $R = \sum_{i=1}^n g[\![x_i]\!] \in \mathscr{R}_0(X;G)$ *such that* $\sum_{i=1}^n g = 0$ *in a geodesic metric space* X*. Then*

$$\mathrm{Fill}_{G,X}(R) = \max_{f_1,\ldots,f_{k+l+m},T} \mathrm{Fill}_{G,T}\left(\sum_{j=1}^{k+l+m} f_{j\#}(\pi_j R)\right), \tag{1.3}$$

where T *ranges over all finite geodesic trees and* 1*-Lipschitz maps* $f_i : X \to T$*. Here* $\pi_i : G \to G_i$ *is the projection onto the* i*th factor and* $\pi_i : \mathscr{R}_*(X;G) \to \mathscr{R}_*(X;G_i)$ *is its induced map.*

As a partial converse to this proposition we point out in Lemma 5.1 that any proper normed Abelian group that can be calibrated with maps into trees, needs to have nonbranching optimal transport plans.

2. Proof of Theorem 1.7

(1)⇒(2). We first consider the situation where $X = \ell_\infty^m$, that is $\mathbb{R}^m$ equipped with the sup-norm. Consider $T = \sum_{i=1}^n g_i [\![x_i]\!] \in \mathscr{R}_0(\ell_\infty^m; G)$, where $\partial T = 0$, meaning $\sum_{i=1}^n g_i = 0$. We also assume without loss of generality that $g_i \neq 0$ and that all the x_i are different.

Claim. *Let $P \in \mathscr{P}_1(\ell_\infty^m; G)$ with $\partial P = T$. For any $\epsilon > 0$ there exist $g_{ij} \in G$, $1 \leq i, j \leq n$, satisfying equations* (NBP)*, and such that*

$$\sum_{i<j} |g_{ij}| d(x_i, x_j) \leq \mathbf{M}(P) + \epsilon.$$

Proof. We start with an arbitrary polyhedral chain $P \in \mathscr{P}_1(\ell_\infty^m; G)$ with $\partial P = T$. In particular, there is an underlying finite, oriented graph with vertex set $V(P) \supseteq \{x_1, \ldots, x_n\}$ and edge set $E(P)$ such that $P = \sum_{e \in E(P)} g_e [\![e]\!]$. We may assume that for each $e \in E(P)$ it holds $g_e \in G \setminus \{0\}$. Fix now $\delta > 0$. Identifying ℓ_∞^m with a subspace of a larger dimensional space ℓ_∞^μ if necessary, we may find a polyhedral chain $P_0 \in \mathscr{P}_1(\ell_\infty^\mu; G)$ and an underlying finite, oriented graph $(V(P_0), E(P_0))$ such that:

- (a) $P_0 = \varphi_\# P$, where $\varphi : \mathrm{spt}(P) \to \ell_\infty^\mu$ is injective and affine on each edge in $E(P)$. Due to the injectivity of φ an underlying graph of P_0 is obtained by $V(P_0) := \varphi(V(P))$ and $E(P_0) := \{[\varphi(v), \varphi(w)] : [v, w] \in E(P)\}$.
- (b) $d(x_i, \varphi(x_i)) \leq \delta$ for all $i = 1, \ldots, n$.
- (c) $\mathbf{M}(P_0) \leq \mathbf{M}(P) + \delta$.
- (d) If $v_1, v_2, v_3 \in V(P_0)$ are different, then $L(v_1, v_2) \cap L(v_1, v_3) = \{v_1\}$, where $L(v, w) = \{v + t(w - v) : t \in \mathbb{R}\}$ is the line through v and w.
- (e) If $v_1, v_2, v_3, v_4 \in V(P_0)$ are different, then $L(v_1, v_2) \cap L(v_3, v_4) = \emptyset$.

If a finite graph has a vertex set that satisfies the above conditions (d) and (e), then we say that it is in *in general position.* If it is clear what the finite graph is that we associate to some $P' \in \mathscr{P}_1(\ell_\infty^\mu; G)$, we may also say that P' is in general position if (d) and (e) hold for the underlying graph. Note that φ above can be constructed already for $\mu = \max\{3, m\}$ by a slight perturbation of the vertices. Set $B := \mathrm{spt}(\partial P_0) = \{x_1', \ldots, x_n'\}$, where $x_i' := \varphi(x_i)$ for all $i = 1, \ldots, n$. Successively, assume that for $\alpha \geq 0$ a polyhedral chain P_α with associated graph $(V(P_\alpha), E(P_\alpha))$ is already constructed. If $V(P_\alpha) \neq B$ then we proceed to construct $P_{\alpha+1} \in \mathscr{P}_1(\ell_\infty^\mu; G)$ and an underlying graph $(V(P_{\alpha+1}), E(P_{\alpha+1}))$ having strictly fewer vertices than $(V(P_\alpha), E(P_\alpha))$, and such that

1. (1) $\partial P_{\alpha+1} = \partial P_0$.
2. (2) $\mathbf{M}(P_{\alpha+1}) \leq \mathbf{M}(P_\alpha)$.
3. (3) $B \subset V(P_{\alpha+1}) \subsetneq V(P_\alpha) \subset V(P_0)$, which implies that $P_{\alpha+1}$ is in general position.

Since $V(P_0)$ is a finite set, this process stops in a finite number of steps and we end up with a polyhedral chain that is supported on straight line segments connecting points in B.

We now describe the construction which achieves the above properties for $P_{\alpha+1}$ and $(V(P_{\alpha+1}), E(P_{\alpha+1}))$. Assume that P_α and its underlying graph are already constructed and that $V(P_\alpha) \neq B$. Pick some $v \in V(P_\alpha) \setminus B$ and denote by $\sum_{i=1}^{l} a_i[\![v, v_i]\!]$ the G-chain obtained by restricting S_α to the union of the edges that contain v. Because $v \notin \mathrm{spt}(\partial P_\alpha)$ it holds that $a_1 + \cdots + a_l = 0$. Since G has nonbranching optimal transport plans there are a_{ij}, $i, j = 1, \ldots, l$, satisfying equations (NBP). Then

$$
\begin{aligned}
\mathbf{M}\Bigg(\sum_{i=1}^{l} a_i[\![v, v_i]\!]\Bigg) &= \sum_{i=1}^{l} |a_i| d(v, v_i) \\
&= \sum_{i=1}^{l}\sum_{j=1}^{l} |a_{ij}| d(v, v_i) \\
&= \sum_{1\le i<j\le l} |a_{ij}| (d(v_j, v) + d(v, v_i)) \\
&\geq \sum_{1\le i<j\le l} |a_{ij}| d(v_j, v_i) \\
&= \mathbf{M}\Bigg(\sum_{1\le i<j\le l} a_{ij}[\![v_j, v_i]\!]\Bigg).
\end{aligned}
$$

In the first and last equality we use the fact that P_α is in general position. Now define

$$
P_{\alpha+1} := S_\alpha - \sum_{i=1}^{l} a_i[\![v, v_i]\!] + \sum_{1\le i<j\le l} a_{ij}[\![v_j, v_i]\!].
$$

The general position assumption guarantees that we can take $V(P_{\alpha+1}) := V(P_\alpha) \setminus \{v\}$ and $E(P_{\alpha+1})$ is obtained from $E(P_\alpha)$ by deleting the edges that contain v and adding the edges $[v_j, v_i]$ for $1 \le i < j \le l$. Since $V(P_{\alpha+1})$ contains only vertices from $V(P_\alpha)$ it is clear that $P_{\alpha+1}$ is also in general position and the estimates above imply that $\mathbf{M}(P_{\alpha+1}) \le \mathbf{M}(P_\alpha)$. From equations (NBP) it follows that $\partial P_{\alpha+1} = \partial P_\alpha = \partial P_0$.

At the end of this iterative procedure we obtain a chain $P' = \sum_{i<j} g_{ij}[\![x'_j, x'_i]\!]$ in $\mathscr{P}_1(\ell_\infty^\mu; G)$ with $\partial P' = \sum_{i=1}^{n} g_i[\![x'_i]\!]$ and hence $\sum_{j=1}^{n} g_{ij} = g_i$ for all $i = 1, \ldots, n$. By construction, $\mathbf{M}(P') \le \mathbf{M}(P_0) \le \mathbf{M}(P) + \delta$ and $d(x'_i, x_i) \le \delta$ for all $i = 1, \ldots, n$. Set $D := \inf_{i\neq j} d(x_i, x_j)$ and assume further that $\delta \le \frac{1}{3}D$. Then

$d(x_i', x_j') \geq \frac{1}{3}D$ and hence

$$\begin{aligned}\sum_{1\leq i<j\leq n} |g_{ij}|d(x_i,x_j) &\leq \sum_{1\leq i<j\leq n} |g_{ij}|(d(x_i',x_j') + 2\delta) \\ &= \mathbf{M}(P') + 2\delta \sum_{1\leq i<j\leq n} |g_{ij}| \\ &\leq \mathbf{M}(P') + 2\delta D^{-1} \sum_{1\leq i<j\leq n} |g_{ij}|D \\ &\leq \mathbf{M}(P') + 6\delta D^{-1} \sum_{1\leq i<j\leq n} |g_{ij}|d(x_i',x_j') \\ &= \mathbf{M}(P')(1 + 6\delta D^{-1})\end{aligned} \tag{2.1}$$

For $\delta > 0$ small enough, the claim follows. □

Now we extend the claim to a general geodesic space X. Let

$$T := \sum_{i=1}^n g_i [\![x_i]\!] \in \mathscr{R}_0(X; G)$$

with $\partial T = 0$ in a general geodesic space X and consider $S \in \mathscr{R}_1(X;G)$ such that $\partial S = T$. Again we can assume that $g_i \neq 0$ for all $1 \leq i \leq n$ and that $x_i \neq x_j$ for different i and j. Let $f : \mathrm{spt}(T) \to \ell_\infty^m$ be an isometric embedding. This can be achieved for example with the Kuratowski embedding, mapping x_i to $(d(x_1, x_i), \ldots, d(x_n, x_i)) \in \ell_\infty^n$ for each $i = 1, \ldots, n$. Then there exists a 1-Lipschitz extension $\bar{f} : X \to \ell_\infty^m$, for example by applying the McShane–Whitney Lipschitz extension theorem to each coordinate function. Thus $\mathbf{M}(\bar{f}_\# S) \leq \mathbf{M}(S)$. Taking a 1-Lipschitz projection onto a box $[-r, r]^m$ that contains $f(\{x_1, \ldots, x_n\})$, we can assume without loss of generality assume that the support of $\bar{f}_\# S$ is contained in $[-r,r]^m$ and is therefore compact. This allows to apply the polyhedral approximation result [5, Theorem 4.2(D)], due to which, for any fixed $\epsilon > 0$ we can find a polyhedral chain $P_\epsilon \in \mathscr{P}_1(\ell_\infty^m; G)$ with $\partial P_\epsilon = f_\# T$ and $\mathbf{M}(P_\epsilon) \leq \mathbf{M}(S) + \epsilon$. Thus by applying the claim to P_ϵ we find $g_{ij} \in G$, $1 \leq i, j \leq n$, for the g_i's as in equations (NBP) such that

$$\sum_{1\leq i<j\leq n} |g_{ij}|d(x_i, x_j) \leq \mathbf{M}(P_\epsilon) + \epsilon \leq \mathbf{M}(S) + 2\epsilon. \tag{2.2}$$

We are now ready to conclude the proof of our first implication. G has optimal transport plans by assumption and thus the infimum in (1.1) is achieved by some coefficients $g_{ij}' \in G$. With $[x_j, x_i]$ we denote an oriented geodesic segment from x_j to x_i in X. The chain $R := \sum_{i<j} g_{ij}' [\![x_j, x_i]\!] \in \mathscr{L}_1(X;G)$ satisfies $\partial R = T$ and $\mathbf{M}(R) \leq \sum_{i<j} |g_{ij}'| d(x_i, x_j)$. If we assume by contradiction that

$\mathrm{Fill}_{G,X}(T) < \mathbf{M}(R)$, then by picking a suitable $S \in \mathscr{R}_1(X;G)$ and $\epsilon > 0$, it follows from (2.2) that there exist some $g_{ij} \in G$ such that

$$\sum_{1\leq i<j\leq n} |g_{ij}| d(x_i, x_j) < \mathbf{M}(S) + \epsilon < \mathbf{M}(R) \leq \sum_{1\leq i<j\leq n} |g'_{ij}| d(x_i, x_j),$$

contradicting the minimality of g'_{ij}.

(2)⇒(1). Given an n-point metric space $(\{x_1, \dots, x_n\}, d)$ and $g_1, \dots, g_n \in G$, let X be the complete graph on the vertices $\{x_1, \dots, x_n\}$ equipped with the geodesic metric d_X that agrees with d on the vertex set. As in the discussion following Definition 1.6, for each admissible solutions of (1.1) we can construct a minimal filling of $T = \sum_{i=1}^n g_i [\![x_i]\!] \in \mathscr{R}_0(X;G)$ in $\mathscr{R}_1(X;G)$ and vice versa. For this note that by the constancy theorem, any filling $S \in \mathscr{R}_1(X;G)$ of T is automatically in $\mathscr{L}_1(X;R)$ and thus $S = \sum_{i<j} g_{ij} [\![x_j, x_i]\!]$ and $\mathbf{M}(S) = \sum_{i<j} |g_{ij}| d_X(x_j, x_i)$ for some g_{ij}. Thus a mass minimal filling of T in $\mathscr{R}_1(X;G)$ gives rise to a minimizer of (1.1). Hence G has optimal transport plans.

Next we show that G has nonbranching optimal transport plans. Let $g_1, \dots, g_n$ be elements in G with $g_1 + \dots + g_n = 0$. Consider the infinite geodesic metric graph (X, d) on the vertex set $V := \{x_1, \dots, x_n\} \sqcup \{c_1, c_2, \dots\}$ and with edges $E := \{\{x_i, x_j\}, \{x_i, c_k\} : i \neq j, k \in \mathbb{N}\}$. The length of the edges is given by $d(x_i, x_j) = 2$ if $i \neq j$ and $d(x_i, c_k) = 1 + \frac{1}{k}$. Consider the chain

$$T := \sum_{i=1}^n g_i [\![x_i]\!] \in \mathscr{R}_0(X;G).$$

By Definition 1.6 and the discussion thereafter, there exist $g_{ij} \in G$ with $g_{ij} = -g_{ji}$, $g_{ii} = 0$, $g_i = \sum_{j=1}^n g_{ij}$ such that the chain $S := \sum_{i<j} g_{ij} [\![x_j, x_i]\!] \in \mathscr{L}_1(X;G)$ satisfies $\partial S = T$ and

$$\sum_{1\leq i<j\leq n} |g_{ij}| d(x_i, x_j) = \mathbf{M}(S) \leq \mathbf{M}(C), \tag{2.3}$$

for all fillings $C \in \mathscr{R}_1(X;G)$ of T. For each $k \in \mathbb{N}$ let $C_k \in \mathscr{R}_1(X;G)$ be the chain given by

$$C_k := \sum_{i=1}^n g_i [\![c_k, x_i]\!],$$

which obviously satisfies $\partial C_k = T$. We set $M := \sum_{i=1}^n |g_i|$. With the definition of d, (2.3) and the triangle inequality we obtain for all $k \in \mathbb{N}$,

$$\sum_{i,j=1}^n |g_{ij}| = \sum_{1\leq i<j\leq n} |g_{ij}| d(x_i, x_j) \leq \mathbf{M}(C_k) = \sum_{i=1}^n |g_i|(1 + \tfrac{1}{k}) = \tfrac{1}{k} M + \sum_{i=1}^n |g_i|.$$

Hence for all i,

$$\sum_{j=1}^{n} |g_{ij}| \leq |g_i| + \tfrac{1}{k} M. \tag{2.4}$$

To justify this, note that if $A_i \leq B_i$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n B_i \leq \epsilon + \sum_{i=1}^n A_i$, then $B_i \leq \epsilon + A_i$ for all i. In the above situation we apply this for $\epsilon = \frac{1}{k} M$, $A_i = |g_i|$ and $B_i = \sum_j |g_{ij}|$. Since (2.4) holds for all k, we obtain that $\sum_j |g_{ij}| = |g_i|$ for all $i = 1, \ldots, n$ and thus G has nonbranching optimal transport plans.

3. Proof of Theorem 1.8

3.1. Product lemma. We first state the product lemma that was mentioned in the introduction.

Lemma 3.1. *Let $\{(G_k, |\cdot|_k)\}_{k \in K}$ be a collection of normed Abelian groups that have nonbranching optimal transport plans. Then the direct sum $\bigoplus_{k \in K} G_k$ with norm given by $|(g_k)_{k \in K}| := \sum_{k \in K} \lambda_k |g_k|_k$, where $\lambda_k > 0$ for $k \in K$ are arbitrary, also have nonbranching optimal transport plans.*

Proof. Let $(g_{k,1})_{k \in K}, \ldots, (g_{k,n})_{k \in K} \in \bigoplus_{k \in K} G_k$ be a finite collection with

$$\sum_{i=1}^{n} (g_{k,i})_{k \in K} = (0_k)_{k \in K}.$$

For each $k \in K$ there are $g_{k,ij} \in G_k$ $i, j \in \{1, \ldots, n\}$ that satisfy equations (NBP) by assumption. Defining the norm as stated, the elements $(g_{k,ij})_{k \in K} \in \bigoplus_{k \in K} G_k$ for $i, j \in \{1, \ldots, n\}$ also satisfy the equations (NBP) for the data $(g_{k,i})_{k \in K}$, $i \in \{1, \ldots, n\}$. With a similar argument we also obtain that $\bigoplus_{k \in K} G_k$ with the given norm has optimal transport plans as in Definition 1.1 if all the $(G_k, |\cdot|_k)$ have optimal transport plans. Hence $\bigoplus_{k \in K} G_k$ has nonbranching optimal transport plans. □

3.2. Discrete groups. Recall that a normed Abelian group $(G, |\cdot|)$ is *discrete*, if the norm induces the discrete topology. This is the case if and only if

$$m_G := \inf\{|g| \,:\, g \in G \setminus \{0\}\}. \tag{3.1}$$

An element g in a normed Abelian group G is called *indecomposable* if whenever $|h| + |g - h| = |g|$ for some $h \in G$, then $h = 0$ or $h = g$. As it is easy to check, equivalent to indecomposability is either one of the following two characterisations:

- for all $h \in G \setminus \{0, g\}$ the inequality $|h| + |g - h| > |g|$ holds.
- for each $n \geq 2$ and any $g_1, \ldots, g_n \in G$, if we have

$$g_1 + \cdots + g_n = g, \quad |g_1| + \cdots + |g_n| = |g|,$$

  then all but one of the elements $g_1, \ldots, g_n$ are equal to 0.

Note that 0 is indecomposable by the above definition. This notion of indecomposability is not related to the classical notion in ring theory, so no confusion should arise.

For $g_1, \ldots, g_n \in G$ we will denote by $\langle g_1, \ldots, g_n \rangle \subset G$ the subgroup generated by the elements $g_1, \ldots, g_n$.

For $h, g \in (G, |\cdot|)$ we write $h \perp \langle g \rangle$ if for all $n \in \mathbb{Z}$ the identity $|ng + h| = |ng| + |h|$ holds.

Lemma 3.2. *If G has nonbranching optimal transport plans and $g \in G \setminus \{0\}$ is indecomposable, then $2g = 0$ or*

$$n|g| = |ng|,$$

for all $n \in \mathbb{N}$. Moreover:

1. *If $|h| + |ng - h| = |ng|$ for some $n \in \mathbb{Z}$, then h is a multiple of g.*
2. *For all $h \in G$ there exists some $n \in \mathbb{Z}$ such that $|h - ng| = \inf_{m \in \mathbb{Z}} |h - mg|$.*
3. *This minimizer n in (2) is unique if $\langle g \rangle = \mathbb{Z}$ and unique modulo 2 if $\langle g \rangle = \mathbb{Z}_2$. Let us denote it by $n(h, g)$.*
4. *$h - n(h, g)g \perp \langle g \rangle$.*

Proof. For some $n \geq 2$ consider the points $g_1 = \cdots = g_n = g$ and $g_{n+1} = -ng$. By assumption there are $g_{i,j}$ for $i, j = 1, \ldots, n+1$ with the property:

$$\begin{cases} g_{i,j} = -g_{j,i} & \text{for all } i, j = 0, \ldots, n+1, \\ g_{i,i} = 0 & \text{for all } \quad i = 1, \ldots, n+1, \\ g_i = \displaystyle\sum_{j=1}^{n+1} g_{i,j} & \text{for all } \quad i = 1, \ldots, n+1, \\ |g_i| = \displaystyle\sum_{j=1}^{n+1} |g_{i,j}| & \text{for all } \quad i = 1, \ldots, n+1. \end{cases}$$

Because g_i is indecomposable for all $i \in \{1, \ldots, n\}$ there is exactly one $j \in \{1, \ldots, n+1\} \setminus \{i\}$ with $g_{i,j} \neq 0$. Hence $g_{i,j} = g_i = g$ and this forces $j = n+1$. Otherwise, $j \leq n$ would imply $g = g_j = g_{j,i} = -g_{i,j} = -g$ contradicting that $2g \neq 0$. Thus $|ng| = |g_{n+1}| = \sum_{j=1}^{n} |g_{n+1,j}| = \sum_{j=1}^{n} |-g| = n|g|$, and the result follows.

Ad (1): Next we show that if $|h|+|ng-h| = |ng|$, then h is a multiple of g. The statement is clear if $n = 1$ since g is indecomposable, and this also settles the case when $2g = 0$. So we can assume that $2g \neq 0$ and hence $n|g| = |ng|$ for $n \geq 1$ by the first part. Let us write $h + h' = ng$ with $|h| + |h'| = |ng|$ and set $g_1 = \cdots = g_n = g$, $g_{n+1} = -h$ and $g_{n+2} = -h'$. Then there are corresponding $g_{i,j} \in G$ as in equations (NBP). The indecomposability of g implies that for all $i = 1, \dots, n$ there is exactly one j with $g_{i,j}$ nonzero and equal g. Moreover, this j is equal $n + 1$ or $n + 2$. Up to a reordering we have $g_{1,n+1} = \cdots = g_{k,n+1} = g$, $g_{k+1,n+2} = \cdots = g_{n,n+2} = g$ and we denote $f := g_{n+1,n+2}$. It follows that $-h = -kg + f$ with $|h| = k|g| + |f|$ and similarly $-h' = (k - n)g - f$ with $|h'| = (n - k)|g| + |f|$. Hence

$$n|g| = |ng| = |h| + |h'| = k|g| + |f| + (n - k)|g| + |f| = n|g| + 2|f|.$$

This implies that $f = 0$ an therefore both h and h' are multiples of g.

Ad (2): Let $h \in G$ and assume first that $2g = 0$. Set $g_1 = g, g_2 = -h$, $g_3 = g+h$. Equations (NBP) and the indecomposability of g imply that $|g+h| = |h| + |g|$ or $|h| = |g| + |g + h|$. In this case it is also clear that (3) holds.

Next assume that $n|g| = |ng|$ for all $n \in \mathbb{N}$. Let $m, n \in \mathbb{Z}$ and assume that $|ng - h| < |mg - h|$. Clearly, $m \neq n$. Consider $g_1 = \cdots = g_s = g$, $g_{s+1} = ng-h$ and $g_{s+2} = h-mg$, where $s := |m-n|$. We fix the corresponding $g_{i,j} \in G$ as in equations (NBP). The indecomposability of g implies that for all $i = 1, \dots, s$ there is exactly one j such that $g_{i,j}$ is nonzero and equal g. Moreover, this j is equal $s + 1$ or $s + 2$. Up to a reordering we therefore have $g_{1,s+1} = \cdots = g_{k,s+1} = g$, $g_{k+1,s+2} = \cdots = g_{s,s+2} = g$ and we denote $f := g_{s+1,s+2}$. It follows that $ng-h = g_{s+1} = -kg+f$, $|ng-h| = k|g|+|f|$ and similarly $h - mg = (k - s)g - f, |h - mg| = (s - k)|g| + |f|$. Hence

$$k|g| + |f| = |ng - h| < |h - mg| = (s - k)|g| + |f|.$$

This implies that $|mg - h| - |ng - h|$ is a positive multiple of $|g|$. Hence the values of $n \mapsto |ng - h|$ are discrete in $\mathbb{R}$ and therefore there exists $n \in \mathbb{Z}$ such that $|ng + h| = \inf_{m\in\mathbb{Z}} |mg + h|$.

Ad (3): Again the statement is trivial for the case $2g = 0$ and we only have to treat the case where $n|g| = |ng|$ for all $n \in \mathbb{N}$. Let $h \in G$ and assume by contradiction that $n, m \in \mathbb{Z}$ are two different minimizers as in (2), i.e. $|ng-h| = |mg - h| = \inf_{k\in\mathbb{Z}} |kg - h|$. We may assume that $m - n > 0$ and consider $g_1 = \cdots = g_{m-n} = g$, $g_{m-n+1} = ng - h$ and $g_{m-n+2} = h - mg$. We fix the corresponding $g_{i,j} \in G$ as in equations (NBP). It follows as before that there is some $k \in \{1, \dots, m - n\}$ and $f \in G$ such that $ng - h = -kg + f$ with $|ng - h| = k|g| + |f|$ and similarly $h - mg = (k - m + n)g - f$ with $|h - mg| = (m - n - k)|g| + |f|$. This forces $k = 0$, otherwise

$$|(n + k)g - h| = |f| < k|g| + |f| = |ng - h|,$$

contradicting the minimality of $|ng - h|$. Hence $m = n$.

Ad (4): In case $2g = 0$, this part is immediate from (2), so we only have to consider the case $\langle g \rangle = \mathbb{Z}$. Given some $h \in G$, the element $h' := h - n(h, g)g$ clearly satisfies $n(h', g) = 0$. We want to show that $|ng - h'| = |ng| + |h'|$ for all $n \in \mathbb{Z}$. This is obvious if $n = 0$. Else consider $g_1 = \dots = g_s = g$, $g_{s+1} = -h'$ and $g_{s+2} = h' - ng$, where $s := |n|$. We fix the corresponding $g_{i,j} \in G$ as in equations (NBP). Again since g is indecomposable, for all $i = 1, \dots, s$ there is exactly one j with $g_{i,j}$ is nonzero and equal g. Moreover, this j is $s+1$ or $s+2$. Up to a reordering, there is some $k \in \{1, \dots, s\}$ and $f \in G$ such that $g_{1,s+1} = \dots = g_{k,s+1} = g$, $g_{k+1,s+2} = \dots = g_{s,s+2} = g$ and $f = g_{s+1,s+2}$. This implies that $-h' = -kg + f$ with $|h'| = k|g| + |f|$ and similarly $h' - ng = (k - s)g - f$ with $|h' - ng| = (s - k)|g| + |f|$. Because $n(h', g) = 0$ it must hold that $k = 0$ and therefore $f = -h'$. Otherwise, $|h' - kg| = |f| < |h'|$. Thus

$$|h' - ng| = (|n| - k)|g| + |f| = |n||g| + |h'| = |ng| + |h'|,$$

and applied to h,

$$|h - n(h, g)g - ng| = |ng| + |h - n(h, g)g|.$$

The above hold for all $n \in \mathbb{Z}$ and hence $h - n(h, g)g \perp \langle g \rangle$. □

Note that the points (1) to (4) of the lemma above are not used for the remaining discussion, but these properties may be useful for a more general classification of normed Abelian groups with nonbranching optimal transport plans.

Next we make a simple observation about the subgroup generated by the indecomposable elements.

Lemma 3.3. *Let $(G, |\cdot|)$ be a normed Abelian group that has nonbranching optimal transport plans. If $g, h \in G$ are indecomposable and $\langle g \rangle \neq \langle h \rangle$, then*

$$|kg + lh| = |kg| + |lh|,$$

for all $k, l \in \mathbb{Z}$ and $\langle g, h \rangle \simeq \langle g \rangle \oplus \langle h \rangle$.

Proof. The statement is clear if $g = 0$,$h = 0$,$k = 0$ or $l = 0$, so we assume that this is not the case. From Lemma 3.2 it follows that the groups generated by g and h are isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2$. Clearly, f is indecomposable if and only if $-f$ is indecomposable. So by replacing g or h by its inverse we can assume that $k, l \geq 1$. In case $\langle g \rangle$ or $\langle h \rangle$ is isomorphic to $\mathbb{Z}_2$ the corresponding integer is assumed to be 1. Consider $g_1 = \dots = g_k = g$, $g_{k+1} = \dots = g_{k+l} = h$ and $g_{k+l+1} = -kg - lh$. We fix the corresponding $g_{i,j} \in G$ as in equations (NBP). Since g and h are indecomposable, for all $i = 1, \dots, k$ there is exactly one j for which $g_{i,j}$ is nonzero and equal g and similarly for $i = k+1, \dots, k+l$. Moreover, this j is equal $k + l + 1$. We therefore have $g_{1,k+l+1} = \cdots = g_{k,k+l+1} = g$, $g_{k+1,k+l+2} = \cdots = g_{k+l,k+l+1} = h$, which implies that $|kg + lh| = k|g| + l|h|$ and hence $|kg + lh| = |kg| + |lh|$ by the triangle inequality.

From $|kg + lh| = k|g| + l|h|$ it follows that if $kg + lh = 0$, then $k = 0$ and $l = 0$. Hence $\langle g\rangle \cap \langle h\rangle = \emptyset$ and hence $\langle g, h\rangle \simeq \langle g\rangle \oplus \langle h\rangle$. □

For a normed Abelian group $(G,|\cdot|)$ we denote by $I_G \subset G$ a choice of a subset such that for any nonzero indecomposable element g either g or $-g$ is in I_G.

Lemma 3.4. *Let $(G, |\cdot|)$ be a normed Abelian group that has nonbranching optimal transport plans. Then*

$$(\langle I_G\rangle, |\cdot|) \text{ is isometrically isomorphic to } \bigoplus_{g\in I_G}^{\ell_1} (\langle g\rangle, |\cdot|)\,,$$

where $\bigoplus^{\ell_1}$ represents the ℓ_1-direct sum, i.e.

$$\left|\sum_{g\in I_G} n_g g\right| = \sum_{g\in I_G} |n_g g|,$$

and $|n_g g| = |n_g||g|$ if $\langle g\rangle = \mathbb{Z}$ and $|n_g g| = |g|$ if $\langle g\rangle = \mathbb{Z}_2$ and n_g is not divisible by 2.

Proof. The proof of Lemma 3.3 can easily be generalized to include any finite collection of indecomposable elements. □

We are now ready to classify discrete normed Abelian groups with nonbranching optimal transport plans.

Proof of Theorem 1.8(1). With Lemma 3.4, the only thing that is left to prove is that the subgroup $H := \langle I_G\rangle$ is equal to all of G. So assume by contradiction that H is not equal to G. As in (3.1) define

$$m_{G\setminus H} := \inf\{|g| \,:\, g \in G \setminus H\}.$$

Let g in $G \setminus H$ be such that $|g| < m_{G\setminus H} + m_G$, which is of course possible because $m_G > 0$. We claim that g is indecomposable. If this would not be the case, then there would exist $x \in G\setminus\{0, g\}$ with $|x|+|g-x| = |g|$. Since g is not in H one of the elements x or $g-x$ is also not in H. Without loss of generality x is not in H. But then

$$|x| = |g| - |g - x| < m_{G\setminus H} + m_G - m_G = m_{G\setminus H}.$$

This contradicts the definition of $m_{G\setminus H}$. Therefore g is indecomposable and hence an element of H, contradicting $g \in G \setminus H$. □

3.3. Finite dimensional normed spaces. It follows directly from Lemma 3.1 that $(\mathbb{R}^n, \|\cdot\|_1)$ has nonbranching optimal transport plans. In this subsection we want to show the converse.

An extreme point in a convex set $C \subset X$ in some Banach space X is a point $p \in C$ that can't be written as $\lambda p_1 + (1-\lambda)p_2$, where $\lambda \in (0,1)$ and $p_1, p_2 \in C \setminus \{p\}$.

Lemma 3.5. *Let p be an extreme point of the closed unit ball $\mathbf{B}_X(0,1)$ in some normed space $(X, \|\cdot\|)$. Then the equations $p = p_1 + \cdots + p_n$ and $\|p\| = \|p_1\| + \cdots + \|p_n\|$ can only hold if $p_1, \ldots, p_n$ are all multiples of p.*

Proof. As an extreme point, p has unit norm. We assume that $n = 2$ and that $p = p_1 + p_2$ and $\|p\| = \|p_1\| + \|p_2\|$, and the case of general n is proved by a simple induction argument. The conclusion of the lemma is clear if $\|p_1\| = 0$ or $\|p_2\| = 0$. If this is not the case, then $p = \lambda\frac{p_1}{\|p_1\|} + (1-\lambda)\frac{p_2}{\|p_2\|}$, where $\lambda = \|p_1\| = \|p\| - \|p_2\| = 1 - \|p_2\|$. By assumption $\lambda \neq 0$ and $\lambda \neq 1$, and therefore $\frac{p_1}{\|p_1\|} = p$ or $\frac{p_2}{\|p_2\|} = p$ since p is an extreme point. But then the equation $p = \lambda\frac{p_1}{\|p_1\|} + (1-\lambda)\frac{p_2}{\|p_2\|}$ implies that $\frac{p_1}{\|p_1\|} = \frac{p_2}{\|p_2\|} = p$. □

Lemma 3.6. *Assume that the normed space $(X, \|\cdot\|)$ has nonbranching optimal transport plans. If $p_1, \ldots, p_n \in \mathbf{B}_X(0,1)$ are linearly independent extreme points and $\lambda_1, \ldots, \lambda_n \in \mathbb{R} \setminus \{0\}$, then*

$$\left\|\sum_{i=1}^n \lambda_i p_i\right\| = \sum_{i=1}^n |\lambda_i|\|p_i\|.$$

Proof. Consider the points $g_1 = \lambda_1 p_1, \ldots, g_n = \lambda_n p_n$, $g_{n+1} = -\sum_{i=1}^n \lambda_i p_i$. By assumption there are $g_{i,j} \in G$ that satisfy equations (NBP). With Lemma 3.5 we conclude that for each i, all the vectors $g_{i,1}, \ldots, g_{i,n}$ are multiples of p_i. Since all the p_i's are linearly independent this implies that $g_{i,j} = 0$ for all $i, j \in \{1, \ldots, n\}$. Hence $g_{i,n+1} = \lambda_i p_i$ and further

$$\left\|\sum_{i=1}^n \lambda_i p_i\right\| = \|g_{n+1}\| = \sum_{i=1}^n \|g_{i,n+1}\| = \sum_{i=1}^n |\lambda_i|\|p_i\|,$$

as claimed. □

Lemma 3.7. *Assume that X is a normed space of dimension n that has nonbranching optimal transport plans, then X is linearly isometric to ℓ_1^n.*

Proof. According to Lemma 3.6 the only thing that needs to be shown is that $\mathbf{B}_X(0,1)$ has at least n linearly independent extreme points. But this is a simple consequence of the Krein–Milman theorem. □

This finishes the proof of Theorem 1.8.

4. Proof of Theorem 1.9 and of Proposition 1.10

We recall that Theorem 1.9 concerns the classification of all normed Abelian groups $(G, |\cdot|)$ that have acyclic nonbranching optimal transport plans. We first show that the groups mentioned in the statement of Theorem 1.9 indeed satisfy the properties stated in Definition 1.4.

Lemma 4.1. *All the groups listed in Theorem 1.9 have acyclic nonbranching optimal transport plans.*

Proof. The proof for $\mathbb{Z}_2$ is straightforward since a transport plan for $g_1 = \cdots = g_{2n} = 1 \in \mathbb{Z}_2$ is obtained for example by the pairing $g_{2i-1,2i} = g_{2i,2i-1} = 1$ for $i = 1, \ldots, n$ and $g_{ij} = 0$ otherwise. Being a disjoint union of edges, the corresponding graph is automatically cycle-free.

The argument for $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is similar. Consider a collection $a_1, \ldots, a_i = (1,1)$, $b_1, \ldots, b_j = (1,0)$, $c_1, \ldots, c_k = (0,1)$ of elements in $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with total sum equal to zero. We may construct a nonbranching optimal transport plan $\{g_{ij}\}$ that verifies equations (NBP) by pairing off elements inside each set $\{a_1, \ldots, a_i\}$, $\{b_1, \ldots, b_j\}$ and $\{c_1, \ldots, c_k\}$ separately. We are then left with the leftover cases where $i, j, k \in \{0, 1\}$. The only nontrivial situation occurs when $i = j = k = 1$ with $a = (1,1), b = (1,0)$ and $c = (0,1)$. In this case the transport plan given by $g_{ab} = (1,0)$, $g_{ac} = (0,1)$ and $g_{bc} = (0,0)$ doesn't produce a cycle.

Finally we assume that $G = \mathbb{R}$. The proof for $G = \mathbb{Z}$ is similar. We prove the statement by induction on the number of elements $g_1, \ldots, g_n \in \mathbb{R}$ with $g_1 + \cdots + g_n = 0$. In case $n = 2$, the problem is trivial. Assume that the statement holds for $n - 1 \geq 2$. Assume that the points $g_1, \ldots, g_n \in \mathbb{R}$ with $g_1 + \cdots + g_n = 0$ are ordered in such a way that $g_i \geq g_{i+1}$. Since all of the points sum to zero, it holds that $g_1 \geq 0 \geq g_n$ and by symmetry we can assume without loss of generality that $g_1 \geq |g_n|$. By induction, we can solve the problem for the points $h_1 := g_1 + g_n, h_2 := g_2, \ldots, h_{n-1} := g_{n-1}$ to obtain a acyclic nonbranching optimal transport plan g_{ij}, $i = 1, \ldots, n-1$. By adding to these elements the elements $-g_{n1} = g_{1n} := -g_n$ and $g_{jn} = g_{nj} := 0$ for $2 \leq j \leq n$ we obtain an acyclic nonbranching optimal transport plan for the original problem. $\square$

The classification of these groups is simplified by first considering the case of only three elements in G. Indeed, in order to obtain the groups in Theorem 1.9 we only have to consider the cases $n = 3$ and $n = 4$ in Definition 1.4.

Lemma 4.2. *If G has acyclic nonbranching optimal transport plans, then G has collinear zero mean triples* (CZT).

Proof. Let $g_1, g_2, g_3 \in G$ with $g_1 + g_2 + g_3 = 0$ and g_{ij} as in Definition 1.4. Since the graph associated with g_{ij} doesn't contain a cylce there is some g_{ij}, say g_{23}, such that $g_{23} = 0$. Then

$$|g_1| = |g_{12}| + |g_{13}| = |g_2 - g_{23}| + |g_3 + g_{23}| = |g_2| + |g_3|.$$

This is precisely what we want. □

In order to classify all the groups with acyclic nonbranching optimal transport plans we first classify the groups with collinear zero mean triples.

4.1. Torsion groups with collinear zero mean triples. We recall that a group G is a *torsion group* if for all $g \in G$ there exists a natural number n such that summing g to itself n times we obtain 0_G, i.e. $ng = 0_G$.

Proposition 4.3 (Classification of torsion groups with (CZT))**.** *The following ones are the only normed Abelian torsion groups that have* (CZT)*, up to rescaling of their norm by a constant factor:*

- $\mathbb{Z}_2$,
- $\mathbb{Z}_4$ *with norm satisfying* $|1| = 1, |2| = 2$,
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *with norm satisfying* $|(1,0)| = 1, |(0,1)| = \alpha, |(1,1)| = 1 + \alpha$ *for any choice of* $\alpha \geq 1$.

Proof. For the groups $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2 \oplus \mathbb{Z}_2$ the determination of the norms satisfying (CZT) follows directly by enumerating the zero mean triples in every case.

Suppose some $\mathbb{Z}_n$ with n odd has some norm $|\cdot|$ with (CZT). Let $a \in \mathbb{Z}_n$ be an element that maximizes $|a|$ and consider the triple a, a, $-2a$ in $\mathbb{Z}_n$. Since n is odd and $a \neq 0$ it must be the case that $|-2a| \neq 0$ and hence none of the inequalities $|a| + |a| = |-2a|$ and $|a| + |-2a| = |a|$ can be satisfied by the maximality of $|a|$.

We next exclude the factors $\mathbb{Z}_n$ for $n = 2^{m+1}, m \geq 2$. In this case, using the (CZT) property we find that $|2| = |1| + |1|$ is the only possible collinearity formula for the triple $1, 1, -2$, and similarly $|2^k| = |2^{k-1}| + |2^{k-1}|$ is the only possible collinearity formula for $2^{k-1}, 2^{k-1}, -2^k$ and $k = 1, \ldots, m$. By induction we find $|2^m| = 2^m|1|$. But as $2(2^m - 1) \equiv -2 \pmod{2^{m+1}}$ we also similarly find $2|2^m - 1| = |2| = 2|1|$ thus the zero mean triple $2^m, 2^m - 1, 1$ has norms proportional to $2^m, 1, 1$, and this contradicts the triangular inequality for $m \geq 2$.

In order to exclude $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, consider $g_1, g_2, g_3 \in G \setminus \{0\}$ such that $|g_1| = \min\{|g| : g \in G \setminus \{0\}\}$, $|g_2| = \min\{|g| : g \in G \setminus \langle g_1 \rangle\}$ and $|g_3| = \min\{|g| : g \in G \setminus \langle g_1, g_2 \rangle\}$. The elements g_1, g_2, g_3 are indecomposable in the sense that whenever g_i, g, h is a zero mean triple, then $|g_i| + |h| = |g|$ or $|g_i| + |g| = |h|$. Since the elements g_1, g_2, g_3 generate G we can express G as the product $\langle g_1 \rangle \oplus \langle g_2 \rangle \oplus \langle g_3 \rangle$ and identify $g_1 = (1, 0, 0)$, $g_2 = (0, 1, 0)$, $g_3 = (0, 0, 1)$. Then the norm of an element (x, y, z), where we chose $\mathbb{Z}$-representatives $x, y, z \in \{0, 1\}$, must be given by $|(x, y, z)| = \alpha x + \beta y + \gamma c$ for some $0 < \alpha \leq \beta \leq \gamma$.

For the collinear triple $(1,1,0),(1,0,1),(0,1,1)$ in particular we have the norms $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$ and we find that the only possible collinearity formula is $2\alpha+\beta+\gamma=\beta+\gamma$, and thus is false. This provides a contradiction to the existence of a norm on $\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ that satisfies (CZT).

As a consequence of the above, the only possible factors in the direct decomposition of G which are still allowed are $\mathbb{Z}_2$, $\mathbb{Z}_4$, and we know that $\mathbb{Z}_2$ can appear at most twice in this product.

It thus remains to exclude the appearance of $\mathbb{Z}_2\oplus\mathbb{Z}_4$ and of $\mathbb{Z}_4\oplus\mathbb{Z}_4$. Suppose that a norm on the group $\mathbb{Z}_2\oplus\mathbb{Z}_4$ had (CZT) and set $\alpha := |(0,2)|$. As the triple $(1,1)$, $(1,1)$, $(0,2)$ has zero mean, we must have by collinearity that $|(1,1)|=\frac{1}{2}|(0,2)|=\frac{1}{2}\alpha$. Similarly $|(0,1)|=\frac{1}{2}\alpha$. Then the triple $(0,1)$, $(1,1)$, $(1,2)$ is of zero mean and thus $|(1,2)|=\frac{1}{2}\alpha+\frac{1}{2}\alpha=\alpha$. But also $(1,0)$, $(0,2)$, $(1,2)$ is of zero mean with $|(0,2)|=|(1,2)|=\alpha$, hence $|(1,0)|=2\alpha$. Finally the zero mean triple $(1,0)$, $(1,1)$, $(0,-1)$ implies the collinearity of the numbers 2α, $\frac{1}{2}\alpha$, $\frac{1}{2}\alpha$, a contradiction. Therefore $\mathbb{Z}_2\oplus\mathbb{Z}_4$ has no norm satisfying (CZT). As $\mathbb{Z}_4\oplus\mathbb{Z}_4$ has a $\mathbb{Z}_2\oplus\mathbb{Z}_4$-subgroup as well, it also has no norm satisfying (CZT). □

4.2. Torsion-free groups with collinear zero mean triples. We say that two normed Abelian groups $(G,|\cdot|_G)$ and $(H,|\cdot|_H)$ are equivalent $(G,|\cdot|_G)\simeq(H,|\cdot|_H)$ if there is a $\lambda>0$ and a group isomorphism $\varphi: G\to H$ such that $|g|_G=\lambda|\varphi(g)|_H$ for all $g\in G$. Also recall that a group is *torsion-free* if there are no $g\in G\setminus\{0_G\}$ and $n\in\mathbb{N}$ with $ng=0_G$. In this subsection we want to prove the following proposition.

Proposition 4.4 (Classification of torsion-free groups with (CZT))**.** *Let $(G,|\cdot|_G)$ be a complete torsion-free normed Abelian group that satisfies* (CZT)*. Then either $G\simeq\mathbb{Z}$ or $G\simeq\mathbb{R}$.*

If not stated otherwise, for the remainder of this subsection $(G,|\cdot|)$ denotes a torsion-free normed Abelian group that has (CZT). As a consequence of the fact that G is torsion-free, for all $g\in G\setminus\{0_G\}$ the subgroup $\langle g\rangle<G$ is isomorphic to $\mathbb{Z}$. We prove first the following:

Lemma 4.5. *Assume that $G=\langle g\rangle$ is an infinite cyclic group and $|\cdot|_G$ is a norm on it. Then $(G,|\cdot|_G)$ has* (CZT) *if and only if for all $n\in\mathbb{N}$ it holds that*

$$|ng|_G = n|g|_G, \tag{4.1}$$

i.e. if and only if $(G,|\cdot|_G)$ is isomorphic to $\mathbb{Z}$ with its Archimedean norm.

Proof. The fact that the group $\mathbb{Z}$ with the usual norm has only zero mean triples which are collinear follows by noting that $a = \pm|a|$ in these cases, and that the zero mean conditions $a + b + c = 0$, for $a, b, c \neq 0$ imply that a, b, c don't have all the same sign.

If $|\cdot|_G$ is a norm on $G \simeq \mathbb{Z}$ for which (CZT) is true and if we have the normalization $|g|_G = 1$, then by using iteratively the condition that the triples $-g$, $-ng$, $(n+1)g$ for $n = 1, 2, \ldots$, are collinear we successively find $|2g|_G = 2$ and for $n \geq 3$ we have $|\pm(n+1)g|_G = n \pm 1$. For $n > 1$ the triple $-2g$, $-(n+1)g$, $(n+1)g$ (and the induction hypothesis $|-(n-1)g|_G = n-1$) shows that $|\pm(n+1)g|_G = n - 1$ is not allowed, thus the only norm with (CZT) is the one satisfying (4.1) as desired. □

For $a, b \in G \setminus \{0_G\}$ we write $a \sim b$ if

$$|a - b| < |a| + |b|.$$

It is clear that $\sim$ is reflexive and symmetric. Next we want to establish that this is indeed an equivalence relation. Note that as a consequence of the triangle inequality, $a \sim b$ if and only if $|a - b| \neq |a| + |b|$.

Lemma 4.6. *Let $a, b, c \in G\setminus\{0_G\}$ and $m, n \in \mathbb{N}$. Then the following properties hold:*

(1) *$a \sim b$ or $a \sim -b$ is satisfied.*
(2) *$a \sim b$ if and only if $ma \sim nb$.*
(3) *If $a \sim b$ and $b \sim c$, then $a \sim c$.*
(4) *Only one of $a \sim b$ or $a \sim -b$ is satisfied.*
(5) *If $a \sim b$, then $a + b \sim a$.*

Proof. We prove (1) by contradiction. We can assume that $a \neq b$, otherwise the statement is trivial. If $a \sim b$ and $a \sim -b$ doesn't hold, the equalities $|a - b| = |a| + |b|$ and $|a + b| = |a| + |b|$ holds and hence $|a - b| = |a + b|$. Consider the zero mean triple $a-b$, $a+b$, $-2a$. Since $|-2a| = |2a| = 2|a| \neq 0$ by Lemma 4.5, none of the equalities $|a-b|+|2a| = |a+b|$ and $|a+b|+|2a| = |a-b|$ hold. Since G has (CZT) we must therefore have that $|a + b| + |a - b| = 2|a|$. But again by our initial assumption we have $|a+b|+|a-b| = 2|a|+2|b| > 2|a|$, as $b \neq 0_G$.

In order prove (2) we first show that $a \sim b$ implies $ma \sim b$ for $m \geq 1$. Note that by Lemma 4.5 it holds that

$$|ma - b| \leq |(m-1)a| + |a - b| < (m-1)|a| + |a| + |b| = m|a| + |b| = |ma| + |b|.$$

By applying the same reasoning to the elements ma and b and some multiplier $n \geq 1$, we find that $a \sim b$ implies $ma \sim nb$ for $m, n \geq 1$. On the other side if

$na \sim mb$, then $nma \sim nmb$ by the first step. And again by Lemma 4.5 setting $k = nm \geq 1$,

$$k|a-b| = |k(a-b)| = |ka-kb| < |ka| + |kb| = k(|a|+|b|).$$

Dividing both sides by k shows (2).

Next, for proving (3) we assume that $a \sim b$ and $b \sim c$ and we desire to prove that $a \sim c$. Because $a \sim b$, it holds that $na \sim b$, i.e. $|na - b| < |na| + |b|$, for all $n \geq 1$ by (2). Since G has (CZT) and $-na$, b, $na - b$ is a zero mean triple, one of the equations $|na-b| + |na| = |b|$ or $|na-b| + |b| = |na|$ must hold. For $n > |b||a|^{-1}$ the first equation can't hold because by the triangle inequality and by Lemma 4.5 we have

$$|b| < 2n|a| - |b| = |na| - |b| + |na| \leq |na - b| + |na|.$$

This similarly applies to the pair (c, b) in place of (a, b). So if n is large enough we therefore have $|na - b| + |b| = |na|$ and $|nc - b| + |b| = |nc|$. Adding these two equalities we obtain by the triangle inequality and Lemma 4.5 that

$$n|c| + n|a| = 2|b| + |na-b| + |nc-b| \geq 2|b| + |na - nc| = 2|b| + n|a-c|.$$

Dividing by n (where n is chosen such that $n > |b| \max\{|a|^{-1}, |c|^{-1}\}$) we obtain that $|a| + |c| > |a - c|$ and therefore $a \sim c$.

(4) is a consequence of (3). Indeed, assume by contradiction that both relations $a \sim b$ and $a \sim -b$ hold. Then it follows from (3) that $b \sim -b$, i.e. $|2b| < |b| + |b|$. But this is not possible since $|2b| = 2|b|$ by Lemma 4.5.

Finally we show (5). If $a \sim b$, then $|a - b| < |a| + |b|$, and by using twice the triangle inequality we find

$$|a+b| + |a| \geq 2|a| + |b| > |a-b| + |a| \geq |b| = |(a+b) - a|,$$

which implies that $a + b \sim a$, as desired. □

The above lemma shows that $\sim$ is an equivalence relation and (1) together with (5) show that $G \setminus \{0_G\}$ is the disjoint union of exactly two equivalence classes ($\{0_G\}$ being the third). Fixing some arbitrary $g_+ \in G \setminus \{0_G\}$, these two classes are given by $G_+ := \{g \in G : g \sim g_+\}$ and $G_- := -G_+$. Consider the map $\varphi : G \to \mathbb{R}$ defined by

$$\varphi(g) := \begin{cases} |g|_G & \text{if } g \in G_+, \\ -|g|_G & \text{if } g \in G_-, \\ 0 & \text{if } g = 0_G. \end{cases}$$

Lemma 4.7. *The map* $\varphi : (G, |\cdot|_G) \to (\mathbb{R}, |\cdot|)$ *is an isometric embedding and a group homomorphism.*

Proof. We will use the notation $|\cdot|$ for the norm $|\cdot|_G$ in the proof, as the only time when the norm on $\mathbb{R}$ intervenes is in the last sentence of the proof. In order to show that φ is a homomorphism we need to show that $\varphi(a+b) = \varphi(a)+\varphi(b)$ and $-\varphi(a) = \varphi(-a)$ for all $a, b \in G$. The second equality is obvious because of Lemma 4.6(4) the relation $a \sim -a$ never holds for $a \neq 0_G$. To prove the first equality we consider only the nontrivial case $a, b \in G \setminus \{0_G\}$. In view of Lemma 4.6(5), if $a \sim b$, then $a + b \sim a \sim b$, and if $a \sim -b$ as well as $a + b \sim a$, then $a + b \sim a \sim -b$. So up to interchanging a and b, either $a + b \sim a \sim b$ or $a + b \sim a \sim -b$. We claim:

(1) $|a + b| = |a| + |b|$ in case $a + b \sim a \sim b$,
(2) $|a + b| = |a| - |b|$ in case $a + b \sim a \sim -b$.

Ad (1): Translating $a + b \sim a$ and $a + b \sim b$ we have $|b| = |(a + b) - a| < |a+b|+|a|$ and similarly $|a| < |a+b|+|b|$. Since G has (CZT) and considering the zero mean triple $a + b$, $-a$, $-b$, one of the following equalities has to hold: $|b| = |a + b| + |a|$, $|a| = |a + b| + |b|$, $|a + b| = |a| + |b|$. Since the first two are excluded we have $|a + b| = |a| + |b|$.

Ad (2): As above we obtain $|b| < |a| + |a + b|$ from $a + b \sim a$ and $|a + b| = |a - (-b)| < |a| + |b|$ from $a \sim -b$. Since G has (CZT) and again considering the triple $a + b$, $-a$, $-b$, we get that $|a| = |b| + |a + b|$.

This shows that φ is a homomorphism. It also follows that φ is an isometric embedding in the sense of metric spaces because $|\varphi(a)| = |a|_G$ by the definition of φ, and $|\varphi(a)-\varphi(b)| = |\varphi(a-b)| = |a-b|_G$ because φ is a homomorphism. □

Recall that a normed Abelian group $(G, |\cdot|)$ is discrete if

$$\inf\{|g| \,:\, g \in G \setminus \{0\}\} > 0.$$

Note that if G is a discrete complete subgroup of $\mathbb{R}$ then G is isomorphic to $\mathbb{Z}$. Indeed if $G \subset \mathbb{R}$ is complete, it is in particular a closed set and there exists an element $g \neq 0$ of smallest norm. If $G \neq \langle g\rangle$, then there would exist some $n \in \mathbb{Z}$ and $g' \in G \setminus \langle g\rangle$ with $|ng - g'| < |g|$ contradict the minimality of $|g|$.

Proof of Proposition 4.4. Because of Lemma 4.7 there is an isometric (in particular injective) homomorphism $\varphi : G \to \mathbb{R}$. If G is not discrete, then $0 \in G$ is an accumulation point and hence the image $\varphi(G)$ is dense in $\mathbb{R}$. Since G is complete, so is $\varphi(G)$ and φ is therefore surjective. This shows that φ is an isometric isomorphism. If G is discrete, then the image of φ must also be discrete and thus G is isomorphic to $\mathbb{Z}$. □

The following example shows that the completeness assumption in Theorem 4.4 is necessary.

Example 4.8. Let $r, s \in \mathbb{R} \setminus \{0\}$ be such that $\frac{r}{s}$ is irrational and define $f : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{R}$ by $f(m,n) = mr + ns$. Now f is an injective homomorphism and the image of f is countable and dense in $\mathbb{R}$. The second statement follows by Hurwitz's theorem, which states that there are infinitely many pairs $(m,n) \in \mathbb{Z} \oplus \mathbb{Z}$ with

$$\left|\frac{m}{n} - \frac{s}{r}\right| < \frac{1}{n^2}.$$

Now the pullback norm $|(m,n)| := |f(m,n)|$ on $\mathbb{Z} \oplus \mathbb{Z}$ has (CZT) but is not complete.

4.3. Conclusion of the classification. Our classification of complete groups with (CZT) is concluded by the following:

Proof of Proposition 1.10. Assume by contradiction that G contains a torsion element $g_T \in G \setminus \{0\}$ and a non-torsion element $g \in G \setminus \{0\}$. By using the classification from Proposition 4.3, and up to taking another element if necessary in case $\langle g_T \rangle \simeq \mathbb{Z}_4$, we can assume that $2g_T = 0$. Together with Proposition 4.4 we obtain that $\langle g_T, g \rangle$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$ and as a subgroup of G we obtain a norm $|\cdot|$ on $\mathbb{Z}_2 \oplus \mathbb{Z}$ that has collinear zero mean triples (CZT).

Like in the proof of Proposition 4.3, for a zero mean triple $a, a, -2a$ with $a, 2a \neq 0$ the only possible collinearity equation is $|a| + |a| = |2a|$. By considering the zero mean triples $(0, 2^k)$, $(0, 2^k)$, $(0, -2^{k+1})$ and $(1, 2^k)$, $(1, 2^k)$, $(0, -2^{k+1})$ we find by induction on k that $2^k|(0,1)| = |(0, 2^k)| = 2^k|(1,1)|$ for all $k \geq 1$. So $(1, 2^k)$, $(0, -2^k)$, $(1, 0)$ form a zero mean triple with norms of the form $2^k\alpha$, $2^k\alpha$, β, where $\alpha := |(1,1)| = |(0,1)|, \beta := |(1,0)|$, which can't be collinear for $2^{k+1} > \frac{\beta}{\alpha}$. Thus $\mathbb{Z}_2 \oplus \mathbb{Z}$ has no norm for which (CZT) holds.

This implies that G is either a torsion group or torsion-free. Both cases have already been classified in Proposition 4.3 and Proposition 4.4. □

With this done we are ready to prove Theorem 1.9, the main theorem of this section.

Proof of Theorem 1.9. Due to Lemma 4.2 and Proposition 1.10, the only possible groups that have acyclic nonbranching optimal transport plans are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{Z}_4$. It is shown in Lemma 4.1 that except for $\mathbb{Z}_4$ all these groups have acyclic nonbranching optimal transport plans. So it remains to exclude $\mathbb{Z}_4$. But $\mathbb{Z}_4$ doesn't have nonbranching optimal transport plans by the classification for discrete groups in Theorem 1.8. □

5. Nonbranching transport and calibrations, and proof of Proposition 1.11

By combining the main result of [11], which gives calibrations for 1-chains with coefficients in $G = \mathbb{Z}_2$, with the classically known calibration/duality available for $G = \mathbb{R}$ and $G = \mathbb{Z}$, we give now a general version of calibrations/Kantorovich duality for groups as in Theorem 1.8.

We note here a fundamental distinction between the theory of calibrations in [11] and other methods appeared in recent works such as [2, 8–10] (papers to which we refer the interested reader for further references in that direction). These notions of calibrations, valid for the case of groups $(\mathbb{Z}^n, |\cdot|)$ or $(\mathbb{R}^n, |\cdot|)$ for some choices of norms, are *essentially using the duality between differential forms and currents*. Such linear duality setting seems to be unable to treat coefficient groups with torsion, as done for $\mathbb{Z}_2$ in [11].

In contrast to previous works, the notion of calibration introduced in [11] and slighthly extended here is *nonlinear*, as it is not based on linear duality applied to the groups of coefficients, but rather connects to *global properties* of minimizers. The present study of branching is parallel to [11], and essentally distinct from previous approaches, again because of the *ability of our framework to treat groups with torsion*. The question of establishing a theory of duality and a notion of global calibrations suited to coefficient groups with torsion, beyond the case $\mathbb{Z}_2$ from [11], or $\mathbb{Z}_2^n$ treated below, remains an interesting open problem.

The basic example from [11, Remark 2.6] indicates that calibrations, i.e. the possibility to re-express the filling problem as a global dual problem defined in terms of maximization among some class of Lipschitz functions, would be prohibited in the cases where the minimum fillings are branched. Note that for the classical branched transport problem, i.e. for the case of the group $(\mathbb{R}, |\cdot|^\alpha)$, $\alpha \in]0, 1[$, the so-called landscape functions, which may be seen as a partial analogue of a calibrations, were introduced in [12, 17]. However, two properties which would be desirable in order to have a global dual problem to the filling problem are missing in that case: First, the fact that only Hölder (and not Lipschitz) regularity holds for the landscape functions indicates that they do not correspond to a true dual variational problem. Second, the fact that a given landscape function is defined in terms of the branched transport minimizers and not in terms of the sources and weights only, indicates that a given landscape function is only a locally dual object, i.e. it will not simultaneously calibrate multiple branched transport minimizers.

Note that for any normed Abelian group G, any geodesic tree T and any $S \in \mathscr{R}_1(T; G)$, it holds that $\partial S = 0$ implies $S = 0$. This follows directly from the homotopy formula for chains, [5, §2.6], and from the fact that $\mathscr{H}^2(f(B)) = 0$ for all Lipschitz maps $f : B \to T$ defined on a Borel set $B \subset \mathbb{R}^2$, which follows for example from [13, Lemma 3.6]. As a consequence, for any $R \in \mathscr{R}_0(T;G)$, the filling length $\mathrm{Fill}_{G,T}(R)$ is achieved by *any* rectifiable filling of R, and in this sense the minimum filling problem in trees is trivialized. We next give a proof of Proposition 1.11 stated in the introduction. It essentially tells that a filling problem with coefficients in $\mathbb{R}^k \oplus \mathbb{Z}^l \oplus \mathbb{Z}_2^k$ can be calibrated by a multivalued map into a tree.

Proof of Proposition 1.11. First note that since we have endowed G with the ℓ_1-norm, we obtain for any choice of f_i and T as in (1.3) and $S \in \mathscr{R}_1(X;G)$ with $\partial S = T$ that

$$\begin{aligned}\mathrm{Fill}_{G,T}\left(\sum_{j=1}^{k+l+m} f_{j\#}(\pi_j R)\right) &\leq \mathbf{M}\left(\sum_{j=1}^{k+l+m} f_{j\#}(\pi_j S)\right)\\ &\leq \sum_{j=1}^{k+l+m} \mathbf{M}(f_{j\#}(\pi_j S))\\ &\leq \sum_{j=1}^{k+l+m} \mathbf{M}(\pi_j S)\\ &= \mathbf{M}(S).\end{aligned}$$

Taking the infimum over all such S, this shows one inequality in (1.3).

To obtain the opposite inequality note that for each $\pi_i R$ we can find a finite geodesic tree T_i and a 1-Lipschitz map $f_i : X \to T_i$ such that $\mathrm{Fill}_{G,X}(\pi_i R) = \mathrm{Fill}_{G,T_i}(f_{i\#}(\pi_i R))$. For $G_i = \mathbb{R}$ or $G_i = \mathbb{Z}$ we can actually take $T_i = \mathbb{R}$, or a closed interval, by Kantorovich-duality (of which a version adapted to the present setting is stated e.g. in [11, Theorem 1.3]). For $G_i = \mathbb{Z}_2$ this follows from the main result [11, Theorem 1.4], respectively, its formulation for chains in [11, Proposition 1.6]. Now one may obtain a finite geodesic tree T by gluing together all the T_i. We can actually manage to glue them to a star-shaped tree such that two different T_i and T_j have enough distance inside T that

$$\mathrm{Fill}_{G,T}\left(\sum_{i=1}^{k+l+m} f_{i\#}(\pi_i R)\right) = \sum_{i=1}^{k+l+m} \mathrm{Fill}_{G,T}\Big(f_{i\#}(\pi_i R)\Big).$$

Hence again using the definition of the ℓ_1-norm on G,

$$\mathrm{Fill}_{G,T}\left(\sum_{i=1}^{k+l+m} f_{i\#}(\pi_i R)\right) = \sum_{i=1}^{k+l+m} \mathrm{Fill}_{G,X}(\pi_i R) = \mathrm{Fill}_{G,X}\left(\sum_{i=1}^{k+l+m} \pi_i R\right) = \mathrm{Fill}_{G,X}(R).$$

This concludes the proof of the proposition. □

There is a partial converse to this statement that generalizes [11, Remark 2.6] and essentially tells that only groups with nonbranching optimal transport plans can be calibrated with maps into trees. In the following Lemma, a tree is a tree T together with a Lipschitz path connected metric d (so we don't assume T to be geodesic). As stated before Proposition 1.11 we get that $\partial S = 0$ implies $S = 0$ in case $S \in \mathscr{R}_1(T; G)$.

Lemma 5.1. *Let G be a normed Abelian group with optimal transport plans. Assume that for any $R = \sum_{i=1}^n g_i [\![x_i]\!] \in \mathscr{R}_0(X; G)$, where X is a geodesic metric space and $\sum_{i=1}^n g_i = 0$, there exists a tree (T, d) and a 1-Lipschitz map $f : X \to T$ such that $\mathrm{Fill}_{G,T}(f_\# R) = \mathrm{Fill}_{G,X}(R)$. Then G has nonbranching optimal transport plans.*

Proof. Consider $g_1, \ldots, g_n \in G \setminus \{0\}$ as in the statement and let (X, d_X) be the geodesic metric space obtained by gluing intervals of length 2 between any two different points of the set $X := \{x_1, \ldots, x_n\}$. Since G has optimal transport plans, there exists $S \in \mathscr{R}_1(X; G)$ with $\partial S = R$ and $\mathbf{M}(S) = \mathrm{Fill}_{G_X}(R)$. With the discussion following Definition 1.6, $S = \sum_{i<j} g_{ij} [\![x_j, x_i]\!]$, where $g_{ij} = -g_{ji}$, $g_{ii} = 0$ and $g_i = \sum_j g_{ij}$. Then

$$
\begin{aligned}
\mathrm{Fill}_{G,X}(R) &= \mathbf{M}(S) \\
&= \sum_{1 \le i < j \le n} |g_{ij}| \ \mathrm{length}([x_j, x_i]) \\
&\ge \sum_{1 \le i < j \le n} |g_{ij}| \ \mathrm{length}([f(x_j), f(x_i)]) \\
&\ge \mathbf{M}(f_\# S) \\
&= \mathrm{Fill}_{G,T}(f_\# R).
\end{aligned}
$$

By assumption, equalities hold and hence f maps each segment $[x_j, x_i]$ injectively and length preserving into T. Because T is uniquely arcwise connected, there is a unique point $x \in T$ such that $x \in [f(x_j), f(x_i)]$ for different i, j and $\mathrm{length}([f(x_j), x]) = 1$. Thus

$$
\sum_{i=1}^n \sum_{j=1}^n |g_{ij}| = \mathbf{M}(S) = \mathbf{M}(f_\# S) = \sum_{i=1}^n |g_i| \ \mathrm{length}([x, f(x_i)]) = \sum_{i=1}^n |g_i|.
$$

Since $|g_i| \le \sum_{j=1}^n |g_{ij}|$ by the triangle inequality, this implies that $|g_i| = \sum_{j=1}^n |g_{ij}|$ for all i. Hence G has nonbranching optimal transport plans. □

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